MATRIX GENERALIZATIONS OF MULTIPLE HYPERGEOMETRIC FUNCTIONS BY USING MATHAI'S MATRIX TRANSFORM TECHNIQUES

By

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IMA Preprint Series # 1943

(November 2003)

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

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A Thesis submitted to the University of Kumaun, Nainital, Uttaranchal, India, for the award of the degree of Doctor of Philosophy in Mathematics

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2003

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF KUMAUN,
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CERTIFICATE

In forwarding the thesis entitled "**Matrix Generalizations of Multiple Hypergeometric Functions By Using The Mathai's Matrix Transform Techniques"** submitted by **Shri Lalit Mohan Upadhyaya** in fulfillment of the requirement for the degree of **Doctor of Philosophy** in Mathematics of the University of Kumaun, Nainital, I hereby certify that he has completed the research work within the period of registration and has put in the attendance as required under the ordinance 6 of the registration. The thesis embodies the results of his investigations carried out during the period he worked as a research scholar with me. It is an original piece of work and can be forwarded to the experts for critical examination.

Signatured
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PREFACE

The present work is the outcome of my studies of several years on functions of matrix arguments, one of the many topics that my teacher Prof. H. S. Dhami had suggested to me. Earlier Prof. Dhami had done some work in this field, especially, his doctoral thesis on the study of the T. M. Macrobert's E-function, and his later work on the application of the theory of functions of matrix arguments to Noncentral Wishart Distributions in multivariate statistical analysis.

When I chose to pursue this field of study, the greatest challenge before me was to search for the concerned literature in this field and its collection. This was perhaps the toughest task. Despite my several efforts when I did not succeed much, one day it suddenly occurred to me 'why not write to Prof. Mathai himself, a pioneer in this field?', and it resulted into the genesis of this work. Needless to say that my very first letter was responded to very promptly by Prof. Mathai, and thus began a long series of correspondence and guidance to me from Prof. Mathai himself, from Canada and also from his short visits to India. I have no hesitation to admit that I could not have been able to do this work without the help and guidance received from Prof Mathai. My continuous correspondence and telephonic talks with him in Canada and India, the literature made available to me by him, especially the reprints of some of his papers which were not available in India and his review and constructive criticism and suggestions on my papers have all led to a better presentation of this work. In fact I owe him a great deal. As I have said earlier that the collection of literature proved to be a tough task for me, I would like to mention here that the 1997 book of Mathai could not become available to me despite my best efforts. I had placed orders with hundred percent advance cost with a number of leadings suppliers of books in Delhi to procure a copy for myself, searched libraries of a number of reputed institutions of India including the library of the Indian Institute of Science, Bangalore, but the book was nowhere available. I had even asked a colleague of mine, who went to the United States in November 2002, to buy a copy of the book for me from there, because it was published there, but all in vain. However, I luckily came across the book in the library of the Indian Statistical Institute in New Delhi and read some portions of the book in the library itself by sitting for a number of hours for some days. Yet, I think I could not derive much from it because if I had a copy of mine I could study it more and that too at ease and leisure and would have been benefited more.

At this juncture I would like to remember with gratitude the help received by me from the staff of the libraries of I. I. Ts. Delhi, Roorkee, Central Science Library, University of Delhi, I. S. I. New Delhi, INSDOC New Delhi, the Indian Institute of Science, Bangalore, and the Center for Mathematical Sciences, Trivandrum. I would like to acknowledge with thanks the help and encouragement received from Prof. M. A. Pathan of A. M. U. Aligarh, and the seminar library of that Department. Especially, I am thankful to Prof. Pathan for the long discussions that I had with him at his home during my about four months of stay in that University. I would like to mention the immense help my colleagues Dr. Ganga Sharan and Prof. Anil Khanduri have lent to me. Dr. Ganga Sharan has helped me a lot in searching the literature, arranging the necessary books and papers, visiting the various institutes, universities and libraries, and he has also provided me his hospitality during my long stay at New Delhi in connection with this research work. Prof. Khanduri and his brother Mr. Sunil Khanduri, M. Sc. (Physics), have helped me a lot in my computer related work, and in removing the defects in the computer system which often appeared frequently. I am also indebted to my former student Shri Jagadish, B. Sc., P. G. D. C. A., whose has been associated with me throughout this venture for about the last two years and has always strived very hard to solve all my computer related problems and he has succeeded in his endeavor. Without his help my papers could not have been published and thesis typing would not have been completed. I am also indebted to Prof. Willard Miller Jr., Department of Mathematics, University of Minnesota, Minneapolis, U.S.A., who, after having gone through my papers, had invited me to participate in the International Conference on "Special Functions in the Digital Age", held at the University of Minnesota, Minneapolis, U. S. A. from July 22, 2002 to Aug. 2, 2002. But unfortunately, I could not visit there owing to the paucity of funds. I cannot repay the love, affection, encouragement and guidance that I have received from my parents during these years and also the support and help from my sisters Smt. Deep Shikha, Km. Taru Shikha and Km. Ambuj Shikha. For my wife Anita, I am not in a position to write anything, she has helped me in correcting the proofs of my papers and thesis and after going through the proofs of the theorems sometimes she has suggested modifications which, I have happily incorporated.

I have no words to express my gratitude to my Guru Prof. Dhami, who has been the sole force behind this work and whom I have been troubling for the past fifteen years with various problems of mine and he has always helped me in the best possible way. He has read each paper, each theorem and I have troubled him and his family members by often visiting his house, staying there and making enormous phone calls to him from Mussoorie and other places of India which I have visited during the course of my studies. Still he has to guide me further in my studies so, I cannot say that my work under him has finished.

Towards the end, I remember the Almighty Mata Jagadamba, to whom this work is dedicated with extreme devotion.

Mussoorie, Lalit Mohan Upadhyaya
Dated: June 02, 2003.

Acknowledgements

I am very much grateful to the following persons in India and abroad who have ungrudgingly lent their help and support to me during the preparation of the present thesis and earlier during the survey and collection of the literature. Some of them have also given me their hospitality during my visits to the various Libraries, Institutes and Universities of the country and during my various stays in different parts of Northern India. I take this opportunity to thank them all from the core of my heart, but for their help this work could not have been completed!

1. Prof. A. M. Mathai, Department of Mathematics and Statistics, McGill University, Montreal, Canada, and Director, Center for Mathematical Sciences, Trivandrum, India.
2. Prof. M. A. Pathan, Chairman, Department of Mathematics, Aligarh Muslim University, Aligarh, U. P.
3. Prof. R. K. Saxena, Emeritus Professor, Department of Mathematics, J. N. V. University, Jodhpur, Rajasthan.
4. Dr. Ganga Sharan, Department of Physics, Municipal Post Graduate College, Mussoorie, Dehradun, Uttaranchal.
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7. Shri Chhotey Lal, Central Building Research Institute, Roorkee, Uttaranchal.
8. Dr. Mohd. Kalimmuddin, Department of Mathematics, Aligarh Muslim University, Aligarh, U. P.
9. Er. Chhavi Prakash Pant, National Thermal Power Corporation, Unchaahar, Raibareilly, U. P.
10. Er. Devendra Joshi, Sector 15, Rohini, New Delhi.
11. Er. Harendra Joshi, R. P. G. Ricoh Ltd., New Delhi.
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13. Er. Himanshu Dhami, Software Enginer, Girerlinerein, Oslo, Norway.
14. Mr. Sunil Khanduri, Khanduri Nivas, Hussainganj, Mussoorie, Uttaranchal.
15. Dr. Lalit Kumar Verma, P. & T., Govt. of India, New Delhi.
16. Er. Ishwar Singh, Delhi Metro Rail Project, New Delhi.
17. Shri Jagadish, Connexions, Kulri, Mussoorie, Uttaranchal.
18. Shri T. C. Pant, Lal Bahadur Shastri National Academy of Administration, Mussoorie, Uttaranchal.
19. Shri Rajeev Kumar Sharma, Lal Bahadur Shastri National Academy of Administration, Mussoorie, Uttaranchal.

PAPERS PUBLISHED FROM THIS THESIS

The following is the list of the papers published from this thesis till the time of its submission. Six more papers, as indicated in the bibliography, are to appear by the end of this year.

1. Upadhyaya Lalit Mohan & Dhami H. S. (Nov. 2001) Matrix Generalizations of Multiple Hypergeometric Functions, # 1818, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
2. Upadhyaya Lalit Mohan & Dhami H. S. (Dec. 2001) On Some Multiple Hypergeometric Functions of Several Matrix Arguments, # 1821, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
3. Upadhyaya Lalit Mohan & Dhami H. S. (Feb. 2002) On Kampé de Fériet and Lauricella Functions of Matrix Arguments-I, # 1832, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
4. Upadhyaya Lalit Mohan & Dhami H. S. (Feb. 2002) On Lauricella and Related Functions of Matrix Arguments-II, # 1836, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
5. Upadhyaya Lalit Mohan & Dhami H. S. (Mar. 2002) Appell's and Humbert's Functions of Matrix Arguments-I, # 1848, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
6. Upadhyaya Lalit Mohan & Dhami H. S. (Apr. 2002) Appell's and Humbert's Functions of Matrix Arguments-II, # 1853, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
7. Upadhyaya Lalit Mohan & Dhami H. S. (May 2002) Humbert's Functions of Matrix Arguments-I, # 1856, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
8. Upadhyaya Lalit Mohan & Dhami H. S. (July 2002) Humbert's Functions of Matrix Arguments-II, # 1865, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
9. Upadhyaya Lalit Mohan & Dhami H. S. (Aug. 2002) Generalized Horn's Functions of Matrix Arguments, # 1876, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
10. Upadhyaya Lalit Mohan & Dhami H. S. (Sep. 2002) On Exton's Generalized Quadruple Hypergeometric Functions and Chandel's Function of Matrix Arguments, # 1882, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
11. Upadhyaya Lalit Mohan & Dhami H. S. (Oct. 2002) On Some Generalized Multiple Hypergeometric Functions of Matrix

Arguments, # 1887, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.

12. Upadhyaya Lalit Mohan & Dhami H. S. (Nov. 2002) Lauricella-Saran Triple Hypergeometric Functions of Matrix Arguments-I, # 1899, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.
13. Upadhyaya Lalit Mohan & Dhami H. S. (May 2003) Exton's Quadruple Hypergeometric Functions of Matrix Arguments-I, #1923, IMA Preprint Series, University of Minnesota, Minneapolis, U. S. A.

CHAPTER I

MATRIX GENERALIZATIONS OF MULTIPLE HYPERGEOMETRIC FUNCTIONS BY USING MATHAI'S MATRIX TRANSFORM TECHNIQUES

(A Brief Historical Survey and Outline of the Programme)

1.1 In the present section I have briefly discussed some concepts and results of matrix algebra, right from the elementary level to the abstract region, which have been used throughout this work.

1.1.1 Matrix (Definition):

A matrix denoted by $A = \left[a_{ij}\right]_{m\times n}$ is an arrangement of mn elements in a rectangular form from any field (usually a complex field). The (i, j)th entry, i.e., the element at the junction of the ith row $(i = 1, \cdots, m)$ and jth column $(j = 1, \cdots, n)$ is denoted by a_{ij}. If $m = n$, the matrix is called a square matrix.

1.1.2 Transpose and Conjugate Transpose of a Matrix:

If $A = \left[a_{ij}\right]_{m\times n}$ be a matrix, then its transpose, denoted by A', is a matrix defined by $A' = \left[a'_{ji}\right]_{n\times m}$, where, $a'_{ji} = a_{ij}$. If a square matrix A is such that, $A' = A$, then it is called a symmetric matrix, and, if $A' = -A$, then A is called a skew-symmetric matrix.

If A^* is such a matrix that the (i, j)th element of it is the same as the complex conjugate of the (j, i)th element of A, then A^* is said to be the conjugate transpose of A, i.e., $A^* = \left[a^*_{ij}\right]_{n\times m}$ where $a^*_{ji} = \overline{a_{ij}}$, $\overline{a_{ji}}$ represents the complex conjugate of a_{ji}. If a square matrix A is such that,

$A^* = A$, then A is called a Hermitian matrix, whereas if $A^* = -A$, then A is termed skew- Hermitian.

Whenever the matrices A and B are such that the products AB and BA are defined, then, $(AB)' = B'A'$ and $(AB)^* = B^*A^*$.

1.1.3 Trace of a Matrix:

For a square matrix $A = \left[a_{ij}\right]_{n\times n}$, the trace of A, denoted by $tr(A)$, is defined to be the sum of the diagonal elements of A, that is,

$$tr(A) = \sum_{i=1}^{n} a_{ii} \tag{1.1}$$

1.1.3.1 Properties of Trace:

If A, B, C be three square matrices of the same order, then,

(i) $tr(A+B) = tr(A) + tr(B)$, (ii) $tr(AB) = tr(BA)$,

(iii) $tr(ABC) = tr(BCA) = tr(CAB) = tr[A'(BC)']$

(iv) and when the matrices are symmetric then, $tr(ABC) = tr(ACB)$.

1.1.4 Determinant of a square matrix:

If $A = \left[a_{ij}\right]_{n\times n}$ be a square matrix of order n, then its determinant, represented by $|A|$, is defined as,

$$|A| = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_n} (-1)^{\rho(i_1,\cdots,i_n)} a_{1i_1} a_{2i_2} \cdots a_{ni_n} \tag{1.2}$$

where, $i_1,\cdots,i_n$ represent the column numbers and $\rho(i_1,\cdots,i_n)$ stands for the number of transpositions needed to bring $(i_1,\cdots,i_n)$ to the natural order $(1, 2, \cdots, n)$.

A matrix A is said to be non-singular if $|A| \neq 0$, otherwise, it is termed as a singular matrix. If A, B, C be three non-singular matrices of order n each and I be the identity (unit) matrix of order n, then,

$$|I \pm AB| = |I \pm BA|; |I \pm ABC| = |I \pm BCA| = |I \pm CAB|.$$

Moreover, whenever, A and B are two square matrices of the same order,

$$|AB| = |BA| = |A||B| = |B||A|.$$

1.1.5 Inverse of a Matrix:

For a square matrix $A = \left[a_{ij}\right]_{n\times n}$, let A_{ij} be the cofactor of a_{ij} in $|A|$, $\operatorname{cof}(A)$ the cofactor matrix of A, then the inverse of A, whenever it exists, is given by,

$$A^{-1} = \frac{1}{|A|}\left[\operatorname{cof}(A)\right]', \qquad |A| \neq 0. \tag{1.3}$$

For two non-singular matrices A and B of the same order, we have, $(AB)^{-1} = B^{-1}A^{-1}$.

1.1.6 Quadratic form and the concept of Positive Definiteness:

If X is a column vector of order $(n \times 1)$ and there is a symmetric matrix A of order n, then, $Q = X'AX$ is called a quadratic form. If $Q > 0 \ \forall X \neq 0$, then Q is called a positive definite quadratic form and A is called a positive definite matrix.

1.1.6.1 Criteria for Positive Definiteness of a Matrix:

(i) A necessary and sufficient condition that a matrix A be positive definite is that all the leading principal minors of A are consistently positive.

(ii) A necessary and sufficient condition that a quadratic form $Q = X'AX$ be positive definite is that all the characteristic roots of A are consistently positive.

(iii) If a matrix A is capable of being expressed as $A = BB'$, for some matrix B, then A is called a gram matrix. A necessary and sufficient condition that a matrix is a Gram matrix is that it is positive definite.

1.1.7 Square Root of a Matrix:

A symmetric positive definite (or a hermitian positive definite) square root of a symmetric positive definite (or a hermitian positive definite) matrix A is $C = P\Lambda^{1/2}P'$ (or $C = P\Lambda^{1/2}P^{*}$) where P is the matrix of normalized eigen vectors of A, Λ is the diagonal matrix of the eigen values of A and

$\Lambda^{1/2}$ denotes the diagonal matrix with the diagonal elements as the positive square roots of the diagonal elements in Λ.

1.1.8 Notations to be used in the thesis:

All the matrices appearing in the present thesis are $(p \times p)$ real symmetric positive definite matrices unless otherwise stated. $A > 0$ will be used to mean that the matrix A is positive definite. $A^{1/2}$ will represent the symmetric square root of the real symmetric positive definite matrix A. When integrating over matrices the following notations will be used:

$\int_X f(X)dX$: integral over X of the scalar function $f(X)$.

$\int_{X>0}$: integral over the positive definite matrix X.

$\int_A^B f(X)dX$: integral over X such that $X > 0, A > 0, B > 0, X - A > 0,$ $B - X > 0.$

$\int_{A<X<B}$: same as above.

$0 < X < I \Rightarrow X > 0, I - X > 0;$ which means that all the eigen values of X are between 0 and 1, and Re(.) denotes the real part of (.).

1.2 Some Results on Jacobians of Matrix Transformations:

Throughout the present work I have frequently used some results of Jacobians of matrix transformations, which will be stated below for the sake of reference. For a thorough discussion of this concept and other results I refer the reader to Mathai [57,73] and the references therein.

Theorem 1.2.1: Let Y and X be $(p \times q)$ matrices of functionally independent real variables. Let A be a $(p \times p)$ non-singular matrix and B a $(p \times q)$ matrix of constants. Then,

$$Y = AX + B \Rightarrow dY = |A|^q \, dX \tag{1.4}$$

Theorem 1.2.2: Let Y and X be $(p \times q)$ matrices of functionally independent real variables. Let A be $(p \times p)$, B be $(q \times q)$ and C be $(p \times q)$ matrices of constants where A and B are non-singular. Then,

$$Y = AXB + C \Rightarrow dY = |A|^{q}|B|^{p}\,dX \tag{1.5}$$

Theorem 1.2.3: Let Y and X be symmetric matrices of $p(p+1)/2$ functionally independent real variables. Let A be a non-singular matrix and C a symmetric matrix of constants. Then,

$$\left.\begin{aligned} &Y = AXA' + C \Rightarrow dY = |A|^{p+1}\,dX, \\ &\text{and, } Y = aX \Rightarrow dY = a^{p(p+1)/2}\,dX \end{aligned}\right\} \tag{1.6}$$

where, a is a scalar quantity.

Theorem 1.2.4: Let X be a $(p \times p)$ non-singular matrix of functionally independent real variables. Then ignoring the sign,

$$\left.\begin{aligned} Y = X^{-1} \Rightarrow dY &= |X|^{-2p}\,dX \text{ for } X \neq X', \\ &= |X|^{-(p+1)}\,dX \text{ for } X = X', \\ &= |X|^{-(p-1)}\,dX \text{ for } X = -X', \\ &= |X|^{-(p+1)}\,dX \text{ for } X \text{ lower or upper triangular.} \end{aligned}\right\} \tag{1.7}$$

Theorem 1.2.5: Let X, A, B be $(p \times p)$ non-singular matrices where A and B are constant matrices and X is a matrix of functionally independent real variables, then,

$$\left.\begin{aligned} Y = AX^{-1}B \Rightarrow dY &= |AB|^{p}\,|X|^{-2p}\,dX \text{ for } X \neq X', \\ &= \left|AX^{-1}\right|^{p+1}\,dX \text{ for } X = X', B = A', \\ &= \left|AX^{-1}\right|^{p-1}\,dX \text{ for } X = -X', B = A'. \end{aligned}\right\} \tag{1.8}$$

Theorem 1.2.6: Let k real $(p \times p)$ matrices $X_1, \cdots, X_k$ be transformed to the $(p \times p)$ matrices $Y_1, \cdots, Y_k$ by the relations,

$$Y_1 = X_1, Y_2 = X_1 + X_2, \cdots, Y_k = X_1 + \cdots + X_k;$$

then,

$$dY_1 \cdots dY_k = dX_1 \cdots dX_k \tag{1.9}$$

1.3 Special Functions of Matrix Arguments:

The study of special functions of matrix arguments, that is, the generalization of the special function of scalar argument(s) case to the matrix argument(s) case seems to have originated probably around 1944 when such functions appeared implicitly in the works of T. W. Anderson and M. A. Girschick [1], while working in multivariate statistical analysis. Thus, the functions of matrix arguments originated in the field of multivariate statistical analysis. But the generalization of the special functions of the scalar arguments to the matrix arguments case was not an easy task owing to the peculiarities and complexities of the matrix algebra as compared to the ordinary algebra of complex numbers.

When generalizing the theory of scalar variables to the matrix variables many types of problems are encountered. If the rectangular matrices are considered then their powers and inverses etc. cannot, in general, be found. Even if one tries to solve these problems by using the concept of the generalized inverse of a matrix, still a unique entity is not obtained. Therefore, it is considered safe to work only with the real symmetric positive definite matrices or the hermitian positive definite matrices.

Many workers have notably contributed to this field. The earlier workers like, Carl S. Herz [22], have used the Laplace transform technique to define the generalized hypergeometric function of a single matrix argument, while, A. G. Constantine [8] and A. T. James [23,24,25] have used the concept of zonal polynomials to define and study the functions of matrix arguments. A. M. Mathai [47,60-63,73] has developed the concept of matrix transform (M-transform) to define and study the functions of matrix arguments. Using this elegant technique Mathai has been able to develop the theory of functions of matrix arguments to a very great extent, which does not seem possible by the use of the earlier two techniques. In particular, he has defined and generalized the integral representations of the Kummer's confluent hypergeometric function ${}_1F_1$, the Gauss's hypergeometric function ${}_2F_1$, the generalized hypergeometric function ${}_rF_s$, the Appell's and the Humbert's functions, the Kampé de Fériet's function and the Lauricella functions of n variables to the matrix arguments case and he has also done much work on the statistical applications of these functions. Among other prominent workers of the field, the names of G. Pederzoli

[64,71,72,74,84], R. K. Saxena and P. L. Sethi [45,46,85,94,95-99], D. G. Kabe [27-29], Yasuko Chikuse [6,7], Kenneth Gross and Donald Richards [19,20,87-92], Jodar Lucas [36-43], H. S. Dhami [9-14], R. K. Kumbhat [31-35], etc. may be mentioned.

The present study aims at an extension of Mathai's work to the more advanced special functions like the Lauricella-Saran triple hypergeometric functions, the Srivastava functions H_A, H_B and H_C, the Exton's twenty one quadruple hypergeometric functions, the Exton's ${}^{(k)}_{(1)}E^{(n)}_D$ and ${}^{(k)}_{(2)}E^{(n)}_D$ functions, the Chandel's ${}^{(k)}_{(1)}E^{(n)}_C$ function, the generalized Horn's functions ${}^{(k)}H^{(n)}_3$ and ${}^{(k)}H^{(n)}_4$ and the generalized Srivastava functions $H^{(n)}_B$ and $H^{(n)}_C$ and also to produce the generalizations of many new results of the functions in which Mathai has already initiated the study. It is hoped that the present study will greatly enrich the existing theory of functions of matrix arguments, which, as Mathai himself has remarked, is a fast developing area of the present day research in mathematics, but still this theory is in its infancy.

Now I give a brief discussion of the various techniques available in the literature for the study of the functions of matrix arguments. There are three different developments available in the literature for the theory of functions of matrix arguments. One approach is through Laplace and inverse Laplace transforms, references can be cited of Bochner [3], Herz[22]. Another is through series representations with the help of zonal polynomials, works of James [23,24,25], Constantine [8] can be cited in this context. The third approach is due to Mathai [47,60-63,73], which involves a generalized matrix transform. The basic ideas in all these approaches will be discussed here briefly.

1.3.1 Laplace transform:

Let $f(x_1,\cdots,x_k)$ be a scalar function of positive variables $x_1,\cdots,x_k$ and $t_1,\cdots,t_k$ be arbitrary parameters then the Laplace transform of f, denoted by $L_T(f), T' = (t_1,\cdots,t_k)$ is defined as follows:

Definition 1.3.1.1: Laplace transform: Multivariable case –

$$L_T(f) = \int_0^\infty \cdots \int_0^\infty e^{-(t_1x_1+\cdots+t_kx_k)} f(x_1,\cdots,x_k)dx_1\cdots dx_k \quad (1.10)$$

Our aim here is to extend this idea to cover the case of matrix variables. Let $f(X)$ be a scalar function of a real symmetric positive definite matrix X of order $(p\times p)$. Consider a $(p\times p)$ matrix T of parameters such that the diagonal elements of T are $t_{jj}, j=1,\cdots,p$ and the non-diagonal elements are $\frac{1}{2}t_{jk}; j,k=1,\cdots,p; j\neq k,$ subject to the condition that $t_{jk}=t_{kj}$. If such matrices X and T are considered then we note that,

$$\mathrm{tr}(XT)=\mathrm{tr}(TX)=\sum_{j\geq k} t_{jk}x_{jk} \tag{1.11}$$

then by using the definition of Laplace transform for the multivariable case (eq.(1.10)) we can define the Laplace transform for the matrix variable case as follows:

Definition 1.3.1.2: Laplace transform: Matrix variable case –

$$L_T(f)=\int_{X>0} e^{-\mathrm{tr}(TX)} f(X)dX \tag{1.12}$$

where $f(X)$ is a scalar function of the real symmetric positive definite matrix X, and T is defined in eq.(1.11). The integral in (1.12) may not exist. When it exists, eq.(1.12) gives the Laplace transform of the function $f(X)$. If $f(X)$ is a density function then $L_{-T}(f)$ is called the moment generating function of $f(X)$ and it is denoted by $E\left[e^{\mathrm{tr}(TX)}\right]$, where E denotes the expected value.

1.3.2 Functions of matrix arguments through Laplace transform:

If $f(X)$ is a scalar function of a $(p\times p)$ real symmetric positive definite matrix X and if $h(T)$ is the Laplace transform of f then the uniqueness of f through $h(T)$ can be established if f and h satisfy some conditions. Let $Z=X+iY, i=\sqrt{-1}$, where X and Y are real matrices, X is symmetric positive definite and belonging to the class of T defined in eq.(1.11), and Z hermitian positive definite. Let $f(Z)$ be a complex analytic function of the $(p\times p)$ matrix Z. Let $f(Z)$ be symmetric in the sense that $f(Z)=f(Q^{*}ZQ), QQ^{*}=I$ for all unitary(or orthonormal when real) $(p\times p)$ matrices Q, where Q^{*} denotes the conjugate transpose of Q. Then f is a function of the p eigenvalues of Z. Also f is a complex analytic function of the p elementary symmetric functions

$$s_1 = \lambda_1 + \cdots + \lambda_p, s_2 = \sum_{jk} \lambda_j \lambda_k, \cdots, s_p = \lambda_1 \lambda_2 \cdots \lambda_p$$

where, $\lambda_1, \cdots, \lambda_p$ are the eigenvalues of Z. Let for A real symmetric positive definite

$$h(Z) = \int_{A=A'>0} e^{-tr(AZ)} f(A) dA \tag{1.13}$$

If the integral is absolutely convergent in some generalized right half plane $R(Z) = X > X_0$ it represents a complex analytic function $h(Z)$. If

$$\int |h(X+iY)| dY < \infty \text{ and } \lim_{|X| \to \infty} \int |h(X+iY)| dY = 0$$

for some $X = X' > X_0 = X_0' > 0$ then Cauchy's inversion formula can be applied to write the inverse Laplace transform, i.e.,

$$\frac{2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}} \int_{Re(Z)=X>X_0} e^{tr(AZ)} h(Z) dZ = \begin{cases} f(A), \text{ for } A > 0. \\ 0, \text{ elsewhere.} \end{cases} \tag{1.14}$$

With the help of the above results one can develop the theory of special functions of matrix arguments. It is assumed that all the special functions appearing here are symmetric in the sense that $f(A) = f(Q^* AQ), QQ^* = I$. A convolution theorem for the Laplace transform, which is available in the literature, can be stated as follows:

Theorem 1.3.2.1: Let f_1 and f_2 be scalar functions of $(p \times p)$ real symmetric positive definite matrices. Let g_1 and g_2 be their Laplace transforms respectively. Let,

$$f_3(X) = \int_0^X f_1(X-S) f_2(S) dS \tag{1.15}$$

where the integration is done over all $S = S' > 0$ and $X - S > 0$. Then the Laplace transform of f_3 is $g_1 g_2$.

1.3.3: Functions of matrix arguments through Zonal Polynomials:

The zonal polynomials associated with a matrix Z are certain symmetric functions in the eigenvalues of Z. A detailed description of the various types of polynomials and the development of the theory is available from Mathai, Provost and Hayakawa [69]. Consider a $(p\times p)$ real symmetric positive definite matrix X. Let V_k be the vector space of homogeneous polynomials $g(X)$ of degree k in the $p(p+1)/2$ different elements of the $(p\times p)$ symmetric matrix X. Consider a congruent transformation $X \rightarrow LXL'$ by a nonsingular $(p\times p)$ matrix L. A subspace $V_s \subset V_k$ is called invariant if $LV_s \subset V_s$ for all nonsingular matrices L. If V_s has no proper invariant subspace it is called an irreducible invariant subspace. It can be shown that V_k decomposes into a direct sum of irreducible invariant subspaces V_K corresponding to each partition $K=(k_1,k_2,\cdots,k_p), k_1+k_2+\cdots+k_p=k$, into not more than p parts. Each subspace contains a unique one dimensional subspace invariant under the orthogonal group of linear transformations. These subspaces are generated by the zonal polynomial $U_K(X)$ which when normalized in a certain fashion give the zonal polynomials $C_K(X)$. Explicit forms of these polynomials are available for small values of k. For large values of k it will be extremely difficult to compute these polynomials. For handling elementary special functions of matrix argument we need a few properties of these zonal polynomials. These will be listed here without proofs since the proofs are too long. One basic result which is an immediate consequence of the definition itself is that when X is a (1×1) matrix, namely, a scalar quantity x,

$$C_K(Z) = z^k \tag{1.16}$$

Hence one can look upon $C_K(X)$ as a generalization of x^k. The exponential function has the following expansion:

$$e^{tr(X)} = \sum_{k=0}^{\infty} \frac{1}{k!}[tr(X)]^k = \sum_{k=0}^{\infty} \sum_{K} \frac{C_K(X)}{k!} \tag{1.17}$$

The binomial expansion is the following: for $I - X > 0$, that is, $X = X' > 0$ and all the eigenvalues of X are between 0 and 1,

$$|I - X|^{-\alpha} = \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha)_K C_K(X)}{k!} \tag{1.18}$$

where,

$$(\alpha)_K = \prod_{j=1}^{p} \left(\alpha - \frac{j-1}{2}\right)_{k_j} \quad \text{with, } K = (k_1, k_2, \cdots, k_p), k_1 + \cdots + k_p = k;$$

$$\int_{O(p)} C_K(H'XHT)dH = C_K(X)C_K(T)/C_K(I) \tag{1.19}$$

where I is the identity matrix, the integral is over the orthogonal group of $(p \times p)$ matrices and dH is the invariant Haar measure.

$$\left.\begin{aligned} &\int_{X=X'>0} e^{-\mathrm{tr}(ZX)} |X|^{\alpha-(p+1)/2} C_K(XT)dX \\ &= |Z|^{-\alpha} C_K(TZ^{-1})\Gamma_p(\alpha, K) \\ &\text{where, } \Gamma_p(\alpha, K) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma(\alpha + k_j - \frac{j-1}{2}) \text{ and for} \\ &\mathrm{Re}(\alpha) > (p-1)/2. \end{aligned}\right\} \tag{1.20}$$

This result is also valid for all complex hermitian positive definite Z. An inverse Laplace transform for the zonal polynomials can be stated as follows:

$$\left.\begin{aligned} &\frac{2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}} \int_{\mathrm{Re}(Z)=X>X_0} e^{\mathrm{tr}(SZ)} |Z|^{-\alpha} C_K(Z)dZ \\ &= \frac{|S|^{\alpha-(p+1)/2} C_K(S)}{\Gamma_p(\alpha, K)} \end{aligned}\right\} \tag{1.21}$$

In a similar fashion the definition of a general hypergeometric function in terms of zonal polynomials can be given.

1.4 The Mathai's Matrix Transform Technique:

This elegant technique was introduced by Mathai [47] in 1978. I will briefly discuss this technique here and use it throughout the present thesis to establish many results in the succeeding chapters.

A generalized matrix transform or M-transform of a function $f(X)$ of a $(p \times p)$ real symmetric positive definite or strictly negative definite matrix X is defined as follows.

$$M_f(s) = \int_{X>0} |X|^{s-(p+1)/2} f(X) dX, \qquad \text{for} X > 0 \qquad (1.22)$$

whenever $M_f(s)$ exists where $f(X)$ is assumed to be a symmetric function in the sense $f(BX) = f(XB) = f(B^{1/2} X B^{1/2})$, for $B = B' > 0$. When $X < 0$ we replace X by $-X$ in eq.(1.22). It is to be noted that when $p = 1$ eq.(1.22) is the Mellin transform of $f(X)$. Since, s is scalar and X is a $(p \times p)$ symmetric matrix with $p(p+1)/2$ variables, $M_f(s)$ is not a multivariable case of the Mellin transform. Hence, for $p > 1$ the transform $M_f(s)$ need not uniquely determine $f(X)$. But in the hypergeometrixc family of functions of matrix arguments one can show that the family of functions $f(X)$ defined by eq.(1.22) have properties analogous to the ones enjoyed by the hypergeometric functions of scalar arguments. Hence we will define hypergeometric functions of matrix arguments by using eq.(1.22) and extend a large number of properties of hypergeometric functions of scalar arguments to those of matrix arguments.

I now state below some results without proof, which have been used by me in the present work in order to prove a number of results.

Theorem 1.4.1: For X and Y real symmetric positive definite $(p \times p)$ matrices and b a scalar,

$$\lim_{b \to \infty} \left| I + \frac{XY}{b} \right|^{-b} = e^{-tr(XY)} \qquad (1.23)$$

Theorem 1.4.2: For α large and ρ finite,

$$\frac{\Gamma_p(\alpha - \rho)}{\Gamma_p(\alpha)} \approx \alpha^{-p\rho} \qquad (1.24)$$

Theorem 1.4.3: Let $f(X)$ be a symmetric function in the sense $f(AX)=f(XA)$ where X and A are $(p\times p)$ real symmetric positive definite matrices. Under interchangeability of limits and integrals,

$$M\left[\lim_{\alpha\to\infty} f\left(\frac{X}{\alpha}\right)\right]=\lim_{\alpha\to\infty}\alpha^{p\rho}M\left[f(X)\right] \tag{1.25}$$

where $M[\ ]$ denotes the M-transform with respect to the parameter ρ.

When extending scalar variable cases to matrix cases sometimes we may be required to use a multiplication formula for the M-transform which is stated below:

Theorem 1.4.4: Let $f_1(X)$ and $f_2(X)$ be symmetric functions in the sense $f_j(XY)=f_j(YX);\ j=1,2;$ where X and Y are $(p\times p)$ real symmetric positive definite matrices. Let the M-transforms with respect to the parameter ρ be $M\left[f_1(X)\right]=g_1(\rho), M\left[f_2(X)\right]=g_2(\rho)$. Let

$$f_3(X)=\int_{Y>0}|Y|^{\beta}f_1(XY)f_2(Y)dY,$$

then the M-transform of f_3 is given by,

$$M\left[f_3(X)\right]=g_1(\rho)g_2\left(\frac{p+1}{2}+\beta-\rho\right) \tag{1.26}$$

Almost all results on scalar variable hypergeometric function can be extended to the corresponding matrix variable case by using the above technique. Some of the cases which do not have direct extensions are the following:

(i) Results which utilize the multiplication formula for the Gamma function. In the scalar case we have,

$$\Gamma(mz)=(2\pi)^{(1-m)/2}m^{mz-1/2}\prod_{j=1}^{m}\Gamma\left(z+\frac{j-1}{m}\right) \tag{1.27}$$

for $m=1,2,\cdots$. For $\Gamma_p(mz), m=1,2,\cdots$ there is no corresponding formula for keeping the same p except for $p=2$. Here the structure is different from that of eq.(1.27) and hence the generalized results will look completely different.

(ii) Results involving fractional powers, higher powers, products etc.
In some of these cases Jacobians of transformations are not available in the literature and when available they are not in a convenient form to rewrite in matrix forms. Even for a simple case $Y = X^2$ when X is a $(p\times p)$ real symmetric matrix,

$$Y = X^2 \Rightarrow dY = 2^p |X| \left\{ \prod_{i<j=1}^{p} |\lambda_i + \lambda_j| \right\} dX \tag{1.28}$$

where $\lambda_1, \cdots, \lambda_p$ are the eigenvalues of X such that $\lambda_i + \lambda_j \neq 0$ for all i and j. It may be noted that we do not have a convenient representation for the right side of eq.(1.28) in terms of Y.

Now I propose to give some basic results concerning the functions of matrix arguments which have been thoroughly utilized in the present thesis.

1.5 The matrix variable cases pertaining to the Gamma and Beta functions and the Dirichlet's integrals for matrix arguments will be stated here.

