EXISTENCE OF PARTIALLY REGULAR SOLUTIONS FOR LANDAU-LIFSHITZ EQUATIONS IN $\mathbb{R}^3$

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IMA Preprint Series # 1945

(November 2003)

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EXISTENCE OF PARTIALLY REGULAR SOLUTIONS FOR LANDAU-LIFSHITZ EQUATIONS IN $\mathbb{R}^3$

CHRISTOF MELCHER

ABSTRACT. We establish existence of partially regular weak solutions for the Landau-Lifshitz equation in three space dimensions for smooth initial data of finite Dirichlet energy. The construction is based on Ginzburg-Landau approximation. The new key ingredient is a nonlocal representation formula for the penalty term that permits to take advantage of the special trilinear structure of the limiting nonlinearity.

Revised Version March 2004

1. INTRODUCTION

The Landau-Lifshitz equation is the basic evolution equation for spin fields in the continuum theory of ferromagnetism as formulated in [18]. In the most simple case, when the spin interaction is modeled on the Dirichlet energy $E(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2$ of the magnetic moment, represented by a direction field u, the Landau-Lifshitz equation can be considered as a hybrid heat and Schrödinger flow for harmonic maps into the unit sphere $\mathbb{S}^2$

$$\alpha\, \partial_t u + \beta\, u \wedge \partial_t u = \Delta u + |\nabla u|^2 u,$$

where $\wedge$ denotes the vector product in $\mathbb{R}^3$. The equation is parabolic when the damping factor α is positive, which is our main assumption. We consider the Cauchy problem for Landau-Lifshitz in case of the three space dimensions. A standard device to construct weak solution is an approximation procedure based on replacing the Dirichlet energy by the Ginzburg-Landau energy $E_\varepsilon(u) = \int_{\mathbb{R}^3} e_\varepsilon(u)\, dx$ with density

$$e_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\big(1-|u|^2\big)^2. \tag{1}$$

Accordingly, the associated approximation for Landau-Lifshitz is given by

$$\alpha\, \partial_t u + \beta\, u \wedge \partial_t u + \nabla E_\varepsilon(u) = 0, \tag{2}$$

where $\nabla E_\varepsilon(u) = -\Delta u - V_\varepsilon(u)u$ is the L^2 gradient. The idea is that, as ε tends to zero, the penalty term $V_\varepsilon(u)u = 1/\varepsilon^2\big(1-|u|^2\big)u$ forces the unconstrained solutions to approach the target manifold $\mathbb{S}^2$. We consider the parameter $\varepsilon > 0$ as a length scale (i.e. it scales with x) which makes equation (2) invariant under parabolic rescaling.

It was shown by Alouges and Soyeur, cf. [1], that the latter approximation yields global weak solutions for Landau-Lifshitz. We show that a sequence of weak solutions for the approximate equation converges with some uniform decay estimate for some suitable energy

I would like to thank Vladimir Šverák for many elucidating discussions on regularity questions and for important suggestions and comments on the subject matter.

average away from a closed set of locally finite 3-dimensional parabolic Hausdorff measure. As a consequence, a weak limit will be partially regular. This is in the same spirit as the existence results for partially regular solutions for the harmonic map heat flow in higher dimensions of Chen and Struwe, cf. [9]. Their argument mainly relies on a monotonicity formula for a certain local energy average in space-time, that, however, is not available in case of Landau-Lifshitz dynamics.

Here, the argument is along the lines of a recent approach of Moser, cf. [19], that is rather based on a spatial substitute for Struwe's monotonicity formula in [23]. Accordingly, Moser proved partial regularity for weak solutions of Landau-Lifshitz equations in small space dimensions that fulfill a certain stability condition, which is an analog of the well-known stationarity condition for weakly harmonic maps. In the context of evolution problems, this notion was first introduced by Feldman [10]. It is not known, however, whether in the dynamic context and for sufficiently regular initial data the Cauchy problem admits solutions in this class or whether partially regular solutions exist at all. This is the question we are concerned with.

The main strategy in [19], see also [20], is to exploit the elliptic problem on suitable time slices where the time derivative is square-integrable in $\mathbb{R}^N$, a term that is (sub-)critical when $N \leq 4$. To cope with the nonlinearity the crucial technical tool is a trilinear Sobolev type estimate due to Feldman, cf. [10], that involves a critical Morrey norm of the spatial gradient. In [19] the requisite estimate arises as a consequence a of a spatial monotonicity type formula that we mentioned above. The crucial new ingredient that makes this strategy work in the approximate context is a nonlocal representation formula for the Ginzburg-Landau penalty term. This formula permits to take advantage of the special trilinear structure of the limiting nonlinearity and yields several distributional estimates for the penalty term. Finally, we derive, like in [19], a decay estimate for the scale invariant local average of Ginzburg-Landau densities or large scales but in an ε-uniform fashion, that eventually implies partial regularity for weak limits.

2. Construction of partially regularity solutions

2.1. The existence result.

Theorem. *For smooth initial data $u_0 : \mathbb{R}^3 \to \mathbb{S}^2$ such that $\nabla u_0 \in L^2(\mathbb{R}^3)$ there exists a global weak solution u for the Landau-Lifshitz equation*

$$\alpha\, \partial_t u + \beta\, u \wedge \partial_t u = \Delta u + |\nabla u|^2\, u$$

that is smooth away from a closed set of locally finite 3-dimensional parabolic Hausdorff measure.

Recall that the parabolic Hausdorff measures arise from the standard notion of Hausdorff measure when the underlying metric d is the parabolic metric

$$d\big((x,t),(y,s)\big) = \max\left\{|x-y|, |t-s|^{1/2}\right\}.$$

In this note, a global weak solution is a measurable map $u : \mathbb{R}^3 \times (0,\infty) \to \mathbb{S}^2$ such that $\partial_t u \in L^2_{loc}(\mathbb{R}^3 \times (0,\infty))$ and $\nabla u \in L^\infty_t L^2_x(\mathbb{R}^3 \times (0,\infty))$ and

$$\int (\alpha\, \partial_t u + \beta\, u \wedge \partial_t u) \cdot \phi + \int \nabla u \cdot \nabla \phi = \int |\nabla u|^2\, u \cdot \phi \tag{3}$$

for every $\phi \in C_0^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)$, and the initial data u_0 is weakly attained.