Theorem 1.5.1: Matrix-variate Gamma: For a $(p\times p)$ real symmetric positive definite matrix X the matrix-variate Gamma, denoted by $\Gamma_p(\alpha)$, is defined by,

$$\Gamma_p(\alpha) = \int_{X>0} e^{-tr(X)} |X|^{\alpha-(p+1)/2} dX, \quad \text{for } Re(\alpha) > (p-1)/2. \tag{1.29}$$

where,

$$\Gamma_p(\alpha) = \pi^{p(p-1)/4} \Gamma(\alpha) \Gamma\left(\alpha - \frac{1}{2}\right) \cdots \Gamma\left(\alpha - \frac{p-1}{2}\right); \quad \text{for } Re(\alpha) > (p-1)/2. \tag{1.30}$$

Theorem 1.5.2: A generalization of the above result for $B = B' > 0$ is,

$$|B|^{-\alpha} \Gamma_p(\alpha) = \int_{X>0} e^{-tr(BX)} |X|^{\alpha-(p+1)/2} dX, \quad \text{for } Re(\alpha) > (p-1)/2. \tag{1.31}$$

Theorem 1.5.3: Matrix-variate Beta (type-1): The type-1 matrix- variate Beta function is defined by the following integral representation:

$$B_p(\alpha,\beta) = \int_0^I |X|^{\alpha-(p+1)/2} |I-X|^{\beta-(p+1)/2} dX = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \quad (1.32)$$

for $\mathrm{Re}(\alpha,\beta) > (p-1)/2$ and $0 < X < I$.

Theorem 1.5.4: Matrix-variate Beta (type-2): The type-2 matrix- variate Beta function is defined by the following integral representation:

$$B_p(\alpha,\beta) = \int_{Y>0} |Y|^{\alpha-(p+1)/2} |I+Y|^{-(\alpha+\beta)} dY = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \quad (1.33)$$

for $\mathrm{Re}(\alpha,\beta) > (p-1)/2$ and $Y=Y' > 0$.

Theorem 1.5.5: Dirichlet integral (type-1) for matrix variables:

$$\left.\begin{aligned}
&\int\cdots(k)\cdots\int |X_1|^{\alpha_1-(p+1)/2} \cdots |X_k|^{\alpha_k-(p+1)/2} \times \\
&|I - X_1 - \cdots - X_k|^{\alpha_{k+1}-(p+1)/2} dX_1 \cdots dX_k \\
&= \frac{\Gamma_p(\alpha_1)\cdots\Gamma_p(\alpha_{k+1})}{\Gamma_p(\alpha_1+\cdots+\alpha_{k+1})} \\
&\text{for } \mathrm{Re}(\alpha_j) > (p-1)/2, 0 < X_j = X_j' < I, j = 1,\cdots,k; \\
&\text{and } 0 < X_1 + \cdots + X_k < I.
\end{aligned}\right\} \quad (1.34)$$

Theorem 1.5.6: Dirichlet integral (type-2) for matrix variables:

$$\begin{aligned}
&\int_{X_1>0}\cdots(k)\cdots\int_{X_k>0} |X_1|^{\alpha_1-(p+1)/2} \cdots |X_k|^{\alpha_k-(p+1)/2} \times \\
&|I + X_1 + \cdots + X_k|^{-(\alpha_1+\cdots+\alpha_{k+1})} dX_1 \cdots dX_k
\end{aligned}$$

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$$= \frac{\Gamma_p(\alpha_1)\cdots\Gamma_p(\alpha_{k+1})}{\Gamma_p(\alpha_1+\cdots+\alpha_{k+1})} \tag{1.35}$$

$$\text{for } \mathrm{Re}(\alpha_j) > (p-1)/2, 0 < X_j = X_j', j = 1, \cdots, k.$$

1.6 Outline of the Programme:

After having a survey of the concerned literature, in this introductory chapter, now I am embarking upon the abstract outcome of the present thesis. The second chapter deals with the ${}_1F_1$, ${}_2F_1$, ${}_rF_s$ and $U(a;b;-Z)$ functions of single matrix argument, while, the Appell's and the Humbert's functions of matrix arguments have been studied in the third chapter. The Kampé de Fériet function, the Lauricella functions and the other related functions of matrix arguments form the subject matter of the fourth chapter. In the fifth chapter the ten Lauricella-Saran triple hypergeometric functions in company with the three Srivastava functions have been studied and some transformation relations and cases of reducibility have also been discussed. I have defined the Exton's twenty one quadruple hypergeometric functions of matrix arguments in the sixth chapter and have established a number of results for these functions including some transformation relations and cases of reducibility. The Exton's ${}^{(k)}_{(1)}E^{(n)}_D$ and ${}^{(k)}_{(2)}E^{(n)}_D$ functions and the Chandel's ${}^{(k)}_{(1)}E^{(n)}_C$ function for the case of matrix arguments have been incorporated in the seventh chapter along with some transformation relations and cases of reducibility. The Horn's ${}^{(k)}H^{(n)}_3$ and ${}^{(k)}H^{(n)}_4$ functions for matrix arguments have been defined in chapter eighth. It is interesting to note that some results of this chapter hold good only for the case $p = 2$ while, others are valid for all the finite values of p. The concluding chapter of the thesis is aimed at describing the generalized Srivastava functions $H^{(n)}_B$ and $H^{(n)}_C$ of matrix arguments and establishment of three results concerning them.

CHAPTER II

FUNCTIONS OF SINGLE MATRIX ARGUMENT

2.1 The functions of single matrix argument have earlier been studied by many workers, viz. Herz [22], James [23-25], Dhami [10,12], Joshi and Joshi [26] and Mathai [45,49,62,63]. In the present chapter I again study some of these functions of matrix arguments which include the ${}_1F_1$, ${}_2F_1$, ${}_rF_s$ and $U(a;b;-Z)$ functions. I have utilized some of the definitions and results of Mathai [62], to establish the results here. A short proof of the well-known Meijer's integral for the function $U(a;d;-X)$ has been given and the confluence of Joshi and Joshi [26] has also been considered again and a proof of it has been given by using the matrix transform (M-transform) technique.

2.2 Preliminary Definitions and Results

Definition 2.2.1: The ${}_rF_s$ function of matrix argument

$$ {}_rF_s = {}_rF_s(a_1,\cdots,a_r;b_1,\cdots,b_s;-X) $$

is defined as that class of functions which has the following M-transform:

$$ M({}_rF_s) = \int_{X>0} |X|^{\rho-(p+1)/2}\, {}_rF_s(a_1,\cdots,a_r;b_1,\cdots,b_s;-X)dX $$

$$ = \frac{\left\{\prod_{j=1}^{s}\Gamma_p(b_j)\right\}\left\{\prod_{j=1}^{r}\Gamma_p(a_j-\rho)\right\}}{\left\{\prod_{j=1}^{r}\Gamma_p(a_j)\right\}\left\{\prod_{j=1}^{s}\Gamma_p(b_j-\rho)\right\}}\Gamma_p(\rho) \tag{2.1} $$

for $\mathrm{Re}(a_1-\rho,\cdots,a_r-\rho,b_1-\rho,\cdots,b_s-\rho,\rho)>(p-1)/2.$

Theorem 2.2.1:

$$ {}_2F_1(a,b;c;-X) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}\int_0^I |Y|^{a-(p+1)/2} \times $$

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$$\left.\begin{array}{l}|I-Y|^{c-a-(p+1)/2}|I+XY|^{-b}\,dY \\ \text{for } \mathrm{Re}(a,c-a)>(p-1)/2.\end{array}\right\} \tag{2.2}$$

Theorem 2.2.2:

$${}_1F_1(a;b;-X)=\frac{\Gamma_p(b)}{\Gamma_p(a)\Gamma_p(b-a)}\int_0^I |T|^{a-(p+1)/2}\times$$

$$|I-T|^{b-a-(p+1)/2}e^{-\mathrm{tr}(XT)}dT \tag{2.3}$$

for $\mathrm{Re}(a,b-a)>(p-1)/2.$

Theorem 2.2.3:

The M-transform of ${}_rF_s={}_rF_s[a_1,\cdots,a_r;b_1,\cdots,b_s;-(X_1+\cdots+X_n)]$ is the following:

$$M({}_rF_s)=\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$

$${}_rF_s[a_1,\cdots,a_r;b_1,\cdots,b_s;-(X_1+\cdots+X_n)]dX_1\cdots dX_n$$

$$=\frac{\left\{\prod_{j=1}^{s}\Gamma_p(b_j)\right\}\left\{\prod_{k=1}^{n}\Gamma_p(\rho_k)\right\}}{\left\{\prod_{i=1}^{r}\Gamma_p(a_i)\right\}}\frac{\left\{\prod_{i=1}^{r}\Gamma_p(a_i-\rho_1-\cdots-\rho_n)\right\}}{\left\{\prod_{j=1}^{s}\Gamma_p(b_j-\rho_1-\cdots-\rho_n)\right\}} \tag{2.4}$$

for $\mathrm{Re}(\rho_k,a_i-\rho_1-\cdots-\rho_n,b_j-\rho_1-\cdots-\rho_n)>(p-1)/2,$ where,

$k=1,\cdots,n;\ j=1,\cdots,s;\ i=1,\cdots,r.$

Now I give my definition for the confluent hypergeometric function $U(a;b;-Z)$ of second kind with matrix argument, which shall be utilized to prove many interesting results concerning this function in this chapter.

Definition 2.2.2:
The confluent hypergeometric function $U(a;b;-Z)$ of second kind with matrix argument is defined as that class of functions for which the M-transform is the following:

$$M[U(a;b;-Z)] = \int_{Z>0} |Z|^{\rho-(p+1)/2} U(a;b;-Z)dZ$$
$$= \frac{\Gamma_p(a-\rho)\Gamma_p[(p+1)/2-b+\rho]\Gamma_p(\rho)}{\Gamma_p(a)\Gamma_p[(p+1)/2+a-b]} \tag{2.5}$$
for $\operatorname{Re}[a-\rho, (p+1)/2-b+\rho, \rho] > (p-1)/2.$

2.3 With the help of the definitions and results of the previous section now I proceed to prove a number of results for the functions already quoted in section 2.1.

Theorem 2.3.1:

$$\left\{\begin{array}{l}
{}_rF_s(a_1,\cdots,a_r;b_1,\cdots,b_s;-X) \\
= \prod_{i=1}^{m}\left\{\dfrac{\Gamma_p(\alpha_i)}{\Gamma_p(a_i)\Gamma_p(\alpha_i-a_i)}\right\}\prod_{j=1}^{n}\left\{\dfrac{\Gamma_p(b_j)}{\Gamma_p(\beta_j)\Gamma_p(b_j-\beta_j)}\right\}\times \\
\int_0^I\cdots\int_0^I \prod_{i=1}^{m}\left\{|U_i|^{a_i-(p+1)/2}|I-U_i|^{\alpha_i-a_i-(p+1)/2}\right\}\times \\
\prod_{j=1}^{n}\left\{|V_j|^{\beta_j-(p+1)/2}|I-V_j|^{b_j-\beta_j-(p+1)/2}\right\}{}_rF_s(\alpha_1,\cdots,\alpha_m, \\
a_{m+1},\cdots,a_r;\beta_1,\cdots,\beta_n,b_{n+1},\cdots,b_s;-V_n^{1/2}\cdots V_1^{1/2}U_m^{1/2}\cdots U_1^{1/2}X\times \\
U_1^{1/2}\cdots U_m^{1/2}V_1^{1/2}\cdots V_n^{1/2})dU_1\cdots dU_m dV_1\cdots dV_n \\
\text{for } \operatorname{Re}(\alpha_i-a_i,a_i,b_j-\beta_j,\beta_j) > (p-1)/2;\ \text{where, } m\le r, n\le s, \\
i=1,\cdots,m;\ j=1,\cdots,n.
\end{array}\right\} \tag{2.6}$$

Proof: Taking the M-transform of the right side of eq.(2.6) with respect to the variable X and the parameter ρ we have,

$$\left\{\begin{array}{l} \int_{X>0} |X|^{\rho-(p+1)/2} {}_rF_s(\alpha_1,\cdots,\alpha_m,a_{m+1},\cdots,a_r;\beta_1,\cdots,\beta_n, \\ b_{n+1},\cdots,b_s;-V_n^{1/2}\cdots V_1^{1/2}U_m^{1/2}\cdots U_1^{1/2}X\times \\ U_1^{1/2}\cdots U_m^{1/2}V_1^{1/2}\cdots V_n^{1/2})dX \end{array}\right\} \quad (2.7)$$

which, under the transformation

$$X_1 = V_n^{1/2}\cdots V_1^{1/2}U_m^{1/2}\cdots U_1^{1/2}XU_1^{1/2}\cdots U_m^{1/2}V_1^{1/2}\cdots V_n^{1/2};\ \text{with,}$$

$$dX_1 = \prod_{j=1}^{n}\left\{\left|V_j\right|^{(p+1)/2}\right\}\prod_{i=1}^{m}\left\{\left|U_i\right|^{(p+1)/2}\right\}dX;\ \text{and,}$$

$$\left|X_1\right| = \prod_{j=1}^{n}\left\{\left|V_j\right|\right\}\prod_{i=1}^{m}\left\{\left|U_i\right|\right\}|X|$$

and using the definition 2.2.1 yields,

$$\begin{array}{l} \prod_{i=1}^{m}\left\{\frac{\Gamma_p(\alpha_i-\rho)}{\Gamma_p(\alpha_i)}\left|U_i\right|^{-\rho}\right\}\prod_{j=1}^{n}\left\{\frac{\Gamma_p(\beta_j)}{\Gamma_p(\beta_j-\rho)}\left|V_j\right|^{-\rho}\right\}\times \\ \frac{\Gamma_p(a_{m+1}-\rho)}{\Gamma_p(a_{m+1})}\cdots\frac{\Gamma_p(a_r-\rho)}{\Gamma_p(a_r)}\frac{\Gamma_p(b_{n+1})}{\Gamma_p(b_{n+1}-\rho)}\cdots\frac{\Gamma_p(b_s)}{\Gamma_p(b_s-\rho)}\Gamma_p(\rho) \end{array} \quad (2.8)$$

Substituting this expression on the right side of eq.(2.6) and integrating out $U_i, V_j (i=1,\cdots,m; j=1,\cdots,n)$ by using a type-1 Beta integral yields $M({}_rF_s)$, as given by eq.(2.1).

Theorem 2.3.2:

$$\int_{T>0} e^{-tr(ST)}|T|^{\nu-(p+1)/2} {}_rF_s(a_1,\cdots,a_r;b_1,\cdots,b_s;$$
$$-T^{1/2}KT^{1/2})dT$$

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$$= \Gamma_p(\nu)|S|^{-\nu} \; {}_{r+1}F_s(a_1,\cdots,a_r,\nu;b_1,\cdots,b_s;-S^{-1/2}KS^{-1/2}) \tag{2.9}$$

for $\mathrm{Re}(\nu) > (p-1)/2, r < s, S > 0;$ or, $r = s, \mathrm{Re}(\nu) > (p-1)/2, S - K > 0.$

Proof: Taking the M-transform of the right side of eq.(2.9) with respect to the variable K and the parameter ρ and applying the transformation $L = S^{-1/2}KS^{-1/2}$, followed by the use of definition (2.2.1) leads to-

$$|S|^{\rho-\nu} \frac{\left\{\prod_{j=1}^{s}\Gamma_p(b_j)\right\}\left\{\prod_{j=1}^{r}\Gamma_p(a_j-\rho)\right\}}{\left\{\prod_{j=1}^{r}\Gamma_p(a_j)\right\}\left\{\prod_{j=1}^{s}\Gamma_p(b_j-\rho)\right\}}\Gamma_p(\nu-\rho)\Gamma_p(\rho) \tag{2.10}$$

The same result is obtained if we take the M-transform of the left side of eq.(2.9) with respect to the variable K and the parameter ρ and apply the transformation $Z = T^{1/2}KT^{1/2}$, use the definition (2.2.1) and finally integrate out T by using a Gamma integral.

Theorem 2.3.3:

$$U(a;b;-X) = \frac{1}{\Gamma_p(a)}\int_{T>0} e^{-\mathrm{tr}(XT)}|T|^{a-(p+1)/2} \times$$
$$|I+T|^{-[(p+1)/2+a-b]}\,dT \tag{2.11}$$

Proof: Taking the M-transform of the right side of eq.(2.11) with respect to the variable X and the parameter ρ, then integrating out X by using a Gamma integral and T by using a type-2 Beta integral yields $M[U(a;b;-X)]$, as given by eq.(2.5).

It is interesting to note that this theorem provides a proof of eq.(2), section2, page 628 of Joshi and Joshi [26], which Joshi and Joshi have taken as their definition of the confluent hypergeometric function of second kind with matrix argument.

Theorem 2.3.4:

(i)
$$\left\{\begin{array}{l} U(a;b;-X)=\dfrac{|X|^{(p+1)/2-b}}{\Gamma_p(a)}\displaystyle\int_{V>0} e^{-tr(V)}\,|V|^{a-(p+1)/2}\times \\ |X+V|^{-[a-b+(p+1)/2]}\,dV \end{array}\right\} \quad (2.12)$$

(ii)
$$\left\{\begin{array}{l} U(a;b;-X)=\dfrac{e^{tr(X)}}{\Gamma_p(a)}\displaystyle\int_{W>I} e^{-tr(XW)}\,|W-I|^{a-(p+1)/2}\times \\ |W|^{b-a-(p+1)/2}\,dW \end{array}\right\} \quad (2.13)$$

(iii)
$$\left\{\begin{array}{l} U(a;b;-X)=\dfrac{2^{p[(p+1)/2-b]}e^{tr(X)/2}}{\Gamma_p(a)}\displaystyle\int_{S>I} e^{-tr(SX)/2}\times \\ |S-I|^{a-(p+1)/2}\,|S+I|^{b-a-(p+1)/2}\,dS \end{array}\right\} \quad (2.14)$$

(iv)
$$\left\{\begin{array}{l} U(a;b;-X)=\dfrac{|D-C|^{b-a}\,e^{tr(X)}}{\Gamma_p(a)}\times \\ \displaystyle\int_C^D e^{-tr[X(D-C)^{1/2}(D-V)^{-1}(D-C)^{1/2}]}\times \\ |D-V|^{-b}\,|V-C|^{a-(p+1)/2}\,dV \\ \text{where, C and D are constant matrices such that } 0<C<V<D. \end{array}\right\} \quad (2.15)$$

Proof: (i) This result can be had from eq.(2.11) by the application of the transformation $V = X^{1/2}TX^{1/2}$. This result is analogous to eq.(4), section 2, page 629 of Joshi and Joshi [26].

(ii) We apply the transformation $W = I + T$, where, $T > 0$, to eq.(2.11) to see this result.

(iii) In eq.(2.11) when we apply the transformation $S = I + 2T$, this result is obtained.

(iv) In eq.(2.13) the application of the transformation, $V = D-(D-C)^{1/2}W^{-1}(D-C)^{1/2}$, followed by the use of theorem 30, page 36, of Mathai [57] leads to this result.

Theorem 2.3.5:

(i)
$$ {}_1F_1(a;b;-X)=\frac{|X|^{(p+1)/2-b}\,\Gamma_p(b)}{\Gamma_p(a)\Gamma_p(b-a)}\int_0^X |V|^{a-(p+1)/2}\times $$
$$ |X-V|^{b-a-(p+1)/2}\,e^{-tr(V)}dV \qquad (2.16) $$
for $\mathrm{Re}(a, b-a) > (p-1)/2.$

(ii)
$$ {}_1F_1(a;b;-X)=e^{-tr(X)}\,{}_1F_1[b-a;b;X] \qquad (2.17) $$

(iii)
$$ \left\{\begin{aligned} &{}_1F_1(a;b;-X)=\frac{2^{p[(p+1)/2-b]}\Gamma_p(b)e^{-tr(X)/2}}{\Gamma_p(a)\Gamma_p(b-a)}\int_{-I}^{I} e^{tr(XS)/2}\times \\ &|I-S|^{a-(p+1)/2}\,|I+S|^{b-a-(p+1)/2}\,dS \end{aligned}\right\} \qquad (2.18) $$

(iv)
$$ \left\{\begin{aligned} &{}_1F_1(a;b;-X)=\frac{2^{p[(p+1)/2-b]}\Gamma_p(b)e^{-tr(X)/2}}{\Gamma_p(a)\Gamma_p(b-a)}\int_{-I}^{I} e^{-tr(XS)/2}\times \\ &|I+S|^{a-(p+1)/2}\,|I-S|^{b-a-(p+1)/2}\,dS \end{aligned}\right\} \qquad (2.19) $$

(v)
$$ \left\{\begin{aligned} &{}_1F_1(a;b;-X)=\frac{|D-C|^{(p+1)/2-b}\,\Gamma_p(b)}{\Gamma_p(a)\Gamma_p(b-a)}\times \\ &\int_C^D e^{-tr[X(D-C)^{-1/2}(U-C)(D-C)^{-1/2}]}\times \\ &|D-U|^{b-a-(p+1)/2}\,|U-C|^{a-(p+1)/2}\,dU \\ &\text{where, C and D are constant matrices such that } 0<C<U<D. \end{aligned}\right\} \qquad (2.20) $$

Proof: (i) By applying the transformation $V = X^{1/2}TX^{1/2}$, to theorem 2.2.2 this result is obtained.

(ii) This result is the extension of the Kummer's first theorem (see for instance, Slater [100], eq.(1.4.1), page 6) for the matrix arguments case and can be had from theorem 2.2.2 by using the transformation $T = I - W$ in it and interpreting the resulting expression in the light of the same theorem.

Similarly the following transformations are to be applied to the theorem 2.2.2 for obtaining the remaining results:

(iii) $S = I - 2T$; (iv) $S = 2T - I$; (v) $U = C + (D - C)^{1/2} T (D - C)^{1/2}$.

Theorem 2.3.6:

$$\left.\begin{aligned}
&\int_{T>0} e^{-\mathrm{tr}(ST)} U(a;c;-T)|T|^{b-(p+1)/2}\, dT \\
&= \frac{\Gamma_p(b)\Gamma_p[(p+1)/2+b-c]}{\Gamma_p[(p+1)/2+a+b-c]} \times {}_2F_1[b,(p+1)/2+b-c; \\
&(p+1)/2+a+b-c;-(S-I)] \\
&\text{for } \mathrm{Re}[b,(p+1)/2+b-c,(p+1)/2+a+b-c] > (p-1)/2.
\end{aligned}\right\} \tag{2.21}$$

Proof: The application of the theorem 2.2.1 to the ${}_2F_1$ function on the right side of eq.(2.21) yields,

$$\begin{aligned}
&{}_2F_1[b,(p+1)/2+b-c;(p+1)/2+a+b-c;-(S-I)] \\
&= \frac{\Gamma_p[(p+1)/2+a+b-c]}{\Gamma_p(b)\Gamma_p[(p+1)/2+a-c]} \int_0^I |Y|^{b-(p+1)/2} \times \\
&|I-Y|^{a-b-(p+1)/2} \left| I + (I-Y)^{-1/2} Y^{1/2} S \times \right. \\
&\left. Y^{1/2}(I-Y)^{-1/2} \right|^{-[(p+1)/2+b-c]} dY
\end{aligned} \tag{2.22}$$

which is to be substituted on the right side of eq.(2.21) for the ${}_2F_1$ function, then we take the M-transform of the modified form of the right side of eq.(2.21) with respect to the variable S and the parameter ρ, consequently, apply the transformation $S_1 = (I-Y)^{-1/2} Y^{1/2} S Y^{1/2} (I-Y)^{-1/2}$ and integrate out S_1 by using a type-2 Beta integral and Y by using a type-1 Beta integral to achieve-

$$\frac{\Gamma_p[(p+1)/2+b-c-\rho]\Gamma_p(a-b+\rho)\Gamma_p(b-\rho)\Gamma_p(\rho)}{\Gamma_p(a)\Gamma_p[(p+1)/2+a-c]} \tag{2.23}$$

The same result is obtained by taking the M-transform of the left side of eq.(2.21) with respect to the variable S and the parameter ρ and subsequently using the definition (2.2.2) in it.

Theorem 2.3.7:

$$\int_{T>0} e^{-tr(ST)} U(a;c;-T)|T|^{b-(p+1)/2}\, dT$$
$$= \frac{\Gamma_p(b)\Gamma_p[(p+1)/2+b-c]}{\Gamma_p[(p+1)/2+a+b-c]} |S|^{-b}\ {}_2F_1[a,b;(p+1)/2+a+b-c; \tag{2.24}$$
$$-(S^{-1}-I)]$$

for $\mathrm{Re}[b,(p+1)/2+b-c,(p+1)/2+a+b-c] > (p-1)/2.$

Proof: This theorem follows in the same manner as the theorem (2.3.6), the ${}_2F_1$ function on the right side of eq.(2.24) is to be replaced by the following expression which is achieved by the application of the theorem (2.2.1) to the ${}_2F_1$ function on the right side of eq.(2.24):

$${}_2F_1[a,b;(p+1)/2+a+b-c;-(S^{-1}-I)]$$
$$= \frac{\Gamma_p[(p+1)/2+a+b-c]}{\Gamma_p(a)\Gamma_p[(p+1)/2+b-c]} \int_0^I |Y|^{a-(p+1)/2} \times$$
$$|I-Y|^{(p+1)/2-c-(p+1)/2} \left| I+(I-Y)^{-1/2} Y^{1/2} S^{-1} \times \right. \tag{2.25}$$
$$\left. Y^{1/2}(I-Y)^{-1/2} \right|^{-b} dY$$

Theorem 2.3.8: Meijer's Integral:

$$U(a;d;-X)=\frac{|X|^{b-a}}{\Gamma_p(b)}\int_{T>0}e^{-tr(XT)}|T|^{b-(p+1)/2}\times \quad (2.26)$$
$${}_2F_1[a,(p+1)/2+a-d;b;-T]dT$$

Proof: Taking the M-transform of the right side of eq.(2.26) with respect to the variable X and the parameter ρ and integrating out X by using a Gamma integral then writing the M-transform of the ${}_2F_1$ function with parameter $(a-\rho)$ we obtain $M[U(a;d;-X)]$ as given by the definition 2.2.2.

Theorem 2.3.9:

$$\int_{X>0}|X|^{s-(p+1)/2}e^{-tr(AX)}U(b;d;-X^{1/2}KX^{1/2})dX$$
$$=\frac{\Gamma_p(s)\Gamma_p[(p+1)/2+s-d]}{\Gamma_p[(p+1)/2+b+s-d]}|A|^{-s}\;{}_2F_1[b,s;(p+1)/2+b+s-d; \quad (2.27)$$
$$-(A^{-1/2}KA^{-1/2}-I)]$$

for $\text{Re}[s,(p+1)/2+s-d,(p+1)/2+b+s-d]>(p-1)/2.$

Proof: We apply the theorem 2.2.1 to replace the ${}_2F_1$ function on the right side of eq.(2.27) by the following integral:

$${}_2F_1[b,s;(p+1)/2+b+s-d;-(A^{-1/2}KA^{-1/2}-I)]$$
$$=\frac{\Gamma_p[(p+1)/2+b+s-d]}{\Gamma_p(b)\Gamma_p[(p+1)/2+s-d]}\int_0^I|Y|^{b-(p+1)/2}\times$$
$$|I-Y|^{(p+1)/2-d-(p+1)/2}\left|I+(I-Y)^{-1/2}Y^{1/2}A^{-1/2}KA^{-1/2}\times \right. \quad (2.28)$$
$$\left. Y^{1/2}(I-Y)^{-1/2}\right|^{-s}dY$$

then taking the M-transform of the right side of eq.(2.27) with respect to the variable K and the parameter ρ, which under the use of the transformation,

$$K_1 = (I-Y)^{-1/2} Y^{1/2} A^{-1/2} K A^{-1/2} Y^{1/2} (I-Y)^{-1/2}$$

and integrating out K_1 by employing a type-2 Beta integral and Y by a type-1 Beta integral lends,

$$|A|^{\rho-s} \frac{\Gamma_p[(p+1)/2-d+\rho]\Gamma_p(s-\rho)\Gamma_p(b-\rho)\Gamma_p(\rho)}{\Gamma_p(b)\Gamma_p[(p+1)/2-d+b]} \qquad (2.29)$$

This result is also achieved by taking the M-transform of the left side of eq.(2.27) with respect to the variable K and the parameter ρ under the transformation $K_2 = X^{1/2} K X^{1/2}$, and using the definition 2.2.2 and integrating out X by the help of a Gamma integral.

It is interesting to note that the left side of eq.(2.27) may be treated as the M-transform of the function $e^{-tr(AX)} U(b; d; -X^{1/2} K X^{1/2})$ with respect to the variable X and the parameter s and on the right side it depends on s, the parameter of the transform!

Theorem 2.3.10: The Confluence of Joshi and Joshi [26]:

$$B_{c-(p+1)/2}(Z) = \lim_{a\to\infty} \Gamma_p[a-c+(p+1)/2] U(a; c; \frac{-Z}{a}) \qquad (2.30)$$

Proof: The above confluence has been stated and proved by Joshi and Joshi [26] as eq.(20), page 633. We provide here an alternative proof for this result by using the M-transform technique.

In the left side of eq.(2.30), $B_{c-(p+1)/2}(Z)$ represents the Bessel function of the second kind with matrix argument. This notation was also used earlier by Herz [22], page 506. We use his results to establish the above confluence.

Putting $l = 0$ in eq.$(5.1')$, page 506 of Herz [22], and noting that according to Herz, $B_\delta(Z) = \dfrac{{}_0G_1[\,;\delta+(p+1)/2; Z]}{\Gamma_p[\delta+(p+1)/2]}$, we have

$$B_\delta(Z) = \int_{\Lambda>0} e^{-\mathrm{tr}(\Lambda+\Lambda^{-1}Z)} |\Lambda|^{-[\delta+(p+1)/2]} d\Lambda$$

which yields,

$$\begin{aligned} M[B_\delta(Z)] &= \int_{Z>0} |Z|^{\rho-(p+1)/2} B_\delta(Z) dZ \\ &= \Gamma_p(\rho)\Gamma_p(\rho-\delta) \end{aligned} \tag{2.31}$$

Now we take the M-transform of the right side of eq.(2.30) with respect to the variable Z and the parameter ρ and observing that,

$$\begin{aligned} & M\left\{ \lim_{a\to\infty} \Gamma_p[a-c+(p+1)/2] U(a;c;\frac{-Z}{a}) \right\} \\ &= \lim_{a\to\infty} a^{p\rho} M\left\{ \Gamma_p[a-c+(p+1)/2] U(a;c;-Z) \right\} \end{aligned} \tag{2.32}$$

from theorem 1.4.3, and utilizing the definition 2.2.2 along with the use of the theorem 1.4.2 leads us to $M[B_{c-(p+1)/2}(Z)]$, as given by eq.(2.31) with $\delta = c-(p+1)/2$.

Theorem 2.3.11:

$$\begin{aligned} & \int_{T>0} e^{-\mathrm{tr}(ST)} |T|^{b-(p+1)/2} {}_1F_1(a;c;-KT) dT \\ &= \Gamma_p(b) |S+K|^{-b} {}_2F_1[c-a, b; c; K(S+K)^{-1}] \end{aligned} \tag{2.33}$$

for $\mathrm{Re}(b) > (p-1)/2$.

Proof: This result can be had by replacing the ${}_1F_1$ function on the left side of eq.(2.33) by the use of the theorem 2.2.2, consequently applying the transformation $Y_1 = I - Y$, and observing that,

$$|S+(I-Y_1)K| = |S+K| \left| I - Y_1 K (S+K)^{-1} \right|$$

and finally using the theorem 2.2.1.

Letting $K \to I$ in the above expression we have the following interesting result:

$$\int_{T>0} e^{-\mathrm{tr}(ST)} |T|^{b-(p+1)/2} {}_1F_1(a;c;-T) dT$$

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$$= \Gamma_p(b)|S+I|^{-b}\ {}_2F_1[c-a, b; c; (S+I)^{-1}] \tag{2.34}$$

for $\mathrm{Re}(b) > (p-1)/2$.

In a similar manner we can establish the following results:

Theorem 2.3.12:

$$\int_{T>0} e^{-\mathrm{tr}(ST)} |T|^{b-(p+1)/2}\ {}_1F_1(a; b; -T^{1/2}KT^{1/2})dT$$

$$= \Gamma_p(b)|S|^{-b}\ {}_1F_0[a;\ ; -S^{-1/2}KS^{-1/2}] \tag{2.35}$$

for $\mathrm{Re}(b) > (p-1)/2$.

Theorem 2.3.13:

$$\int_{T>0} e^{-\mathrm{tr}(ST)} |T|^{a-(p+1)/2}\ {}_0F_1(\ ; b; -T^{1/2}KT^{1/2})dT$$

$$= \Gamma_p(a)|S|^{-a}\ {}_1F_1[a; b; -S^{-1/2}KS^{-1/2}] \tag{2.36}$$

for $\mathrm{Re}(a) > (p-1)/2$.

Theorem 2.3.14:

$$\int_{T>0} e^{-\mathrm{tr}(ST)} |T|^{a-(p+1)/2}\ {}_0F_1(\ ; a; -T^{1/2}KT^{1/2})dT$$

$$= \Gamma_p(a)|S|^{-a}\ e^{-\mathrm{tr}(KS^{-1})} \tag{2.37}$$

for $\mathrm{Re}(a) > (p-1)/2$.

Theorem 2.3.15:

$$\int_{X>0} e^{-\mathrm{tr}(AX)} |X|^{s-(p+1)/2}\ {}_1F_1(b; d; -X^{1/2}KX^{1/2})dX$$

$$= \Gamma_p(s)|A|^{-s}\ {}_2F_1[b, s; d; -A^{-1/2}KA^{-1/2}] \tag{2.38}$$

for $\mathrm{Re}(s) > (p-1)/2$.

Theorem 2.3.16:

$$\int_{X>0} e^{-\mathrm{tr}(X)} |X|^{s-(p+1)/2} {}_0F_1(\ ;b;-X^{1/2}YX^{1/2})dX$$

$$= \Gamma_p(s) \times {}_1F_1[s;b;-Y] \tag{2.39}$$

for $\mathrm{Re}(s) > (p-1)/2$.

Theorem 2.3.17:

$$\int_{X>0} e^{-\mathrm{tr}(KX)} |X|^{\rho-(p+1)/2} {}_rF_s(a_1,\cdots,a_r;b_1,\cdots,b_s;$$

$$-X^{1/2}YX^{1/2})dX \tag{2.40}$$

$$= \Gamma_p(\rho)|K|^{-\rho} {}_{r+1}F_s(a_1,\cdots,a_r,\rho;b_1,\cdots,b_s;-K^{-1/2}YK^{-1/2})$$

CHAPTER III

APPELL'S AND HUMBERT'S FUNCTIONS OF MATRIX ARGUMENTS

3.1 Definitions

In this section I first quote the Mathai's definitions of the four Appell's and the seven Humbert's functions of matrix arguments which will be utilized by me in this and the forthcoming chapters of this thesis for proving a number of results.