A well-known construction due to Rivière, cf. [21], shows that partial regularity for weak solutions cannot hold in general. As mentioned in the introduction, for space dimension three and four, however, Moser showed that every weak solution that exhibits a certain stability condition is partially regular, cf. [19]. Our theorem on the other hand tells that, from the viewpoint of existence, the Cauchy problem is well posed in the class of partially regular maps. Let us point out, however, that the present argument does not permit to establish uniqueness of weak solutions in the class of partially regular maps. We also remark, that, unlike in [19], the argument we will use does not imply the analog result in four space dimensions, cf. Section 3.3.

Recall that stronger partial regularity results hold in case of two space dimensions. In fact, for smooth initial data, any weak solution for Landau-Lifshitz that satisfies the energy inequality, is smooth except at finitely many points in space-time where energy concentrates as shown by Chen, Ding, and Guo [4, 5], Ding and Guo [14, 15], and Harpes [17]. Indeed, the well-known arguments for partial regularity for the harmonic map heat flow, i.e. Struwe's construction [22] and Freire's uniqueness result [12], carry over.

2.2. **Ginzburg-Landau approximation.** For notational convenience we introduce, for any $\varepsilon > 0$, the potential

$$V_\varepsilon(u) = \frac{1}{\varepsilon^2}\left(1 - |u|^2\right),$$

with the convention that $V(u) = V_1(u)$. Also let $R(u) \in \mathbb{R}^{3\times 3}$ denote the linear mapping given by $\xi \mapsto R(u)\xi = \alpha\,\xi + \beta\, u \wedge \xi$, where $\alpha > 0$ and $\alpha^2 + \beta^2 = 1$. Notice that this situation can always be reached by rescaling time suitably. Moreover, if $|u| = 1$ then $R(u)$ is a rotation. Then the approximate Landau-Lifshitz equation can be written as

$$R(u)\partial_t u = \Delta u + V_\varepsilon(u)\, u. \tag{4}$$

Galerkin's method provides a global weak solution. Moreover, a simple Maximum principle type argument shows that we have an a priori bound on the modulus $|u| \leq 1$, and in particular $V_\varepsilon(u) \geq 0$, for any such weak solution of (4), cf. [1]. The energy inequality implies an L^2 bound for $\partial_t u$, and standard parabolic estimates imply a local L^2 bound for $D_x^2 u$. We summarize:

Proposition 1. *For any $\varepsilon > 0$ and for smooth initial data $u_0 : \mathbb{R}^3 \to \mathbb{S}^2$ with $\nabla u_0 \in L^2(\mathbb{R}^3)$ equation (4) has a global weak solution $u = u^{(\varepsilon)}$ such that $\partial_t u$ and $D_x^2 u \in L^2_{loc}(\mathbb{R}^3 \times (0,\infty))$, $\nabla u \in L_t^\infty L_x^2(\mathbb{R}^3 \times (0,\infty))$, and $|u| \leq 1$ almost everywhere.*

Following the arguments in [1], one easily passes to the limit $\varepsilon \to 0$. Indeed, for smooth initial data $u_0 : \mathbb{R}^3 \to \mathbb{S}^2$ of finite Dirichlet energy, the energy inequality implies a uniform local H^1 bound in space-time for such solutions $u = u^{(\varepsilon)}$. Observe that $u \wedge R(u)\,\partial_t u = \nabla \cdot \big(u \wedge \nabla u\big)$ for any $\varepsilon > 0$, while $|u|^2 \to 1$ locally in L^2. Thus, the identity $(u \wedge \nabla u) \cdot \nabla(u \wedge \phi) = \nabla u \cdot \nabla \phi - |\nabla u|^2 u \cdot \phi$, that holds true for $u \in H^1$ with $|u| = 1$, ensures that a weak limit satisfies the Landau-Lifshitz equation in the sense of (3).

2.3. **Uniform small energy regularity.**

Notation. We denote by $B_r(x)$ the ball of radius r centered at $x \in \mathbb{R}^3$ in space. Let $z = (x,t) \in \mathbb{R}^3 \times (0,\infty)$ denote space-time variables. For parabolic cylinders $P_r(z) = B_r(x) \times (t-r^2, t)$ in $\mathbb{R}^3 \times (0,\infty)$ and locally integrable maps u, $(u)_{P_r} = \fint_{P_r(z)} u$ denotes the average. For maps $u \in H^1\big(P_r(z)\big)$ we introduce the following local average

$$\mathcal{E} = \mathcal{E}(u,r,z,\varepsilon) = r^2 \fint_{P_r(z)} e_\varepsilon(u)$$

of Ginburg-Landau densities

$$e_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\big(1-|u|^2\big)^2.$$

Notice that $\mathcal{E}$ is invariant under parabolic rescaling when ε is considered as a length scale. The main goal is to prove the following ε-uniform energy decay result for weak solutions $u = u^{(\varepsilon)}$ in the regime of large radii $\rho \gtrsim \varepsilon$ that is sufficient to establish regularity for a weak limit.

Theorem. *For every Hölder exponent $\gamma \in (0,1)$ there exist a universal constant $C(\alpha,\gamma) > 0$, an aspect ratio $\kappa > 0$, and an energy threshold $\mathcal{E}_0 > 0$ such that, whenever $\mathcal{E}(u,r,z,\varepsilon) < \mathcal{E}_0$, then for every $z \in P_{r/2}(z_0)$ and $\varepsilon/\kappa < \rho < r/2$ the estimate*

$$\rho^2 \fint_{P_\rho(z)} e_\varepsilon(u) + \rho^4 \fint_{P_\rho(z)} |\partial_t u|^2 \le C(\alpha,\gamma)\,\left(\frac{\rho}{r}\right)^{2\gamma} \mathcal{E}_0$$

holds true for suitable weak solutions $u = u^{(\varepsilon)}$ of the approximate Landau-Lifshitz equation.

The statement is a consequence of a decay estimate for $\mathcal{E}(u,r,z,\varepsilon)$, proved in Section 5, Proposition 3, and the local energy inequality in Section 3.1, Lemma 2.