3.1.1 The Appell's function F_1 of matrix arguments

$$F_1 = F_1(a, b, b'; c; -X, -Y)$$

is defined as that class of functions which has the following matrix transform:

$$M(F_1) = \int_{X>0}\int_{Y>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \times$$

$$F_1(a, b, b'; c; -X, -Y)dXdY$$

$$= \frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1-\rho_2)\Gamma_p(b-\rho_1)\Gamma_p(b'-\rho_2)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-\rho_1-\rho_2)} \qquad (3.1)$$

for Re $(a-\rho_1-\rho_2, b-\rho_1, b'-\rho_2, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2.$

3.1.2 $$F_2 = F_2(a, b, b'; c, c'; -X, -Y)$$

$$M(F_2) = \int_{X>0}\int_{Y>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \times$$

$$F_2(a, b, b'; c, c'; -X, -Y)dXdY$$

$$= \frac{\Gamma_p(c)\Gamma_p(c')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1-\rho_2)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-\rho_1)\Gamma_p(c'-\rho_2)} \times$$

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$$\Gamma_p(b-\rho_1)\Gamma_p(b'-\rho_2)$$

for Re $(a-\rho_1-\rho_2, b-\rho_1, b'-\rho_2, c-\rho_1, c'-\rho_2, \rho_1, \rho_2) > (p-1)/2.$ (3.2)

3.1.3 $F_3 = F_3(a, a', b, b'; c; -X, -Y)$

$$M(F_3) = \int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times F_3(a, a', b, b'; c; -X, -Y)dXdY$$

$$= \frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1)\Gamma_p(a'-\rho_2)}{\Gamma_p(a)\Gamma_p(a')\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-\rho_1-\rho_2)}\times \quad (3.3)$$

$$\Gamma_p(b-\rho_1)\Gamma_p(b'-\rho_2)$$

for Re $(a-\rho_1, a'-\rho_2, b-\rho_1, b'-\rho_2, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2.$

3.1.4 $F_4 = F_4(a, b; c, c'; -X, -Y)$

$$M(F_4) = \int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times F_4(a, b; c, c'; -X, -Y)dXdY$$

$$= \frac{\Gamma_p(c)\Gamma_p(c')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1-\rho_2)\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c-\rho_1)\Gamma_p(c'-\rho_2)} \quad (3.4)$$

for Re $(a-\rho_1-\rho_2, b-\rho_1-\rho_2, c-\rho_1, c'-\rho_2, \rho_1, \rho_2) > (p-1)/2.$

3.1.5 The Humbert's function Φ_1 of matrix arguments

$$\Phi_1 = \Phi_1(a, b; c; -X, -Y)$$

is defined as that class of functions for which the matrix transform is the following:

$$M(\Phi_1) = \int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$

$$\Phi_1(a, b; c; -X, -Y)dXdY$$

$$= \frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1-\rho_2)\Gamma_p(b-\rho_1)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c-\rho_1-\rho_2)} \tag{3.5}$$

for $\mathrm{Re}\,(a-\rho_1-\rho_2, b-\rho_1, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2.$

3.1.6 $$\Phi_2 = \Phi_2(b, b'; c; -X, -Y)$$

$$M(\Phi_2) = \int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$

$$\Phi_2(b, b'; c; -X, -Y)dXdY$$

$$= \frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(b-\rho_1)\Gamma_p(b'-\rho_2)}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-\rho_1-\rho_2)} \tag{3.6}$$

for $\mathrm{Re}\,(b-\rho_1, b'-\rho_2, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2.$

3.1.7 $$\Phi_3 = \Phi_3(b; c; -X, -Y)$$

$$M(\Phi_3) = \int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$

$$\Phi_3(b; c; -X, -Y)dXdY$$

$$= \frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(b-\rho_1)}{\Gamma_p(b)\Gamma_p(c-\rho_1-\rho_2)} \tag{3.7}$$

for $\mathrm{Re}\,(b-\rho_1, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2.$

3.1.8 $$\Psi_1 = \Psi_1(a,b;c,c';-X,-Y)$$

$$M(\Psi_1) = \int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$\Psi_1(a,b;c,c';-X,-Y)dXdY$$
$$= \frac{\Gamma_p(c)\Gamma_p(c')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1-\rho_2)\Gamma_p(b-\rho_1)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c-\rho_1)\Gamma_p(c'-\rho_2)} \tag{3.8}$$

for Re $(a-\rho_1-\rho_2, b-\rho_1, c-\rho_1, c'-\rho_2, \rho_1, \rho_2) > (p-1)/2$.

3.1.9 $$\Psi_2 = \Psi_2(a;c,c';-X,-Y)$$

$$M(\Psi_2) = \int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$\Psi_2(a;c,c';-X,-Y)dXdY$$
$$= \frac{\Gamma_p(c)\Gamma_p(c')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1-\rho_2)}{\Gamma_p(a)\Gamma_p(c-\rho_1)\Gamma_p(c'-\rho_2)} \tag{3.9}$$

for Re $(a-\rho_1-\rho_2, c-\rho_1, c'-\rho_2, \rho_1, \rho_2) > (p-1)/2$.

3.1.10 $$\Xi_1 = \Xi_1(a,a',b;c;-X,-Y)$$

$$M(\Xi_1) = \int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$\Xi_1(a,a',b;c;-X,-Y)dXdY$$
$$= \frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1)\Gamma_p(a'-\rho_2)\Gamma_p(b-\rho_1)}{\Gamma_p(a)\Gamma_p(a')\Gamma_p(b)\Gamma_p(c-\rho_1-\rho_2)} \tag{3.10}$$

for Re $(a-\rho_1, a'-\rho_2, b-\rho_1, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2$.

3.1.11 $$\Xi_2 = \Xi_2(a,b;c;-X,-Y)$$

$$M(\Xi_2) = \int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$\Xi_2(a,b;c;-X,-Y)dXdY$$
$$=\frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1)\Gamma_p(b-\rho_1)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c-\rho_1-\rho_2)} \quad (3.11)$$

for $\mathrm{Re}\,(a-\rho_1, b-\rho_1, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2.$

3.2 In this section I quote some results of Mathai for the functions defined in the previous section, which will be required for proving the results in this and the subsequent chapters.

Theorem 3.2.1:

$$F_1(a,b,b';c;-X,-Y)$$
$$=\frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-b-b')}\iint|U_1|^{b-(p+1)/2}|U_2|^{b'-(p+1)/2}\times$$
$$|I-U_1-U_2|^{c-b-b'-(p+1)/2}\times \quad (3.12)$$
$$\left|I+U_1^{1/2}XU_1^{1/2}+U_2^{1/2}YU_2^{1/2}\right|^{-a}dU_1dU_2$$

for $\mathrm{Re}(b,b',c-b-b') > (p-1)/2.$

Theorem 3.2.2:

$$F_1(a,b,b';c;-X,-Y)$$
$$=\frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}\int_0^I|V|^{a-(p+1)/2}|I-V|^{c-a-(p+1)/2}\times$$

Continued to the next page … … … … … …

$$\left|I + V^{1/2} X V^{1/2}\right|^{-b} \left|I + V^{1/2} Y V^{1/2}\right|^{-b'} dV \qquad (3.13)$$

for $\mathrm{Re}(a, c-a) > (p-1)/2.$

Theorem 3.2.3:

$$F_2(a, b, b'; c, c'; -X, -Y)$$

$$= \frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-b)\Gamma_p(c'-b')} \int_0^I \int_0^I |U_1|^{b-(p+1)/2} \times$$

$$|U_2|^{b'-(p+1)/2} |I - U_1|^{c-b-(p+1)/2} |I - U_2|^{c'-b'-(p+1)/2} \times \qquad (3.14)$$

$$\left|I + U_1^{1/2} X U_1^{1/2} + U_2^{1/2} Y U_2^{1/2}\right|^{-a} dU_1 dU_2$$

for $\mathrm{Re}(b, b', c-b, c'-b') > (p-1)/2.$

Theorem 3.2.4:

$$F_3(a, a', b, b'; c; -X, -Y)$$

$$= \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-b-b')} \iint |V_1|^{b-(p+1)/2} |V_2|^{b'-(p+1)/2} \times$$

$$|I - V_1 - V_2|^{c-b-b'-(p+1)/2} \left|I + V_1^{1/2} X V_1^{1/2}\right|^{-a} \times \qquad (3.15)$$

$$\left|I + V_2^{1/2} Y V_2^{1/2}\right|^{-a'} dV_1 dV_2$$

for $\mathrm{Re}(b, b', c-b-b') > (p-1)/2$.

Theorem 3.2.5:

$$\Phi_1(a, b; c; -X, -Y)$$

$$= \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \int_0^I |V|^{a-(p+1)/2} |I-V|^{c-a-(p+1)/2} \times \tag{3.16}$$

$$\left|I + V^{1/2} X V^{1/2}\right|^{-b} e^{-\mathrm{tr}(VY)} dV$$

for $\mathrm{Re}(a, c-a) > (p-1)/2$.

Theorem 3.2.6:

$$\Phi_2(b, b'; c; -X, -Y)$$

$$= \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-b-b')} \iint |V_1|^{b-(p+1)/2} |V_2|^{b'-(p+1)/2} \times \tag{3.17}$$

$$|I - V_1 - V_2|^{c-b-b'-(p+1)/2} e^{-\mathrm{tr}(V_1X + V_2Y)} dV_1 dV_2$$

for $\mathrm{Re}(b, b', c-b-b') > (p-1)/2$.

Theorem 3.2.7:

$$\Xi_1(a, b', b; c; -X, -Y)$$

$$= \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-b-b')} \iint |V_1|^{b-(p+1)/2} |V_2|^{b'-(p+1)/2} \times \tag{3.18}$$

$$|I - V_1 - V_2|^{c-b-b'-(p+1)/2} \left|I + V_1^{1/2} X V_1^{1/2}\right|^{-a} e^{-\mathrm{tr}(V_2Y)} dV_1 dV_2$$

for $\mathrm{Re}(b, b', c-b-b') > (p-1)/2$.

3.3 In this section we prove some results for the Appell's functions F_1, F_3 and F_4 by using the definitions of the previous section. The proofs of only a small number of results are being given here. Many results have been proved by me in my papers [107-110] listed in the bibliography. Only the results have been listed here for the sake of illustration.

Theorem 3.3.1:

$$F_1(\alpha,\beta,\beta';\gamma;-X,-Y)$$

$$=\frac{\Gamma_p(\gamma)}{\Gamma_p(\beta)\Gamma_p(\beta')\Gamma_p(\gamma-\beta-\beta')}\int_0^I\int_0^I|U|^{\beta+\beta'-(p+1)/2}|V|^{\beta'-(p+1)/2}\times$$

$$|I-U|^{\gamma-\beta-\beta'-(p+1)/2}|I-V|^{\beta-(p+1)/2}\times \tag{3.19}$$

$$\left|I+(I-V)^{1/2}U^{1/2}XU^{1/2}(I-V)^{1/2}+V^{1/2}U^{1/2}YU^{1/2}V^{1/2}\right|^{-\alpha}dUdV$$

for $0<U<I,\ 0<V<I$ and for $\operatorname{Re}(\beta,\beta',\gamma-\beta-\beta')>(p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(3.19) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we obtain,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\left|I+(I-V)^{1/2}U^{1/2}XU^{1/2}(I-V)^{1/2}+V^{1/2}U^{1/2}YU^{1/2}V^{1/2}\right|^{-\alpha}dXdY \tag{3.20}$$

Making use of the transformations,

$$X_1=(I-V)^{1/2}U^{1/2}XU^{1/2}(I-V)^{1/2},\ Y_1=V^{1/2}U^{1/2}YU^{1/2}V^{1/2}$$

with, $dX_1=|I-V|^{(p+1)/2}|U|^{(p+1)/2}dX, dY_1=|V|^{(p+1)/2}\times$

$|U|^{(p+1)/2}dY;$ and, $|X_1|=|I-V||U||X|, |Y_1|=|V||U||Y|$

in the above expression and then integrating out the variables X_1 and Y_1 by using a type-2 Dirichlet integral we get,

$$|U|^{-\rho_1-\rho_2}\,|V|^{-\rho_2}\,|I-V|^{-\rho_1}\,\frac{\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)}{\Gamma_p(\alpha)} \tag{3.21}$$

Substituting this expression on the right side of eq.(3.19) and then integrating out the variables U and V in the resulting expression by using a type-1 Beta integral generates $M(F_1)$ as given by eq.(3.1).

Theorem 3.3.2:

$$\begin{aligned}
&|P|^{-\beta'}\,F_1(\alpha,\beta,\beta';\gamma;-X,-P^{-1/2}YP^{-1/2})\\
&=\frac{1}{\Gamma_p(\beta')}\int_{T>0}e^{-\mathrm{tr}(PT)}\,|T|^{\beta'-(p+1)/2}\times\\
&\Phi_1(\alpha,\beta;\gamma;-X,-T^{1/2}YT^{1/2})dT
\end{aligned} \tag{3.22}$$

for $\mathrm{Re}\,(\beta')>(p-1)/2.$

Theorem 3.3.3:

$$\begin{aligned}
&|P|^{-\beta'}\,F_2(\alpha,\beta,\beta';\gamma,\gamma';-X,-P^{-1/2}YP^{-1/2})\\
&=\frac{1}{\Gamma_p(\beta')}\int_{T>0}e^{-\mathrm{tr}(PT)}\,|T|^{\beta'-(p+1)/2}\times\\
&\Psi_1(\alpha,\beta;\gamma,\gamma';-X,-T^{1/2}YT^{1/2})dT
\end{aligned} \tag{3.23}$$

for $\mathrm{Re}\,(\beta')>(p-1)/2.$

Theorem 3.3.4:

$$F_3(\alpha,\alpha',\beta,\beta';\gamma+\gamma';-X,-Y)$$

$$=\frac{\Gamma_p(\gamma+\gamma')}{\Gamma_p(\gamma)\Gamma_p(\gamma')}\int_0^I |U|^{\gamma-(p+1)/2}|I-U|^{\gamma'-(p+1)/2}\times \tag{3.24}$$

$$_2F_1(\alpha,\beta;\gamma;-U^{1/2}XU^{1/2})\,_2F_1[\alpha',\beta';\gamma';-(I-U)^{1/2}Y(I-U)^{1/2}]dU$$

for $0<U<I$ and for $\operatorname{Re}(\gamma,\gamma')>(p-1)/2.$

Theorem 3.3.5: For $p=2$,

$$|P|^{-\alpha'}F_3[\alpha,(\alpha'+1)/2,\beta,(2\alpha'+1)/4;\gamma;-X,-4P^{-1}YP^{-1}]$$

$$=\frac{1}{\Gamma_p(\alpha')}\int_{T>0}e^{-\operatorname{tr}(PT)}|T|^{\alpha'-(p+1)/2}\Xi_2(\alpha,\beta;\gamma;-X,-TYT')dT \tag{3.25}$$

where $\operatorname{Re}(\alpha')>(p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(3.25) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we obtain,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$\Xi_2(\alpha,\beta;\gamma;-X,-TYT')dXdY \tag{3.26}$$

Applying the transformation $Y_1=TYT'$, with, $dY_1=|T|^{p+1}dY$ and $|Y_1|=|T|^2|Y|$ to the above expression and then using the definition (3.1.11) produces,

$$|T|^{-2\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1)\Gamma_p(\beta-\rho_1)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \tag{3.27}$$

Substituting this expression on the right side of eq.(3.25) and then integrating out T in the resulting expression by using a Gamma integral leads us to,

$$|P|^{-(\alpha'-2\rho_2)}\frac{\Gamma_p(\gamma)\Gamma_p(\alpha-\rho_1)\Gamma_p(\beta-\rho_1)\Gamma_p(\alpha'-2\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)\Gamma_p(\alpha')} \quad (3.28)$$

Now taking the M-transform of the left side of eq.(3.25) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|P|^{-\alpha'}\times \quad (3.29)$$
$$F_3[\alpha,(\alpha'+1)/2,\beta,(2\alpha'+1)/4;\gamma;-X,-4P^{-1}YP^{-1}]dXdY$$

Making use of the transformation,

$$Y_2=4P^{-1}YP^{-1},\text{ with, }dY_2=4^{p(p+1)/2}|P|^{-(p+1)}dY$$

$$\text{and }|Y_2|=4^p|P|^{-2}|Y|$$

in the above expression and then using the definition (3.1.3) along with the observation that for $p=2$,

$$4^{-p\rho_2}\frac{\Gamma_p[(\alpha'+1)/2-\rho_2]\Gamma_p[(2\alpha'+1)/4-\rho_2]}{\Gamma_p[(\alpha'+1)/2]\Gamma_p[(2\alpha'+1)/4]}=\frac{\Gamma_p(\alpha'-2\rho_2)}{\Gamma_p(\alpha')} \quad (3.30)$$

from eq.(6.13) page 84 of Mathai [62], finally leads us to the same result as in eq.(3.28). It is to be noted that this result is different from the corresponding result in the scalar case.

Theorem 3.3.6:

$$|P|^{-\alpha}F_4(\alpha,\beta;\gamma,\gamma';-P^{-1/2}XP^{-1/2},-P^{-1/2}YP^{-1/2})$$
$$=\frac{1}{\Gamma_p(\alpha)}\int_{T>0}e^{-tr(PT)}|T|^{\alpha-(p+1)/2}\times \quad (3.31)$$
$$\Psi_2(\beta;\gamma,\gamma';-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dT$$

for $\mathrm{Re}(\alpha)>(p-1)/2$.

Theorem 3.3.7:

$$|P|^{-\alpha} F_1(\alpha,\beta,\beta';\gamma;-P^{-1/2}XP^{-1/2},-P^{-1/2}YP^{-1/2})$$
$$=\frac{1}{\Gamma_p(\alpha)}\int_{T>0} e^{-\mathrm{tr}(PT)}|T|^{\alpha-(p+1)/2}\times \tag{3.32}$$
$$\Phi_2(\beta,\beta';\gamma;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dT$$

for $\mathrm{Re}(\alpha)>(p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(3.32) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$\Phi_2(\beta,\beta';\gamma;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dXdY \tag{3.33}$$

Applying the transformations,

$$X_1=T^{1/2}XT^{1/2},\ Y_1=T^{1/2}YT^{1/2},\ \text{with, } dX_1=|T|^{(p+1)/2}dX,$$
$$dY_1=|T|^{(p+1)/2}dY,\ \text{and, } |X_1|=|T||X|, |Y_1|=|T||Y|;$$

to the above expression and then using the definition (3.1.6) yields,

$$|T|^{-\rho_1-\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\beta')\Gamma_p(\gamma-\rho_1-\rho_2)} \tag{3.34}$$

Substituting this expression on the right side of eq.(3.32) and then integrating out T in the resulting expression by using a Gamma integral gives

$$|P|^{-(\alpha-\rho_1-\rho_2)}\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2)\Gamma_p(\rho_1)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\beta')\Gamma_p(\gamma-\rho_1-\rho_2)}\times \tag{3.35}$$
$$\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)$$

Now, taking the M-transform of the left side of eq.(3.32) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we obtain

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|P|^{-\alpha}F_1(\alpha,\beta,\beta^{'};\gamma;$$
$$-P^{-1/2}XP^{-1/2},-P^{-1/2}YP^{-1/2})dXdY \tag{3.36}$$

which, under the transformations,

$$X_2 = P^{-1/2}XP^{-1/2}, Y_2 = P^{-1/2}YP^{-1/2}, \text{ with, } dX_2 = |P|^{-(p+1)/2}dX,$$

$$dY_2 = |P|^{-(p+1)/2}dY, \text{ and, } |X_2| = |P|^{-1}|X|, |Y_2| = |P|^{-1}|Y|;$$

and then using eq.(3.1) produces the same result as in eq.(3.35) above.

Theorem 3.3.8:

$$F_3(\alpha,\alpha^{'},\beta,\beta^{'};\alpha+\alpha^{'};-X,-Y)$$
$$=\frac{\Gamma_p(\alpha+\alpha^{'})}{\Gamma_p(\alpha)\Gamma_p(\alpha^{'})}\int_0^I|U|^{\alpha-(p+1)/2}|I-U|^{\alpha^{'}-(p+1)/2}\times$$
$$\left|I+U^{1/2}XU^{1/2}\right|^{-\beta}\left|I+(I-U)^{1/2}Y(I-U)^{1/2}\right|^{-\beta^{'}}dU \tag{3.37}$$

for $\operatorname{Re}(\alpha,\alpha^{'})>(p-1)/2$, and $0<U<I$.

Proof: Taking the M-transform of the right side of eq.(3.37) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we acquire

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\left|I+U^{1/2}XU^{1/2}\right|^{-\beta}\times$$
$$\left|I+(I-U)^{1/2}Y(I-U)^{1/2}\right|^{-\beta^{'}}dXdY \tag{3.38}$$

this expression on the application of the transformations,

$$X_1 = U^{1/2}XU^{1/2}, Y_1 = (I-U)^{1/2}Y(I-U)^{1/2},$$

and then integrating out X_1 and Y_1 by employing a type-2 Beta integral generates,

$$|U|^{-\rho_1}|I-U|^{-\rho_2}\frac{\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\beta')} \quad (3.39)$$

Putting back this expression on the right side of eq.(3.37) and then integrating out U in the ensuing expression by utilizing a type-1 Beta integral generates $M(F_3)$ as given by eq.(3.3).

Theorem 3.3.9:

$$F_3(\alpha,\alpha',\beta,\beta';\beta+\beta';-X,-Y)$$

$$=\frac{\Gamma_p(\beta+\beta')}{\Gamma_p(\beta)\Gamma_p(\beta')}\int_0^I|U|^{\beta-(p+1)/2}|I-U|^{\beta'-(p+1)/2}\times \quad (3.40)$$

$$\left|I+U^{1/2}XU^{1/2}\right|^{-\alpha}\left|I+(I-U)^{1/2}Y(I-U)^{1/2}\right|^{-\alpha'}dU$$

for $\mathrm{Re}\,(\beta,\beta')>(p-1)/2,$ and $0<U<I.$

Theorem 3.3.10:

$$|P|^{-\beta'}F_3(\alpha,\alpha',\beta,\beta';\gamma;-X,-P^{-1/2}YP^{-1/2})$$

$$=\frac{1}{\Gamma_p(\beta')}\int_{T>0}e^{-\mathrm{tr}(PT)}|T|^{\beta'-(p+1)/2}\times \quad (3.41)$$

$$\Xi_1(\alpha,\alpha',\beta;\gamma;-X,-T^{1/2}YT^{1/2})dT$$

for $\mathrm{Re}\,(\beta')>(p-1)/2.$

Theorem 3.3.11:

$$F_4(\alpha,\beta;\gamma,\gamma';-X,-Y)=\frac{\Gamma_p(\gamma)\Gamma_p(\gamma')}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma'-\beta)}\times$$

$$\iint |U|^{\alpha-(p+1)/2}|V|^{\beta-(p+1)/2}|I-U|^{\gamma-\alpha-(p+1)/2}\times$$

$$|I-V|^{\gamma'-\beta-(p+1)/2}\times$$

$$\left|I-(I-V)^{-1/2}V^{1/2}U^{1/2}XU^{1/2}V^{1/2}(I-V)^{-1/2}\right|^{\gamma'-\beta-(p+1)/2}\times$$

$$\left|I-(I-U)^{-1/2}V^{1/2}U^{1/2}YU^{1/2}V^{1/2}(I-U)^{-1/2}\right|^{\gamma-\alpha-(p+1)/2}dUdV$$

for $0<U<I$ and $0<V<I$, and for $\mathrm{Re}\,(\alpha,\beta,\gamma-\alpha,\gamma'-\beta)>(p-1)/2.$ (3.42)

Proof: Taking the M-transform of the right side of eq.(3.42) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we achieve

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\left|I-(I-V)^{-1/2}V^{1/2}U^{1/2}XU^{1/2}V^{1/2}(I-V)^{-1/2}\right|^{\gamma'-\beta-(p+1)/2}\times \qquad (3.43)$$

$$\left|I-(I-U)^{-1/2}V^{1/2}U^{1/2}YU^{1/2}V^{1/2}(I-U)^{-1/2}\right|^{\gamma-\alpha-(p+1)/2}dXdY$$

Now, applying the transformations

$$X_1=(I-V)^{-1/2}V^{1/2}U^{1/2}XU^{1/2}V^{1/2}(I-V)^{-1/2},$$

$$Y_1=(I-U)^{-1/2}V^{1/2}U^{1/2}YU^{1/2}V^{1/2}(I-U)^{-1/2},$$

with, $dX_1 = |I-V|^{-(p+1)/2} |V|^{(p+1)/2} |U|^{(p+1)/2} dX,$

$dY_1 = |I-U|^{-(p+1)/2} |V|^{(p+1)/2} |U|^{(p+1)/2} dY,$

and, $|X_1| = |I-V|^{-1} |V||U||X|, |Y_1| = |I-U|^{-1} |V||U||Y|;$

to the last expression and then integrating out X_1 and Y_1 by using a type-1 Beta integral yields

$$|U|^{-\rho_1-\rho_2} |V|^{-\rho_1-\rho_2} |I-V|^{\rho_1} |I-U|^{\rho_2} \times$$

$$\frac{\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma'-\beta)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\gamma-\alpha+\rho_2)\Gamma_p(\gamma'-\beta+\rho_1)} \tag{3.44}$$

Substituting this expression on the right side of eq.(3.42) and then integrating out U and V in the consequent expression by utilization of a type-1 Beta integral leads to $M(F_4)$ as given by eq.(3.4).

3.4 This section deals with theorems for the Humbert's functions of matrix arguments. As in the previous section, proofs of only some representative cases will be given and other results will be stated without proof as they hve been proved in my papers [109,110] cited in the bibliography.

Theorem 3.4.1:

$$|P|^{-\beta} \Xi_1(\alpha, \alpha', \beta; \gamma; -P^{-1/2} X P^{-1/2}, -Y)$$

$$= \frac{1}{\Gamma_p(\beta)} \int_{T>0} e^{-tr(PT)} |T|^{\beta-(p+1)/2} \times \tag{3.45}$$

$$\Phi_2(\alpha, \alpha'; \gamma; -T^{1/2} X T^{1/2}, -Y) dT$$

for $Re(\beta) > (p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(3.45) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we obtain

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times \Phi_2(\alpha,\alpha';\gamma;-T^{1/2}XT^{1/2},-Y)dXdY \tag{3.46}$$

Applying the transformation,

$$X_1=T^{1/2}XT^{1/2},\ \text{with } dX_1=|T|^{(p+1)/2}\,dX \text{ and } |X_1|=|T||X|$$

and then using eq.(3.6), the above expression yields,

$$|T|^{-\rho_1}\frac{\Gamma_p(\gamma)\Gamma_p(\alpha-\rho_1)\Gamma_p(\alpha'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\alpha')\Gamma_p(\gamma-\rho_1-\rho_2)} \tag{3.47}$$

Substituting this expression on the right side of eq.(3.45) and then integrating out T in the resulting expression by using a Gamma integral generates

$$|P|^{-(\beta-\rho_1)}\frac{\Gamma_p(\gamma)\Gamma_p(\alpha-\rho_1)\Gamma_p(\alpha'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\beta-\rho_1)}{\Gamma_p(\beta)\Gamma_p(\alpha)\Gamma_p(\alpha')\Gamma_p(\gamma-\rho_1-\rho_2)} \tag{3.48}$$

Now, taking the M-transform of the left side of eq.(3.45) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we have

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|P|^{-\beta}\times \Xi_1(\alpha,\alpha',\beta;\gamma;-P^{-1/2}XP^{-1/2},-Y)dXdY \tag{3.49}$$

which under the transformation

$$X_2=P^{-1/2}XP^{-1/2},\ \text{with } dX_2=|P|^{-(p+1)/2}\,dX \text{ and } |X_2|=|P|^{-1}|X|$$

then using eq.(3.10) yields the same result as in eq.(3.48) above.

Theorem 3.4.2:

$$|P|^{-\gamma}\left|I+P^{-1/2}XP^{-1/2}\right|^{-\beta}\left|I+P^{-1/2}YP^{-1/2}\right|^{-\beta'}$$

Continued to the next page … … … … … …

$$= \frac{1}{\Gamma_p(\gamma)} \int_{T>0} e^{-\mathrm{tr}(PT)} |T|^{\gamma-(p+1)/2} \times$$

$$\Phi_2(\beta,\beta';\gamma;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dT \qquad (3.50)$$

for $\mathrm{Re}(\gamma) > (p-1)/2$.

Theorem 3.4.3:

$$|P|^{-\beta'} \Phi_2(\beta,\beta';\gamma;-X,-P^{-1/2}YP^{-1/2})$$

$$= \frac{1}{\Gamma_p(\beta')} \int_{T>0} e^{-\mathrm{tr}(PT)} |T|^{\beta'-(p+1)/2} \times \qquad (3.51)$$

$$\Phi_3(\beta;\gamma;-X,-T^{1/2}YT^{1/2})dT$$

for $\mathrm{Re}(\beta') > (p-1)/2$.

Theorem 3.4.4: For $p = 2$,

$$|P|^{-\alpha} \Xi_1[(\alpha+1)/2,\beta,(2\alpha+1)/4;\gamma;-4P^{-1}YP^{-1},-X]$$

$$= \frac{1}{\Gamma_p(\alpha)} \int_{T>0} e^{-\mathrm{tr}(PT)} |T|^{\alpha-(p+1)/2} \times \qquad (3.52)$$

$$\Phi_3(\beta;\gamma;-X,-TYT')dT$$

where, $\mathrm{Re}(\alpha) > (p-1)/2$.

Theorem 3.4.5:

$$|P|^{-\beta} \Psi_1(\alpha,\beta;\gamma,\gamma';-P^{-1/2}XP^{-1/2},-Y)$$

$$= \frac{1}{\Gamma_p(\beta)} \int_{T>0} e^{-\mathrm{tr}(PT)} |T|^{\beta-(p+1)/2} \times$$

Continued to the next page … … … … … …

$$\Psi_2(\alpha;\gamma,\gamma';-T^{1/2}XT^{1/2},-Y)dT \tag{3.53}$$

for $\mathrm{Re}(\beta) > (p-1)/2$.

Theorem 3.4.6:

$$|P|^{-\alpha}\,\Xi_2(\alpha,\beta;\gamma;-P^{-1/2}XP^{-1/2},-Y)$$

$$=\frac{1}{\Gamma_p(\alpha)}\int_{T>0} e^{-\mathrm{tr}(PT)}\,|T|^{\alpha-(p+1)/2}\times \tag{3.54}$$

$$\Phi_3(\beta;\gamma;-T^{1/2}XT^{1/2},-Y)dT$$

for $\mathrm{Re}\,(\alpha) > (p-1)/2$.

Theorem 3.4.7:

$$|P|^{-\gamma}\left|I+P^{-1/2}XP^{-1/2}\right|^{-\beta}e^{-\mathrm{tr}(P^{-1}Y)}$$

$$=\frac{1}{\Gamma_p(\gamma)}\int_{T>0} e^{-\mathrm{tr}(PT)}\,|T|^{\gamma-(p+1)/2}\times \tag{3.55}$$

$$\Phi_3(\beta;\gamma;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dT$$

for $\mathrm{Re}\,(\gamma) > (p-1)/2$.

Theorem 3.4.8:

$$|P|^{-\alpha}\,\Phi_1(\alpha,\beta;\gamma;-P^{-1/2}XP^{-1/2},-P^{-1/2}YP^{-1/2})$$

$$=\frac{1}{\Gamma_p(\alpha)}\int_{T>0} e^{-\mathrm{tr}(PT)}\,|T|^{\alpha-(p+1)/2}\times \tag{3.56}$$

$$\Phi_3(\beta;\gamma;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dT$$

for $\mathrm{Re}\,(\alpha) > (p-1)/2$.

Theorem 3.4.9:

$$|P|^{-\alpha'}\,\Xi_1(\alpha,\alpha',\beta;\gamma;-X,-P^{-1/2}YP^{-1/2})$$

$$=\frac{1}{\Gamma_p(\alpha')}\int_{T>0} e^{-\mathrm{tr}(PT)}\,|T|^{\alpha'-(p+1)/2}\times \tag{3.57}$$

$$\Xi_2(\alpha,\beta;\gamma;-X,-T^{1/2}YT^{1/2})dT$$

for $\mathrm{Re}\,(\alpha') > (p-1)/2.$

3.5: In this section some transformation relations and cases of reducibility of the Appell's and Humbert's functions of matrix arguments shall be established.

Theorem 3.5.1:

(i) $$F_1(a,b,b';c;-X,-X) = {}_2F_1(a,b+b';c;-X) \tag{3.58}$$

(ii) $$F_1(a,b,b';c;-X,I) = \frac{\Gamma_p(c)\Gamma_p(c-a-b')}{\Gamma_p(c-a)\Gamma_p(c-b')}\times \tag{3.59}$$

$${}_2F_1(a,b;c-b';-X)$$

(iii) $$F_1(a,b,b';c;I,I) = \frac{\Gamma_p(c)\Gamma_p(c-a-b-b')}{\Gamma_p(c-a)\Gamma_p(c-b-b')} \tag{3.60}$$

where it is being assumed that all the Γ_p 's involved are defined.

Proof: (i) This result follows by putting $Y = X$ in theorem 3.2.2 and consequently using the theorem 2.2.1.
(ii) For obtaining this result we let $Y \to -I$ in theorem 3.2.2 subsequently using the theorem 2.2.1.
(iii). Letting $X \to -I$ in eq.(3.59), followed by the use of the theorem (2.3.2) page 39 of Mathai [62] generates this result. Alternatively, we can let both X and Y approach to $-I$ in theorem 3.2.2 and then use a type-1 Beta integral to see the result.

Theorem 3.5.2:

(i) $$F_1(a,b,b';c;-X,-Y)=|I+X|^{-a}\,F_1[a,c-b-b',b';c;$$
$$(I+X)^{-1/2}X(I+X)^{-1/2},-(I+X)^{-1/2}(Y-X)(I+X)^{-1/2}] \quad (3.61)$$

where $Y-X>0$.

(ii) $$F_1(a,b,b';c;-X,-Y)=|I+Y|^{-a}\,F_1[a,b,c-b-b';c;$$
$$-(I+Y)^{-1/2}(X-Y)(I+Y)^{-1/2},(I+Y)^{-1/2}Y(I+Y)^{-1/2}] \quad (3.62)$$

where $X-Y>0$.

(iii) $$F_1(a,b,b';c;-X,-Y)=|I+X|^{-b}|I+Y|^{-b'}\,F_1[c-a,b,b';c;$$
$$(I+X)^{-1/2}X(I+X)^{-1/2},(I+Y)^{-1/2}Y(I+Y)^{-1/2}] \quad (3.63)$$

(iv) $$F_1(a,b,b';c;-X,-Y)=|I+X|^{c-a-b}|I+Y|^{-b'}\times$$
$$F_1[c-a,c-b-b',b';c;-X, \quad (3.64)$$
$$(I+X)^{1/2}(I+Y)^{-1/2}Y(I+Y)^{-1/2}(I+X)^{1/2}-X]$$

(v) $$F_1(a,b,b';c;-X,-Y)=|I+X|^{-b}|I+Y|^{c-a-b'}\times$$
$$F_1[c-a,b,c-b-b';c;(I+Y)^{1/2}(I+X)^{-1/2}X\times \quad (3.65)$$
$$(I+X)^{-1/2}(I+Y)^{1/2}-Y,-Y]$$

Proof: To prove this theorem we first give two definitions of the F_1 function through integral representations:

$$F_1(a,b,b';c;-X,-Y)=\frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-b-b')}\times$$

Continued to the next page … … … … …

$$\iint |U_1|^{b-(p+1)/2} |U_2|^{b'-(p+1)/2} |I-U_1-U_2|^{c-b-b'-(p+1)/2} \times$$

$$\left|I + X^{1/2} U_1 X^{1/2} + Y^{1/2} U_2 Y^{1/2}\right|^{-a} dU_1 dU_2 \tag{3.66}$$

for Re $(b, b', c-b-b') > (p-1)/2$.

Also,

$$F_1(a, b, b'; c; -X, -Y) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \int_0^I |V|^{a-(p+1)/2} \times$$

$$|I-V|^{c-a-(p+1)/2} \left|I + X^{1/2} V X^{1/2}\right|^{-b} \left|I + Y^{1/2} V Y^{1/2}\right|^{-b'} dV \tag{3.67}$$

for Re $(a, c-a) > (p-1)/2$.

To prove the result in eq.(3.61) we apply the transformations

$$V_1 = I - U_1 - U_2, V_2 = U_2; \text{ so that, } dV_1 dV_2 = dU_1 dU_2,$$

to eq.(3.66) and by using the concept of symmetry, as has been assumed by previous workers like Herz [22] page 478 and Mathai [60] page 516; we observe that,

$$\left|I + X^{1/2}(I - V_1 - V_2)X^{1/2} + Y^{1/2} V_2 Y^{1/2}\right|^{-a} = |I+X|^{-a} \times$$

$$\Big|I - (I+X)^{-1/2} X^{1/2} V_1 X^{1/2} (I+X)^{-1/2} +$$

$$(I+X)^{-1/2} (Y-X)^{1/2} V_2 (Y-X)^{1/2} (I+X)^{-1/2}\Big|^{-a}$$

where, $Y - X > 0$. The desired result then follows immediately after a suitable interpretation of the resulting expression in the light of eq.(3.66).

The result of eq.(3.62) also follows similarly from eq.(3.66).

The result in eq.(3.63) can be proved by using eq.(3.67) by observing that,

$$\left|I + X^{1/2} V X^{1/2}\right| = |I+X| \left|I - (I+X)^{-1/2} X^{1/2} (I-V) X^{1/2} (I+X)^{-1/2}\right|$$

and a similar expression for $\left|I + Y^{1/2} V Y^{1/2}\right|$ and then applying the transformation $V_1 = I - V$ and suitably interpreting the resulting expression as per eq.(3.67).

The result in eq.(3.64) is obtained by applying eq.(3.61) to the F_1 function on the right side of eq.(3.63) and similarly, the result in eq.(3.65) follows from eqs.(3.62) and (3.63).