Proof of Theorem 1. Let $u_k = u^{(\varepsilon_k)}$ be a sequence of suitable weak solutions for approximate Landau-Lifshitz equations as in Proposition 1 such that $\varepsilon_k \downarrow 0$ and with weak local H^1 limit u that weakly solves the full Landau-Lifshitz equation. We introduce the singular set

$$\Sigma = \bigcap_{r>0} \left\{ z \in \mathbb{R}^3 \times (0,\infty) : \liminf_{k\to\infty} \mathcal{E}\left(u_k, r, z, \varepsilon_k\right) \ge \mathcal{E}_0 \right\}.$$

A standard covering argument, cf. e.g. [13], shows that Σ has locally finite 3-dimensional parabolic Hausdorff measure. Invoking Theorem 2.3, one easily shows that Σ has an open complement. We fix some $\gamma > 1/2$. By weak lower semicontinuity we find that for any $z_0 \notin \Sigma$ there is some radius $r > 0$ such that

$$\rho^2 \fint_{P_\rho(z)} |\nabla u|^2 + \rho^4 \fint_{P_\rho(z)} |\partial_t u|^2 \le C(\alpha,\gamma)\,\left(\frac{\rho}{r}\right)^{2\gamma} \mathcal{E}_0$$

for any $z \in P_{r/2}(z_0)$ and $0 < \rho < r/2$. Observe that, by means of the parabolic version of Morrey's lemma, cf. [8], this implies parabolic γ-Hölder continuity of u in $P_{r/2}(z_0)$. As pointed out in [19], the latter decay estimate implies in turn, for Hölder continuous u, a local bound for the spatial gradient ∇u away from Σ. Indeed, Feldman's argument in [10] Lemma 21 can be adapted by utilizing estimates for fundamental solutions for general parabolic systems with Hölder continuous coefficients as constructed in [11]. Higher regularity eventually follows from a bootstrap argument.

3. Preliminary estimates

3.1. **Monotonicity and energy inequality.** In [19] the following monotonicity type formula (Lemma 1) and local energy inequality (Lemma 2) for the full Landau-Lifshitz equation are a consequence of a certain stability condition that was primarily introduced by Feldman, cf. [10], in the context of harmonic map heat flows. For the approximate problem we can, in view of the regularity assertion in Proposition 1, derive them by means of suitable multipliers.

The function $x \cdot \nabla u$ is an admissible multiplier for (2) on almost every time slice $\mathbb{R}^3 \times \{t\}$. We deduce a Pohozaev-type identity

$$\nabla \cdot \Big(x \cdot T^\varepsilon[u] \Big) = \text{trace}\, T^\varepsilon[u] - \big(x \cdot \nabla u \big)\, R(u)\, \partial_t u$$

where $T^\varepsilon_{ij}[u] = e_\varepsilon(u)\, \delta_{ij} - \partial_i u \cdot \partial_j u$ denotes the energy-momentum tensor. Consequently, if $t > 0$ is such that $u(t) \in L^2(\mathbb{R}^3)$, then integration over $B_r(0)$ and an approximation argument shows that

$$\frac{d}{dr}\Big(r^2 \fint_{B_r(0)} e_\varepsilon\big(u(t)\big) \Big) \geq -r^2 \fint_{B_r(0)} |\partial_t u(t)||\nabla u(t)|.$$

By means of Young's inequality and the monotonicity property of the function $r \mapsto r^4 - \int_{B_r(0)} |\partial_t u(t)|^2$, which holds true in space dimension four and less, we find the following:

Lemma 1. *There is a universal constant $C > 0$ such that, for every $\rho < r$ and for almost every $t > 0$, the following estimate holds true*

$$\rho^2 \fint_{B_\rho(x)} e_\varepsilon(u(t)) \leq C\, r^2 \fint_{B_r(x)} e_\varepsilon(u(t)) + C\, r^4 \fint_{B_r(x)} |\partial_t u(t)|^2.$$

Likewise, multiplying (2) by the admissible test function $\varphi^2 \partial_t u$, for any cut-off function $\varphi \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))$, yields, after integration by parts and using Young's inequality, the following local energy inequality:

Lemma 2. *There is a universal constant $C > 0$ such that for almost every $t > 0$ the following estimate holds true*

$$\frac{d}{dt} \int_{\mathbb{R}^3} e_\varepsilon(u(t))\, \varphi^2 + \frac{\alpha}{2} \int_{\mathbb{R}^3} |\partial_t u(t)|^2 \varphi^2 \leq \frac{C}{\alpha} \int_{\mathbb{R}^3} e_\varepsilon(u(t)) \Big(|\nabla \varphi|^2 + |\partial_t \varphi^2| \Big).$$

In particular, there is a universal constant $C(\alpha) > 0$ such that for any $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times (0,\infty)$ and $r > 0$ such that $P_r(z_0) \subset \mathbb{R}^3 \times (0,\infty)$ and $t_0 - (r/2)^2 \leq t \leq t_0$

$$\fint_{B_{r/2}(x_0)} e_\varepsilon(u(t)) + r^2 \fint_{P_{r/2}(z_0)} |\partial_t u|^2 + \leq C(\alpha) \fint_{P_r(z_0)} e_\varepsilon(u).$$

3.2. **Suitable time slices.** The following trick is used in [19] for the handling of suitable time slices. Since $\partial_t u$ is square integrable over $P_r(z_0)$ we can, for given volume fraction $\lambda \in (0,1)$, decompose the interval $\big(t_0 - r^2, t_0\big)$ into two disjoint measurable subsets S and Λ where the good set

$$S = \Big\{ t \in \big(t_0 - r^2, t_0\big) : \lambda \fint_{B_r(x_0)} |\partial_t u(t)|^2 \leq \fint_{P_r(z_0)} |\partial_t u|^2 \Big\}$$

indicates suitable time slices corresponding to a volume fraction λ. Notice that $|\Lambda| \leq \lambda\, r^2$. Hence we obtain from Lemma 2

$$r^2 \fint_{B_{r/2}(x_0)} |\partial_t u(t)|^2 \leq \frac{C(\alpha)}{\lambda} \fint_{P_r(z_0)} e_\varepsilon(u) \quad \text{whenever } \ t \in S \tag{5}$$

for some universal constant $C(\alpha) > 0$. Then, invoking Lemma 1, we finally obtain the following estimate for the scale invariant spatial energy average:

Lemma 3. *There is a universal constant $C(\alpha) > 0$ such that, on a suitable time slice corresponding to the volume fraction $\lambda \in (0,1)$ within the parabolic cylinder $P_r(z_0)$, the estimate*

$$\rho^2 \fint_{B_\rho(x)} e_\varepsilon(u(t)) \leq \frac{C(\alpha)}{\lambda}\, r^2 \fint_{P_r(z_0)} e_\varepsilon(u)$$

holds true for every $x \in B_{r/4}(x_0)$ and $0 < \rho < r/4$.

3.3. **Pointwise estimate on the modulus.** We will need a pointwise estimate for the deviation of approximate solutions u from the target $\mathbb{S}^2$ in terms of the local energy average. For this purpose we will make use of the estimate below, which follows from a slight modification of the arguments in the proof of Lemma III.3. in [2], that we now recall:

Lemma 4. *Suppose that $u \in C^{1/2}(B_\varepsilon(x))$ with $|u| \leq 1$ and $[u]_{C^{1/2}} \leq M/\sqrt{\varepsilon}$, then there is a universal constant $c(M) > 0$ only depending on M such that*

$$\big(1 - |u(x)|^2\big)^5 \leq c(M) \fint_{B_\varepsilon(x)} \big(1 - |u|^2\big)^2.$$

Proof. Since $|u(x)| \leq 1$, it is obviously sufficient to derive an analog bound for the quantity $1 - |u(x)|$: Let $\gamma = \frac{\varepsilon}{4M^2}\big(1 - |u(x)|\big)^2$. Then we have for any $y \in B_\gamma \cap B_\varepsilon(x)$

$$|u(y)| - |u(x)| \leq [u]_{C^{1/2}}\, |x-y|^{1/2} \leq \frac{M}{\sqrt{\varepsilon}} |y-x|^{1/2} \leq \frac{1}{2}(1 - |u(x)|).$$

Thus we infer that $|u(y)| \leq \frac{1}{2}\big(1 + |u(x)|\big)$ for any $y \in B_\gamma \cap B_\varepsilon(x)$.

In case $\gamma < \varepsilon$ we have $\int_{B_\gamma(x)} \big(1 - |u|^2\big)^2 \leq \int_{B_\varepsilon(x)} \big(1 - |u|^2\big)^2$ and therefore

$$\int_{B_\varepsilon(x)} \big(1 - |u|^2\big)^2 \geq \int_{B_\gamma(x)} \big(1 - |u|\big)^2 \geq \frac{1}{4}\, |B_\gamma|\, \big(1 - |u(x)|\big)^2,$$

and the claim follows from the above choice of γ.

If otherwise $\varepsilon \leq \gamma$, then $|u(y)| \leq \frac{1}{2}\big(1 + |u(x)|\big)$ for all $y \in B_\varepsilon(x)$, thus

$$\int_{B_\varepsilon(x)} \big(1 - |u|^2\big)^2 \geq \int_{B_\varepsilon(x)} \big(1 - |u|\big)^2 \geq \frac{1}{4}\, |B_\varepsilon|\, \big(1 - |u(x)|\big)^2.$$

□

Let us now consider suitable weak solution $u = u^{(\varepsilon)}$ of (4) defined on $P_r(z_0) \subset \mathbb{R}^3 \times (0,\infty)$. First, we derive spatial Hölder estimates on suitable time slices $B_r(x_0) \times \{t\}$, where the time derivative is square integrable.

Lemma 5. *Suppose that $0 < \varepsilon < r/4$. Then, on a suitable time slice within $P_r(z_0)$, the map $x \mapsto u(x,t)$ is uniformly Hölder continuous in $B_{r/4}(x_0)$ with*

$$\big[u(t)\big]_{C^{1/2}} \leq M/\sqrt{\varepsilon} \quad \textit{in} \quad B_{r/4}(x_0),$$

where the constant $M > 0$ only depends on the average $r^4 \fint_{B_{r/2}(x_0)} |\partial_t u(t)|^2$.

Proof. We fix any $x \in B_{r/4}(x_0)$ and a suitable time t. Then by assumption $B_\varepsilon(x) \subset B_{r/2}(x_0)$. Since u is uniformly bounded, it is sufficient to show that $[u]_{C^{1/2}} \leq M/\sqrt{\varepsilon}$ in $B_{\varepsilon/2}(x)$, where M is independent of x and ε. After translation we may assume that $(x,t) = (0,0)$. Then the rescaled map $v(y,s) = u(\varepsilon y, \varepsilon^2 s)$ defined on $B_1(0) \times I(0)$, for some sufficiently small time interval $I(0)$, satisfies, at $s = 0$, the estimate

$$\int_{B_1(0)} |\partial_s v|^2 \, dy = \varepsilon \int_{B_\varepsilon(0)} |\partial_t u|^2 \leq c\, r^4 \fint_{B_r(x_0)} |\partial_t u|^2. \tag{6}$$

Moreover, the map $x \mapsto v(x,0)$ is a weak solution of a Poisson equation

$$\Delta\, v = R(v)\,\partial_s v - V(v)\, v \quad \text{in} \quad B_1(0) \times \{0\}$$

with square integrable right hand side, having an L^2 estimate like (6) plus a constant. It follows from Friedrich's lemma and Sobolev embedding that v has a uniform (exponent $1/2$) Hölder bound M in $B_{1/2}(0)$, and the proof follows after rescaling. □

Remark 1. *Notice that the above elliptic estimate becomes critical for higher space dimensions, and Hölder estimates might fail in that case.*

Recall that on a suitable time slice corresponding to a volume fraction $\lambda \in (0,1)$, the energy inequality (Lemma 2) implies the bound

$$r^4 \fint_{B_{r/2}(x_0)} |\partial_t u(t)|^2 \leq \frac{C(\alpha)}{\lambda}\, r^2 \fint_{P_r(z_0)} e_\varepsilon(u).$$

Thus, Lemma 5 shows that Lemma 4 applies in the present context, since the above expression will remain uniformly bounded by our later choice of parameters. In order to utilize Lemma 4 we observe that in case $\varepsilon < r/4$

$$\fint_{B_\varepsilon(x)} \left(1 - |u|^2\right)^2 = \varepsilon^2 \fint_{B_\varepsilon(x)} \frac{1}{\varepsilon^2}\left(1 - |u|^2\right)^2 \leq \sup_{0<\rho<r/4} \rho^2 \fint_{B_\rho(x)} e_\varepsilon(u).$$

In combination with Lemma 3 we infer the following pointwise estimate:

Proposition 2. *For every $\alpha > 0$ there is a continuous non-decreasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ such that for $0 < \varepsilon < r/4$ and on a suitable time slice corresponding to a volume fraction $\lambda \in (0,1)$*

$$1 - |u(x,t)|^2 \leq \omega\Big(\mathcal{E}(u,r,z_0,\varepsilon)/\lambda\Big) \quad \textit{for every} \quad x \in B_{r/4}(x_0).$$