Theorem 3.5.3:

(i)
$$F_2(a,b,b';c,c';-X,-Y) = |I+X|^{-a} F_2[a, c-b, b'; c, c'; (I+X)^{-1/2} X (I+X)^{-1/2}, -(I+X)^{-1/2} Y (I+X)^{-1/2}] \tag{3.68}$$

(ii)
$$F_2(a,b,b';c,c';-X,-Y) = |I+Y|^{-a} F_2[a, b, c-b'; c, c'; -(I+Y)^{-1/2} X (I+Y)^{-1/2}, (I+Y)^{-1/2} Y (I+Y)^{-1/2}] \tag{3.69}$$

(iii)
$$F_2(a,b,b';c,c';-X,-Y) = |I+X+Y|^{-a} F_2[a, c-b, c'-b'; c, c'; (I+X+Y)^{-1/2} X (I+X+Y)^{-1/2}, (I+X+Y)^{-1/2} Y (I+X+Y)^{-1/2}] \tag{3.70}$$

Proof: To prove this theorem we define the function F_2 through an integral representation:

$$F_2(a,b,b';c,c';-X,-Y) = \frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-b)\Gamma_p(c'-b')} \times$$

$$\int_0^I \int_0^I |U_1|^{b-(p+1)/2} |U_2|^{b'-(p+1)/2} |I-U_1|^{c-b-(p+1)/2} \times$$

Continued to the next page … … … … …

$$\left|I-U_2\right|^{c'-b'-(p+1)/2}\left|I+X^{1/2}U_1X^{1/2}+Y^{1/2}U_2Y^{1/2}\right|^{-a}dU_1dU_2 \quad (3.71)$$

for Re $(b, b', c-b, c'-b') > (p-1)/2$.

The result in eq.(3.68) is obtained by applying the transformation, $V_1 = I - U_1$ to eq.(3.71) and observing that,

$$\left|I+X^{1/2}(I-V_1)X^{1/2}+Y^{1/2}U_2Y^{1/2}\right| = |I+X|\times$$

$$\left|I-(I+X)^{-1/2}X^{1/2}V_1X^{1/2}(I+X)^{-1/2}+\right.$$

$$\left.(I+X)^{-1/2}Y^{1/2}U_2Y^{1/2}(I+X)^{-1/2}\right|$$

and then interpreting the ensuing expression in view of eq.(3.71). The result in eq.(3.69) also follows similarly from eq.(3.71) while, the result in eq.(3.70) is a combination of the results in eqs.(3.68) and (3.69).

Theorem 3.5.4:

(i) $$\Phi_1(a,b;c;I,-Y) = \frac{\Gamma_p(c)\Gamma_p(c-b-a)}{\Gamma_p(c-a)\Gamma_p(c-b)} \times {}_1F_1(a;c-b;-Y) \quad (3.72)$$

(ii) $$\begin{aligned}\Phi_1(a,b;c;-X,-Y) &= e^{-tr(Y)}|I+X|^{-b}\times \\ &\Phi_1[c-a,b;c;(I+X)^{-1/2}X(I+X)^{-1/2},Y]\end{aligned} \quad (3.73)$$

(iii) $$\Phi_2(b,b';c;-X,-Y) = e^{-tr(X)}\Phi_2(c-b-b',b';c;X,X-Y) \quad (3.74)$$

$$= e^{-tr(Y)}\Phi_2(b,c-b-b';c;Y-X,Y) \quad (3.75)$$

Proof: (i) The result in eq.(3.72) follows by making $X \to -I$ in eq.(3.16) followed by the use of the theorem 2.2.2.

(ii) Consider eq.(3.16) along with the observation,

$$\left|I + V^{1/2} X V^{1/2}\right| = \left|I + XV\right| = \left|I + X^{1/2} V X^{1/2}\right| \tag{3.76}$$

Now, applying the transformation $U = I - V$ in conjunction with the observation

$$\left|I + X^{1/2}(I - U)X^{1/2}\right| = \left|I + X\right|\left|I - (I + X)^{-1/2} X^{1/2} U X^{1/2} (I + X)^{-1/2}\right|$$

the desired result follows immediately by a suitable interpretation of the resulting expression in the light of eq.(3.16) together with eq.(3.76).

(iii) The result in eq. (3.74) follows by employing the transformations

$$U_1 = I - V_1 - V_2, U_2 = V_2, \text{ so that, } dV_1 dV_2 = dU_1 dU_2$$

in eq.(3.17) and suitably interpreting the ensuing expression as per eq.(3.17). The result of eq.(3.75) also follows similarly from eq.(3.17).

CHAPTER IV

KAMPÉ DE FÉRIET'S FUNCTION, LAURICELLA AND OTHER RELATED FUNCTIONS OF MATRIX ARGUMENTS

4.1 Definitions

In this section I first quote the Mathai's definitions of the Kampé de Fériet's function, the four Lauricella functions and the $\Phi_2^{(n)}$- function of matrix arguments.

4.1.1 The Kampé de Fériet's function

$$F_{s:m;n}^{r:q;k} = F_{s:m;n}^{r:q;k}\left[\begin{matrix}(a_r):(b_q);(c_k); \\ (\alpha_s):(\beta_m);(\gamma_n);\end{matrix} -X, -Y\right]$$

of matrix arguments is defined as that class of functions for which the M-transform is the following:

$$M\left(F_{s:m;n}^{r:q;k}\right) = \int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2} \times$$

$$F_{s:m;n}^{r:q;k}\left[\begin{matrix}(a_r):(b_q);(c_k); \\ (\alpha_s):(\beta_m);(\gamma_n);\end{matrix} -X, -Y\right] dXdY$$

$$= \frac{\{\prod_{j=1}^{s}\Gamma_p(\alpha_j)\}\{\prod_{j=1}^{m}\Gamma_p(\beta_j)\}\{\prod_{j=1}^{n}\Gamma_p(\gamma_j)\}\{\prod_{j=1}^{r}\Gamma_p(a_j-\rho_1-\rho_2)\}}{\{\prod_{j=1}^{r}\Gamma_p(a_j)\}\{\prod_{j=1}^{q}\Gamma_p(b_j)\}\{\prod_{j=1}^{k}\Gamma_p(c_j)\}\{\prod_{j=1}^{s}\Gamma_p(\alpha_j-\rho_1-\rho_2)\}}$$

$$\times \frac{\{\prod_{j=1}^{q}\Gamma_p(b_j-\rho_1)\}\{\prod_{j=1}^{k}\Gamma_p(c_j-\rho_2)\}\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\{\prod_{j=1}^{m}\Gamma_p(\beta_j-\rho_1)\}\{\prod_{j=1}^{n}\Gamma_p(\gamma_j-\rho_2)\}} \qquad (4.1)$$

for $\mathrm{Re}(\rho_1, \rho_2, a_j - \rho_1 - \rho_2, j = 1, \cdots, r; \alpha_j - \rho_1 - \rho_2, j = 1, \cdots, s;$
$b_j - \rho_1, j = 1, \cdots, q; \beta_j - \rho_1, j = 1, \cdots, m; c_j - \rho_2, j = 1, \cdots, k; \gamma_j - \rho_2,$
$j = 1, \cdots, n) > (p-1)/2.$

4.1.2 The Lauricella function

$F_A^{(n)} = F_A^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; -X_1, \cdots, -X_n)$
of matrix arguments is defined as that class of functions which has the following matrix transform:

$$M(F_A^{(n)}) = \int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1 - (p+1)/2} \cdots |X_n|^{\rho_n - (p+1)/2}$$
$$\times F_A^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; -X_1, \cdots, -X_n) dX_1 \cdots dX_n$$
$$= \frac{\{\prod_{j=1}^{n} \Gamma_p(c_j)\}\{\prod_{j=1}^{n} \Gamma_p(b_j - \rho_j)\}\Gamma_p(a - \rho_1 - \cdots - \rho_n)\{\prod_{j=1}^{n} \Gamma_p(\rho_j)\}}{\Gamma_p(a)\{\prod_{j=1}^{n} \Gamma_p(b_j)\}\{\prod_{j=1}^{n} \Gamma_p(c_j - \rho_j)\}} \quad (4.2)$$

for $\mathrm{Re}(b_j - \rho_j, c_j - \rho_j, \rho_j, a - \rho_1 - \cdots - \rho_n) > (p-1)/2;\ j = 1, \cdots, n.$

4.1.3 $\qquad F_B^{(n)} = F_B^{(n)}(a_1, \cdots, a_n, b_1, \cdots, b_n; c; -X_1, \cdots, -X_n)$

$$M(F_B^{(n)}) = \int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1 - (p+1)/2} \cdots |X_n|^{\rho_n - (p+1)/2} \times$$
$$F_B^{(n)}(a_1, \cdots, a_n, b_1, \cdots, b_n; c; -X_1, \cdots, -X_n) dX_1 \cdots dX_n \quad (4.3)$$
$$= \frac{\Gamma_p(c) \prod_{j=1}^{n} \{\Gamma_p(a_j - \rho_j)\Gamma_p(b_j - \rho_j)\Gamma_p(\rho_j)\}}{\prod_{j=1}^{n} \{\Gamma_p(a_j)\Gamma_p(b_j)\}\Gamma_p(c - \rho_1 - \cdots - \rho_n)}$$

for $\operatorname{Re}(a_j-\rho_j, b_j-\rho_j, \rho_j, c-\rho_1-\cdots-\rho_n) > (p-1)/2;$

$j=1,\cdots,n.$

4.1.4 $F_C^{(n)} = F_C^{(n)}(a,b;c_1,\cdots,c_n;-X_1,\cdots,-X_n)$

$$M(F_C^{(n)}) = \int_{X_1>0}\cdots\int_{X_n>0} |X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}$$

$$\times F_C^{(n)}(a,b;c_1,\cdots,c_n;-X_1,\cdots,-X_n)dX_1\cdots dX_n$$

$$= \frac{\{\prod_{j=1}^{n}\Gamma_p(c_j)\}\Gamma_p(a-\rho_1-\cdots-\rho_n)\Gamma_p(b-\rho_1-\cdots-\rho_n)}{\Gamma_p(a)\Gamma_p(b)\{\prod_{j=1}^{n}\Gamma_p(c_j-\rho_j)\}}\times$$

$$\{\prod_{j=1}^{n}\Gamma_p(\rho_j)\}$$

for $\operatorname{Re}(c_j-\rho_j, \rho_j, a-\rho_1-\cdots-\rho_n, b-\rho_1-\cdots-\rho_n)$

$> (p-1)/2;\ j=1,\cdots,n.$ (4.4)

4.1.5 $F_D^{(n)} = F_D^{(n)}(a,b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)$

$$M(F_D^{(n)}) = \int_{X_1>0}\cdots\int_{X_n>0} |X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}$$

$$\times F_D^{(n)}(a,b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)dX_1\cdots dX_n \quad (4.5)$$

$$= \frac{\Gamma_p(c)\prod_{j=1}^{n}\{\Gamma_p(b_j-\rho_j)\Gamma_p(\rho_j)\}\Gamma_p(a-\rho_1-\cdots-\rho_n)}{\Gamma_p(a)\prod_{j=1}^{n}\{\Gamma_p(b_j)\}\Gamma_p(c-\rho_1-\cdots-\rho_n)}$$

for $\operatorname{Re}(b_j - \rho_j, \rho_j, a - \rho_1 - \cdots - \rho_n, c - \rho_1 - \cdots - \rho_n) > (p-1)/2;$

$j = 1, \cdots, n.$

4.1.6 The $\Phi_2^{(n)}$ - function of matrix arguments

$\Phi_2^{(n)} = \Phi_2^{(n)}(b_1, \cdots, b_n; c; -X_1, \cdots, -X_n)$

is defined as that class of functions which has the following matrix transform:

$$M(\Phi_2^{(n)}) = \int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2}$$

$$\times \Phi_2^{(n)}(b_1, \cdots, b_n; c; -X_1, \cdots, -X_n) dX_1 \cdots dX_n \tag{4.6}$$

$$= \frac{\Gamma_p(c) \prod_{j=1}^{n} \{\Gamma_p(b_j - \rho_j)\Gamma_p(\rho_j)\}}{\prod_{j=1}^{n} \{\Gamma_p(b_j)\} \Gamma_p(c - \rho_1 - \cdots - \rho_n)}$$

for $\operatorname{Re}(b_j - \rho_j, \rho_j, c - \rho_1 - \cdots - \rho_n) > (p-1)/2;\ j = 1, \cdots, n.$

4.2 Further Definitions

In this section I give my definitions of the $\Psi_A^{(n)}, \Xi_1^{(n)}$ and $\Phi_D^{(n)}$ functions of matrix arguments.

4.2.1 The $\Psi_A^{(n)}$ - function of matrix arguments

$$\Psi_A^{(n)} = \Psi_A^{(n)}(\alpha, \beta_1, \cdots, \beta_{n-1}; \gamma_1, \cdots, \gamma_n; -X_1, \cdots, -X_n)$$

is defined as that class of functions for which the M-transform is the following:

$$M(\Psi_A^{(n)}) = \int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2}$$

$$\times \Psi_A^{(n)}(\alpha, \beta_1, \cdots, \beta_{n-1}; \gamma_1, \cdots, \gamma_n; -X_1, \cdots, -X_n) dX_1 \cdots dX_n$$

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$$= \frac{\Gamma_p(\alpha-\rho_1-\cdots-\rho_n)}{\Gamma_p(\alpha)} \frac{\{\prod_{i=1}^{n-1}\Gamma_p(\beta_i-\rho_i)\}}{\{\prod_{i=1}^{n-1}\Gamma_p(\beta_i)\}} \frac{\{\prod_{j=1}^{n}\Gamma_p(\gamma_j)\Gamma_p(\rho_j)\}}{\{\prod_{j=1}^{n}\Gamma_p(\gamma_j-\rho_j)\}}$$

for $\mathrm{Re}(\alpha-\rho_1-\cdots-\rho_n, \beta_i-\rho_i, \gamma_j-\rho_j, \rho_j) > (p-1)/2;$ (4.7)

$i=1,\cdots,n-1;\ j=1,\cdots,n.$

4.2.2 $\Xi_1^{(n)} = \Xi_1^{(n)}(a_1,\cdots,a_n,b_1,\cdots,b_{n-1};c;-X_1,\cdots,-X_n)$

$$M(\Xi_1^{(n)}) = \int_{X_1>0}\cdots\int_{X_n>0} |X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}$$

$$\times\Xi_1^{(n)}(a_1,\cdots,a_n,b_1,\cdots,b_{n-1};c;-X_1,\cdots,-X_n)dX_1\cdots dX_n$$

$$= \frac{\{\prod_{i=1}^{n}\Gamma_p(a_i-\rho_i)\}}{\{\prod_{i=1}^{n}\Gamma_p(a_i)\}} \frac{\{\prod_{j=1}^{n-1}\Gamma_p(b_j-\rho_j)\}}{\{\prod_{j=1}^{n-1}\Gamma_p(b_j)\}} \frac{\Gamma_p(c)\{\prod_{i=1}^{n}\Gamma_p(\rho_i)\}}{\Gamma_p(c-\rho_1-\cdots-\rho_n)} \qquad (4.8)$$

for $\mathrm{Re}(a_i-\rho_i, b_j-\rho_j, c-\rho_1-\cdots-\rho_n, \rho_i) > (p-1)/2;$

$i=1,\cdots,n;\ j=1,\cdots,n-1.$

4.2.3 $\Phi_D^{(n)} = \Phi_D^{(n)}(a,b_1,\cdots,b_{n-1};c;-X_1,\cdots,-X_n)$

$$M(\Phi_D^{(n)}) = \int_{X_1>0}\cdots\int_{X_n>0} |X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}$$

$$\times\Phi_D^{(n)}(a,b_1,\cdots,b_{n-1};c;-X_1,\cdots,-X_n)dX_1\cdots dX_n$$

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$$=\frac{\Gamma_p(a-\rho_1-\cdots-\rho_n)}{\Gamma_p(a)}\frac{\{\prod_{i=1}^{n-1}\Gamma_p(b_i-\rho_i)\}}{\{\prod_{i=1}^{n-1}\Gamma_p(b_i)\}}\frac{\Gamma_p(c)\{\prod_{j=1}^{n}\Gamma_p(\rho_j)\}}{\Gamma_p(c-\rho_1-\cdots-\rho_n)}$$

for $\mathrm{Re}(a-\rho_1-\cdots-\rho_n, b_i-\rho_i, c-\rho_1-\cdots-\rho_n, \rho_j) >$ (4.9)

$(p-1)/2; i=1,\cdots,n-1;\ j=1,\cdots,n.$

4.3 The Kampé de Fériet's Function of Matrix Arguments

Theorem 4.3.1:

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}\alpha:\beta,\lambda;\beta',\lambda'; \\ \gamma:\ \ \mu;\ \ \mu';\end{matrix} -X,-Y\right]$$

$$=\frac{\Gamma_p(\mu)\Gamma_p(\mu')}{\Gamma_p(\lambda)\Gamma_p(\mu-\lambda)\Gamma_p(\lambda')\Gamma_p(\mu'-\lambda')}\times$$

$$\int_0^I\int_0^I|U|^{\lambda-(p+1)/2}|V|^{\lambda'-(p+1)/2}|I-U|^{\mu-\lambda-(p+1)/2}\times$$

$$|I-V|^{\mu'-\lambda'-(p+1)/2}F_1(\alpha,\beta,\beta';\gamma;$$

$$-U^{1/2}XU^{1/2},-V^{1/2}YV^{1/2})dUdV$$

for $\mathrm{Re}(\lambda,\lambda',\mu-\lambda,\mu'-\lambda')>(p-1)/2.$ (4.10)

Proof: From definition (4.1.1) we deduce that,

$$M(F^{1:2;2}_{1:1;1})=\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}\alpha:\beta,\lambda;\beta',\lambda'; \\ \gamma:\ \mu;\ \mu';\end{matrix} -X,-Y\right]dXdY$$

$$=\frac{\Gamma_p(\gamma)\Gamma_p(\mu)\Gamma_p(\mu')\Gamma_p(\alpha-\rho_1-\rho_2)\Gamma_p(\beta-\rho_1)\Gamma_p(\lambda-\rho_1)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\lambda)\Gamma_p(\beta')\Gamma_p(\lambda')\Gamma_p(\gamma-\rho_1-\rho_2)\Gamma_p(\mu-\rho_1)}\times$$

$$\frac{\Gamma_p(\beta'-\rho_2)}{\Gamma_p(\mu'-\rho_2)}\Gamma_p(\lambda'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2) \tag{4.11}$$

for $\mathrm{Re}(\rho_1,\rho_2,\alpha-\rho_1-\rho_2,\beta-\rho_1,\lambda-\rho_1,\beta'-\rho_2,\lambda'-\rho_2,$

$\gamma-\rho_1-\rho_2,\mu-\rho_1,\mu'-\rho_2)>(p-1)/2.$

Now taking the M-transform of the right side of eq.(4.10) with respect to the variables X, Y and the parameters ρ_1,ρ_2 respectively, we have,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$F_1(\alpha,\beta,\beta';\gamma;-U^{1/2}XU^{1/2},-V^{1/2}YV^{1/2})dXdY \tag{4.12}$$

Applying the transformations,

$X_1=U^{1/2}XU^{1/2}, Y_1=V^{1/2}YV^{1/2}$ (implying thereby $dX_1=$ $|U|^{(p+1)/2}\,dX, dY_1=|V|^{(p+1)/2}\,dY,$ and $|X_1|=|U||X|, |Y_1|=|V||Y|$)

in the expression (4.12) and then making use of Mathai's definition of M-transform of an Appell's function F_1 we get,

$$|U|^{-\rho_1}|V|^{-\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\beta')\Gamma_p(\gamma-\rho_1-\rho_2)}\times$$
$$\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2) \tag{4.13}$$

Substituting this expression on the right side of eq.(4.10) and integrating out the variables U and V in the resulting expression by using a type-1 Beta integral we finally obtain $M(F^{1:2;2}_{1:1;1})$ as given by eq.(4.11).

4.4 The Lauricella Functions of Matrix Arguments

Theorem 4.4.1:

$$F_A^{(n)}(\alpha,\beta_1,\cdots,\beta_n;\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(\beta_n)}\int_{T>0}e^{-tr(T)}|T|^{\beta_n-(p+1)/2}\Psi_A^{(n)}(\alpha,\beta_1,\cdots,\beta_{n-1}; \tag{4.14}$$
$$\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_{n-1},-T^{1/2}X_nT^{1/2})dT$$

for $\mathrm{Re}(\beta_n)>(p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(4.14) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$
$$\Psi_A^{(n)}(\alpha,\beta_1,\cdots,\beta_{n-1};\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_{n-1}, \tag{4.15}$$
$$-T^{1/2}X_nT^{1/2})dX_1\cdots dX_n$$

which on applying the transformation

$Y_n=T^{1/2}X_nT^{1/2}$ (with $dY_n=|T|^{(p+1)/2}dX_n$ and $|Y_n|=|T||X_n|$)

and then using definition (4.2.1) yields

$$|T|^{-\rho_n}\frac{\Gamma_p(\alpha-\rho_1-\cdots-\rho_n)}{\Gamma_p(\alpha)}\frac{\{\prod_{i=1}^{n-1}\Gamma_p(\beta_i-\rho_i)\}}{\{\prod_{i=1}^{n-1}\Gamma_p(\beta_i)\}}\frac{\{\prod_{j=1}^{n}\Gamma_p(\gamma_j)\Gamma_p(\rho_j)\}}{\{\prod_{j=1}^{n}\Gamma_p(\gamma_j-\rho_j)\}} \tag{4.16}$$

Substituting this expression on the right side of eq.(4.14) and then integrating out T in the resulting expression by using a Gamma integral produces $M(F_A^{(n)})$ as given by eq.(4.2).

Theorem 4.4.2:

$$F_A^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; -X_1, \cdots, -X_n)$$
$$= \frac{1}{\Gamma_p(a)\Gamma_p(b_1)\cdots\Gamma_p(b_n)} \int_{S>0}\int_{T_1>0}\cdots\int_{T_n>0} e^{-\mathrm{tr}(S+T_1+\cdots+T_n)} \times$$
$$|S|^{a-(p+1)/2} |T_1|^{b_1-(p+1)/2} \cdots |T_n|^{b_n-(p+1)/2} \times \tag{4.17}$$
$${}_0F_1(\ ;c_1; -S^{1/2}T_1^{1/2}X_1T_1^{1/2}S^{1/2}) \cdots {}_0F_1(\ ;c_n;$$
$$-S^{1/2}T_n^{1/2}X_nT_n^{1/2}S^{1/2})\,dS\,dT_1\cdots dT_n$$

for $\mathrm{Re}(a, b_1, \cdots, b_n) > (p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(4.17) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we have

$$\int_{X_1>0}\cdots\int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$
$${}_0F_1(\ ;c_1; -S^{1/2}T_1^{1/2}X_1T_1^{1/2}S^{1/2}) \cdots \times \tag{4.18}$$
$${}_0F_1(\ ;c_n; -S^{1/2}T_n^{1/2}X_nT_n^{1/2}S^{1/2})\,dX_1\cdots dX_n$$

Applying the transformations

$$Y_j = S^{1/2}T_j^{1/2}X_jT_j^{1/2}S^{1/2}, \text{ whence, } dY_j = |S|^{(p+1)/2} \times$$
$$|T_j|^{(p+1)/2}\, dX_j \text{ and } |Y_j| = |S||T_j||X_j| \text{ for } j = 1, \cdots, n;$$

to the expression (4.18) and then using the M-transform of a ${}_0F_1$ -function gives us

$$|S|^{-\rho_1-\cdots-\rho_n} |T_1|^{-\rho_1} \cdots |T_n|^{-\rho_n} \frac{\Gamma_p(c_1)\Gamma_p(\rho_1)}{\Gamma_p(c_1-\rho_1)} \cdots \frac{\Gamma_p(c_n)\Gamma_p(\rho_n)}{\Gamma_p(c_n-\rho_n)} \tag{4.19}$$

Substituting this expression on the right side of eq.(4.17) and then integrating out the variables $S, T_1, \cdots, T_n$ in the resulting expression by using a Gamma integral produces $M(F_A^{(n)})$ as given by eq.(4.2).

Theorem 4.4.3:

$$F_B^{(n)}(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n; \gamma; -X_1, \cdots, -X_n)$$

$$= \frac{1}{\Gamma_p(\beta_n)} \int_{T>0} e^{-\mathrm{tr}(T)} |T|^{\beta_n - (p+1)/2} \times \tag{4.20}$$

$$\Xi_1^{(n)}(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_{n-1}; \gamma; -X_1, \cdots, -X_{n-1}, -T^{1/2} X_n T^{1/2}) dT$$

for $\mathrm{Re}(\beta_n) > (p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(4.20) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we obtain,

$$\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1 - (p+1)/2} \cdots |X_n|^{\rho_n - (p+1)/2} \times$$

$$\Xi_1^{(n)}(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_{n-1}; \gamma; -X_1, \cdots, -X_{n-1}, -T^{1/2} X_n T^{1/2}) \times \tag{4.21}$$

$$dX_1 \cdots dX_n$$

On applying the same transformation as in the theorem 4.4.1 and using the definition (4.2.2) the above expression yields

$$|T|^{-\rho_n} \frac{\{\prod_{i=1}^{n} \Gamma_p(\alpha_i - \rho_i)\}}{\{\prod_{i=1}^{n} \Gamma_p(\alpha_i)\}} \frac{\{\prod_{j=1}^{n-1} \Gamma_p(\beta_j - \rho_j)\}}{\{\prod_{j=1}^{n-1} \Gamma_p(\beta_j)\}} \frac{\Gamma_p(\gamma)\{\prod_{i=1}^{n} \Gamma_p(\rho_i)\}}{\Gamma_p(\gamma - \rho_1 - \cdots - \rho_n)} \tag{4.22}$$

Substituting this expression on the right side of eq.(4.20) and integrating out T in the resulting expression by using a Gamma integral generates $M(F_B^{(n)})$ as given by eq.(4.3).

Theorem 4.4.4:

$$F_B^{(n)}(a_1,\cdots,a_n,b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(a_1)\Gamma_p(b_1)\cdots\Gamma_p(a_n)\Gamma_p(b_n)}\int_{S_1>0}\cdots(2n)\cdots\int_{T_n>0}\times$$
$$e^{-tr(S_1+T_1+\cdots+S_n+T_n)}|S_1|^{a_1-(p+1)/2}|T_1|^{b_1-(p+1)/2}\cdots\times \tag{4.23}$$
$$|S_n|^{a_n-(p+1)/2}|T_n|^{b_n-(p+1)/2}\,{}_0F_1(\ ;c;-T_1^{1/2}S_1^{1/2}X_1S_1^{1/2}T_1^{1/2}$$
$$-\cdots-T_n^{1/2}S_n^{1/2}X_nS_n^{1/2}T_n^{1/2})dS_1dT_1\cdots dS_ndT_n$$

for $\mathrm{Re}(a_i,b_i)>(p-1)/2,\ i=1,\cdots,n.$

Proof: Taking the M-transform of the right side of eq.(4.23) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\,{}_0F_1(\ ;c;$$
$$-T_1^{1/2}S_1^{1/2}X_1S_1^{1/2}T_1^{1/2}-\cdots-T_n^{1/2}S_n^{1/2}X_nS_n^{1/2}T_n^{1/2})dX_1\cdots dX_n \tag{4.24}$$

Making use of the transformations

$$Y_i=T_i^{1/2}S_i^{1/2}X_iS_i^{1/2}T_i^{1/2},\text{ so that, }dY_i=|T_i|^{(p+1)/2}\times$$
$$|S_i|^{(p+1)/2}dX_i,\text{ and, }|Y_i|=|T_i||S_i||X_i|\text{ for }i=1,\cdots,n;$$

in the expression (4.24) and then using the theorem 2.2.3 yields

$$|S_1|^{-\rho_1}|T_1|^{-\rho_1}\cdots|S_n|^{-\rho_n}|T_n|^{-\rho_n}\frac{\Gamma_p(c)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)}{\Gamma_p(c-\rho_1-\cdots-\rho_n)} \tag{4.25}$$

Substituting this expression on the right side of eq. (4.23) and then integrating out the variables $S_1,T_1,\cdots,S_n,T_n$ in the resulting expression by using a Gamma integral we obtain $M(F_B^{(n)})$ as given by eq.(4.3).

Theorem 4.4.5:

$$F_C^{(n)}(\alpha,\beta;\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(\alpha)}\int_{T>0} e^{-tr(T)}|T|^{\alpha-(p+1)/2}\Psi_2^{(n)}(\beta;\gamma_1,\cdots,\gamma_n; \qquad (4.26)$$
$$-T^{1/2}X_1T^{1/2},\cdots,-T^{1/2}X_nT^{1/2})dT$$
for $\mathrm{Re}(\alpha)>(p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(4.26) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we have

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times \qquad (4.27)$$
$$\Psi_2^{(n)}(\beta;\gamma_1,\cdots,\gamma_n;-T^{1/2}X_1T^{1/2},\cdots,-T^{1/2}X_nT^{1/2})dX_1\cdots dX_n$$

Making use of the transformations

$$Y_j=T^{1/2}X_jT^{1/2},\ \text{so that,}\ dY_j=|T|^{(p+1)/2}\,dX_j\ \text{and,}$$
$$|Y_j|=|T||X_j|\ \text{for}\ j=1,\cdots,n;$$

in the above expression and then applying the definition (6.5) page 79 of Mathai [62], the outcome is

$$|T|^{-\rho_1-\cdots-\rho_n}\frac{\{\prod_{j=1}^{n}\Gamma_p(\gamma_j)\Gamma_p(\rho_j)\}\Gamma_p(\beta-\rho_1-\cdots-\rho_n)}{\Gamma_p(\beta)\{\prod_{j=1}^{n}\Gamma_p(\gamma_j-\rho_j)\}} \qquad (4.28)$$

Substituting this expression on the right side of eq.(4.26) and then integrating out T in the ensuing equation by employing a Gamma integral leads to $M(F_C^{(n)})$ as given by eq.(4.4).

Theorem 4.4.6:

$$F_D^{(n)}(\alpha,\beta_1,\cdots,\beta_n;\gamma;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(\beta_n)}\int_{T>0} e^{-tr(T)}|T|^{\beta_n-(p+1)/2}\Phi_D^{(n)}(\alpha,\beta_1,\cdots,\beta_{n-1};\gamma; \tag{4.29}$$
$$-X_1,\cdots,-X_{n-1},-T^{1/2}X_nT^{1/2})dT$$

for $\mathrm{Re}(\beta_n)>(p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(4.29) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$
$$\Phi_D^{(n)}(\alpha,\beta_1,\cdots,\beta_{n-1};\gamma;-X_1,\cdots,-X_{n-1},-T^{1/2}X_nT^{1/2})dX_1\cdots dX_n \tag{4.30}$$

On applying the same transformation as in theorem (4.4.1) and then using the definition (4.2.3) the expression (4.30) leads us to

$$|T|^{-\rho_n}\frac{\Gamma_p(\alpha-\rho_1-\cdots-\rho_n)\{\prod_{i=1}^{n-1}\Gamma_p(\beta_i-\rho_i)\}\Gamma_p(\gamma)\{\prod_{j=1}^{n}\Gamma_p(\rho_j)\}}{\Gamma_p(\alpha)\{\prod_{i=1}^{n-1}\Gamma_p(\beta_i)\}\Gamma_p(\gamma-\rho_1-\cdots-\rho_n)} \tag{4.31}$$

Substituting this expression on the right side of eq.(4.29) and integrating out T in the resulting expression by using a Gamma integral gives $M(F_D^{(n)})$ in conformity with eq.(4.5).

Theorem 4.4.7:

$$F_D^{(n)}(a,b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(a)\Gamma_p(b_1)\cdots\Gamma_p(b_n)}\int_{T>0}\int_{S_1>0}\cdots\int_{S_n>0}e^{-tr(T+S_1+\cdots+S_n)}\times$$

Continued to the next page … … … … …

$$\times|T|^{a-(p+1)/2}|S_1|^{b_1-(p+1)/2}\cdots|S_n|^{b_n-(p+1)/2}\,{}_0F_1(\ ;c;$$
$$-T^{1/2}S_1^{1/2}X_1S_1^{1/2}T^{1/2}-\cdots-T^{1/2}S_n^{1/2}X_nS_n^{1/2}T^{1/2})dTdS_1\cdots dS_n \quad (4.32)$$

for $\mathrm{Re}(a,b_1,\cdots,b_n)>(p-1)/2.$

Proof: This theorem can be proved in a similar fashion as theorem (4.4.4) above. The transformations to be used are $Y_i=T^{1/2}S_i^{1/2}X_iS_i^{1/2}T^{1/2}$, for $i=1,\cdots,n$. Finally $M(F_D^{(n)})$ is obtained, as given by eq. (4.5).

4.5 The $\Phi_D^{(n)}$ and $\Phi_2^{(n)}$ Functions of Matrix Arguments

Theorem 4.5.1:

$$\Phi_D^{(n)}(a,b_1,\cdots,b_{n-1};c;-X_1,\cdots,-X_n)$$
$$=\frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}\int_0^I|U|^{a-(p+1)/2}|I-U|^{c-a-(p+1)/2}\times$$
$$\left|I+U^{1/2}X_1U^{1/2}\right|^{-b_1}\cdots\left|I+U^{1/2}X_{n-1}U^{1/2}\right|^{-b_{n-1}}e^{-\mathrm{tr}(UX_n)}dU \quad (4.33)$$

for $\mathrm{Re}(a,c-a)>(p-1)/2$ and $0<U<I$.

Proof: Taking the M-transform of the right side of eq.(4.33) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we obtain,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$
$$\left|I+U^{1/2}X_1U^{1/2}\right|^{-b_1}\cdots\left|I+U^{1/2}X_{n-1}U^{1/2}\right|^{-b_{n-1}}\times \quad (4.34)$$
$$e^{-\mathrm{tr}(UX_n)}dX_1\cdots dX_n$$

Applying the transformations,

$$Y_j = U^{1/2} X_j U^{1/2}, \text{ with } dY_j = |U|^{(p+1)/2} dX_j \text{ and } |Y_j| = |U||X_j|,$$

for $j = 1, \cdots, n-1$; to the expression (4.34) and then integrating out the variables $Y_1, \cdots, Y_{n-1}$ by using a type-2 Beta integral and the variable X_n by using a Gamma integral we are led to,

$$|U|^{-\rho_1 - \cdots - \rho_n} \frac{\Gamma_p(b_1 - \rho_1)\Gamma_p(\rho_1)}{\Gamma_p(b_1)} \cdots \frac{\Gamma_p(b_{n-1} - \rho_{n-1})\Gamma_p(\rho_{n-1})}{\Gamma_p(b_{n-1})} \times \Gamma_p(\rho_n) \tag{4.35}$$

Substituting this expression on the right side of eq.(4.33) and integrating out the variable U in the resulting expression by using a type-1 Beta integral yields $M(\Phi_D^{(n)})$ as given by eq.(4.9).

Theorem 4.5.2:

$$\Phi_2^{(n)}(b_1, \cdots, b_n; c; -X_1, \cdots, -X_n)$$

$$= \frac{2^{p(p-1)/2}\Gamma_p(c)}{(2\pi i)^{p(p+1)/2}} \int e^{tr(S)} |S|^{-c} \left|I + X_1 S^{-1}\right|^{-b_1} \cdots \times \left|I + X_n S^{-1}\right|^{-b_n} dS \tag{4.36}$$

where $S = S_1 + iS_2$, S_1 and S_2 are real matrices with $S_1 = S_1' > 0$ and it being assumed that $S^{-1} = VV'$ with $V'X_jV > 0$, $j = 1, \cdots, n$ and $i = \sqrt{-1}$.

Proof: Taking the M-transform of the right side of eq.(4.36) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we have,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\left|I+X_1S^{-1}\right|^{-b_1}$$
$$\cdots\left|I+X_nS^{-1}\right|^{-b_n}\,dX_1\cdots dX_n \tag{4.37}$$

Now observing that,

$$\left|I+X_jS^{-1}\right|=\left|I+X_jVV'\right|=\left|I+V'X_jV\right| \text{ for } j=1,\cdots,n; |S|^{-1}=|V|^2;$$

and making use of the transformations

$$Y_j=V'X_jV \text{ (with } dY_j=|V|^{p+1}dX_j \text{ and } \left|Y_j\right|=|V|^2\left|X_j\right|) \text{ for } j=1,\cdots,n;$$

and then integrating out the variables $Y_1,\cdots,Y_n$ by using a type-2 Beta integral, the last expression renders

$$|S|^{\rho_1+\cdots+\rho_n}\frac{\Gamma_p(b_1-\rho_1)\Gamma_p(\rho_1)}{\Gamma_p(b_1)}\cdots\frac{\Gamma_p(b_n-\rho_n)\Gamma_p(\rho_n)}{\Gamma_p(b_n)} \tag{4.38}$$

Substituting this expression on the right of eq.(4.36) and using eq.(2.5.11) page 49 of Mathai [62] in the resulting expression produces $M(\Phi_2^{(n)})$, as given by eq.(4.6).