4. Representation formula and trilinear estimate

The goal is to represent the nonlinearity $V_\varepsilon(u)\, u$, that arises from the Ginzburg-Landau approximation, on a suitable time slice in terms of the well-known anti-symmetric bilinear mapping

$$b^{ij}_\alpha(u,p) = u^i p^j_\alpha - u^j p^i_\alpha \tag{7}$$

and some error terms. For this purpose we introduce the singularly perturbed but uniformly invertible elliptic operator $\mathcal{L}_\varepsilon = \mathbf{1} - \frac{1}{2}\,\varepsilon^2\,\Delta$. Notice the uniform spatial L^2 boundedness of the inverse $\|\mathcal{L}_\varepsilon^{-1} v\|_{L^2(\mathbb{R}^3)} \leq \|v\|_{L^2(\mathbb{R}^3)}$ by means of Plancherel. Basic manipulations yield the following representation formula for $V_\varepsilon(u)\, u$:

Lemma 6. *For a sufficiently regular solution $u = u^{(\varepsilon)}$ of (4) the identity*

$$\mathcal{L}_\varepsilon\Big(V_\varepsilon(u)\,u\Big) = b(u,\nabla u)\nabla u - \nabla\cdot\left(\frac{3}{2}\big(1-|u|^2\big)\nabla u\right) + A(u)\,\partial_t u \tag{8}$$

$$\textit{where}\quad \mathcal{L}_\varepsilon = \mathbf{1} - \frac{1}{2}\,\varepsilon^2\,\Delta \quad\textit{and}\quad A(u) = \big(1-|u|^2\big)\ R(u) + \alpha\, u\otimes u,$$

holds true in the sense of distributions.

Proof. Since (8) is scale invariant we may assume that $\varepsilon = 1$. We have

$$\Delta\big(V(u)\,u\big) = V(u)\,\Delta u - 4(u\cdot\nabla u)\nabla u - 2|\nabla u|^2 u - 2(u\cdot\Delta u)\,u,$$

where $V(u) = V_1(u)$. On the other hand, invoking the equation, we have

$$\begin{aligned} 2(u\cdot\nabla u)\,\nabla u &= -\nabla\cdot\big[\big(1-|u|^2\big)\nabla u\big] - |V(u)|^2 u + V(u)\,R(u)\,\partial_t u,\\ (u\cdot\Delta u)\,u &= |V(u)|^2 u - V(u)\,u + \alpha\,(u\cdot\partial_t u)\,u,\\ V(u)\,\Delta u &= V(u)\,R(u)\,\partial_t u - |V(u)|^2 u,\\ |\nabla u|^2 u &= b(u,\nabla u)\nabla u + (u\cdot\nabla u)\nabla u. \end{aligned}$$

With the last identity the above expansion for $\Delta\big(V(u)\,u\big)$ becomes

$$V(u)\,\Delta u - 2\,b(u,\nabla u)\nabla u - 6\,(u\cdot\nabla u)\nabla u - 2\,(u\cdot\Delta u)\,u.$$

Substituting the other identities yields the formula to be proved. □

Obviously, the Lemma can be applied on suitable time slices as well, and the formula holds in the sense of spatial distributions at a given time when $\partial_t u$ is locally integrable. For space-time arguments, however, we introduce the singularly perturbed parabolic operator $\mathcal{P}_\varepsilon = \mathbf{1} + \frac{1}{2}\,\varepsilon^2\big(\alpha\partial_t - \Delta\big)$. Then, in view of the identity $2\,(u\cdot\partial_t u)\ u = \big(1-|u|^2\big)\ \partial_t u - \varepsilon^2\,\partial_t\,(V_\varepsilon(u)\,u)$, formula (8) can be written in a more compact form, and we obtain a parabolic version of the Lemma.

Lemma 7. *The following identity holds true in the sense of distributions:*

$$\mathcal{P}_\varepsilon\Big(V_\varepsilon(u)\,u\Big) = b(u,\nabla u)\nabla u - \frac{3}{2}\nabla\cdot\Big(V(u)\nabla u\Big) + V(u)\ \left(R(u) + \frac{\alpha}{2}\right)\ \partial_t u.$$

Remark 2. *With the same notation as above, the corresponding scalar identity*

$$\mathcal{P}_\varepsilon V_\varepsilon(u) = |\nabla u|^2 + \frac{1}{\varepsilon^2}\big(1-|u|^2\big)^2$$

holds true in the sense of distributions. In case of heat flows, i.e. if $\beta = 0$, and in the elliptic context, analog identities were used before, see e.g. [8], [7], *mainly in connection with the elliptic or parabolic maximum principle.*

One major problem in using Ginzburg-Landau approximation is to find suitable (local) estimates for the nonlinearity $V_\varepsilon(u)\,u$ in terms of the (local) energy. The representation formula and local L^2 bounds for time derivatives, cf. Lemma 2, provide us with a distributional bound that holds uniformly on large scales:

Lemma 8. *In the regime $\varepsilon/r \le \kappa$ the following estimate holds true for some universal constant $C > 0$ and for every test function $\phi \in C_0^\infty(P_{r/2}(z))$*

$$|\langle \mathcal{P}_\varepsilon^*\phi, V_\varepsilon(u)\,u\rangle| \le C\left((1+\kappa)\,\|\phi\|_{L^\infty} + \varepsilon\,\|\nabla\phi\|_{L^\infty}\right)\ \int_{P_r(z)} e_\varepsilon(u).$$

This estimate proves helpful when a blow-up analysis is being performed. The main purpose of our representation formula, however, is to make use of the following trilinear estimate due to Feldman [10].

Lemma 9. *Suppose that* $f \in H^1(\mathbb{R}^N)$*,* $g \in L^2(\mathbb{R}^N;\mathbb{R}^N)$*, and* $h \in H^1(\mathbb{R}^N)$ *such that* $\nabla \cdot g \in L^2(\mathbb{R}^N)$ *and* h *satisfies*

$$\sup_{B_r} r^2 \fint_{B_r} |\nabla h|^2 \le A^2 < \infty,$$

then

$$|\int f\, g \cdot \nabla h| \le C\, A\Big(\|\nabla f\|_{L^2}\|g\|_{L^2} + \|f\|_{L^2}\|\nabla \cdot g\|_{L^2}\Big).$$

Following [10, 19] this can now be applied in the present approximate context to estimate the leading term in formula (8): Suppose u is a suitable weak solution of (4) on $P = P_1(0)$. Recall that from Lemma 3

$$r^2 \fint_{B_r(x)} |\nabla u(t)|^2 \le \frac{C(\alpha)}{\lambda} \int_P e_\varepsilon(u) \quad \text{for} \quad x \in B_{1/4}(0) \quad \text{and} \quad r \in (0, 1/4).$$

By reflection and after suitably cutting off at infinity we find extensions $\bar{u} = \bar{u}(t) \in H^1(\mathbb{R}^3;\mathbb{R}^3)$ with $\bar{u} = u$ on $B_{1/4}(0)$ that have the requisite critical Morrey space bound on suitable time slices

$$r^2 \fint_{B_r(x)} |\nabla \bar{u}|^2 \le \frac{C(\alpha)}{\lambda} \int_P e_\varepsilon(u) \quad \text{for every} \quad x \in \mathbb{R}^3 \quad \text{and} \quad r > 0$$

with an appropriately modified but uniform constant $C(\alpha)$ Moreover, in view of (4), we observe that the identity

$$\nabla \cdot b(u, \nabla u) = u \otimes R(u)\, \partial_t u - R(u)\, \partial_t u \otimes u \tag{9}$$

holds in the sense of spatial distributions for almost every $t \in (-1, 0)$. We fix some cut-off functions $\zeta \in C_0^\infty(B_{1/4}(0))$ with $\zeta = 1$ on $B_{1/6}(0)$. Since $|u| \le 1$ and $\partial_t u$ is square integrable on suitable time slices, we obtain from (5) and Lemma 2 that

$$\big\|\nabla \cdot \big[b(u, \nabla u)\, \zeta\big]\big\|_{L^2}^2 \le C\, \|\zeta\, \partial_t u\|_{L^2}^2 + C\, \|\nabla \zeta\, \nabla u\|_{L^2}^2 \le \frac{C(\alpha)}{\lambda} \int_P e_\varepsilon(u),$$

and that a similar bound holds for $\|b(u, \nabla u)\, \zeta\big\|_{L^2}$. We conclude:

Lemma 10. *There is a universal constant* $C = C(\alpha) > 0$ *such that on a suitable time slice corresponding to a volume fraction* $\lambda \in (0, 1)$ *the estimate*

$$|\int_{\mathbb{R}^3} b(u, \nabla u)\, \nabla u\, \psi| \le \frac{C(\alpha)}{\lambda} \int_P e_\varepsilon(u)\ \|\psi\|_{H^1}$$

holds true for every $\psi \in H_0^1\big(B_{1/6}\big)$*.*

5. Energy improvement on large scales

Proposition 3. *For every* $\gamma \in (0, 1)$ *there is a number* $\theta \in (0, 1)$*, an aspect ratio* $\kappa > 0$ *and an energy threshold* $\mathcal{E}_0 > 0$ *with the property that, whenever* $\varepsilon/r < \kappa$*, and* $\mathcal{E}(u, r, z, \varepsilon) < \mathcal{E}_0$*, then the energy improves like*

$$\mathcal{E}(u, \theta\, r, z, \varepsilon) \le \theta^{2\gamma}\ \mathcal{E}(u, r, z, \varepsilon).$$

The proof of Proposition 3 extends the arguments in [19] to the present situation. The main adapting tool is the representation formula (8) that allows to make use of the crucial trilinear estimate in Lemma 10. By translation and scale invariance of the Landau-Lifshitz equation it is sufficient to consider the special case when $z = 0$ and $r = 1$. The combination of weak energy improvement (Lemma 11) and a hybrid inequality (Lemma 12) implies, cf. [16, 19], the desired energy improvement.

Lemma 11. *There is a universal constant $C_0 > 0$ such that for every $\theta \in (0, 1/4)$ there are numbers $\varepsilon(\theta) > 0$ and $\mathcal{E}_0(\theta) > 0$ such that whenever $\varepsilon < \varepsilon(\theta)$ then $⨍_{P_1} e_\varepsilon < \mathcal{E}(\theta)$ implies*

$$⨍_{P_\theta} |u - (u)_{P_\theta}|^2 \le C_0\, \theta^2 ⨍_{P_1} e_\varepsilon(u), \tag{10}$$

where $(u)_{P_\theta} = ⨍_{P_\theta} u$ denotes the average.

Proof. If the statement would be false then, for every constant $c > 0$, there would exist a number $\theta \in (0, 1/4)$, such that for every $\varepsilon_0 > 0$ we would find a sequence (ε_k) with $\varepsilon_k \in (0, \varepsilon_0)$ and a corresponding sequence (u_k) of weak solutions such that

$$\int_P e_{\varepsilon_k}(u_k) = \mathcal{E}_k \to 0 \quad \text{but} \quad ⨍_{P_\theta} |u_k - (u_k)_{P_\theta}|^2\, dx > c\, \mathcal{E}_k\, \theta^2.$$

We consider blow-up functions $v_k = \mathcal{E}_k^{-1/2}\big(u_k - (u_k)_{P_\theta}\big)$. By Lemma 2 and Poincaré's inequality, (v_k) is uniformly bounded in $H^1(P_{1/2})$, independently of ε_0. Passing to a subsequence we may assume that $\varepsilon_k \to \varepsilon$, $u_k \to \bar{u}$ in $H^1(P_1)$ where $\bar{u} \in \mathbb{S}^2$ is constant, while $v_k \rightharpoonup v$ weakly in $H^1(P_{1/2})$ and strongly in $L^2(P_{1/2})$. Moreover, for every $k \in \mathbb{N}$, it holds

$$R(u_k)\, \partial_t v_k - \Delta v_k = \mathcal{E}_k^{-1/2} V_{\varepsilon_k}(u_k)\, u_k \quad \text{in} \quad P_{1/2}.$$

To pass to the limit we apply the parabolic operator $\mathcal{P}_{\varepsilon_k}$, see Lemma 7, to the equation. Using Lemma 8, we find that the limit function v satisfies a linear fourth order parabolic equation with constant coefficients

$$\mathcal{P}_\varepsilon\Big(R(\bar{u})\, \partial_t - \Delta\Big) v = 0 \quad \text{in} \quad P_{1/2}. \tag{11}$$

Clearly $⨍_{P_\theta} v = 0$ is preserved. We claim that this implies, when ε_0 is sufficiently small, the existence of a universal constant $C > 0$ such that

$$⨍_{P_\theta} |v|^2 \le C\, \theta^2 \quad \text{for any} \quad \theta \in (0, 1/4). \tag{12}$$