Theorem 4.5.3:

$$|P|^{-\gamma}\left|I+P^{-1/2}X_1P^{-1/2}\right|^{-\beta_1}\cdots\left|I+P^{-1/2}X_nP^{-1/2}\right|^{-\beta_n}$$
$$=\frac{1}{\Gamma_p(\gamma)}\int_{T>0}e^{-tr(PT)}|T|^{\gamma-(p+1)/2}\Phi_2^{(n)}(\beta_1,\cdots,\beta_n;\gamma; \tag{4.39}$$
$$-T^{1/2}X_1T^{1/2},\cdots,-T^{1/2}X_nT^{1/2})dT$$

for $\mathrm{Re}(\gamma)>(p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(4.39) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$
$$\Phi_2^{(n)}(\beta_1,\cdots,\beta_n;\gamma;-T^{1/2}X_1T^{1/2},\cdots,-T^{1/2}X_nT^{1/2})dX_1\cdots dX_n \tag{4.40}$$

Applying the transformations

$$Y_i=T^{1/2}X_iT^{1/2} \text{ with } dY_i=|T|^{(p+1)/2}dX_i \text{ and } |Y_i|=|T||X_i|,$$

for $i=1,\cdots,n;$ in the above expression and then using eq.(4.6), we obtain,

$$|T|^{-\rho_1-\cdots-\rho_n}\frac{\Gamma_p(\gamma)\prod_{i=1}^{n}\{\Gamma_p(\beta_i-\rho_i)\Gamma_p(\rho_i)\}}{\prod_{i=1}^{n}\{\Gamma_p(\beta_i)\}\Gamma_p(\gamma-\rho_1-\cdots-\rho_n)} \tag{4.41}$$

Substituting this expression on the right side of eq.(4.39) and integrating out T in the resulting expression by using a Gamma integral produces,

$$|P|^{-(\gamma-\rho_1-\cdots-\rho_n)}\frac{\prod_{i=1}^{n}\{\Gamma_p(\beta_i-\rho_i)\Gamma_p(\rho_i)\}}{\prod_{i=1}^{n}\{\Gamma_p(\beta_i)\}} \tag{4.42}$$

Now taking the M-transform of the left side of eq.(4.39) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$
$$|P|^{-\gamma}\left|I+P^{-1/2}X_1P^{-1/2}\right|^{-\beta_1}\cdots\left|I+P^{-1/2}X_nP^{-1/2}\right|^{-\beta_n}dX_1\cdots dX_n \tag{4.43}$$

On using the transformations

$$Z_i=P^{-1/2}X_iP^{-1/2} \text{ with } dZ_i=|P|^{-(p+1)/2}dX_i \text{ and } |Z_i|=|P|^{-1}|X_i|$$

for $i=1,\cdots,n$; in the above expression and integrating out the variables $Z_1,\cdots,Z_n$ by using a type-2 Beta integral the outcome is the same as in eq.(4.42) above.

Theorem 4.5.4: Cases concerning transformation and reducibility:

(i)
$$\begin{aligned}&\Phi_D^{(n)}(a,b_1,\cdots,b_{n-1};c;-X_1,\cdots,-X_n)\\&=e^{-\mathrm{tr}(X_n)}\left|I+X_1\right|^{-b_1}\cdots\left|I+X_{n-1}\right|^{-b_{n-1}}\Phi_D^{(n)}[c-a,b_1,\cdots,\\&b_{n-1};c;(I+X_1)^{-1/2}X_1(I+X_1)^{-1/2},\cdots,(I+X_{n-1})^{-1/2}X_{n-1}\times\\&(I+X_{n-1})^{-1/2},X_n]\end{aligned}\tag{4.44}$$

(ii)
$$\begin{aligned}&\Phi_D^{(n)}(a,b_1,\cdots,b_{n-1};c;-X,\cdots(n-1)\cdots,-X,-X_n)\\&=\Phi_1(a,b_1+\cdots+b_{n-1};c;-X,-X_n)\end{aligned}\tag{4.45}$$

(iii)
$$\begin{aligned}&\Phi_D^{(n)}(a,b_1,\cdots,b_{n-1};c;I,\cdots(n-1)\cdots,I,-X_n)\\&=\frac{\Gamma_p(c)\Gamma_p(c-b_1-\cdots-b_{n-1}-a)}{\Gamma_p(c-a)\Gamma_p(c-b_1-\cdots-b_{n-1})}\times\\&{}_1F_1(a;c-b_1-\cdots-b_{n-1};-X_n)\end{aligned}\tag{4.46}$$

Proof: (i) To prove the result in eq.(4.44) we define the $\Phi_D^{(n)}$ function through an integral representation:

$$\begin{aligned}&\Phi_D^{(n)}(a,b_1,\cdots,b_{n-1};c;-X_1,\cdots,-X_n)\\&=\frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}\int_0^I|U|^{a-(p+1)/2}|I-U|^{c-a-(p+1)/2}\times\\&\left|I+X_1^{1/2}UX_1^{1/2}\right|^{-b_1}\cdots\left|I+X_{n-1}^{1/2}UX_{n-1}^{1/2}\right|^{-b_{n-1}}e^{-\mathrm{tr}(UX_n)}dU\end{aligned}\tag{4.47}$$

for $\mathrm{Re}(a,c-a)>(p-1)/2$ and $0<U<I$.

Now the result in eq.(4.44) follows from eq.(4.47) by applying the transformation $I - U = V$ and by observing that,

$$\left| I + X_i^{1/2}(I - V)X_i^{1/2} \right| = \left| I + X_i \right| \left| I - (I + X_i)^{-1/2} X_i^{1/2} V X_i^{1/2} (I + X_i)^{-1/2} \right|$$

for $i = 1, \cdots, n-1$; and then suitably interpreting the resulting expression in the light of eq.(4.47).

(ii) The result in eq.(4.45) follows by putting $X_1 = \cdots = X_{n-1} = X$ in eq.(4.33) and then using eq.(3.16).

(iii) This result can be obtained by letting $X_1 \to -I, \cdots, X_{n-1} \to -I$ in eq.(4.33) and then utilizing the eq.(2.3).

CHAPTER V

THE LAURICELLA- SARAN AND THE SRIVASTAVA'S TRIPLE HYERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENTS

5.1 Definitions: The Lauricella-Saran Functions

Lauricella, in 1893, had studied the multiple hypergeometric functions. He also gave the properties of the four triple hypergeometric functions $F_A^{(3)}, F_B^{(3)}, F_C^{(3)}$ and $F_D^{(3)}$ and conjectured the existence of ten other such functions, where all these fourteen functions are complete and of the second order. Saran [93] in 1954 introduced and studied the remaining ten functions, thus completing the Lauricella's conjectured set of functions. This section contains my definitions of the ten Lauricella-Saran functions of matrix arguments.

5.1.1 The Lauricella-Saran triple hypergeometric function F_E of matrix arguments

$$F_E = F_E(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_3; -X, -Y, -Z)$$

is defined as that class of functions which has the following matrix transform:

$$M(F_E) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2} F_E(a_1, a_1, a_1, b_1, b_2, b_2; c_1, c_2, c_3; -X, -Y, -Z)\times$$
$$dXdYdZ \tag{5.1}$$
$$= \frac{\Gamma_p(a_1-\rho_1-\rho_2-\rho_3)}{\Gamma_p(a_1)}\frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_2-\rho_3)}{\Gamma_p(b_2)}\times$$
$$\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)}\frac{\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}$$

for $\mathrm{Re}(a_1-\rho_1-\rho_2-\rho_3, b_1-\rho_1, b_2-\rho_2-\rho_3, c_1-\rho_1, c_2-\rho_2,$
$c_3-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2.$

5.1.2 $F_F = F_F(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;-X,-Y,-Z)$

$$M(F_F) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}\,|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}\, F_F(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;-X,-Y,-Z)\times$$
$$dXdYdZ$$
$$=\frac{\Gamma_p(a_1-\rho_1-\rho_2-\rho_3)}{\Gamma_p(a_1)}\frac{\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)}\times$$
$$\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)}\frac{\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_2-\rho_2-\rho_3)}$$

for $\mathrm{Re}(a_1-\rho_1-\rho_2-\rho_3, b_1-\rho_1-\rho_3, b_2-\rho_2, c_1-\rho_1,$
$c_2-\rho_2-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2.$ (5.2)

5.1.3 $F_G = F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$

$$M(F_G) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}\,|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}\, F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)\times$$
$$dXdYdZ \tag{5.3}$$
$$=\frac{\Gamma_p(a_1-\rho_1-\rho_2-\rho_3)}{\Gamma_p(a_1)}\frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)}\times$$
$$\frac{\Gamma_p(b_3-\rho_3)}{\Gamma_p(b_3)}\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)}\frac{\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_2-\rho_2-\rho_3)}$$

for $\mathrm{Re}(a_1-\rho_1-\rho_2-\rho_3, b_1-\rho_1, b_2-\rho_2, b_3-\rho_3, c_1-\rho_1,$
$c_2-\rho_2-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2.$

5.1.4 $F_K = F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; -X, -Y, -Z)$

$$M(F_K) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; -X, -Y, -Z) \times$$
$$dXdYdZ$$
$$= \frac{\Gamma_p(a_1-\rho_1)}{\Gamma_p(a_1)} \frac{\Gamma_p(a_2-\rho_2-\rho_3)\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(a_2)\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)} \times$$
$$\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)} \frac{\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)}$$

for $\mathrm{Re}(a_1-\rho_1, a_2-\rho_2-\rho_3, b_1-\rho_1-\rho_3, b_2-\rho_2, c_1-\rho_1,$
$c_2-\rho_2, c_3-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2.$ (5.4)

5.1.5 $F_M = F_M(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)$

$$M(F_M) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} F_M(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z) \times$$
$$dXdYdZ$$
$$= \frac{\Gamma_p(a_1-\rho_1)}{\Gamma_p(a_1)} \frac{\Gamma_p(a_2-\rho_2-\rho_3)}{\Gamma_p(a_2)} \frac{\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)} \times$$
$$\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)} \frac{\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_2-\rho_2-\rho_3)}$$

for $\mathrm{Re}(a_1-\rho_1, a_2-\rho_2-\rho_3, b_1-\rho_1-\rho_3, b_2-\rho_2, c_1-\rho_1,$
$c_2-\rho_2-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2.$ (5.5)

5.1.6 $F_N = F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)$

$$M(F_N) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z) \times$$
$$dXdYdZ$$
$$= \frac{\Gamma_p(a_1-\rho_1)\Gamma_p(a_2-\rho_2)\Gamma_p(a_3-\rho_3)\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(a_3)\Gamma_p(b_1)} \times$$
$$\frac{\Gamma_p(b_2-\rho_2)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(b_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2-\rho_3)}$$
for Re $(a_i-\rho_i, b_1-\rho_1-\rho_3, b_2-\rho_2, c_1-\rho_1, c_2-\rho_2-\rho_3, \rho_i)$
$> (p-1)/2, \ i = 1, 2, 3.$ (5.6)

5.1.7 $F_P = F_P(a_1, a_2, a_1, b_1, b_1, b_2; c_1, c_2, c_2; -X, -Y, -Z)$

$$M(F_P) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} F_P(a_1, a_2, a_1, b_1, b_1, b_2; c_1, c_2, c_2; -X, -Y, -Z) \times$$
$$dXdYdZ$$
$$= \frac{\Gamma_p(a_1-\rho_1-\rho_3)\Gamma_p(a_2-\rho_2)\Gamma_p(b_1-\rho_1-\rho_2)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_1)} \times$$
$$\frac{\Gamma_p(b_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(b_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2-\rho_3)}$$
for Re $(a_1-\rho_1-\rho_3, a_2-\rho_2, b_1-\rho_1-\rho_2, b_2-\rho_3, c_1-\rho_1,$
$c_2-\rho_2-\rho_3, \rho_i) > (p-1)/2, \ i = 1, 2, 3.$ (5.7)

5.1.8 $F_R = F_R(a_1, a_2, a_1, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)$

$$M(F_R) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \times$$
$$|Z|^{\rho_3 - (p+1)/2} F_R(a_1, a_2, a_1, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z) \times$$
$$dXdYdZ$$
$$= \frac{\Gamma_p(a_1 - \rho_1 - \rho_3)\Gamma_p(a_2 - \rho_2)\Gamma_p(b_1 - \rho_1 - \rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_1)} \times$$
$$\frac{\Gamma_p(b_2 - \rho_2)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(b_2)\Gamma_p(c_1 - \rho_1)\Gamma_p(c_2 - \rho_2 - \rho_3)}$$
for Re $(a_1 - \rho_1 - \rho_3, a_2 - \rho_2, b_1 - \rho_1 - \rho_3, b_2 - \rho_2, c_1 - \rho_1,$
$c_2 - \rho_2 - \rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2.$ (5.8)

5.1.9 $F_S = F_S(a_1, a_2, a_2, b_1, b_2, b_3; c_1, c_1, c_1; -X, -Y, -Z)$

$$M(F_S) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \times$$
$$|Z|^{\rho_3 - (p+1)/2} F_S(a_1, a_2, a_2, b_1, b_2, b_3; c_1, c_1, c_1; -X, -Y, -Z) \times$$
$$dXdYdZ$$
$$= \frac{\Gamma_p(a_1 - \rho_1)\Gamma_p(a_2 - \rho_2 - \rho_3)\Gamma_p(b_1 - \rho_1)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_1)} \times$$
$$\frac{\Gamma_p(b_2 - \rho_2)\Gamma_p(b_3 - \rho_3)\Gamma_p(c_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1 - \rho_1 - \rho_2 - \rho_3)}$$
for Re $(a_1 - \rho_1, a_2 - \rho_2 - \rho_3, b_1 - \rho_1, b_2 - \rho_2, b_3 - \rho_3, c_1 - \rho_1$
$-\rho_2 - \rho_3, \rho_i) > (p-1)/2,\ i = 1, 2, 3.$ (5.9)

5.1.10 $F_T = F_T(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; -X, -Y, -Z)$

$$M(F_T) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} F_T(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; -X, -Y, -Z) \times$$
$$dXdYdZ$$
$$= \frac{\Gamma_p(a_1-\rho_1)}{\Gamma_p(a_1)} \frac{\Gamma_p(a_2-\rho_2-\rho_3)}{\Gamma_p(a_2)} \frac{\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)} \times$$
$$\frac{\Gamma_p(c_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_1-\rho_1-\rho_2-\rho_3)} \tag{5.10}$$

for $\mathrm{Re}(a_1-\rho_1, a_2-\rho_2-\rho_3, b_1-\rho_1-\rho_3, b_2-\rho_2,$
$c_1-\rho_1-\rho_2-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2.$

5.2 Definitions: The Srivastava Functions

5.2.1 The Srivastava function H_A of matrix arguments

$$H_A = H_A(a, b, b'; c, c'; -X, -Y, -Z)$$

is defined as that class of functions which has the following matrix transform:

$$M(H_A) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} H_A(a, b, b'; c, c'; -X, -Y, -Z) dXdYdZ$$
$$= \frac{\Gamma_p(a-\rho_1-\rho_3)}{\Gamma_p(a)} \frac{\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(b)} \frac{\Gamma_p(b'-\rho_2-\rho_3)}{\Gamma_p(b')} \frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1)} \times$$
$$\frac{\Gamma_p(c')}{\Gamma_p(c'-\rho_2-\rho_3)} \Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)$$

for $\mathrm{Re}\ (a-\rho_1-\rho_3, b-\rho_1-\rho_2, b'-\rho_2-\rho_3, c-\rho_1, c'-\rho_2-\rho_3,$
$\rho_1, \rho_2, \rho_3) > (p-1)/2.$ (5.11)

5.2.2 $\quad H_B = H_B(a, b, b'; c_1, c_2, c_3; -X, -Y, -Z)$

$$M(H_B) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} H_B(a, b, b'; c_1, c_2, c_3; -X, -Y, -Z)dXdYdZ$$
$$= \frac{\Gamma_p(a-\rho_1-\rho_3)}{\Gamma_p(a)} \frac{\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(b)} \frac{\Gamma_p(b'-\rho_2-\rho_3)}{\Gamma_p(b')} \frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)} \times \tag{5.12}$$
$$\frac{\Gamma_p(c_2)}{\Gamma_p(c_2-\rho_2)} \frac{\Gamma_p(c_3)}{\Gamma_p(c_3-\rho_3)} \Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)$$

for Re $(a-\rho_1-\rho_3, b-\rho_1-\rho_2, b'-\rho_2-\rho_3, c_1-\rho_1, c_2-\rho_2, c_3-\rho_3,$
$\rho_1, \rho_2, \rho_3) > (p-1)/2.$

5.2.3 $\quad H_C = H_C(a, b, b'; c; -X, -Y, -Z)$

$$M(H_C) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} H_C(a, b, b'; c; -X, -Y, -Z)dXdYdZ$$
$$= \frac{\Gamma_p(a-\rho_1-\rho_3)}{\Gamma_p(a)} \frac{\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(b)} \frac{\Gamma_p(b'-\rho_2-\rho_3)}{\Gamma_p(b')} \times \tag{5.13}$$
$$\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3)} \Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)$$

for Re $(a-\rho_1-\rho_3, b-\rho_1-\rho_2, b'-\rho_2-\rho_3, c-\rho_1-\rho_2-\rho_3,$
$\rho_1, \rho_2, \rho_3) > (p-1)/2.$

5.3 This section contains a number of results for the Lauricella-Saran functions. We prove only some of the results, others can be proved on similar lines.

Theorem 5.3.1:

$$F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; -X, -Y, -Z)$$

$$= \frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(a_1)\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(c_1 - a_1)\Gamma_p(c_2 - b_2)\Gamma_p(c_3 - b_1)} \times$$

$$\int_0^I\int_0^I\int_0^I |U|^{a_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_1-(p+1)/2} \times$$

$$|I-U|^{c_1-a_1-(p+1)/2}|I-V|^{c_2-b_2-(p+1)/2} \times$$

$$|I-W|^{c_3-b_1-(p+1)/2}\left|I+U^{1/2}XU^{1/2}\right|^{-b_1}\left|I+V^{1/2}YV^{1/2}\right|^{-a_2} \times$$

$$\Big|I+(I+V^{1/2}YV^{1/2})^{-1/2}(I+U^{1/2}XU^{1/2})^{-1/2}W^{1/2}ZW^{1/2} \times$$

$$(I+U^{1/2}XU^{1/2})^{-1/2}(I+V^{1/2}YV^{1/2})^{-1/2}\Big|^{-a_2}\, dUdVdW \tag{5.14}$$

for $\mathrm{Re}(c_1 - a_1, c_2 - b_2, c_3 - b_1, a_1, b_1, b_2) > (p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(5.14) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 respectively, we get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2} \times$$

$$|Z|^{\rho_3-(p+1)/2}\left|I+U^{1/2}XU^{1/2}\right|^{-b_1}\left|I+V^{1/2}YV^{1/2}\right|^{-a_2} \times$$

$$\Big|I+(I+V^{1/2}YV^{1/2})^{-1/2}(I+U^{1/2}XU^{1/2})^{-1/2}W^{1/2}ZW^{1/2} \times$$

Continued to the next page … … … … …

$$(I+U^{1/2}XU^{1/2})^{-1/2}(I+V^{1/2}YV^{1/2})^{-1/2}\Big|^{-a_2}\, dXdYdZ \quad (5.15)$$

Applying the transformations,

$X_1 = U^{1/2}XU^{1/2}, Y_1 = V^{1/2}YV^{1/2}, Z_1 = (I+V^{1/2}YV^{1/2})^{-1/2} \times$
$(I+U^{1/2}XU^{1/2})^{-1/2} W^{1/2}ZW^{1/2}(I+U^{1/2}XU^{1/2})^{-1/2} \times$
$(I+V^{1/2}YV^{1/2})^{-1/2};$ so that, $Z_1 = (I+Y_1)^{-1/2}(I+X_1)^{-1/2}W^{1/2} \times$
$ZW^{1/2}(I+X_1)^{-1/2}(I+Y_1)^{-1/2}; dX_1 = |U|^{(p+1)/2}\, dX,$
$dY_1 = |V|^{(p+1)/2}\, dY, dZ_1 = |I+Y_1|^{-(p+1)/2}\, |I+X_1|^{-(p+1)/2} \times$
$|W|^{(p+1)/2}\, dZ;$ and, $|X_1| = |U||X|, |Y_1| = |V||Y|, |Z_1| = |I+Y_1|^{-1} \times$
$|I+X_1|^{-1}|W||Z|;$

to the expression (5.15) and integrating out the variables X_1, Y_1, Z_1 by using a type-2 Beta integral produces,

$$|U|^{-\rho_1}\, |V|^{-\rho_2}\, |W|^{-\rho_3}\, \frac{\Gamma_p(\rho_1)\Gamma_p(b_1-\rho_1-\rho_3)\Gamma_p(\rho_2)}{\Gamma_p(b_1-\rho_3)} \times \frac{\Gamma_p(a_2-\rho_2-\rho_3)\Gamma_p(\rho_3)}{\Gamma_p(a_2)} \quad (5.16)$$

which on substituting on the right side of eq.(5.14) and integrating out U, V and W by using a type-1 Beta integral leads to $M(F_K)$ as given by eq.(5.4).

Theorem 5.3.2:

$$
\begin{aligned}
&F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)\\
&=\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)}\times\\
&\iiint |U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}\times\\
&|I-U|^{c_1-b_1-(p+1)/2}|I-V-W|^{c_2-b_2-b_3-(p+1)/2}\times\\
&\left|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}ZW^{1/2}\right|^{-a_1}dUdVdW
\end{aligned}
\tag{5.17}
$$

for $\mathrm{Re}(c_1-b_1,c_2-b_2-b_3,b_1,b_2,b_3)>(p-1)/2.$

Theorem 5.3.3:

$$
\begin{aligned}
&F_E(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;-X,-Y,-Z)\\
&=\frac{1}{\Gamma_p(a_1)}\int_{U>0}e^{-\mathrm{tr}(U)}|U|^{a_1-(p+1)/2}\,{}_1F_1(b_1;c_1;-U^{1/2}XU^{1/2})\times\\
&\Psi_2(b_2;c_2,c_3;-U^{1/2}YU^{1/2},-U^{1/2}ZU^{1/2})dU
\end{aligned}
\tag{5.18}
$$

for $\mathrm{Re}(a_1)>(p-1)/2.$

Theorem 5.3.4:

$$
\begin{aligned}
&F_F(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;-X,-Y,-Z)\\
&=\frac{1}{\Gamma_p(a_1)\Gamma_p(b_1)}\int_{R_1>0}\int_{R_2>0}e^{-\mathrm{tr}(R_1+R_2)}|R_1|^{a_1-(p+1)/2}\times\\
&|R_2|^{b_1-(p+1)/2}\,{}_0F_1(\,;c_1;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2})\times\\
&\Phi_3(b_2;c_2;-R_1^{1/2}YR_1^{1/2},-R_2^{1/2}R_1^{1/2}ZR_1^{1/2}R_2^{1/2})dR_1dR_2
\end{aligned}
\tag{5.19}
$$

$$\text{for } \operatorname{Re}(a_1, b_1) > (p-1)/2.$$

Proof: Taking the M-transform of the right side of eq.(5.19) with respect to the variables X,Y,Z and the parameters ρ_1, ρ_2, ρ_3 respectively, we have

$$\int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2} \times$$
$${}_0F_1(\,;c_1;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2})\Phi_3(b_2;c_2;-R_1^{1/2}YR_1^{1/2}, \tag{5.20}$$
$$-R_2^{1/2}R_1^{1/2}ZR_1^{1/2}R_2^{1/2})dXdYdZ$$

Making use of the transformations

$$X_1 = R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2}, Y_1 = R_1^{1/2}YR_1^{1/2}, Z_1 = R_2^{1/2}R_1^{1/2}ZR_1^{1/2}R_2^{1/2};$$

in the expression (5.20) and then writing the M-transforms of the ${}_0F_1$ and Φ_3 - functions leads us to

$$|R_1|^{-\rho_1-\rho_2-\rho_3}|R_2|^{-\rho_1-\rho_3}\frac{\Gamma_p(c_1)\Gamma_p(\rho_1)\Gamma_p(c_2)\Gamma_p(b_2-\rho_2)}{\Gamma_p(c_1-\rho_1)\Gamma_p(b_2)} \times$$
$$\frac{\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_2-\rho_2-\rho_3)} \tag{5.21}$$

Substituting this expression on the right side of eq.(5.19) and then integrating out the variables R_1 and R_2 in the resulting expression by using a Gamma integral gives $M(F_F)$ as given by eq.(5.2).

Theorem 5.3.5:

$$F_F(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;-X,-Y,-Z)$$
$$=\frac{\Gamma_p(c_2)}{\Gamma_p(b_2)\Gamma_p(c_2-b_2)}\int_0^I |T|^{b_2-(p+1)/2}|I-T|^{c_2-b_2-(p+1)/2} \times$$

Continued to the next page … … … … …

$$\left|I+T^{1/2}YT^{1/2}\right|^{-a_1} F_4[a_1, b_1; c_1, c_2-b_2; -(I+T^{1/2}YT^{1/2})^{-1/2}\times$$

$$X(I+T^{1/2}YT^{1/2})^{-1/2}, -(I+T^{1/2}YT^{1/2})^{-1/2}(I-T)^{1/2}Z\times \tag{5.22}$$

$$(I-T)^{1/2}(I+T^{1/2}YT^{1/2})^{-1/2}]dT$$

for $\mathrm{Re}(b_2, c_2-b_2) > (p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(5.22) with respect to the variables X,Y,Z and the parameters ρ_1, ρ_2, ρ_3 respectively, we get

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times$$

$$\left|I+T^{1/2}YT^{1/2}\right|^{-a_1} F_4[a_1, b_1; c_1, c_2-b_2; -(I+T^{1/2}YT^{1/2})^{-1/2}X \tag{5.23}$$

$$\times(I+T^{1/2}YT^{1/2})^{-1/2}, -(I+T^{1/2}YT^{1/2})^{-1/2}(I-T)^{1/2}Z\times$$

$$(I-T)^{1/2}(I+T^{1/2}YT^{1/2})^{-1/2}]dXdYdZ$$

The application of the transformations

$$Y_1 = T^{1/2}YT^{1/2}, X_1 = (I+T^{1/2}YT^{1/2})^{-1/2}X(I+T^{1/2}YT^{1/2})^{-1/2},$$

$$Z_1 = (I+T^{1/2}YT^{1/2})^{-1/2}(I-T)^{1/2}Z(I-T)^{1/2}(I+T^{1/2}YT^{1/2})^{-1/2},$$

to the above expression and then writing the M-transform of an F_4- function and integrating out Y_1 by using a type-2 Beta integral yields

$$|T|^{-\rho_2}|I-T|^{-\rho_3}\frac{\Gamma_p(\rho_2)\Gamma_p(a_1-\rho_1-\rho_2-\rho_3)\Gamma_p(c_1)}{\Gamma_p(a_1)\Gamma_p(b_1)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-b_2-\rho_3)}\times \tag{5.24}$$

$$\Gamma_p(c_2-b_2)\Gamma_p(b_1-\rho_1-\rho_3)\Gamma_p(\rho_1)\Gamma_p(\rho_3)$$

Substituting this expression on the right side of eq.(5.22) and then integrating out the variable T in the resulting expression by using a type -1 Beta integral yields $M(F_F)$ as given by eq.(5.2) above.

Theorem 5.3.6:

$$F_M(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;-X,-Y,-Z)$$
$$=\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(a_1)\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(c_1-a_1)\Gamma_p(c_2-b_2-b_1)}\times$$
$$\iint\int_0^I |U|^{b_2-(p+1)/2}|V|^{b_1-(p+1)/2}|T|^{a_1-(p+1)/2}\times$$
$$|I-T|^{c_1-a_1-(p+1)/2}\left|I+T^{1/2}XT^{1/2}\right|^{-b_1}\times$$
$$|I-U-V|^{c_2-b_2-b_1-(p+1)/2}\Big|I+U^{1/2}YU^{1/2}+$$
$$(I+T^{1/2}XT^{1/2})^{-1/2}V^{1/2}ZV^{1/2}(I+T^{1/2}XT^{1/2})^{-1/2}\Big|^{-a_2}\times$$
$$dUdVdT$$
$$\text{where, } T=T'>0, 0<T<I;\ U=U'>0, V=V'>0, \tag{5.25}$$
$$0<U+V<I, \text{and for } \operatorname{Re}(a_1,b_1,b_2,c_1-a_1,c_2-b_2-b_1)>(p-1)/2.$$

Theorem 5.3.7:

$$F_M(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;-X,-Y,-Z)$$
$$=\frac{1}{\Gamma_p(b_1)}\int_{R>0}e^{-tr(R)}|R|^{b_1-(p+1)/2}\,{}_1F_1(a_1;c_1;-R^{1/2}XR^{1/2})\times \tag{5.26}$$
$$\Phi_1(a_2,b_2;c_2;-Y,-R^{1/2}ZR^{1/2})dR$$
for $\operatorname{Re}(b_1)>(p-1)/2.$

Theorem 5.3.8:

$$F_T(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_1,c_1;-X,-Y,-Z)$$

$$=\frac{\Gamma_p(c_1)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-a_1-a_2)}\iint|U|^{a_1-(p+1)/2}|V|^{a_2-(p+1)/2}$$

$$\times|I-U-V|^{c_1-a_1-a_2-(p+1)/2}\left|I+U^{1/2}XU^{1/2}+V^{1/2}ZV^{1/2}\right|^{-b_1} \tag{5.27}$$

$$\times\left|I+V^{1/2}YV^{1/2}\right|^{-b_2}\,dUdV$$

for $\operatorname{Re}(a_1,a_2,c_1-a_1-a_2)>(p-1)/2;$ where, $U>0,V>0$ and $0<U+V<I.$

Theorem 5.3.9:

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$$

$$=\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)}\times$$

$$\int_0^I\int_0^I\int_0^I|U|^{b_1-(p+1)/2}|S|^{b_2+b_3-(p+1)/2}|T|^{b_3-(p+1)/2}\times$$

$$|I-U|^{c_1-b_1-(p+1)/2}|I-S|^{c_2-b_2-b_3-(p+1)/2}\times$$

$$|I-T|^{b_2-(p+1)/2}\left|I+U^{1/2}XU^{1/2}+(I-T)^{1/2}S^{1/2}YS^{1/2}\times\right.$$

$$\left.(I-T)^{1/2}+T^{1/2}S^{1/2}ZS^{1/2}T^{1/2}\right|^{-a_1}\,dUdSdT \tag{5.28}$$

where $0 < U < I$, $0 < S < I$, $0 < T < I$, and for $\mathrm{Re}(b_1, b_2, b_3,$

$c_1 - b_1, c_2 - b_2 - b_3) > (p-1)/2.$

Theorem 5.3.10:

$$F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; -X, -Y, -Z)$$

$$= \frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_2)\Gamma_p(c_1 - a_1)\Gamma_p(c_2 - b_2)\Gamma_p(c_3 - a_2)} \times$$

$$\int_0^I \int_0^I \int_0^I |U|^{a_1 - (p+1)/2} |V|^{a_2 - (p+1)/2} |T|^{b_2 - (p+1)/2} \times$$

$$|I - U|^{c_1 - a_1 - (p+1)/2} |I - V|^{c_3 - a_2 - (p+1)/2} \times$$

$$|I - T|^{c_2 - b_2 - (p+1)/2} \left| I + T^{1/2} Y T^{1/2} \right|^{-a_2} \Big| I + U^{1/2} X U^{1/2}$$

$$+ (I + T^{1/2} Y T^{1/2})^{-1/2} V^{1/2} Z V^{1/2} (I + T^{1/2} Y T^{1/2})^{-1/2} \Big|^{-b_1} \times$$

$$dUdVdT \tag{5.29}$$

for $\mathrm{Re}(c_1 - a_1, c_2 - b_2, c_3 - a_2, a_1, a_2, b_2) > (p-1)/2.$

Theorem 5.3.11:

$$F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)$$

$$= \frac{1}{\Gamma_p(a_3)\Gamma_p(b_1)\Gamma_p(b_2)} \int_{R_1 > 0} \int_{R_2 > 0} \int_{R_3 > 0} e^{-\mathrm{tr}(R_1 + R_2 + R_3)} \times$$

$$|R_1|^{a_3 - (p+1)/2} |R_2|^{b_1 - (p+1)/2} |R_3|^{b_2 - (p+1)/2} {}_1F_1(a_1; c_1; \tag{5.30}$$

$$-R_2^{1/2} X R_2^{1/2}) \Phi_3(a_2; c_2; -R_3^{1/2} Y R_3^{1/2}, -R_2^{1/2} R_1^{1/2} Z R_1^{1/2} R_2^{1/2}) \times$$

$$dR_1 dR_2 dR_3$$

$$\text{for Re }(a_3, b_1, b_2) > (p-1)/2.$$

Theorem 5.3.12:

$$F_P(a_1, a_2, a_1, b_1, b_1, b_2; c_1, c_2, c_2; -X, -Y, -Z)$$
$$= \frac{1}{\Gamma_p(a_1)\Gamma_p(b_1)} \int_{R_1>0}\int_{R_2>0} e^{-\mathrm{tr}(R_1+R_2)} \left|R_1\right|^{a_1-(p+1)/2} \times$$
$$\left|R_2\right|^{b_1-(p+1)/2} {}_0F_1(\,;c_1; -R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2}) \times \qquad (5.31)$$
$$\Phi_2(a_2, b_2; c_2; -R_2^{1/2}YR_2^{1/2}, -R_1^{1/2}ZR_1^{1/2})dR_1 dR_2$$
$$\text{for Re }(a_1, b_1) > (p-1)/2.$$

Theorem 5.3.13:

$$F_S(a_1, a_2, a_2, b_1, b_2, b_3; c_1, c_1, c_1; -X, -Y, -Z)$$
$$= \frac{1}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)} \int_{R_1>0} \cdots (5) \cdots \int_{R_5>0} \times$$
$$e^{-\mathrm{tr}(R_1+\cdots+R_5)} \left|R_1\right|^{a_1-(p+1)/2} \left|R_2\right|^{a_2-(p+1)/2} \times$$
$$\left|R_3\right|^{b_1-(p+1)/2} \left|R_4\right|^{b_2-(p+1)/2} \left|R_5\right|^{b_3-(p+1)/2} \times \qquad (5.32)$$
$${}_0F_1(\,;c_1; -R_3^{1/2}R_1^{1/2}XR_1^{1/2}R_3^{1/2} - R_4^{1/2}R_2^{1/2}YR_2^{1/2}R_4^{1/2}$$
$$-R_5^{1/2}R_2^{1/2}ZR_2^{1/2}R_5^{1/2})dR_1 \cdots dR_5$$
$$\text{for Re }(a_1, a_2, b_1, b_2, b_3) > (p-1)/2.$$

Theorem 5.3.14:

$$F_G(a, a, a, b_1, b_2, b_3; c_1, c_2, c_2; -X, -Y, -Z)$$

Continued to the next page … ... … … …

$$= \frac{1}{\Gamma_p(a)} \int_{U>0} e^{-tr(U)} |U|^{a-(p+1)/2} {}_1F_1(b_1; c_1; -U^{1/2} X U^{1/2}) \times$$

$$\Phi_2(b_2, b_3; c_2; -U^{1/2} Y U^{1/2}, -U^{1/2} Z U^{1/2}) dU \tag{5.33}$$

for Re $(a) > (p-1)/2$.

Theorem 5.3.15:

$$F_R(a_1, a_2, a_1, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)$$

$$= \frac{1}{\Gamma_p(a_1)\Gamma_p(b_1)} \int_{R_1>0} \int_{R_2>0} e^{-tr(R_1+R_2)} |R_1|^{a_1-(p+1)/2} \times$$

$$|R_2|^{b_1-(p+1)/2} {}_0F_1(; c_1; -R_2^{1/2} R_1^{1/2} X R_1^{1/2} R_2^{1/2}) \Xi_2(a_2, b_2; c_2; \tag{5.34}$$

$$-Y, -R_2^{1/2} R_1^{1/2} Z R_1^{1/2} R_2^{1/2}) dR_1 dR_2$$

for Re $(a_1, b_1) > (p-1)/2$.

Theorem 5.3.16:

$$F_G(a, a, a, b_1, b_2, b_3; c_1, c_2, c_2; -X, -Y, -Z)$$

$$= \frac{1}{\Gamma_p(b_2)\Gamma_p(b_3)} \int_{S>0} \int_{T>0} e^{-tr(S+T)} |S|^{b_2-(p+1)/2} \times$$

$$|T|^{b_3-(p+1)/2} \Psi_1(a, b_1; c_1, c_2; -X, -S^{1/2} Y S^{1/2} \tag{5.35}$$

$$-T^{1/2} Z T^{1/2}) dS dT$$

for Re$(b_2, b_3) > (p-1)/2$.

5.4 Some results are being given in this section for the Srivastava's triple hypergeomeric functions of matrix arguments.