This would provide a contradiction, since by assumption $⨍_{P_\theta} |v|^2 \ge c\, \theta^2$ where $c > 0$ could be any constant. To prove estimate (12), we recall that in case $\varepsilon = 0$ inequality (12) is a standard estimates for second order parabolic systems with constant coefficients (cf. [3]). But since the set of possible blow-up limits v has a uniform H^1 bound in $P_{1/2}$ and is relatively compact in $L^2(P_{1/2})$, an estimate like (12) still holds true, at least in our context, when ε is sufficiently small, as shown by a simple contradiction argument similar to the one we just have performed. □

Lemma 12. *For any $\delta \in (0, 1)$ there is a constant $C(\delta) > 0$, an aspect ratio $\eta(\delta) > 0$, and an energy threshold $\mathcal{E}_1(\delta) > 0$ such that, whenever $\varepsilon/\theta < \eta(\delta)$ and $\theta^2 ⨍_{P_\theta} e_\varepsilon(u) < \mathcal{E}_1(\delta)$, then the following estimate holds true:*

$$\theta^2 ⨍_{P_{\theta/8}} e_\varepsilon(u) \le \delta\, \theta^2 ⨍_{P_\theta} e_\varepsilon(u) + \frac{C(\delta)}{\delta} ⨍_{P_\theta} |u - (u)_{P_\theta}|^2$$

Proof. After rescaling we may assume that $\theta = 1$. Let $\varphi \in C_0^\infty(P_{1/4})$ be a cut-off function such that $\varphi = 1$ on $P_{1/8}$ and $\varphi = 0$ on $P_{1/6}$. For $P = P_1(0)$ we consider the local energy $\mathcal{E} = ⨍_P e_\varepsilon(u)$. We single out suitable time slices and an exceptional set $\Lambda \subset (0,1)$ of prescribed measure $\lambda = \lambda(\delta)$ to be determined later.

Step 1: We first derive an estimate for the Dirichlet part of the energy. Using $\psi = \big(u - (u)_P\big)\,\varphi^2 \in H_0^1(P)$ as a test function we get on a suitable time slice $B = B_1(0) \times \{t\}$:

$$\int_B |\nabla u|^2 \varphi^2 \leq 2 \int_B |u-(u)_P|^2 |\nabla\varphi|^2 + \int_B \Big(R(u)\,\partial_t u \varphi\Big)\Big(\big(u-(u)_P\big)\varphi\Big)$$
$$+ \int_B \Big(V_\varepsilon(u)\,u\,\varphi\Big)\Big(\big(u-(u)_P\big)\varphi\Big) = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

In the following, $C = C(\alpha)$ will be used to denote any constant that only depends on the parabolicity parameter $\alpha > 0$. Using the estimate for $\partial_t u$ on suitable time slices (5) and Young's inequality we get

$$|\mathrm{II}| \leq \frac{\delta}{16} \int_P e_\varepsilon(u) + \frac{C}{\lambda\delta} \int_B |u-(u)_P|^2. \tag{13}$$

In order to estimate III by using the (elliptic) representation formula (8) for the Ginzburg-Landau term, we insert the operator $\mathcal{L}_\varepsilon$. Then

$$\mathrm{III} = \int_{\mathbb{R}^3} \Big(\mathcal{L}_\varepsilon\big(V_\varepsilon(u)\,u\big)\;\varphi\Big)\Big(\mathcal{L}_\varepsilon^{-1}\big[\big(u-(u)_P\big)\varphi\big]\Big) + \text{localization terms}.$$

Let us recall that $\mathcal{L}_\varepsilon$ is uniformly invertible on $L^2(\mathbb{R}^3)$ with operator norm $\|\mathcal{L}_\varepsilon^{-1}\| \leq 1$. Moreover, $\mathcal{L}_\varepsilon$ and its inverse commute with ∇-operations.
The localization terms are easily estimated using Young's inequality. Indeed,

$$\text{localization terms} = \Big|\int_{\mathbb{R}^3} \Big(\big[\Delta,\varphi\big]\big(V(u)\,u\big)\Big)\Big(\mathcal{L}_\varepsilon^{-1}\big[\big(u-(u)_P\big)\,\varphi\big]\Big)\Big|$$
$$\leq \frac{\delta}{32}\int_P e_\varepsilon(u) + \frac{C}{\lambda\delta}\int_B |u-(u)_P|^2, \tag{14}$$

where $[\cdot,\cdot]$ denotes the commutator. The term that involves the anti-symmetric mapping $b(u,\nabla u)$ can be estimated like in [19] by applying Lemma 10. Using that

$$\big\|\mathcal{L}_\varepsilon^{-1}\big[\big(u-(u)_P\big)\big]\varphi\big\|_{H^1} \leq C\sqrt{\mathcal{E}} + C\,\|u-(u)_P\|_{L^2(B)}$$

we obtain from Young's inequality

$$\Big|\int_{\mathbb{R}^3}\Big(b(u,\nabla u)\,\nabla u\Big)\Big(\varphi\,\mathcal{L}_\varepsilon^{-1}\big[\big(u-(u)_P\big)\varphi\big]\Big)\Big|$$
$$\leq \frac{C}{\lambda}\left(\mathcal{E}^{3/2} + \int_B |u-(u)_P|^2\right). \tag{15}$$

To estimate the divergence term we apply, after integration by parts, Young's inequality and Proposition 2

$$\frac{3}{2}\,\Big|\int_{\mathbb{R}^3}\Big((1-|u|^2)\nabla u\;\varphi\Big)\Big(\nabla\mathcal{L}_\varepsilon^{-1}\big[\big(u-(u)_P\big)\;\varphi\big]\Big)\Big|$$
$$\leq 3\,\omega\,(\mathcal{E}/\lambda)\;\Big(\int_B |\nabla u|^2\,\varphi^2 + C\int_B |u-(u)_P|^2\Big). \tag{16}$$

We can arrange that the latter prefactor is less than 1/2 when $\mathcal{E}/\lambda$ is small enough, see below. The terms that involve time derivatives have a bound like (13). Indeed, with the notation in (8), estimate (5) implies

$$
\begin{aligned}
(17)\qquad & \Big|\int_{\mathbb{R}^3} \Big(A(u)\,\partial_t u\,\varphi\Big)\Big(\mathcal{L}_\varepsilon^{-1}\big[\big(u-(u)_P)\varphi\big]\Big)\Big| \\
&\leq \frac{\delta}{32}\int_P e_\varepsilon(u) + \frac{C}{\lambda\delta}\int_B |u-(u)_P|^2.
\end{aligned}
$$