Theorem 5.4.1:

$$H_A(a,b,b';c,c';-X,-Y,-Z)$$

$$=\frac{1}{\Gamma_p(a)\Gamma_p(b)}\int_{S>0}\int_{T>0} e^{-tr(S+T)}|T|^{a-(p+1)/2}|S|^{b-(p+1)/2}\times$$

$${}_0F_1(\,;c;-T^{1/2}S^{1/2}XS^{1/2}T^{1/2})\,{}_1F_1[b';c';-(S^{1/2}YS^{1/2} \tag{5.36}$$

$$+T^{1/2}ZT^{1/2})]dSdT$$

for $\mathrm{Re}(a,b) > (p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(5.36) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively, we get

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times$$

$${}_0F_1(\,;c;-T^{1/2}S^{1/2}XS^{1/2}T^{1/2})\,{}_1F_1[b';c';-(S^{1/2}YS^{1/2} \tag{5.37}$$

$$+T^{1/2}ZT^{1/2})]dXdYdZ$$

Applying the transformations,

$$X_1 = T^{1/2}S^{1/2}XS^{1/2}T^{1/2},\ Y_1 = S^{1/2}YS^{1/2},\ Z_1 = T^{1/2}ZT^{1/2};$$

and writing the M-transforms of the ${}_0F_1$ and the ${}_1F_1$ functions yields,

$$|T|^{-\rho_1-\rho_3}|S|^{-\rho_1-\rho_2}\frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(b'-\rho_2-\rho_3)}{\Gamma_p(c'-\rho_2-\rho_3)\Gamma_p(c-\rho_1)\Gamma_p(b')}\times \tag{5.38}$$

$$\Gamma_p(c')\Gamma_p(\rho_2)\Gamma_p(\rho_3)$$

Substituting this expression on the right side of eq.(5.36) and integrating out S and T by using a Gamma integral leads to $M(H_A)$ as given by eq.(5.11).

Theorem 5.4.2:

$$H_B(a,b,b';c_1,c_2,c_3;-X,-Y,-Z)$$

$$=\frac{1}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(b')}\int_{S_1>0}\int_{S_2>0}\int_{S_3>0} e^{-\mathrm{tr}(S_1+S_2+S_3)}\times$$

$$|S_1|^{a-(p+1)/2}|S_2|^{b-(p+1)/2}|S_3|^{b'-(p+1)/2}\times \tag{5.39}$$

$${}_0F_1(\,;c_1;-S_2^{1/2}S_1^{1/2}XS_1^{1/2}S_2^{1/2})\,{}_0F_1(\,;c_2;-S_3^{1/2}S_2^{1/2}YS_2^{1/2}S_3^{1/2})\times$$

$${}_0F_1(\,;c_3;-S_3^{1/2}S_1^{1/2}ZS_1^{1/2}S_3^{1/2})dS_1dS_2dS_3$$

for $\mathrm{Re}(a,b,b')>(p-1)/2.$

Theorem 5.4.3:

$$H_C(a,b,b';c;-X,-Y,-Z)$$

$$=\frac{1}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(b')}\int_{S_1>0}\int_{S_2>0}\int_{S_3>0} e^{-\mathrm{tr}(S_1+S_2+S_3)}\times$$

$$|S_1|^{a-(p+1)/2}|S_2|^{b-(p+1)/2}|S_3|^{b'-(p+1)/2}\times \tag{5.40}$$

$${}_0F_1(\,;c;-S_2^{1/2}S_1^{1/2}XS_1^{1/2}S_2^{1/2}-S_3^{1/2}S_2^{1/2}YS_2^{1/2}S_3^{1/2}$$

$$-S_3^{1/2}S_1^{1/2}ZS_1^{1/2}S_3^{1/2})dS_1dS_2dS_3$$

for $\mathrm{Re}(a,b,b')>(p-1)/2.$

Theorem 5.4.4:

$$H_B(a,b,b';c_1,c_2,c_3;-X,-Y,-Z)$$

Continued to the next page … … … … …

$$=\frac{1}{\Gamma_p(a)\Gamma_p(b)}\int_{S>0}\int_{T>0} e^{-tr(S+T)}|S|^{a-(p+1)/2}|T|^{b-(p+1)/2}\times$$

$${}_0F_1(\,;c_1;-T^{1/2}S^{1/2}XS^{1/2}T^{1/2})\Psi_2(b';c_2,c_3;-T^{1/2}YT^{1/2},\qquad (5.41)$$

$$-S^{1/2}ZS^{1/2})dSdT$$

for $\mathrm{Re}(a,b)>(p-1)/2$.

5.5: This section deals with some transformation cases and cases of reducibility of the Lauricella-Saran triple hypergeometric functions of matrix arguments.

Theorem 5.5.1: A transformation theorem:

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$$

$$\text{(i)}=|I+X|^{-a_1}\,F_G[a_1,a_1,a_1,c_1-b_1,b_2,b_3;c_1,c_2,c_2;(I+X)^{-1/2}\times\qquad (5.42)$$

$$X(I+X)^{-1/2},-(I+X)^{-1/2}Y(I+X)^{-1/2},-(I+X)^{-1/2}Z(I+X)^{-1/2}]$$

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$$

$$=|I+Y|^{-a_1}\,F_G[a_1,a_1,a_1,b_1,c_2-b_2-b_3,b_3;c_1,c_2,c_2;$$

$$\text{(ii)}\qquad -(I+Y)^{-1/2}X(I+Y)^{-1/2},(I+Y)^{-1/2}Y(I+Y)^{-1/2},\qquad (5.43)$$

$$-(I+Y)^{-1/2}(Z-Y)(I+Y)^{-1/2}]$$

where, $Z-Y>0$.

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$$

$$\text{(iii)}=|I+X+Z|^{-a_1}\,F_G[a_1,a_1,a_1,c_1-b_1,b_2,c_2-b_2-b_3;c_1,c_2,c_2;$$

$$(I+X+Z)^{-1/2}X(I+X+Z)^{-1/2},-(I+X+Z)^{-1/2}(Y-Z)\times$$

Continued to the next page … … … … …

$$(I+X+Z)^{-1/2}, (I+X+Z)^{-1/2} Z (I+X+Z)^{-1/2}] \tag{5.44}$$

where, $Y - Z > 0$.

Proof: To prove this theorem we define F_G through an integral representation:

$$\begin{aligned}
&F_G(a_1, a_1, a_1, b_1, b_2, b_3; c_1, c_2, c_2; -X, -Y, -Z) \\
&= \frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)} \times \\
&\iiint |U|^{b_1-(p+1)/2} |V|^{b_2-(p+1)/2} |W|^{b_3-(p+1)/2} \times \\
&|I-U|^{c_1-b_1-(p+1)/2} |I-V-W|^{c_2-b_2-b_3-(p+1)/2} \times \\
&\left| I + X^{1/2} U X^{1/2} + Y^{1/2} V Y^{1/2} + Z^{1/2} W Z^{1/2} \right|^{-a_1} dUdVdW
\end{aligned} \tag{5.45}$$

for $\mathrm{Re}(c_1 - b_1, c_2 - b_2 - b_3, b_1, b_2, b_3) > (p-1)/2$.

which has been obtained from eq.(5.17) by using the assumptions of symmetry which have earlier been used by Herz [22], page 478, and Mathai [60], page 516, and Mathai [62], page 54.

(i) This result can be had from eq.(5.45) by observing that

$$\begin{aligned}
&\left| I + X^{1/2} U X^{1/2} + Y^{1/2} V Y^{1/2} + Z^{1/2} W Z^{1/2} \right| \\
&= |I+X| \Big| I - (I+X)^{-1/2} X^{1/2} (I-U) X^{1/2} (I+X)^{-1/2} + (I+X)^{-1/2} \times \\
&Y^{1/2} V Y^{1/2} (I+X)^{-1/2} + (I+X)^{-1/2} Z^{1/2} W Z^{1/2} (I+X)^{-1/2} \Big|
\end{aligned}$$

and applying the transformation $U_1 = I - U$, and interpreting the expression thus obtained in terms of eq.(5.45).

(ii) We apply the transformations $V_1 = I - V - W, W_1 = W$ to eq.(5.45) and observe that

$$\left|I + X^{1/2} U X^{1/2} + Y^{1/2}(I - V_1 - W_1) Y^{1/2} + Z^{1/2} W_1 Z^{1/2}\right|$$

$$= |I + Y| \Big| I + (I+Y)^{-1/2} X^{1/2} U X^{1/2} (I+Y)^{-1/2} - (I+Y)^{-1/2} \times$$

$$Y^{1/2} V_1 Y^{1/2} (I+Y)^{-1/2} + (I+Y)^{-1/2} (Z-Y)^{1/2} W_1 (Z-Y)^{1/2} (I+Y)^{-1/2} \Big|$$

and suitably interpret eq.(5.45) to see this result.

(iii) This result is a combination of the previous two results. The transformations to be used are $U_1 = I - U, V_1 = V, W_1 = I - V - W$ along with the observation

$$\left|I + X^{1/2}(I - U_1) X^{1/2} + Y^{1/2} V_1 Y^{1/2} + Z^{1/2}(I - V_1 - W_1) Z^{1/2}\right|$$

$$= |I + X + Z| \Big| I - (I+X+Z)^{-1/2} X^{1/2} U_1 X^{1/2} (I+X+Z)^{-1/2} +$$

$$(I+X+Z)^{-1/2} (Y-Z)^{1/2} V_1 (Y-Z)^{1/2} (I+X+Z)^{-1/2}$$

$$-(I+X+Z)^{-1/2} Z^{1/2} W_1 Z^{1/2} (I+X+Z)^{-1/2} \Big|$$

Theorem 5.5.2: A case of reducibility:

$$\begin{aligned} &F_G(a_1, a_1, a_1, b_1, b_2, b_3; c_1, c_2, c_2; -X, -Y, -Y) \\ &= F_2(a_1, b_1, b_2 + b_3; c_1, c_2; -X, -Y) \end{aligned} \qquad (5.46)$$

Proof: We put $Z = Y$ in eq.(5.17) and on the basis of symmetry assumptions (as used earlier by Herz [22] and Mathai [60]) observing that

$$\left|I + U^{1/2} X U^{1/2} + V^{1/2} Y V^{1/2} + W^{1/2} Y W^{1/2}\right|$$

Continued to the next page … … … … …

$$=\left|I+U^{1/2}XU^{1/2}+(V+W)^{1/2}Y(V+W)^{1/2}\right|$$

then applying the transformations,

$V_1 = V, W_1 = V + W;$ with, $dV_1 dW_1 = dVdW,$ and, $0 < V_1 < W_1 < I;$ and integrating out V_1 by using a type-1 Beta integral and finally employing eq.(3.14), this result is achieved.

Theorem 5.5.3: A limiting case:

$$\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,b_1,b_2,b_3;c_1,c_2,c_2;\frac{-X}{\alpha}\frac{-Y}{\alpha},\frac{-Z}{\alpha})$$

$$=\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)}\times$$

$$\iiint |U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}\times \tag{5.47}$$

$$|I-U|^{c_1-b_1-(p+1)/2}|I-V-W|^{c_2-b_2-b_3-(p+1)/2}\times$$

$$e^{-tr(UX+VY+WZ)}dUdVdW$$

for $\mathrm{Re}(c_1-b_1,c_2-b_2-b_3,b_1,b_2,b_3)>(p-1)/2.$

Proof: This result is obtained from eq.(5.17) by an application of eq.(1.23) to it.

Theorem 5.5.4: A case of reducibility:

$$\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,b_1,b_2,b_3;c_1,c_2,c_2;\frac{-X}{\alpha},\frac{-Y}{\alpha},\frac{-Y}{\alpha}) \tag{5.48}$$

$$={}_1F_1(b_1;c_1;-X)\,{}_1F_1(b_2+b_3;c_2;-Y)$$

Proof: Setting $Z=Y$ in eq.(5.47), then applying the transformations, $W_1 = V, W_2 = V + W;$ and integrating out W_1 by the help of a type-1 Beta integral and utilizing eq.(2.3) the desired result can be had.

Theorem 5.5.5:

(i)
$$\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,b_1,b_2,b_3;c_1,c_2,c_2;-\frac{X}{\alpha},-\frac{Y}{\alpha},-\frac{Z}{\alpha})$$
$$= e^{-tr(X)} \lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,c_1-b_1,b_2,b_3;c_1,c_2,c_2;\frac{X}{\alpha},-\frac{Y}{\alpha},-\frac{Z}{\alpha}) \qquad (5.49)$$

(ii)
$$= e^{-tr(Y)} \lim_{\alpha\to\infty} F_G[\alpha,\alpha,\alpha,b_1,c_2-b_2-b_3,b_3;c_1,c_2,c_2; -\frac{X}{\alpha},\frac{Y}{\alpha},-\frac{(Z-Y)}{\alpha}] \qquad (5.50)$$

(iii)
$$\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,b_1,b_2,b_3;c_1,c_2,c_2;-\frac{X}{\alpha},-\frac{Y}{\alpha},-\frac{Y}{\alpha})$$
$$= e^{-tr(Y)}\,{}_1F_1(b_1;c_1;-X)\,{}_1F_1(c_2-b_2-b_3;c_2;Y) \qquad (5.51)$$

(iv)
$$\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,b_1,b_2,b_3;c_1,c_2,c_2;-\frac{X}{\alpha},-\frac{Y}{\alpha},-\frac{Z}{\alpha})$$
$$= e^{-tr(Z)} \lim_{\alpha\to\infty} F_G[\alpha,\alpha,\alpha,b_1,b_2,c_2-b_2-b_3;c_1,c_2,c_2; -\frac{X}{\alpha},-\frac{(Y-Z)}{\alpha},\frac{Z}{\alpha}] \qquad (5.52)$$

(v)
$$= e^{-tr(X+Y)} \lim_{\alpha\to\infty} F_G[\alpha,\alpha,\alpha,c_1-b_1,c_2-b_2-b_3,b_3;c_1,c_2,c_2; \frac{X}{\alpha},\frac{Y}{\alpha},-\frac{(Z-Y)}{\alpha}] \qquad (5.53)$$

(vi)
$$= e^{-tr(X+Z)} \lim_{\alpha\to\infty} F_G[\alpha,\alpha,\alpha,c_1-b_1,b_2,c_2-b_2-b_3;c_1,c_2,c_2; \frac{X}{\alpha},-\frac{(Y-Z)}{\alpha},\frac{Z}{\alpha}] \qquad (5.54)$$

Proof: To prove this theorem we will use eq.(5.47).

(i) This result is obtained by the application of the transformation $I-U=U_1$ to the above equation.

(ii) We utilize the transformations $V_1 = I-V-W, W_1 = W;$ with, $dV_1dW_1 = dVdW;$ in eq.(5.47) to see this result.

(iii) This result is obtained by putting $Z = Y$ in eq.(5.50), afterwards employing the transformation $W_2 = (I - V_1)^{-1/2} W_1 (I - V_1)^{-1/2}$ followed by integrating out of W_2 by using a type-1 Beta integral and interpreting the consequent expression in the light of eq.(2.3).

(iv) To deduce this result we apply the transformations, $V_1 = V, W_1 = I - V - W;$ to eq.(5.47), and suitably interpret the ensuing expression as per the same equation.

(v) This result follows by employing the transformations $U_1 = I - U,$ $V_1 = I - V - W, W_1 = W;$ to eq.(5.47).

(vi) We have this result in a similar fashion as eq.(5.53) is deduced.

Theorem 5.5.6: A transformation theorem:

$$F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; -X, -Y, -Z)$$
$$= |I + X|^{-b_1} F_K[c_1 - a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; \tag{5.55}$$
$$(I + X)^{-1/2} X (I + X)^{-1/2}, -Y, -(I + X)^{-1/2} Z (I + X)^{-1/2}]$$

Proof: In order to prove this theorem we first define the function F_K through an integral representation:

$$F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; -X, -Y, -Z)$$
$$= \frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_2)\Gamma_p(c_1 - a_1)\Gamma_p(c_2 - b_2)\Gamma_p(c_3 - a_2)} \times$$
$$\int_0^I \int_0^I \int_0^I |U|^{a_1 - (p+1)/2} |V|^{a_2 - (p+1)/2} |T|^{b_2 - (p+1)/2} \times$$
$$|I - U|^{c_1 - a_1 - (p+1)/2} |I - V|^{c_3 - a_2 - (p+1)/2} \times$$
$$|I - T|^{c_2 - b_2 - (p+1)/2} \left| I + Y^{1/2} T Y^{1/2} \right|^{-a_2} \left| I + X^{1/2} U X^{1/2} + \right.$$

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$$(I+Y^{1/2}TY^{1/2})^{-1/2}Z^{1/2}VZ^{1/2}(I+Y^{1/2}TY^{1/2})^{-1/2}\Big|^{-b_1}\times$$

$$dUdVdT \tag{5.56}$$

for $\mathrm{Re}\,(a_1, a_2, b_2, c_1 - a_1, c_2 - b_2, c_3 - a_2) > (p-1)/2.$

The desired result simply follows by the application of the transformation $U_1 = I - U$, to the above equation and interpreting the consequent expression in accordance with eq.(5.56).

Theorem 5.5.7:

(i)
$$\lim_{\alpha\to\infty} F_T(a_1, a_2, a_2, \alpha, \alpha, \alpha; c_1, c_1, c_1; \frac{-X}{\alpha}, \frac{-Y}{\alpha}, \frac{-Z}{\alpha}) \tag{5.57}$$
$$= \Phi_2(a_1, a_2; c_1; -X, -Y-Z)$$

(ii)
$$F_T(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; -X, -Y, -Z)$$
$$= |I+X|^{-b_1} F_T[c_1 - a_1 - a_2, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1;$$
$$(I+X)^{-1/2}X(I+X)^{-1/2}, -Y, -(I+X)^{-1/2}(Z-X)(I+X)^{-1/2}] \tag{5.58}$$

where, $Z - X > 0$.

Proof: (i) The result stated here is the limiting case of eq.(5.27)and an application of eq.(3.17).

(ii) To prove this result we first define the function F_T through an integral representation:

$$F_T(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; -X, -Y, -Z)$$
$$= \frac{\Gamma_p(c_1)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1 - a_1 - a_2)} \iint |U|^{a_1-(p+1)/2} |V|^{a_2-(p+1)/2}$$
$$\times |I-U-V|^{c_1-a_1-a_2-(p+1)/2} \left|I + X^{1/2}UX^{1/2} + Z^{1/2}VZ^{1/2}\right|^{-b_1}$$

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$$\times\left|I + Y^{1/2} V Y^{1/2}\right|^{-b_2} \, dU dV$$

for $\operatorname{Re}(a_1, a_2, c_1 - a_1 - a_2) > (p-1)/2;$ where, $U > 0, V > 0$ (5.59)

and $0 < U + V < I.$

The desired result is obtained by applying the transformations $U_1 = I - U - V,\ V_1 = V;$ to the above equation and observing that,

$$\left|I + X^{1/2}(I - U_1 - V_1)X^{1/2} + Z^{1/2} V_1 Z^{1/2}\right|$$

$$= |I + X| \left|I - (I + X)^{-1/2} X^{1/2} U_1 X^{1/2} (I + X)^{-1/2} + \right.$$

$$\left.(I + X)^{-1/2} (Z - X)^{1/2} V_1 (Z - X)^{1/2} (I + X)^{-1/2}\right|$$

and interpreting the result so obtained in the light of eq.(5.59).

CHAPTER VI

THE EXTON'S TWENTY ONE QUADRUPLE HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENTS

6.1 Exton [18] in 1972 defined and studied some properties of the hypergeometric functions of four variables. Prior to him no specific study of the quadruple hypergeometric functions had been made, except the four Lauricella functions $F_A^{(4)}, F_B^{(4)}, F_C^{(4)}$ and $F_D^{(4)}$ and some of their limiting cases. In the present chapter I have defined these twenty one quadruple hypergeometric functions for the matrix arguments case and established a number of results for these functions.

6.2 Definitions

The following are the definitions of the Exton's twenty one quadruple hypergeometric functions with matrix arguments.

6.2.1 The Exton's K_1 function of matrix arguments,

$$K_1 = K_1(a,a,a,a;b,b,b,c;d,e_1,e_2,d;-X,-Y,-Z,-T)$$

is defined as that class of functions which has the following matrix transform:

$$\begin{aligned} M(K_1) &= \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times \\ &|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_1(a,a,a,a;b,b,b,c;d,e_1,e_2,d; \\ &-X,-Y,-Z,-T)dXdYdZdT \\ &= \frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b-\rho_1-\rho_2-\rho_3)}{\Gamma_p(b)}\frac{\Gamma_p(c-\rho_4)}{\Gamma_p(c)}\times \quad (6.1) \\ &\frac{\Gamma_p(d)}{\Gamma_p(d-\rho_1-\rho_4)}\frac{\Gamma_p(e_1)}{\Gamma_p(e_1-\rho_2)}\frac{\Gamma_p(e_2)}{\Gamma_p(e_2-\rho_3)}\times \\ &\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4) \end{aligned}$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2-\rho_3, c-\rho_4,$
$d-\rho_1-\rho_4, e_1-\rho_2, e_2-\rho_3, \rho_i) > (p-1)/2,$ where, $i=1,\cdots,4.$

6.2.2 $\quad K_2 = K_2(a,a,a,a;b,b,b,c;d_1,d_2,d_3,d_4;-X,-Y,-Z,-T)$

$$M(K_2) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2} K_2(a,a,a,a;b,b,b,c;$$

$$d_1,d_2,d_3,d_4;-X,-Y,-Z,-T)dXdYdZdT$$

$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b-\rho_1-\rho_2-\rho_3)}{\Gamma_p(b)}\frac{\Gamma_p(c-\rho_4)}{\Gamma_p(c)}\times$$

$$\frac{\Gamma_p(d_1)}{\Gamma_p(d_1-\rho_1)}\frac{\Gamma_p(d_2)}{\Gamma_p(d_2-\rho_2)}\frac{\Gamma_p(d_3)}{\Gamma_p(d_3-\rho_3)}\frac{\Gamma_p(d_4)}{\Gamma_p(d_4-\rho_4)}\times$$

$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2-\rho_3, c-\rho_4,$ (6.2)

$d_i-\rho_i, \rho_i) > (p-1)/2,$ where, $i=1,\cdots,4.$

6.2.3 $\quad K_3 = K_3(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_2,c_1;-X,-Y,-Z,-T)$

$$M(K_3) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2} K_3(a,a,a,a;b_1,b_1,b_2,b_2;$$

$$c_1,c_2,c_2,c_1;-X,-Y,-Z,-T)dXdYdZdT$$

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$$= \frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)} \frac{\Gamma_p(b_1-\rho_1-\rho_2)}{\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_3-\rho_4)}{\Gamma_p(b_2)} \times$$

$$\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1-\rho_4)} \frac{\Gamma_p(c_2)}{\Gamma_p(c_2-\rho_2-\rho_3)} \Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4) \qquad (6.3)$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b_1-\rho_1-\rho_2, b_2-\rho_3-\rho_4, c_1-\rho_1-\rho_4,$

$c_2-\rho_2-\rho_3, \rho_i) > (p-1)/2,$ where, $i = 1, \cdots, 4.$

6.2.4 $\quad K_4 = K_4(a,a,a,a;b_1,b_1,b_2,b_2;c,d_1,d_2,c;-X,-Y,-Z,-T)$

$$M(K_4) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$

$$|Z|^{\rho_3-(p+1)/2} |T|^{\rho_4-(p+1)/2} K_4(a,a,a,a;b_1,b_1,b_2,b_2;$$

$$c,d_1,d_2,c;-X,-Y,-Z,-T)dXdYdZdT$$

$$= \frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)} \frac{\Gamma_p(b_1-\rho_1-\rho_2)}{\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_3-\rho_4)}{\Gamma_p(b_2)} \times$$

$$\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_4)} \frac{\Gamma_p(d_1)}{\Gamma_p(d_1-\rho_2)} \frac{\Gamma_p(d_2)}{\Gamma_p(d_2-\rho_3)} \times$$

$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

for Re$(a-\rho_1-\rho_2-\rho_3-\rho_4, b_1-\rho_1-\rho_2, b_2-\rho_3-\rho_4, c-\rho_1-\rho_4,$ (6.4)

$d_1-\rho_2, d_2-\rho_3, \rho_i) > (p-1)/2,$ where, $i = 1, \cdots, 4.$

6.2.5 $\quad K_5 = K_5(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;-X,-Y,-Z,-T)$

$$M(K_5) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2}\,|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}\,|T|^{\rho_4-(p+1)/2}\,K_5(a,a,a,a;b_1,b_1,b_2,b_2;$$
$$c_1,c_2,c_3,c_4;-X,-Y,-Z,-T)dXdYdZdT$$
$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b_1-\rho_1-\rho_2)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_3-\rho_4)}{\Gamma_p(b_2)}$$
$$\times\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)}\frac{\Gamma_p(c_2)}{\Gamma_p(c_2-\rho_2)}\frac{\Gamma_p(c_3)}{\Gamma_p(c_3-\rho_3)}\frac{\Gamma_p(c_4)}{\Gamma_p(c_4-\rho_4)}\times$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$
for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4, b_1-\rho_1-\rho_2, b_2-\rho_3-\rho_4, c_1-\rho_1,$
$c_2-\rho_2, c_3-\rho_3, c_4-\rho_4, \rho_i) > (p-1)/2$, where, $i = 1,\cdots,4.$ (6.5)

6.2.6 $K_6 = K_6(a,a,a,a;b,b,c_1,c_2;e,d,d,d;-X,-Y,-Z,-T)$

$$M(K_6) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2}\,|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}\,|T|^{\rho_4-(p+1)/2}\,K_6(a,a,a,a;b,b,c_1,c_2;e,d,d,d;$$
$$-X,-Y,-Z,-T)dXdYdZdT$$
$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(b)}\frac{\Gamma_p(c_1-\rho_3)}{\Gamma_p(c_1)}\times$$
$$\frac{\Gamma_p(c_2-\rho_4)}{\Gamma_p(c_2)}\frac{\Gamma_p(e)}{\Gamma_p(e-\rho_1)}\frac{\Gamma_p(d)}{\Gamma_p(d-\rho_2-\rho_3-\rho_4)}\times$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$
for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2, c_1-\rho_3,$
$c_2-\rho_4, e-\rho_1, d-\rho_2-\rho_3-\rho_4, \rho_i) > (p-1)/2$, where, $i = 1,\cdots,4.$ (6.6)

6.2.7 $K_7 = K_7(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_1,d_2;-X,-Y,-Z,-T)$

$$M(K_7)=\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_7(a,a,a,a;b,b,c_1,c_2;$$
$$d_1,d_2,d_1,d_2;-X,-Y,-Z,-T)dXdYdZdT$$
$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(b)}\frac{\Gamma_p(c_1-\rho_3)}{\Gamma_p(c_1)}\times$$
$$\frac{\Gamma_p(c_2-\rho_4)}{\Gamma_p(c_2)}\frac{\Gamma_p(d_1)}{\Gamma_p(d_1-\rho_1-\rho_3)}\frac{\Gamma_p(d_2)}{\Gamma_p(d_2-\rho_2-\rho_4)}\times$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2, c_1-\rho_3, c_2-\rho_4,$
$d_1-\rho_1-\rho_3, d_2-\rho_2-\rho_4, \rho_i) > (p-1)/2,$ where, $i=1,\cdots,4.$ (6.7)

6.2.8 $K_8 = K_8(a,a,a,a;b,b,c_1,c_2;d,e_1,d,e_2;-X,-Y,-Z,-T)$

$$M(K_8)=\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_8(a,a,a,a;b,b,c_1,c_2;d,e_1,d,e_2;$$
$$-X,-Y,-Z,-T)dXdYdZdT$$
$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(b)}\frac{\Gamma_p(c_1-\rho_3)}{\Gamma_p(c_1)}\times$$
$$\frac{\Gamma_p(c_2-\rho_4)}{\Gamma_p(c_2)}\frac{\Gamma_p(d)}{\Gamma_p(d-\rho_1-\rho_3)}\frac{\Gamma_p(e_1)}{\Gamma_p(e_1-\rho_2)}\frac{\Gamma_p(e_2)}{\Gamma_p(e_2-\rho_4)}\times$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2, c_1-\rho_3, c_2-\rho_4,$
$d-\rho_1-\rho_3, e_1-\rho_2, e_2-\rho_4, \rho_i) > (p-1)/2,$ where, $i=1,\cdots,4.$ (6.8)

6.2.9 $\quad K_9 = K_9(a,a,a,a;b,b,c_1,c_2;e_1,e_2,d,d;-X,-Y,-Z,-T)$

$$M(K_9) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2}\,|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}\,|T|^{\rho_4-(p+1)/2}\,K_9(a,a,a,a;b,b,c_1,c_2;$$

$$e_1,e_2,d,d;-X,-Y,-Z,-T)dXdYdZdT$$

$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(b)}\frac{\Gamma_p(c_1-\rho_3)}{\Gamma_p(c_1)}\times$$

$$\frac{\Gamma_p(c_2-\rho_4)}{\Gamma_p(c_2)}\frac{\Gamma_p(e_1)}{\Gamma_p(e_1-\rho_1)}\frac{\Gamma_p(e_2)}{\Gamma_p(e_2-\rho_2)}\frac{\Gamma_p(d)}{\Gamma_p(d-\rho_3-\rho_4)}\times$$

$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2, c_1-\rho_3, c_2-\rho_4,$

$e_1-\rho_1, e_2-\rho_2, d-\rho_3-\rho_4, \rho_i) > (p-1)/2,$ where, $i=1,\cdots,4.$ (6.9)

6.2.10 $\quad K_{10} = K_{10}(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_3,d_4;-X,-Y,-Z,-T)$

$$M(K_{10}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2}\,|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}\,|T|^{\rho_4-(p+1)/2}\,K_{10}(a,a,a,a;b,b,c_1,c_2;$$

$$d_1,d_2,d_3,d_4;-X,-Y,-Z,-T)dXdYdZdT$$

$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(b)}\frac{\Gamma_p(c_1-\rho_3)}{\Gamma_p(c_1)}\times$$

$$\frac{\Gamma_p(c_2-\rho_4)}{\Gamma_p(c_2)}\frac{\Gamma_p(d_1)}{\Gamma_p(d_1-\rho_1)}\frac{\Gamma_p(d_2)}{\Gamma_p(d_2-\rho_2)}\frac{\Gamma_p(d_3)}{\Gamma_p(d_3-\rho_3)}\times$$

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$$\frac{\Gamma_p(d_4)}{\Gamma_p(d_4-\rho_4)}\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2, c_1-\rho_3, c_2-\rho_4,$ (6.10)

$d_i-\rho_i, \rho_i) > (p-1)/2,$ where, $i = 1, \cdots, 4.$

6.2.11 $K_{11} = K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$

$$M(K_{11}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;$$
$$-X,-Y,-Z,-T)dXdYdZdT$$
$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)}\times$$
$$\frac{\Gamma_p(b_3-\rho_3)}{\Gamma_p(b_3)}\frac{\Gamma_p(b_4-\rho_4)}{\Gamma_p(b_4)}\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3)}\frac{\Gamma_p(d)}{\Gamma_p(d-\rho_4)}\times$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b_i-\rho_i, c-\rho_1-\rho_2-\rho_3,$

$d-\rho_4, \rho_i) > (p-1)/2,$ where, $i = 1, \cdots, 4.$ (6.11)

6.2.12

$K_{12} = K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;-X,-Y,-Z,-T)$

$$M(K_{12}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;$$
$$c_1,c_1,c_2,c_2;-X,-Y,-Z,-T)dXdYdZdT$$

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$$= \frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)} \frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)} \times$$

$$\frac{\Gamma_p(b_3-\rho_3)}{\Gamma_p(b_3)} \frac{\Gamma_p(b_4-\rho_4)}{\Gamma_p(b_4)} \frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1-\rho_2)} \frac{\Gamma_p(c_2)}{\Gamma_p(c_2-\rho_3-\rho_4)} \times$$

$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4) \tag{6.12}$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b_1-\rho_1, b_2-\rho_2, b_3-\rho_3, b_4-\rho_4,$
$c_1-\rho_1-\rho_2, c_2-\rho_3-\rho_4, \rho_i) > (p-1)/2$, where, $i = 1, \cdots, 4$.

6.2.13 $K_{13} = K_{13}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,d_1,d_2;-X,-Y,-Z,-T)$

$$M(K_{13}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} |T|^{\rho_4-(p+1)/2} K_{13}(a,a,a,a;b_1,b_2,b_3,b_4;$$
$$c,c,d_1,d_2;-X,-Y,-Z,-T) dX dY dZ dT$$

$$= \frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)} \left\{ \prod_{i=1}^{4} \frac{\Gamma_p(b_i-\rho_i)}{\Gamma_p(b_i)} \Gamma_p(\rho_i) \right\} \times$$

$$\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2)} \frac{\Gamma_p(d_1)}{\Gamma_p(d_1-\rho_3)} \frac{\Gamma_p(d_2)}{\Gamma_p(d_2-\rho_4)}$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b_i-\rho_i, \rho_i, c-\rho_1-\rho_2, d_1-\rho_3,$
$d_2-\rho_4) > (p-1)/2$, where, $i = 1, \cdots, 4$. (6.13)

6.2.14 $K_{14} = K_{14}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;-X,-Y,-Z,-T)$

$$M(K_{14}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} |T|^{\rho_4-(p+1)/2} K_{14}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;$$
$$-X,-Y,-Z,-T) dX dY dZ dT$$

$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3)}{\Gamma_p(a)\Gamma_p(c_3)}\frac{\Gamma_p(b-\rho_1-\rho_4)}{\Gamma_p(b)}\frac{\Gamma_p(c_1-\rho_2)}{\Gamma_p(c_1)}\frac{\Gamma_p(c_2-\rho_3)}{\Gamma_p(c_2)}\times$$

$$\frac{\Gamma_p(c_3-\rho_4)\Gamma_p(d)}{\Gamma_p(d-\rho_1-\rho_2-\rho_3-\rho_4)}\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4) \qquad (6.14)$$

for Re $(a-\rho_1-\rho_2-\rho_3, b-\rho_1-\rho_4, c_1-\rho_2, c_2-\rho_3, c_3-\rho_4,$

$d-\rho_1-\rho_2-\rho_3-\rho_4, \rho_i) > (p-1)/2$, where, $i=1,\cdots,4$.

6.2.15 $\quad K_{15}=K_{15}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;-X,-Y,-Z,-T)$

$$M(K_{15})=\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_{15}(a,a,a,b_5;b_1,b_2,b_3,b_4;$$

$$c,c,c,c;-X,-Y,-Z,-T)dXdYdZdT$$

$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3)}{\Gamma_p(a)}\frac{\Gamma_p(b_5-\rho_4)}{\Gamma_p(b_5)}\left\{\prod_{i=1}^{4}\frac{\Gamma_p(b_i-\rho_i)}{\Gamma_p(b_i)}\Gamma_p(\rho_i)\right\}\times$$

$$\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3-\rho_4)}$$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3, b_5-\rho_4, c-\rho_1-\rho_2-\rho_3-\rho_4, b_i-\rho_i, \rho_i)$ (6.15)

$>(p-1)/2$, where, $i=1,\cdots,4$.

6.2.16 $\quad K_{16}=K_{16}(a_1,a_2,a_3,a_4;b;-X,-Y,-Z,-T)$

$$M(K_{16})=\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

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$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_{16}(a_1,a_2,a_3,a_4;b;$$
$$-X,-Y,-Z,-T)dXdYdZdT$$
$$=\frac{\Gamma_p(a_1-\rho_1-\rho_2)}{\Gamma_p(a_1)}\frac{\Gamma_p(a_2-\rho_1-\rho_3)}{\Gamma_p(a_2)}\frac{\Gamma_p(a_3-\rho_2-\rho_4)}{\Gamma_p(a_3)}\times$$
$$\frac{\Gamma_p(a_4-\rho_3-\rho_4)}{\Gamma_p(a_4)}\frac{\Gamma_p(b)}{\Gamma_p(b-\rho_1-\rho_2-\rho_3-\rho_4)}\times \tag{6.16}$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$
for Re $(a_1-\rho_1-\rho_2, a_2-\rho_1-\rho_3, a_3-\rho_2-\rho_4, a_4-\rho_3-\rho_4,$
$b-\rho_1-\rho_2-\rho_3-\rho_4, \rho_i) > (p-1)/2$, where, $i=1,\cdots,4$.

6.2.17 $K_{17}=K_{17}(a_1,a_2,a_3,b_1,b_2;c;-X,-Y,-Z,-T)$

$$M(K_{17})=\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_{17}(a_1,a_2,a_3,b_1,b_2;$$
$$c;-X,-Y,-Z,-T)dXdYdZdT$$
$$=\frac{\Gamma_p(a_1-\rho_1-\rho_2)}{\Gamma_p(a_1)}\frac{\Gamma_p(a_2-\rho_1-\rho_3)}{\Gamma_p(a_2)}\frac{\Gamma_p(a_3-\rho_2-\rho_3)}{\Gamma_p(a_3)}\times$$
$$\frac{\Gamma_p(b_1-\rho_4)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_4)}{\Gamma_p(b_2)}\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3-\rho_4)}\times$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$
for Re$(a_1-\rho_1-\rho_2, a_2-\rho_1-\rho_3, a_3-\rho_2-\rho_3, b_1-\rho_4,$ (6.17)
$b_2-\rho_4, c-\rho_1-\rho_2-\rho_3-\rho_4, \rho_i) > (p-1)/2$, where, $i=1,\cdots,4$.

6.2.18 $$K_{18} = K_{18}(a_1, a_2, a_3, b_1, b_2; c; -X, -Y, -Z, -T)$$

$$M(K_{18}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$

$$|Z|^{\rho_3-(p+1)/2} |T|^{\rho_4-(p+1)/2} K_{18}(a_1, a_2, a_3, b_1, b_2;$$

$$c; -X, -Y, -Z, -T) dX dY dZ dT$$

$$= \frac{\Gamma_p(a_1-\rho_1-\rho_2)}{\Gamma_p(a_1)} \frac{\Gamma_p(a_2-\rho_1-\rho_4)}{\Gamma_p(a_2)} \frac{\Gamma_p(a_3-\rho_2-\rho_3)}{\Gamma_p(a_3)} \times$$

$$\frac{\Gamma_p(b_1-\rho_3)}{\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_4)}{\Gamma_p(b_2)} \frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3-\rho_4)} \times$$

$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

for $\mathrm{Re}(a_1-\rho_1-\rho_2, a_2-\rho_1-\rho_4, a_3-\rho_2-\rho_3, b_1-\rho_3,$ (6.18)

$b_2-\rho_4, c-\rho_1-\rho_2-\rho_3-\rho_4, \rho_i) > (p-1)/2$, where, $i = 1, \cdots, 4$.

6.2.19 $$K_{19} = K_{19}(a_1, a_2, b_1, b_2, b_3, b_4; c; -X, -Y, -Z, -T)$$

$$M(K_{19}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$

$$|Z|^{\rho_3-(p+1)/2} |T|^{\rho_4-(p+1)/2} K_{19}(a_1, a_2, b_1, b_2, b_3, b_4; c;$$

$$-X, -Y, -Z, -T) dX dY dZ dT$$

$$= \frac{\Gamma_p(a_1-\rho_1-\rho_2)}{\Gamma_p(a_1)} \frac{\Gamma_p(a_2-\rho_1-\rho_3)}{\Gamma_p(a_2)} \frac{\Gamma_p(b_1-\rho_2)}{\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_3)}{\Gamma_p(b_2)} \times$$

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$$\frac{\Gamma_p(b_3-\rho_4)}{\Gamma_p(b_3)}\frac{\Gamma_p(b_4-\rho_4)}{\Gamma_p(b_4)}\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3-\rho_4)}\times$$

$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4) \tag{6.19}$$

for $\mathrm{Re}(a_1-\rho_1-\rho_2, a_2-\rho_1-\rho_3, b_1-\rho_2, b_2-\rho_3, b_3-\rho_4,$

$b_4-\rho_4, c-\rho_1-\rho_2-\rho_3-\rho_4, \rho_1, \rho_2, \rho_3, \rho_4) > (p-1)/2.$

6.2.20

$$K_{20} = K_{20}(a_1, a_1, b_3, b_4; b_1, b_2, a_2, a_2; c, c, c, c; -X, -Y, -Z, -T)$$

$$M(K_{20}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2} K_{20}(a_1, a_1, b_3, b_4; b_1, b_2, a_2, a_2;$$

$$c, c, c, c; -X, -Y, -Z, -T)dXdYdZdT$$

$$=\frac{\Gamma_p(a_1-\rho_1-\rho_2)}{\Gamma_p(a_1)}\frac{\Gamma_p(a_2-\rho_3-\rho_4)}{\Gamma_p(a_2)}\frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)}\times$$

$$\frac{\Gamma_p(b_3-\rho_3)}{\Gamma_p(b_3)}\frac{\Gamma_p(b_4-\rho_4)}{\Gamma_p(b_4)}\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3-\rho_4)}\times$$

$$\times\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

for $\mathrm{Re}(a_1-\rho_1-\rho_2, a_2-\rho_3-\rho_4, b_i-\rho_i, \rho_i,$

$c-\rho_1-\rho_2-\rho_3-\rho_4) > (p-1)/2, \text{ where } i = 1, \cdots, 4.$ (6.20)

6.2.21 $K_{21} = K_{21}(a, a, b_6, b_5; b_1, b_2, b_3, b_4; c, c, c, c; -X, -Y, -Z, -T)$

$$M(K_{21}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2} K_{21}(a, a, b_6, b_5; b_1, b_2, b_3, b_4;$$

$$c, c, c, c; -X, -Y, -Z, -T)dXdYdZdT$$

$$=\frac{\Gamma_p(a-\rho_1-\rho_2)}{\Gamma_p(a)}\frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)}\frac{\Gamma_p(b_3-\rho_3)}{\Gamma_p(b_3)}\times$$

$$\frac{\Gamma_p(b_4-\rho_4)}{\Gamma_p(b_4)}\frac{\Gamma_p(b_6-\rho_3)}{\Gamma_p(b_6)}\frac{\Gamma_p(b_5-\rho_4)}{\Gamma_p(b_5)}\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3-\rho_4)}$$

$$\times\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4) \tag{6.21}$$

for $\mathrm{Re}(a-\rho_1-\rho_2, b_6-\rho_3, b_5-\rho_4, b_i-\rho_i, \rho_i,$
$c-\rho_1-\rho_2-\rho_3-\rho_4) > (p-1)/2,$ where $i=1,\cdots,4.$

6.3 A number of results are being proved in this section regarding the functions defined in the previous section. At least one result will be stated for each of the above functions and proofs of only a few representative results are being given, other results follow similarly.

Theorem 6.3.1:

$$K_5(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(a)}\int_{S>0}e^{-\mathrm{tr}(S)}|S|^{a-(p+1)/2}\Psi_2(b_1;c_1,c_2;-S^{1/2}XS^{1/2},$$
$$-S^{1/2}YS^{1/2})\Psi_2(b_2;c_3,c_4;-S^{1/2}ZS^{1/2},-S^{1/2}TS^{1/2})dS \tag{6.22}$$

for $\mathrm{Re}(a) > (p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(6.22) with respect to the variables X,Y,Z,T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively, we get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\Psi_2(b_1;c_1,c_2;-S^{1/2}XS^{1/2}, \tag{6.23}$$
$$-S^{1/2}YS^{1/2})\Psi_2(b_2;c_3,c_4;-S^{1/2}ZS^{1/2},-S^{1/2}TS^{1/2})dXdYdZdT$$

Applying the transformations,

$X_1 = S^{1/2} X S^{1/2}, Y_1 = S^{1/2} Y S^{1/2}, Z_1 = S^{1/2} Z S^{1/2}, T_1 = S^{1/2} T S^{1/2};$

with, $dX_1 = |S|^{(p+1)/2} dX, dY_1 = |S|^{(p+1)/2} dY, dZ_1 = |S|^{(p+1)/2} dZ,$

$dT_1 = |S|^{(p+1)/2} dT;$ and, $|X_1| = |S||X|, |Y_1| = |S||Y|, |Z_1| = |S||Z|,$

$|T_1| = |S||T|;$ to the above expression and then writing the M-transforms of the two involved Ψ_2- functions, we obtain

$$|S|^{-\rho_1-\rho_2-\rho_3-\rho_4} \frac{\Gamma_p(b_1-\rho_1-\rho_2)}{\Gamma_p(b_1)} \frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)} \frac{\Gamma_p(c_2)}{\Gamma_p(c_2-\rho_2)} \times$$

$$\frac{\Gamma_p(c_3)}{\Gamma_p(c_3-\rho_3)} \frac{\Gamma_p(c_4)}{\Gamma_p(c_4-\rho_4)} \frac{\Gamma_p(b_2-\rho_3-\rho_4)}{\Gamma_p(b_2)} \times \tag{6.24}$$

$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

Substituting this expression on the right side of eq.(6.22) and then integrating out S in the resulting expression by using a Gamma integral gives $M(K_5)$ as given by eq.(6.5).

Theorem 6.3.2:

$$K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;-X,-Y,-Z,-T)$$

$$= \frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(c_1-b_1-b_2)\Gamma_p(c_2-b_3-b_4)}$$

$$\times \int\int\int\int |U|^{b_1-(p+1)/2} |V|^{b_2-(p+1)/2} |W|^{b_3-(p+1)/2} \times$$

$$|S|^{b_4-(p+1)/2} |I-U-V|^{c_1-b_1-b_2-(p+1)/2} \times$$

$$|I-W-S|^{c_2-b_3-b_4-(p+1)/2} \Big| I + U^{1/2} X U^{1/2} + V^{1/2} Y V^{1/2}$$

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$$+W^{1/2}ZW^{1/2}+S^{1/2}TS^{1/2}\Big|^{-a} dUdVdWdS \tag{6.25}$$

for $\mathrm{Re}(b_i, c_1-b_1-b_2, c_2-b_3-b_4)>(p-1)/2$, where $i=1,\cdots,4.$

Proof: Taking the M-transform of the right side of eq.(6.25) with respect to the variables X,Y,Z,T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively, we obtain

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\Big|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2} \tag{6.26}$$
$$+W^{1/2}ZW^{1/2}+S^{1/2}TS^{1/2}\Big|^{-a} dXdYdZdT$$

On making use of the transformations,

$X_1=U^{1/2}XU^{1/2}, Y_1=V^{1/2}YV^{1/2}, Z_1=W^{1/2}ZW^{1/2}, T_1=S^{1/2}TS^{1/2};$

in the above expression and then integrating out X_1, Y_1, Z_1, T_1 by using a type-2 Dirichlet integral yields

$$|U|^{-\rho_1}|V|^{-\rho_2}|W|^{-\rho_3}|S|^{-\rho_4}\frac{\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)}{\Gamma_p(a)}\times \tag{6.27}$$
$$\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)$$

Substituting this expression on the right side of eq.(6.25) and then integrating out the variables U,V and W,S in the resulting expression by using a type-1 Dirichlet integral generates $M(K_{12})$ as given by eq.(6.12).

Theorem 6.3.3:

$$K_{16}(a_1,a_2,a_3,a_4;b;-X,-Y,-Z,-T)$$

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$$= \frac{1}{\Gamma_p(a_2)\Gamma_p(a_3)} \int_{S_1>0} \int_{S_2>0} e^{-\mathrm{tr}(S_1+S_2)} \left|S_1\right|^{a_2-(p+1)/2} \times$$

$$\left|S_2\right|^{a_3-(p+1)/2} \Phi_2(a_1, a_4; b; -S_1^{1/2} X S_1^{1/2} - S_2^{1/2} Y S_2^{1/2}, \tag{6.28}$$

$$-S_1^{1/2} Z S_1^{1/2} - S_2^{1/2} T S_2^{1/2}) dS_1 dS_2$$

for $\mathrm{Re}(a_2, a_3) > (p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(6.28) with respect to the variables X,Y,Z,T and the parameters $\rho_1, \rho_2, \rho_3, \rho_4$ respectively, we have

$$\int_{X>0} \int_{Y>0} \int_{Z>0} \int_{T>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$

$$|Z|^{\rho_3-(p+1)/2} |T|^{\rho_4-(p+1)/2} \Phi_2(a_1, a_4; b; -S_1^{1/2} X S_1^{1/2} \tag{6.29}$$

$$-S_2^{1/2} Y S_2^{1/2}, -S_1^{1/2} Z S_1^{1/2} - S_2^{1/2} T S_2^{1/2}) dX dY dZ dT$$

Making use of the transformations,

$$X_1 = S_1^{1/2} X S_1^{1/2}, Y_1 = S_2^{1/2} Y S_2^{1/2}, Z_1 = S_1^{1/2} Z S_1^{1/2}, T_1 = S_2^{1/2} T S_2^{1/2};$$

in the above equation and then applying the following transformations in the resulting expression so obtained,

$$X_2 = X_1, Y_2 = X_1 + Y_1; Z_2 = Z_1, T_2 = Z_1 + T_1; \text{ with, } dX_1 dY_1 = dX_2 dY_2, \text{ and, } dZ_1 dT_1 = dZ_2 dT_2; \text{ where, } 0 < X_2 < Y_2 \text{ and} \tag{6.30}$$

$0 < Z_2 < T_2$, followed by first, integrating out of X_2 and Z_2 by utilizing a type-1 Beta integral, and afterwards writing the M-transform of a Φ_2 function in the consequent expression leads us to,

$$\left|S_1\right|^{-\rho_1-\rho_3} \left|S_2\right|^{-\rho_2-\rho_4} \frac{\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)}{\Gamma_p(a_1)\Gamma_p(a_4)} \times$$

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$$\frac{\Gamma_p(b)\Gamma_p(a_1-\rho_1-\rho_2)\Gamma_p(a_4-\rho_3-\rho_4)}{\Gamma_p(b-\rho_1-\rho_2-\rho_3-\rho_4)} \tag{6.31}$$

Putting back this expression on the right side of eq.(6.28) and integrating out S_1 and S_2 by employing a Gamma integral generates $M(K_{16})$ as given by eq.(6.16) above.

Theorem 6.3.4:

$$\begin{aligned}
&K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)\\
&=\frac{1}{\Gamma_p(b_1)\cdots\Gamma_p(b_4)}\int_{S_1>0}\cdots\int_{S_4>0}e^{-tr(S_1+\cdots+S_4)}\times\\
&|S_1|^{b_1-(p+1)/2}\cdots|S_4|^{b_4-(p+1)/2}\Psi_2(a;c,d:-S_1^{1/2}XS_1^{1/2}\\
&-S_2^{1/2}YS_2^{1/2}-S_3^{1/2}ZS_3^{1/2},-S_4^{1/2}TS_4^{1/2})dS_1\cdots dS_4
\end{aligned} \tag{6.32}$$

for $\mathrm{Re}\,(b_i)>(p-1)/2,\ i=1,\cdots,4.$

Proof: Taking the M-transform of the right side of eq.(6.32) with respect to the variables X,Y,Z,T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively, we achieve

$$\begin{aligned}
&\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times\\
&|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\Psi_2(a;c,d:-S_1^{1/2}XS_1^{1/2}\\
&-S_2^{1/2}YS_2^{1/2}-S_3^{1/2}ZS_3^{1/2},-S_4^{1/2}TS_4^{1/2})dXdYdZdT
\end{aligned} \tag{6.33}$$

The application of the transformations

$$X_1=S_1^{1/2}XS_1^{1/2},Y_1=S_2^{1/2}YS_2^{1/2},Z_1=S_3^{1/2}ZS_3^{1/2},T_1=S_4^{1/2}TS_4^{1/2}; \tag{6.34}$$

to the above expression followed by the use of another set of transformations,

$$X_2 = X_1, Y_2 = X_1 + Y_1, Z_2 = X_1 + Y_1 + Z_1; \text{ with, } dX_1dY_1dZ_1 = dX_2dY_2dZ_2; \text{ where, } 0 < X_2 < Y_2 < Z_2;$$

and then first integrating out X_2 and Y_2 one-by-one and in order by using a type-1 Beta integral, afterwards, invoking the M-transform of a Ψ_2 function gives

$$|S_1|^{-\rho_1}\cdots|S_4|^{-\rho_4}\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_4)\Gamma_p(a-\rho_1-\cdots-\rho_4)\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(a)\Gamma_p(d-\rho_4)\Gamma_p(c-\rho_1-\rho_2-\rho_3)} \quad (6.35)$$

Substituting this expression on the right side of eq.(6.32) and integrating out $S_1,\cdots,S_4$ by the help of a Gamma integral gives $M(K_{11})$ as given by eq.(6.11).

Theorem 6.3.5:

$$K_{20}(a_1, a_1, b_3, b_4; b_1, b_2, a_2, a_2; c, c, c, c; -X, -Y, -Z, -T)$$
$$= \frac{1}{\Gamma_p(b_1)\cdots\Gamma_p(b_4)}\int_{S_1>0}\cdots\int_{S_4>0} e^{-tr(S_1+\cdots+S_4)} \times$$
$$|S_1|^{b_1-(p+1)/2}\cdots|S_4|^{b_4-(p+1)/2}\,\Phi_2(a_1, a_2; c : -S_1^{1/2}XS_1^{1/2} \quad (6.36)$$
$$-S_2^{1/2}YS_2^{1/2}, -S_3^{1/2}ZS_3^{1/2} - S_4^{1/2}TS_4^{1/2})dS_1\cdots dS_4$$

for $\mathrm{Re}\,(b_i) > (p-1)/2,\ i = 1,\cdots,4.$

Theorem 6.3.6:

$$K_{21}(a, a, b_6, b_5; b_1, b_2, b_3, b_4; c, c, c, c; -X, -Y, -Z, -T)$$

$$= \frac{1}{\Gamma_p(a)\left\{\prod_{i=1}^{6}\Gamma_p(b_i)\right\}}\int_{R>0}\int_{S_1>0}\cdots\int_{S_6>0} e^{-tr(R+\sum_{i=1}^{6} S_i)} \times$$

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$$|R|^{a-(p+1)/2}\left\{\prod_{i=1}^{6}|S_i|^{b_i-(p+1)/2}\right\}{}_0F_1(\,;c;-S_1^{1/2}R^{1/2}XR^{1/2}S_1^{1/2}$$

$$-S_2^{1/2}R^{1/2}YR^{1/2}S_2^{1/2}-S_6^{1/2}S_3^{1/2}ZS_3^{1/2}S_6^{1/2}-S_5^{1/2}S_4^{1/2}TS_4^{1/2}S_5^{1/2})\times \tag{6.37}$$

$$dR\left\{\prod_{i=1}^{6}dS_i\right\}$$

for $\mathrm{Re}(a,b_i)>(p-1)/2,\ i=1,\cdots,6.$

Proof: Taking the M-transform of the right side of eq.(6.37) with respect to the variables X, Y, Z, T and the parameters $\rho_1, \rho_2, \rho_3, \rho_4$ respectively, we obtain

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}{}_0F_1(\,;c;-S_1^{1/2}R^{1/2}XR^{1/2}S_1^{1/2} \tag{6.38}$$

$$-S_2^{1/2}R^{1/2}YR^{1/2}S_2^{1/2}-S_6^{1/2}S_3^{1/2}ZS_3^{1/2}S_6^{1/2}-S_5^{1/2}S_4^{1/2}TS_4^{1/2}S_5^{1/2})$$

$$\times dXdYdZdT$$

The application of the transformations

$$X_1=S_1^{1/2}R^{1/2}XR^{1/2}S_1^{1/2},\ Y_1=S_2^{1/2}R^{1/2}YR^{1/2}S_2^{1/2},$$

$$Z_1=S_6^{1/2}S_3^{1/2}ZS_3^{1/2}S_6^{1/2},\ T_1=S_5^{1/2}S_4^{1/2}TS_4^{1/2}S_5^{1/2};\ \text{with,}$$

$$dX_1=|S_1|^{(p+1)/2}|R|^{(p+1)/2}dX,\ dY_1=|S_2|^{(p+1)/2}|R|^{(p+1)/2}dY,$$

$$dZ_1=|S_6|^{(p+1)/2}|S_3|^{(p+1)/2}dZ,\ dT_1=|S_5|^{(p+1)/2}|S_4|^{(p+1)/2}dT;$$

and, $|X_1|=|S_1||R||X|,|Y_1|=|S_2||R||Y|,|Z_1|=|S_6||S_3||Z|,|T_1|=|S_5||S_4||T|;$

to the last equation followed by the use of the eq.(2.4) produces,

$$\left\{\prod_{i=1}^{4}\left|S_i\right|^{-\rho_i}\right\}\left|S_5\right|^{-\rho_4}\left|S_6\right|^{-\rho_3}\left|R\right|^{-\rho_1-\rho_2}\frac{\left\{\prod_{i=1}^{4}\Gamma_p(\rho_i)\right\}\Gamma_p(c)}{\Gamma_p(c-\rho_1-\cdots-\rho_4)} \qquad (6.39)$$

Replacing this expression on the right side of eq.(6.37), subsequently integrating out the variables of integration by using a Gamma integral generates $M(K_{21})$ in agreement with eq.(6.21).

Theorem 6.3.7:

$$K_{13}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,d_1,d_2;-X,-Y,-Z,-T)$$
$$=\frac{\Gamma_p(c)\Gamma_p(d_1)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(c-b_1-b_2)\Gamma_p(d_1-b_3)}\times$$
$$\frac{\Gamma_p(d_2)}{\Gamma_p(d_2-b_4)}\int\!\!\int\!\!\int\!\!\int|U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}\times$$
$$|W|^{b_3-(p+1)/2}|S|^{b_4-(p+1)/2}|I-W|^{d_1-b_3-(p+1)/2}\times$$
$$|I-S|^{d_2-b_4-(p+1)/2}|I-U-V|^{c-b_1-b_2-(p+1)/2}\times$$
$$\left|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}ZW^{1/2}+S^{1/2}TS^{1/2}\right|^{-a}\times$$
$$dUdVdWdS \qquad (6.40)$$

for $\text{Re}\,(b_1,b_2,b_3,b_4,c-b_1-b_2,d_1-b_3,d_2-b_4)>(p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(6.40) with respect to the variables X,Y,Z,T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively, we have

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\times$$

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$$\left|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}ZW^{1/2}+S^{1/2}TS^{1/2}\right|^{-a}\times \quad (6.41)$$

$$dXdYdZdT$$

Applying the transformations

$$X_1=U^{1/2}XU^{1/2},\ Y_1=V^{1/2}YV^{1/2},\ Z_1=W^{1/2}ZW^{1/2},\ T_1=S^{1/2}TS^{1/2};$$

to the above expression and then integrating out the variables X_1, Y_1, Z_1, T_1 by employing a type-2 Dirichlet integral we achieve

$$|U|^{-\rho_1}|V|^{-\rho_2}|W|^{-\rho_3}|S|^{-\rho_4}\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\times \quad (6.42)$$

$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

Substituting this expression on the right side of eq.(6.40) and integrating out W and S by using a type-1 Beta integral and U and V by a type-1 Dirichlet integral we are led to $M(K_{13})$ as given by eq.(6.13).

Theorem 6.3.8:

$$K_9(a,a,a,a;b,b,c_1,c_2;e_1,e_2,d,d;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(a)}\int_{S>0}e^{-tr(S)}|S|^{a-(p+1)/2}\Psi_2(b;e_1,e_2;-S^{1/2}XS^{1/2}, \quad (6.43)$$

$$-S^{1/2}YS^{1/2})\Phi_2(c_1,c_2;d;-S^{1/2}ZS^{1/2},-S^{1/2}TS^{1/2})dS$$

for $\mathrm{Re}\,(a)>(p-1)/2$.

Theorem 6.3.9:

$$K_7(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_1,d_2;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(b)\Gamma_p(c_1)\Gamma_p(c_2)}\int_{S_1>0}\int_{S_2>0}\int_{S_3>0}e^{-tr(S_1+S_2+S_3)}\times$$

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$$\left|S_1\right|^{b-(p+1)/2}\left|S_2\right|^{c_1-(p+1)/2}\left|S_3\right|^{c_2-(p+1)/2}\Psi_2(a;d_1,d_2;$$

$$-S_1^{1/2}XS_1^{1/2}-S_2^{1/2}ZS_2^{1/2},-S_1^{1/2}YS_1^{1/2}-S_3^{1/2}TS_3^{1/2})dS_1dS_2dS_3 \quad (6.44)$$

for Re $(b,c_1,c_2)>(p-1)/2$.

Theorem 6.3.10:

$$K_1(a,a,a,a;b,b,b,c;d,e_1,e_2,d;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(a)\Gamma_p(b)}\int_{R_1>0}\int_{R_2>0}e^{-tr(R_1+R_2)}\left|R_1\right|^{a-(p+1)/2}\times$$

$$\left|R_2\right|^{b-(p+1)/2}{}_0F_1(\,;e_1;-R_2^{1/2}R_1^{1/2}YR_1^{1/2}R_2^{1/2})\times \quad (6.45)$$

$${}_0F_1(\,;e_2;-R_2^{1/2}R_1^{1/2}ZR_1^{1/2}R_2^{1/2})\Phi_3(c;d;-R_1^{1/2}TR_1^{1/2},$$

$$-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2})dR_1dR_2$$

for Re $(a,b)>(p-1)/2$.

Theorem 6.3.11:

$$K_8(a,a,a,a;b,b,c_1,c_2;d,e_1,d,e_2;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c_1)}\int_{R_1>0}\int_{R_2>0}\int_{R_3>0}e^{-tr(R_1+R_2+R_3)}\times$$

$$\left|R_1\right|^{a-(p+1)/2}\left|R_2\right|^{b-(p+1)/2}\left|R_3\right|^{c_1-(p+1)/2}\times \quad (6.46)$$

$${}_0F_1(\,;d;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2}-R_3^{1/2}R_1^{1/2}ZR_1^{1/2}R_3^{1/2})\times$$

$${}_0F_1(\,;e_1;-R_2^{1/2}R_1^{1/2}YR_1^{1/2}R_2^{1/2})\,{}_1F_1(c_2;e_2;-R_1^{1/2}TR_1^{1/2})\times$$

$$dR_1dR_2dR_3$$

$$\text{for Re }(a,b,c_1) > (p-1)/2.$$

Theorem 6.3.12:

$$K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;-X,-Y,-Z,-T)$$

$$= \frac{1}{\Gamma_p(a)} \int_{S>0} e^{-tr(S)} |S|^{a-(p+1)/2}\, \Phi_2(b_1,b_2;c_1;-S^{1/2}XS^{1/2}, \tag{6.47}$$

$$-S^{1/2}YS^{1/2})\Phi_2(b_3,b_4;c_2;-S^{1/2}ZS^{1/2},-S^{1/2}TS^{1/2})dS$$

for $\mathrm{Re}(a) > (p-1)/2$.

Theorem 6.3.13:

$$K_{14}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;-X,-Y,-Z,-T)$$

$$= \frac{\Gamma_p(d)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(d-a-b)} \int_0^I \int_0^I |U|^{a-(p+1)/2}\, |V|^{b-(p+1)/2} \times$$

$$|I-U|^{d-a-(p+1)/2}\, |I-V|^{d-a-b-(p+1)/2} \left| I+U^{1/2}XU^{1/2} \right|^{-b} \times$$

$$\left| I+U^{1/2}YU^{1/2} \right|^{-c_1} \left| I+U^{1/2}ZU^{1/2} \right|^{-c_2} \left| I+V^{1/2}(I-U)^{1/2} \times \right.$$

$$\left. (I+U^{1/2}XU^{1/2})^{-1/2}T(I+U^{1/2}XU^{1/2})^{-1/2}(I-U)^{1/2}V^{1/2} \right|^{-c_3} \times$$

$$dUdV \tag{6.48}$$

for $\mathrm{Re}(a,b,d-a-b) > (p-1)/2$.

Theorem 6.3.14:

$$K_{10}(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_3,d_4;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(a)}\int_{S>0} e^{-\mathrm{tr}(S)}|S|^{a-(p+1)/2}\Psi_2(b;d_1,d_2;-S^{1/2}XS^{1/2},$$

$$-S^{1/2}YS^{1/2})\,{}_1F_1(c_1;d_3;-S^{1/2}ZS^{1/2})\,{}_1F_1(c_2;d_4;-S^{1/2}TS^{1/2})dS \tag{6.49}$$

for $\mathrm{Re}(a)>(p-1)/2$.

Theorem 6.3.15:

$$K_2(a,a,a,a;b,b,b,c;d_1,d_2,d_3,d_4;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(a)\Gamma_p(b)}\int_{R_1>0}\int_{R_2>0} e^{-\mathrm{tr}(R_1+R_2)}|R_1|^{a-(p+1)/2}\times$$

$$|R_2|^{b-(p+1)/2}\;{}_0F_1(\,;d_1;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2})\times$$

$${}_0F_1(\,;d_2;-R_2^{1/2}R_1^{1/2}YR_1^{1/2}R_2^{1/2})\,{}_0F_1(\,;d_3;$$

$$-R_2^{1/2}R_1^{1/2}ZR_1^{1/2}R_2^{1/2})\,{}_1F_1(c;d_4;-R_1^{1/2}TR_1^{1/2})dR_1dR_2$$

for $\mathrm{Re}(a,b)>(p-1)/2$. (6.50)

Theorem 6.3.16:

$$K_{19}(a_1,a_2,b_1,b_2,b_3,b_4;c;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(a_1)\Gamma_p(a_2)\left\{\prod_{i=1}^{4}\Gamma_p(b_i)\right\}}\int_{R_1>0}\cdots(6)\cdots\int_{S_4>0}\times$$

Continued to the next page … … … … …

$$e^{-\mathrm{tr}(R_1+R_2+\left\{\sum_{i=1}^{4} S_i\right\})} \left|R_1\right|^{a_1-(p+1)/2} \left|R_2\right|^{a_2-(p+1)/2} \times$$

$$\left\{\prod_{i=1}^{4} \left|S_i\right|^{b_i-(p+1)/2}\right\} {}_0F_1(\,;c;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2}$$

$$-S_1^{1/2}R_1^{1/2}YR_1^{1/2}S_1^{1/2}-S_2^{1/2}R_2^{1/2}ZR_2^{1/2}S_2^{1/2} \tag{6.51}$$

$$-S_4^{1/2}S_3^{1/2}TS_3^{1/2}S_4^{1/2})dR_1dR_2\left\{\prod_{i=1}^{4} dS_i\right\}$$

for $\mathrm{Re}(a_1, a_2, b_i) > (p-1)/2;$ where, $i = 1, \cdots, 4.$

Theorem 6.3.17:

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

$$= \frac{\Gamma_p(c)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c-b_1-b_2-b_3)} \int\int\int |U|^{b_1-(p+1)/2} \times$$

$$|V|^{b_2-(p+1)/2} |W|^{b_3-(p+1)/2} \times$$

$$|I-U-V-W|^{c-b_1-b_2-b_3-(p+1)/2} \times$$

$$\left|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}ZW^{1/2}\right|^{-a} {}_2F_1[a,b_4;d;$$

$$-(I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}ZW^{1/2})^{-1/2}\,T\times \tag{6.52}$$

$$(I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}ZW^{1/2})^{-1/2}]dUdVdW$$

where, $U = U' > 0, V = V' > 0, W = W' > 0, 0 < U + V + W < I$,

and for $\mathrm{Re}(b_1, b_2, b_3, c - b_1 - b_2 - b_3) > (p-1)/2$.

Theorem 6.3.18:

$$K_{13}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,d_1,d_2;-X,-Y,-Z,-T)$$

$$= \frac{1}{\Gamma_p(a)} \int_{S>0} e^{-\mathrm{tr}(S)} |S|^{a-(p+1)/2}\, \Phi_2(b_1, b_2; c; -S^{1/2}XS^{1/2}, \tag{6.53}$$

$$-S^{1/2}YS^{1/2})\, {}_1F_1(b_3; d_1; -S^{1/2}ZS^{1/2})\, {}_1F_1(b_4; d_2; -S^{1/2}TS^{1/2})\, dS$$

for $\mathrm{Re}(a) > (p-1)/2$.

Theorem 6.3.19:

$$K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;-X,-Y,-Z,-T)$$

$$= \frac{1}{\left\{\prod_{i=1}^{4} \Gamma_p(b_i)\right\}} \int_{S_1>0} \cdots (4) \cdots \int_{S_4>0} e^{-\mathrm{tr}(\sum_{i=1}^{4} S_i)} \times$$

$$\left\{\prod_{i=1}^{4} |S_i|^{b_i-(p+1)/2}\right\} \Psi_2(a; c_1, c_2; -S_1^{1/2}XS_1^{1/2} - S_2^{1/2}YS_2^{1/2}, \tag{6.54}$$

$$-S_3^{1/2}ZS_3^{1/2} - S_4^{1/2}TS_4^{1/2}) \left\{\prod_{i=1}^{4} dS_i\right\}$$

for $\mathrm{Re}(b_i) > (p-1)/2$, where, $i = 1, \cdots, 4$.

Theorem 6.3.20:

$$K_{15}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;-X,-Y,-Z,-T)$$

Continued to the next page … … … … …

$$= \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(b_5)\Gamma_p(c-a-b_5)} \iint |U|^{a-(p+1)/2} |V|^{b_5-(p+1)/2} \times$$

$$|I-U|^{c-a-(p+1)/2} |I-V|^{c-a-b_5-(p+1)/2} \times$$

$$\left|I + U^{1/2} X U^{1/2}\right|^{-b_1} \left|I + U^{1/2} Y U^{1/2}\right|^{-b_2} \left|I + U^{1/2} Z U^{1/2}\right|^{-b_3} \times \tag{6.55}$$

$$\left|I + V^{1/2}(I-U)^{1/2} T (I-U)^{1/2} V^{1/2}\right|^{-b_4} \, dUdV$$

for $\mathrm{Re}(a, b_5, c-a-b_5) > (p-1)/2$, where, $0 < U < I$ and $0 < V < I$.

Theorem 6.3.21:

$$K_4(a,a,a,a;b_1,b_1,b_2,b_2;c,d_1,d_2,c;-X,-Y,-Z,-T)$$

$$= \frac{1}{\Gamma_p(a)\Gamma_p(b_1)\Gamma_p(b_2)} \int_{R_1>0}\int_{R_2>0}\int_{R_3>0} e^{-\mathrm{tr}(R_1+R_2+R_3)} \times$$

$$|R_1|^{a-(p+1)/2} |R_2|^{b_1-(p+1)/2} |R_3|^{b_2-(p+1)/2} \times$$

$${}_0F_1(\,;c;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2} - R_3^{1/2}R_1^{1/2}TR_1^{1/2}R_3^{1/2}) \times$$

$${}_0F_1(\,;d_1;-R_2^{1/2}R_1^{1/2}YR_1^{1/2}R_2^{1/2})\,{}_0F_1(\,;d_2;$$

$$-R_3^{1/2}R_1^{1/2}ZR_1^{1/2}R_3^{1/2})dR_1dR_2dR_3 \tag{6.56}$$

for $\mathrm{Re}(a, b_1, b_2) > (p-1)/2$.

Theorem 6.3.22:

$$K_{17}(a_1,a_2,a_3,b_1,b_2;c;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(a_3)\Gamma_p(b_1)\Gamma_p(b_2)}\int_{R_1>0}\cdots(5)\cdots\int_{S_2>0}\times$$

$$e^{-\mathrm{tr}(\sum_{i=1}^{3} R_i+S_1+S_2)}\left\{\prod_{i=1}^{3}\left|R_i\right|^{a_i-(p+1)/2}\right\}\left|S_1\right|^{b_1-(p+1)/2}\times$$

$$\left|S_2\right|^{b_2-(p+1)/2}\ {}_0F_1(\ ;c;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2}$$

$$-R_3^{1/2}R_1^{1/2}YR_1^{1/2}R_3^{1/2}-R_3^{1/2}R_2^{1/2}ZR_2^{1/2}R_3^{1/2}-S_2^{1/2}S_1^{1/2}TS_1^{1/2}S_2^{1/2})\times$$

$$\left\{\prod_{i=1}^{3}dR_i\right\}dS_1dS_2 \tag{6.57}$$

for $\mathrm{Re}(a_1,a_2,a_3,b_1,b_2)>(p-1)/2.$

Theorem 6.3.23:

$$K_{18}(a_1,a_2,a_3,b_1,b_2;c;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(a_1)\Gamma_p(b_1)\Gamma_p(b_2)}\int_{S_1>0}\int_{S_2>0}\int_{S_3>0}e^{-\mathrm{tr}(S_1+S_2+S_3)}\times$$

$$\left|S_1\right|^{a_1-(p+1)/2}\left|S_2\right|^{b_1-(p+1)/2}\left|S_3\right|^{b_2-(p+1)/2}\Phi_2(a_2,a_3;c; \tag{6.58}$$

$$-S_1^{1/2}XS_1^{1/2}-S_3^{1/2}TS_3^{1/2},-S_1^{1/2}YS_1^{1/2}-S_2^{1/2}ZS_2^{1/2})dS_1dS_2dS_3$$

for $\mathrm{Re}(a_1,b_1,b_2)>(p-1)/2.$

Theorem 6.3.24:

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

$$= \frac{1}{\Gamma_p(a)} \int_{S>0} e^{-tr(S)} |S|^{a-(p+1)/2} \Phi_2^{(3)}(b_1,b_2,b_3;c;-S^{1/2}XS^{1/2},$$

$$-S^{1/2}YS^{1/2},-S^{1/2}ZS^{1/2})\,{}_1F_1(b_4;d;-S^{1/2}TS^{1/2})dS \tag{6.59}$$

for $\mathrm{Re}(a) > (p-1)/2$.