Summarizing (14), (15), (16), and (17), we find

$$
|\mathrm{III}| \leq \frac{1}{2}\int_B |\nabla u|^2\,\varphi^2\,dx + \frac{\delta}{16}\int_P e_\varepsilon(u) + C\,\frac{\sqrt{\mathcal{E}}}{\lambda}\,\mathcal{E} + \frac{C}{\lambda\delta}\int_B |u-(u)_P|^2.
$$

Summing up the bounds for I, II, and III, time integration gives

$$
\frac{1}{2}\int_P |\nabla u|^2\varphi^2 \leq \left(\frac{\delta}{8} + C\,\frac{\sqrt{\mathcal{E}}}{\lambda} + \lambda\right)\mathcal{E} + \frac{C}{\lambda}\,\frac{1}{\delta}\int_P |u-(u)_P|^2.
$$

Finally, a suitable choice of $\lambda(\delta) \lesssim \delta$, $\mathcal{E}(\delta) \lesssim \delta^4$, while $\mathcal{E}(\delta)/\lambda(\delta) \ll 1$, and $C(\delta) \gtrsim 1/\lambda(\delta)$ makes the first coefficient smaller than $\delta/4$ and implies the desired bound for the Dirichlet part of the energy.

Step 2. We show that the latter estimate implies a bound for the entire energy. Indeed, using $\psi = \big(1-|u|^2\big)\,u\,\varphi^2 \in H_0^1(P)$ as a test function and integration by parts gives

$$
\begin{aligned}
&\int_P \big(1-|u|^2\big)\,|\nabla u|^2\,\varphi^2 - \frac{1}{4}\int_P \big(1-|u|^2\big)^2\big(\alpha\partial_t + \Delta\big)\,\varphi^2 \\
&-2\int_P |u\cdot\nabla u|^2\varphi^2 + \frac{1}{\varepsilon^2}\int_P \big(1-|u|^2\big)^3\varphi^2 = \int_P \frac{1}{\varepsilon^2}\big(1-|u|^2\big)^2\,\varphi^2.
\end{aligned}
$$

Using Proposition 2 the last term on the left hand side is bounded by $\omega(\mathcal{E}/\lambda)\,\varepsilon^{-2}\int_P \big(1-|u|^2\big)^2\varphi^2$, and it can be absorbed when the quotient $\mathcal{E}/\lambda$ is sufficiently small. We find that

$$
\frac{1}{2}\int_P \frac{1}{\varepsilon^2}\big(1-|u|^2\big)^2\,\varphi^2 \leq \int_P |\nabla u|^2\,\varphi^2 + C\,\varepsilon^2\int_P \frac{1}{\varepsilon^2}\big(1-|u|^2\big)^2.
$$

Assuming $\varepsilon \leq \eta(\delta) \lesssim \sqrt{\delta}$, the last coefficient can be made smaller than δ. Hence, in view of what we have shown in Step 1, the proof is complete. $\square$

Proof of Proposition 3. Since the choice of parameters ε, θ, and δ is restricted by means of the assumption of our lemmata, we perform the arguments, that follow closely the lines of [16], in some detail. After translation and rescaling, we may assume that $z=0$ and $r=1$. Let us adopt the the notation of Lemma 11 and Lemma 12. First we fix a moderately small candidate $\theta \in (0,1/4)$ such that $8^k\theta = 1$ for some integer $k = k(\theta)$. We observe that the choice of $\delta = \theta^{3/k}$ implies $\delta = 1/512$. Finally, we consider (renormalized) $\varepsilon \in \big(0,\kappa(\theta)\big)$, where

$$
\kappa(\theta) = \min_{0\leq i\leq k}\,\min\Big\{\varepsilon\big(8^i\theta\big), 8^i\theta\;\eta\big(\delta\big)\Big\} > 0
$$

is well defined and positive for every such θ. For $\rho \in (0,1)$ we denote the local energy average by $\mathcal{E}(u,\varepsilon,\rho) = \rho^2 \fint_{P_\rho} e_\varepsilon(u)$. Under the condition that $\mathcal{E}(u,\varepsilon,8^{i+1}\theta) < \mathcal{E}_1(\delta)$ and $\mathcal{E}(u,\varepsilon,1) < \mathcal{E}_0(8^{i+1}\theta)$, we can apply Lemma 12 and Lemma 11. We find that

$$
\mathcal{E}\big(u,\varepsilon,8^i\theta\big) \leq \delta\,\mathcal{E}\big(u,\varepsilon,8^{i+1}\theta\big) + C_0\,\frac{C(\delta)}{\delta}\,\mathcal{E}\big(u,\varepsilon,1\big) \quad \text{for } \varepsilon < \kappa(\theta).
$$

Hence it follows by induction that, if, for $c(\delta) \lesssim \delta / C(\delta)$, we choose

$$\mathcal{E}(u,\varepsilon,1) < \min_{0 \le i \le k} \min \Big\{ \mathcal{E}_0\big(8^i\theta\big), c(\delta)\,\mathcal{E}_1\big(\delta\big) \Big\},$$

then we have automatically $\mathcal{E}\big(u,\varepsilon,8^i\theta\big) \le \mathcal{E}(u,\varepsilon,1)$ for every $0 \le i \le k$. Thus, for sufficiently small energy $\mathcal{E}(u,\varepsilon,1)$, we are in the position to iterate Lemma 12 and Lemma 11

$$\begin{aligned} \mathcal{E}\big(u,\varepsilon,\theta\big) &\le \delta\,\mathcal{E}(u,\varepsilon,8\theta) + C_0\,C(\delta)\,\left(\frac{\theta}{\delta}\right)^2 8^2\,\delta\,\mathcal{E}(u,\varepsilon,1) \le \cdots \le \\ &\le \delta^k\,\mathcal{E}(u,\varepsilon,8^k\theta) + \frac{C_0\,C(\delta)}{1-64\delta}\left(\frac{\theta}{\delta}\right)^2 \mathcal{E}(u,\varepsilon,1) \quad \text{for } \varepsilon < \kappa(\theta). \end{aligned}$$

We recall that, according to our definitions, $\mathcal{E}(u,\varepsilon,8^k\theta) = \mathcal{E}(u,\varepsilon,1)$ while $\delta^k = \theta^3$ and $64\delta = 1/8$. Thus we observe that $\mathcal{E}(u,\varepsilon,\theta) \le \theta^{2\gamma}\,\mathcal{E}(u,\varepsilon,1)$ can be achieved by choosing θ sufficiently small, proving the proposition.

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