Theorem 6.3.25:

$$K_{14}(a,a,a,c_3;b,c_1,c_2,b;d,d,d,d;-X,-Y,-Z,-T)$$

$$= \frac{1}{\Gamma_p(b)\Gamma_p(c_1)\Gamma_p(c_2)} \int_{S_1>0}\int_{S_2>0}\int_{S_3>0} e^{-tr(S_1+S_2+S_3)} \times$$

$$|S_1|^{b-(p+1)/2} |S_2|^{c_1-(p+1)/2} |S_3|^{c_2-(p+1)/2} \Phi_2(a,c_3;d; \tag{6.60}$$

$$-S_1^{1/2}XS_1^{1/2} - S_2^{1/2}YS_2^{1/2} - S_3^{1/2}ZS_3^{1/2}, -S_1^{1/2}TS_1^{1/2})dS_1dS_2dS_3$$

for $\mathrm{Re}(b,c_1,c_2) > (p-1)/2$.

Theorem 6.3.26:

$$K_{15}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;-X,-Y,-Z,-T)$$

$$= \frac{1}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)} \int_{S_1>0}\int_{S_2>0}\int_{S_3>0}\int_{S_4>0} \times$$

$$e^{-tr(S_1+S_2+S_3+S_4)} \left\{ \prod_{i=1}^{4} |S_i|^{b_i-(p+1)/2} \right\} \Phi_2(a,b_5;c; \tag{6.61}$$

$$-S_1^{1/2}XS_1^{1/2} - S_2^{1/2}YS_2^{1/2} - S_3^{1/2}ZS_3^{1/2}, -S_4^{1/2}TS_4^{1/2})dS_1dS_2dS_3dS_4$$

for $\operatorname{Re}(b_i) > (p-1)/2$, where, $i = 1, \cdots, 4$.

Theorem 6.3.27:

$$K_3(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_2,c_1;-X,-Y,-Z,-T)$$

$$= \frac{1}{\Gamma_p(b_1)\Gamma_p(b_2)} \int_{S_1>0}\int_{S_2>0} e^{-\operatorname{tr}(S_1+S_2)} \left|S_1\right|^{b_1-(p+1)/2} \times$$

$$\left|S_2\right|^{b_2-(p+1)/2} \Psi_2(a;c_1,c_2:-S_1^{1/2}XS_1^{1/2} - S_2^{1/2}TS_2^{1/2}, \tag{6.62}$$

$$-S_1^{1/2}YS_1^{1/2} - S_2^{1/2}ZS_2^{1/2})dS_1dS_2$$

for $\operatorname{Re}(b_1, b_2) > (p-1)/2$.

Theorem 6.3.28:

$$K_6(a,a,a,a;b,b,c_1,c_2;e,d,d,d;-X,-Y,-Z,-T)$$

$$= \frac{1}{\Gamma_p(b)\Gamma_p(c_1)\Gamma_p(c_2)} \int_{S_1>0}\int_{S_2>0}\int_{S_3>0} e^{-\operatorname{tr}(S_1+S_2+S_3)} \times$$

$$\left|S_1\right|^{b-(p+1)/2} \left|S_2\right|^{c_1-(p+1)/2} \left|S_3\right|^{c_2-(p+1)/2} \Psi_2(a;e,d; \tag{6.63}$$

$$-S_1^{1/2}XS_1^{1/2}, -S_1^{1/2}YS_1^{1/2} - S_2^{1/2}ZS_2^{1/2} - S_3^{1/2}TS_3^{1/2})dS_1dS_2dS_3$$

for $\operatorname{Re}(b, c_1, c_2) > (p-1)/2$.

Theorem 6.3.29:

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

Continued to the next page … … … … …

$$
= \frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(d-b_4)\Gamma_p(c-b_1-b_2-b_3)} \times
$$

$$
\iint\iint |U|^{b_1-(p+1)/2} |V|^{b_2-(p+1)/2} |W|^{b_3-(p+1)/2} \times
$$

$$
|S|^{b_4-(p+1)/2} |I-U-V-W|^{c-b_1-b_2-b_3-(p+1)/2} \times
$$

$$
|I-S|^{d-b_4-(p+1)/2} \Big| I + U^{1/2} X U^{1/2} + V^{1/2} Y V^{1/2}
$$

$$
+ W^{1/2} Z W^{1/2} + S^{1/2} T S^{1/2} \Big|^{-a} \, dU dV dW dS
$$

$$
\text{where, } U = U' > 0, V = V' > 0, W = W' > 0, 0 < U + V + W < I,
$$

$$
0 < S < I, \text{ and for } \mathrm{Re}(b_i, c - b_1 - b_2 - b_3, d - b_4) > (p-1)/2, \tag{6.64}
$$

$$
i = 1, \cdots, 4.
$$

6.4 Some transformation relations and cases of reducibility are being discussed in this section.

Theorem 6.4.1: A case of reducibility:

(i)
$$
\lim_{\alpha\to\infty} K_{12}(\alpha,\alpha,\alpha,\alpha; b_1, b_2, b_3, b_4; c_1, c_1, c_2, c_2; \frac{-X}{\alpha}, \frac{-Y}{\alpha}, \frac{-Z}{\alpha}, \frac{-T}{\alpha}) \tag{6.65}
$$

$$
= \Phi_2(b_1, b_2; c_1; -X, -Y)\Phi_2(b_3, b_4; c_2; -Z, -T)
$$

(ii)
$$
\lim_{\alpha\to\infty} K_{12}(\alpha,\alpha,\alpha,\alpha; b_1, b_2, b_3, b_4; c_1, c_1, c_2, c_2; \frac{-X}{\alpha}, \frac{-X}{\alpha}, \frac{-Z}{\alpha}, \frac{-Z}{\alpha}) \tag{6.66}
$$

$$
= {}_1F_1(b_1 + b_2; c_1; -X)\, {}_1F_1(b_3 + b_4; c_2; -Z)
$$

Proof: (i). This result follows by putting $a = \alpha$ in eq.(6.25) and replacing X by X/α, etc. and then proceeding to the limit as $\alpha \to \infty$ keeping in mind the eq.(1.23), and finally using the theorem 3.2.6 in the resulting expression.

(ii). Replacing the two Φ_2- functions in eq. (6.65) by their integral representations as given by eq.(3.17), we get

$$\lim_{\alpha\to\infty} K_{12}(\alpha,\alpha,\alpha,\alpha;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;$$
$$\frac{-X}{\alpha},\frac{-Y}{\alpha},\frac{-Z}{\alpha},\frac{-T}{\alpha})$$
$$=\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(c_1-b_1-b_2)\Gamma_p(c_2-b_3-b_4)}$$
$$\times\int\int\int\int|U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}\times \tag{6.67}$$
$$|S|^{b_4-(p+1)/2}|I-U-V|^{c_1-b_1-b_2-(p+1)/2}\times$$
$$|I-W-S|^{c_2-b_3-b_4-(p+1)/2}e^{-tr(UX+VY+WZ+ST)}\times$$
$$dUdVdWdS$$

The result in eq.(6.66) is obtained by putting Y=X and T=Z in eq. (6.67) and then applying the transformations,

$$U_1=U, V_1=U+V, W_1=W, S_1=W+S;\ \text{with}\ dU_1dV_1=dUdV,$$

$$dW_1dS_1=dWdS,\ \text{where},\ 0<U_1<V_1<I, 0<W_1<S_1<I;$$

to it and then integrating out U_1 and W_1 in the resulting expression by using a type-1 Beta integral which leads to the desired result in the light of eq.(2.3).

Theorem 6.4.2: A case of reducibility:

(i) $$\lim_{\alpha\to\infty} K_{13}(\alpha,\alpha,\alpha,\alpha;b_1,b_2,b_3,b_4;c,c,d_1,d_2;\frac{-X}{\alpha},\frac{-Y}{\alpha},\frac{-Z}{\alpha},\frac{-T}{\alpha}) \tag{6.68}$$
$$={}_1F_1(b_3;d_1;-Z)\,{}_1F_1(b_4;d_2;-T)\,\Phi_2(b_1,b_2;c;-X,-Y)$$

(ii) $$\lim_{\alpha\to\infty} K_{13}(\alpha,\alpha,\alpha,\alpha;b_1,b_2,b_3,b_4;c,c,d_1,d_2;\frac{-X}{\alpha},\frac{-X}{\alpha},\frac{-Z}{\alpha},\frac{-T}{\alpha}) \tag{6.69}$$
$$={}_1F_1(b_3;d_1;-Z)\,{}_1F_1(b_4;d_2;-T)\,{}_1F_1(b_1+b_2;c;-X)$$

Proof: (i) This result is a limiting case of eq.(6.40), in which use of eq.(1.23) and eqs.(2.3) and (3.17) has been made.

(ii) The result stated here is a particular case of eq.(6.68), which can be obtained by putting $Y = X$ in the said equation.

Theorem 6.4.3: A case of reducibility:

(i)
$$\lim_{\alpha\to\infty} K_{11}(\alpha,\alpha,\alpha,\alpha;b_1,b_2,b_3,b_4;c,c,c,d;\frac{-X}{\alpha},\frac{-Y}{\alpha},\frac{-Z}{\alpha},\frac{-T}{\alpha})$$
$$=\frac{\Gamma_p(c)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c-b_1-b_2-b_3)}\times {}_1F_1(b_4;d;-T)\times \tag{6.70}$$
$$\iiint |U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}\times$$
$$|I-U-V-W|^{c-b_1-b_2-b_3-(p+1)/2}e^{-tr(UX+VY+WZ)}\times$$
$$dUdVdW$$
for $\mathrm{Re}(b_i, c-b_1-b_2-b_3) > (p-1)/2, i = 1,\cdots,3.$

(ii)
$$\lim_{\alpha\to\infty} K_{11}(\alpha,\alpha,\alpha,\alpha;b_1,b_2,b_3,b_4;c,c,c,d;\frac{-X}{\alpha},\frac{-X}{\alpha},\frac{-X}{\alpha},\frac{-T}{\alpha}) \tag{6.71}$$
$$= {}_1F_1(b_4;d;-T)\,{}_1F_1(b_1+b_2+b_3;c;-X)$$

Proof: (i) This result is a limiting case of eq.(6.64) in which use of eq.(1.23) and eq.(2.3) has been made.
(ii) Putting $Z = Y = X$ in eq.(6.70) and applying the transformations, $W_1 = U, W_2 = U+V, W_3 = U+V+W;$ to it and then integrating out W_1 and W_2 one-by-one and in order by using a type-1 Beta integral followed by the application of eq.(2.3) leads to this result.

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Theorem 6.4.4: A transformation theorem:

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

$$=\left|I+X\right|^{-a}K_{11}[a,a,a,a;c-b_1-b_2-b_3,b_2,b_3,b_4;c,c,c,d;$$

(i) $$(I+X)^{-1/2}X(I+X)^{-1/2},-(I+X)^{-1/2}(Y-X)(I+X)^{-1/2}, \quad (6.72)$$

$$-(I+X)^{-1/2}(Z-X)(I+X)^{-1/2},-(I+X)^{-1/2}T(I+X)^{-1/2}]$$

where, $Y-X>0$ and $Z-X>0$.

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

$$=\left|I+Y\right|^{-a}K_{11}[a,a,a,a;b_1,c-b_1-b_2-b_3,b_3,b_4;c,c,c,d;$$

(ii) $$-(I+Y)^{-1/2}(X-Y)(I+Y)^{-1/2},(I+Y)^{-1/2}Y(I+Y)^{-1/2}, \quad (6.73)$$

$$-(I+Y)^{-1/2}(Z-Y)(I+Y)^{-1/2},-(I+Y)^{-1/2}T(I+Y)^{-1/2}]$$

where, $Z-Y>0$ and $X-Y>0$.

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

$$=\left|I+Z\right|^{-a}K_{11}[a,a,a,a;b_1,b_2,c-b_1-b_2-b_3,b_4;c,c,c,d;$$

(iii) $$-(I+Z)^{-1/2}(X-Z)(I+Z)^{-1/2},-(I+Z)^{-1/2}(Y-Z)(I+Z)^{-1/2}, \quad (6.74)$$

$$(I+Z)^{-1/2}Z(I+Z)^{-1/2},-(I+Z)^{-1/2}T(I+Z)^{-1/2}]$$

where, $X-Z>0$ and $Y-Z>0$.

(iv)
$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$
$$=|I+T|^{-a}K_{11}[a,a,a,a;b_1,b_2,b_3,d-b_4;c,c,c,d;$$
$$-(I+T)^{-1/2}X(I+T)^{-1/2},-(I+T)^{-1/2}Y(I+T)^{-1/2},$$
$$-(I+T)^{-1/2}Z(I+T)^{-1/2},(I+T)^{-1/2}T(I+T)^{-1/2}] \tag{6.75}$$

(v)
$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$
$$=|I+X+T|^{-a}K_{11}[a,a,a,a;c-b_1-b_2-b_3,b_2,b_3,d-b_4;$$
$$c,c,c,d;(I+X+T)^{-1/2}X(I+X+T)^{-1/2},-(I+X+T)^{-1/2}(Y-X)\times$$
$$(I+X+T)^{-1/2},-(I+X+T)^{-1/2}(Z-X)(I+X+T)^{-1/2},$$
$$(I+X+T)^{-1/2}T(I+X+T)^{-1/2}] \tag{6.76}$$

where, $Y-X>0$ and $Z-X>0$.

Proof: To prove this theorem we define the function K_{11} through an integral representation, which is obtained from eq.(6.64) by the use of the assumption of symmetry of the hypergeometric function in its matrix arguments as has earlier been done by Herz [22] and Mathai [60,62]:

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$
$$=\frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(d-b_4)\Gamma_p(c-b_1-b_2-b_3)}\times$$
$$\iint\iint|U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}\times$$
$$|S|^{b_4-(p+1)/2}|I-U-V-W|^{c-b_1-b_2-b_3-(p+1)/2}\times$$

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$$|I-S|^{d-b_4-(p+1)/2}\Big|I+X^{1/2}UX^{1/2}+Y^{1/2}VY^{1/2}$$

$$+Z^{1/2}WZ^{1/2}+T^{1/2}ST^{1/2}\Big|^{-a}\ dUdVdWdS$$

where, $U=U'>0, V=V'>0, W=W'>0, 0<U+V+W<I,$ (6.77)

$0<S<I,$ and for $\mathrm{Re}(b_i, c-b_1-b_2-b_3, d-b_4)>(p-1)/2,$

$i=1,\cdots,4.$

(i) This result is obtained by the application of the transformations $U_1=I-U-V-W, V_1=V, W_1=W;$ to eq.(6.77) and observing that

$$\Big|I+X^{1/2}(I-U_1-V_1-W_1)X^{1/2}+Y^{1/2}V_1Y^{1/2}+Z^{1/2}W_1Z^{1/2}$$

$$+T^{1/2}ST^{1/2}\Big|=|I+X|\Big|I-(I+X)^{-1/2}X^{1/2}U_1X^{1/2}(I+X)^{-1/2}+$$

$$(I+X)^{-1/2}(Y-X)^{1/2}V_1(Y-X)^{1/2}(I+X)^{-1/2}+(I+X)^{-1/2}\times$$

$$(Z-X)^{1/2}W_1(Z-X)^{1/2}(I+X)^{-1/2}+(I+X)^{-1/2}T^{1/2}ST^{1/2}(I+X)^{-1/2}\Big|$$

then interpreting the resulting expression as a K_{11} as per eq.(6.77). The results in eq.(6. 73) and (6.74) follow similarly.

(iv) On observing that

$$\Big|I+X^{1/2}UX^{1/2}+Y^{1/2}VY^{1/2}+Z^{1/2}WZ^{1/2}+T^{1/2}ST^{1/2}\Big|$$

$$=|I+T|\Big|I+(I+T)^{-1/2}X^{1/2}UX^{1/2}(I+T)^{-1/2}+(I+T)^{-1/2}Y^{1/2}\times$$

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$$VY^{\frac{1}{2}}(I+T)^{-\frac{1}{2}}+(I+T)^{-\frac{1}{2}}Z^{\frac{1}{2}}WZ^{\frac{1}{2}}(I+T)^{-\frac{1}{2}}-$$
$$\left.(I+T)^{-\frac{1}{2}}T^{\frac{1}{2}}(I-S)T^{\frac{1}{2}}(I+T)^{-\frac{1}{2}}\right|$$

and applying the transformation $S_1 = I - S$, the result of eq.(6.75) is obtained by a suitable interpretation of the consequent expression in the light of eq.(6.77).

The result of eq.(6.76) is a combination of the results of eqs.(6.72) and (6.75). Two similar results of the type of eq.(6.76) in the variables Y and T and Z and T also exist.

Theorem 6.4.5: A case of reducibility:

(i)
$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Y,-T) = F_G(a,a,a,b_4,b_1,b_2+b_3;d,c,c;-T,-X,-Y) \tag{6.78}$$

(ii)
$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-Y,-Y,-Y,-T) = F_2(a,b_4,b_1+b_2+b_3;d,c;-T,-Y) \tag{6.79}$$

Proof: (i) In eq.(6.64) we put $Z = Y$ and observe that,

$$\left|I+U^{\frac{1}{2}}XU^{\frac{1}{2}}+V^{\frac{1}{2}}YV^{\frac{1}{2}}+W^{\frac{1}{2}}YW^{\frac{1}{2}}+S^{\frac{1}{2}}TS^{\frac{1}{2}}\right|$$

$$=\left|I+U^{\frac{1}{2}}XU^{\frac{1}{2}}+(V+W)^{\frac{1}{2}}Y(V+W)^{\frac{1}{2}}+S^{\frac{1}{2}}TS^{\frac{1}{2}}\right|$$

then applying the transformations, $V_1 = V, W_1 = V + W;$ and integrating out V_1 by using a type-1 beta integral and comparing the resulting expression with eq.(5.17) produces this result.

(ii) This result follows by putting $X = Y$ in eq.(6.78) followed by the use of eq.(5.46).

CHAPTER VII

THE EXTON'S ${}^{(k)}_{(1)}E^{(n)}_{D}$ & ${}^{(k)}_{(2)}E^{(n)}_{D}$ AND THE CHANDEL'S ${}^{(k)}_{(1)}E^{(n)}_{C}$ FUNCTIONS OF MATRIX ARGUMENTS

7.1 Exton [17,18] has given two functions ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(2)}E^{(n)}_{D}$ which, according to him are the generalizations of certain of the quadruple hypergeometric functions discussed by him in [18]. Chandel [5] has also given a similar function ${}^{(k)}_{(1)}E^{(n)}_{C}$. The purpose of this chapter is to define and study these three functions for the case of matrix arguments.

7.2 Definitions

7.2.1 The Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$ function of matrix arguments

$$ {}^{(k)}_{(1)}E^{(n)}_{D} = {}^{(k)}_{(1)}E^{(n)}_{D}(a, b_1, \cdots, b_n; c, c'; -X_1, \cdots, -X_n) $$

is defined as that class of functions which has the following matrix transform:

$$ M\left[{}^{(k)}_{(1)}E^{(n)}_{D} \right] = \int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} $$

$$ \times {}^{(k)}_{(1)}E^{(n)}_{D}(a, b_1, \cdots, b_n; c, c'; -X_1, \cdots, -X_n) dX_1 \cdots dX_n $$

$$ = \frac{\Gamma_p(a-\rho_1-\cdots-\rho_n)\Gamma_p(b_1-\rho_1)\cdots\Gamma_p(b_n-\rho_n)\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(a)\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-\rho_1-\cdots-\rho_k)} \times \qquad (7.1) $$

$$ \frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)}{\Gamma_p(c'-\rho_{k+1}-\cdots-\rho_n)} $$

for $\mathrm{Re}(a-\rho_1-\cdots-\rho_n, c-\rho_1-\cdots-\rho_k, c'-\rho_{k+1}-\cdots-\rho_n,$
$b_i-\rho_i, \rho_i) > (p-1)/2,$ where, $i=1,\cdots,n.$

7.2.2 $\quad {}_{(2)}^{(k)}E_D^{(n)} = {}_{(2)}^{(k)}E_D^{(n)}(a, a', b_1, \cdots, b_n; c; -X_1, \cdots, -X_n)$

$$M\left[{}_{(2)}^{(k)}E_D^{(n)}\right] = \int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}$$

$$\times{}_{(2)}^{(k)}E_D^{(n)}(a, a', b_1, \cdots, b_n; c; -X_1, \cdots, -X_n)dX_1\cdots dX_n$$

$$=\frac{\Gamma_p(a-\rho_1-\cdots-\rho_k)\Gamma_p(a'-\rho_{k+1}-\cdots-\rho_n)\Gamma_p(b_1-\rho_1)\cdots}{\Gamma_p(a)\Gamma_p(a')\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-\rho_1-\cdots-\rho_n)}\times \tag{7.2}$$

$$\Gamma_p(b_n-\rho_n)\Gamma_p(c)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)$$

for $\mathrm{Re}(a-\rho_1-\cdots-\rho_k, c-\rho_1-\cdots-\rho_n, a'-\rho_{k+1}-\cdots-\rho_n,$
$b_i-\rho_i, \rho_i) > (p-1)/2,$ where, $i=1,\cdots,n.$

7.2.3 The Chandel's ${}_{(1)}^{(k)}E_C^{(n)}$ function of matrix arguments,

$${}_{(1)}^{(k)}E_C^{(n)} = {}_{(1)}^{(k)}E_C^{(n)}(a, a', b; c_1, \cdots, c_n; -X_1, \cdots, -X_n)$$

is defined as that class of functions for which the M-transform is as depicted below:

$$M\left[{}_{(1)}^{(k)}E_C^{(n)}\right] = \int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}$$

$$\times{}_{(1)}^{(k)}E_C^{(n)}(a, a', b; c_1, \cdots, c_n; -X_1, \cdots, -X_n)dX_1\cdots dX_n$$

$$=\frac{\Gamma_p(a-\rho_1-\cdots-\rho_k)\Gamma_p(a'-\rho_{k+1}-\cdots-\rho_n)\Gamma_p(b-\rho_1-\cdots-\rho_n)}{\Gamma_p(a)\Gamma_p(a')\Gamma_p(b)\Gamma_p(c_1-\rho_1)\cdots\Gamma_p(c_n-\rho_n)}\times$$

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$$\Gamma_p(c_1)\cdots\Gamma_p(c_n)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)$$

$$\text{for } \mathrm{Re}(a-\rho_1-\cdots-\rho_k, b-\rho_1-\cdots-\rho_n, a'-\rho_{k+1}-\cdots-\rho_n, \quad (7.3)$$
$$c_i-\rho_i, \rho_i) > (p-1)/2, \text{ where, } i=1,\cdots,n.$$

7.3 Nine results are being established in this section, three for the function ${}^{(k)}_{(1)}E^{(n)}_{D}$, four for the function ${}^{(k)}_{(2)}E^{(n)}_{D}$ and two for the function ${}^{(k)}_{(1)}E^{(n)}_{C}$.

Theorem 7.3.1:

$${}^{(k)}_{(1)}E^{(n)}_{D}(a, b_1,\cdots,b_n; c, c'; -X_1,\cdots,-X_n)$$

$$= \frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-b_1-\cdots-b_k)\Gamma_p(c'-b_{k+1}-\cdots-b_n)} \times$$

$$\int\cdots(n)\cdots\int |U_1|^{b_1-(p-1)/2}\cdots|U_n|^{b_n-(p-1)/2}\times$$

$$|I-U_1-\cdots-U_k|^{c-b_1-\cdots-b_k-(p+1)/2}\times$$

$$|I-U_{k+1}-\cdots-U_n|^{c'-b_{k+1}-\cdots-b_n-(p+1)/2}\left|I+U_1^{1/2}X_1U_1^{1/2}+\right.$$

$$\left.\cdots+U_n^{1/2}X_nU_n^{1/2}\right|^{-a} dU_1\cdots dU_k dU_{k+1}\cdots dU_n$$

$$\text{for } U_i>0, 0<U_1+\cdots+U_k<I, 0<U_{k+1}+\cdots+U_n<I, \text{ and} \qquad (7.4)$$

$$\text{for } \mathrm{Re}(b_i, c-b_1-\cdots-b_k, c'-b_{k+1}-\cdots-b_n) > (p-1)/2; i=1,\cdots,n.$$

Proof: Taking the M-transform of the right side of eq.(7.4) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we have,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$
$$\left|I+U_1^{1/2}X_1U_1^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2}\right|^{-a}dX_1\cdots dX_n \tag{7.5}$$

Applying the transformations, $Y_i=U_i^{1/2}X_iU_i^{1/2}$, for $i=1,\cdots,n;$ to the above expression and integrating out $Y_i\,(i=1,\cdots,n)$, in the resulting expression by using a type-2 Dirichlet integral, we obtain

$$|U_1|^{-\rho_1}\cdots|U_n|^{-\rho_n}\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\Gamma_p(a-\rho_1-\cdots-\rho_n)}{\Gamma_p(a)} \tag{7.6}$$

which, on substitution on the right side of eq.(7.4) and integrating out the variables $U_1,\cdots,U_k$ and $U_{k+1},\cdots,U_n$ by using a type-1 Dirchlet integral yields $M\left[{}_{(1)}^{(k)}E_D^{(n)}\right]$ as given by eq.(7.1).

Theorem 7.3.2:

$$\begin{aligned}&{}_{(1)}^{(k)}E_D^{(n)}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)\\&=\frac{1}{\Gamma_p(a)}\int_{U>0}e^{-tr(U)}|U|^{a-(p+1)/2}\,\Phi_2^{(k)}(b_1,\cdots,b_k;c;\\&-U^{1/2}X_1U^{1/2},\cdots,-U^{1/2}X_kU^{1/2})\Phi_2^{(n-k)}(b_{k+1},\cdots,b_n;c';\\&-U^{1/2}X_{k+1}U^{1/2},\cdots,-U^{1/2}X_nU^{1/2})dU\end{aligned} \tag{7.7}$$

for $\mathrm{Re}(a)>(p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(7.7) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$
$$\Phi_2^{(k)}(b_1,\cdots,b_k;c;-U^{1/2}X_1U^{1/2},\cdots,-U^{1/2}X_kU^{1/2})\times \tag{7.8}$$
$$\Phi_2^{(n-k)}(b_{k+1},\cdots,b_n;c';-U^{1/2}X_{k+1}U^{1/2},\cdots,-U^{1/2}X_nU^{1/2})\times$$
$$dX_1\cdots dX_k dX_{k+1}\cdots dX_n$$

On making use of the transformations, $Y_i=U^{1/2}X_iU^{1/2}$, for $i=1,\cdots,n$; and eq.(4.6) in the last expression leads to

$$|U|^{-\rho_1-\cdots-\rho_n}\frac{\Gamma_p(b_1-\rho_1)\cdots\Gamma_p(b_n-\rho_n)\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-\rho_1-\cdots-\rho_k)}\times \tag{7.9}$$
$$\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)}{\Gamma_p(c'-\rho_{k+1}-\cdots-\rho_n)}$$

Putting back this expression on the right side of eq.(7.7) and integrating out U by using a Gamma integral generates $M\left[{}^{(k)}_{(1)}E^{(n)}_{D}\right]$ as given by eq.(7.1).

Theorem 7.3.3:

$${}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)}\int_{U_1>0}\cdots\int_{U_n>0}e^{-\mathrm{tr}(U_1+\cdots+U_n)}\times$$
$$|U_1|^{b_1-(p-1)/2}\cdots|U_n|^{b_n-(p-1)/2}\,\Psi_2(a;c,c';-U_1^{1/2}X_1U_1^{1/2}- \tag{7.10}$$
$$\cdots-U_k^{1/2}X_kU_k^{1/2},-U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}-\cdots-U_n^{1/2}X_nU_n^{1/2})\times$$
$$dU_1\cdots dU_n$$

for $\mathrm{Re}(b_1,\cdots,b_n)>(p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(7.10) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$
$$\Psi_2(a;c,c';-U_1^{1/2}X_1U_1^{1/2}-\cdots-U_k^{1/2}X_kU_k^{1/2},-U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2} \quad (7.11)$$
$$-\cdots-U_n^{1/2}X_nU_n^{1/2})dX_1\cdots dX_n$$

Applying the following transformations in succession

(i) $$Y_j=U_j^{1/2}X_jU_j^{1/2},\ \text{for } j=1,\cdots,n;$$

(ii) $$Z_1=Y_1, Z_2=Y_1+Y_2,\cdots,Z_k=Y_1+\cdots+Y_k;\ \text{and,}$$
$$Z_{k+1}=Y_{k+1}, Z_{k+2}=Y_{k+1}+Y_{k+2},\cdots,Z_n=Y_{k+1}+\cdots+Y_n;$$

to the above expression, then integrating out the variables $Z_1,\cdots,Z_{k-1}$ and $Z_{k+1},\cdots,Z_{n-1}$ one-by-one and in order by using a type-1 Beta integral and on writing the M-transform of a Ψ_2 function, we achieve

$$|U_1|^{-\rho_1}\cdots|U_n|^{-\rho_n}\frac{\Gamma_p(a-\rho_1-\cdots-\rho_n)\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(a)\Gamma_p(c-\rho_1-\cdots-\rho_k)}\times$$
$$\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)}{\Gamma_p(c'-\rho_{k+1}-\cdots-\rho_n)} \quad (7.12)$$

Putting back this expression on the right side of eq.(7.10) and integrating out $U_1,\cdots,U_n$ by employing a Gamma integral produces $M\left[{}_{(1)}^{(k)}E_D^{(n)}\right]$ as given by eq.(7.1).

Theorem 7.3.4:

$$ {}_{(2)}^{(k)}E_D^{(n)}(a, a', b_1, \cdots, b_n; c; -X_1, \cdots, -X_n) $$

$$ = \frac{\Gamma_p(c)}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c - b_1 - \cdots - b_n)} \int\cdots(n)\cdots\int \times $$

$$ \left|U_1\right|^{b_1-(p+1)/2} \cdots \left|U_n\right|^{b_n-(p+1)/2} \times $$

$$ \left|I - U_1 - \cdots - U_n\right|^{c-b_1-\cdots-b_n-(p+1)/2} \times $$

$$ \left|I + U_1^{1/2} X_1 U_1^{1/2} + \cdots + U_k^{1/2} X_k U_k^{1/2}\right|^{-a} \left|I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \right. $$

$$ \left. \cdots + U_n^{1/2} X_n U_n^{1/2}\right|^{-a'} dU_1 \cdots dU_n \tag{7.13} $$

where, $U_i' = U_i > 0$ and $0 < U_1 + \cdots + U_n < I$ and for $\mathrm{Re}(b_i,$
$c - b_1 - \cdots - b_n) > (p-1)/2; i = 1, \cdots, n.$

Proof: Taking the M-transform of the right side of eq.(7.13) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we have,

$$ \int_{X_1>0} \cdots \int_{X_n>0} \left|X_1\right|^{\rho_1-(p+1)/2} \cdots \left|X_n\right|^{\rho_n-(p+1)/2} \times $$

$$ \left|I + U_1^{1/2} X_1 U_1^{1/2} + \cdots + U_k^{1/2} X_k U_k^{1/2}\right|^{-a} \times \tag{7.14} $$

$$ \left|I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots + U_n^{1/2} X_n U_n^{1/2}\right|^{-a'} dX_1 \cdots dX_n $$

Making use of the transformations, $Y_j = U_j^{1/2} X_j U_j^{1/2}$, for $j = 1, \cdots, n$; in the last expression, subsequently, integrating out $Y_1, \cdots, Y_k$ and $Y_{k+1}, \cdots, Y_n$ by the help of a type-2 Dirichlet integral we are led to,

$$|U_1|^{-\rho_1} \cdots |U_n|^{-\rho_n} \frac{\Gamma_p(a - \rho_1 - \cdots - \rho_k)\Gamma_p(a' - \rho_{k+1} - \cdots - \rho_n)}{\Gamma_p(a)\Gamma_p(a')} \tag{7.15}$$

$$\times \Gamma_p(\rho_1) \cdots \Gamma_p(\rho_n)$$

Substituting this expression on the right side of eq.(7.13) and integrating out $U_1, \cdots, U_n$ by using a type-1 Dirichlet integral we have $M\left[{}^{(k)}_{(2)}E^{(n)}_{D} \right]$ as given by eq.(7.2).

Theorem 7.3.5:

$${}^{(k)}_{(2)}E^{(n)}_{D}(a, a', b_1, \cdots, b_n; c; -X_1, \cdots, -X_n)$$

$$= \frac{1}{\Gamma_p(a)\Gamma_p(a')} \int_{U>0} \int_{V>0} e^{-\mathrm{tr}(U+V)} |U|^{a-(p+1)/2} \times$$

$$|V|^{a'-(p+1)/2} \Phi_2^{(n)}(b_1, \cdots, b_n; c; -U^{1/2} X_1 U^{1/2}, \cdots, -U^{1/2} X_k U^{1/2}, \tag{7.16}$$

$$-V^{1/2} X_{k+1} V^{1/2}, \cdots, -V^{1/2} X_n V^{1/2}) dU dV$$

for $\mathrm{Re}(a, a') > (p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(7.16) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we get,

$$\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1 - (p+1)/2} \cdots |X_n|^{\rho_n - (p+1)/2} \times$$

Continued to the next page … … … … …

$$\Phi_2^{(n)}(b_1,\cdots,b_n;c;-U^{1/2}X_1U^{1/2},\cdots,-U^{1/2}X_kU^{1/2},$$
$$-V^{1/2}X_{k+1}V^{1/2},\cdots,-V^{1/2}X_nV^{1/2})dX_1\cdots dX_n \tag{7.17}$$

Applying the transformations,

$$Y_i = U^{1/2}X_iU^{1/2}, Y_j = V^{1/2}X_jV^{1/2};\ \text{for } i = 1,\cdots,k;\ j = k+1,\cdots,n;$$

to the last expression and then using eq.(4.6) yields,

$$|U|^{-\rho_1-\cdots-\rho_k}\,|V|^{-\rho_{k+1}-\cdots-\rho_n}\,\frac{\Gamma_p(b_1-\rho_1)\cdots\Gamma_p(b_n-\rho_n)}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)}\times$$
$$\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\cdots-\rho_n)}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n) \tag{7.18}$$

Substituting this expression on the right side of eq.(7.16) and integrating out U and V by using a Gamma integral generates $M\left[{}_{(2)}^{(k)}E_D^{(n)}\right]$ as given by eq.(7.2).

Theorem 7.3.6:

$${}_{(2)}^{(k)}E_D^{(n)}(a,a',b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)$$
$$=\frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(a')\Gamma_p(c-a-a')}\iint|U|^{a-(p+1)/2}\,|V|^{a'-(p+1)/2}\times$$
$$|I-U-V|^{c-a-a'-(p+1)/2}\left|I+U^{1/2}X_1U^{1/2}\right|^{-b_1}\cdots$$
$$\left|I+U^{1/2}X_kU^{1/2}\right|^{-b_k}\left|I+V^{1/2}X_{k+1}V^{1/2}\right|^{-b_{k+1}}\cdots$$

Continued to the next page … … … … …

$$\left|I + V^{1/2} X_n V^{1/2}\right|^{-b_n} dUdV$$

where, $U = U' > 0, V = V' > 0,$ and $0 < U + V < I$ and (7.19)

for $\mathrm{Re}(a, a', c - a - a') > (p-1)/2.$

Proof: We take the M-transform of the right side of eq.(7.19) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively in order to get

$$\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1 - (p+1)/2} \cdots |X_n|^{\rho_n - (p+1)/2} \times$$

$$\left|I + U^{1/2} X_1 U^{1/2}\right|^{-b_1} \cdots \left|I + U^{1/2} X_k U^{1/2}\right|^{-b_k} \times \tag{7.20}$$

$$\left|I + V^{1/2} X_{k+1} V^{1/2}\right|^{-b_{k+1}} \cdots \left|I + V^{1/2} X_n V^{1/2}\right|^{-b_n} dX_1 \cdots dX_n$$

On using the transformations

$$Y_i = U^{1/2} X_i U^{1/2}, Y_j = V^{1/2} X_j V^{1/2};\ \text{for } i = 1, \cdots k,\ j = k+1, \cdots, n;$$

in the above expression and integrating out $Y_1, \cdots, Y_n$ by using a type-2 Beta integral gives,

$$|U|^{-\rho_1 - \cdots - \rho_k} |V|^{-\rho_{k+1} - \cdots - \rho_n} \frac{\Gamma_p(b_1 - \rho_1) \cdots \Gamma_p(b_n - \rho_n)}{\Gamma_p(b_1) \cdots \Gamma_p(b_n)} \times \tag{7.21}$$

$$\Gamma_p(\rho_1) \cdots \Gamma_p(\rho_n)$$

Replacing this expression on the right side of eq.(7.19) and integrating out U and V by utilizing a type-1 Dirichlet integral produces $M\left[{}^{(k)}_{(2)}E_D^{(n)}\right]$ as given by eq.(7.2).

Theorem 7.3.7:

$$
{}_{(2)}^{(k)}E_D^{(n)}(a,a',b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)
$$

$$
=\frac{1}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)}\int_{U_1>0}\cdots\int_{U_n>0}e^{-tr(U_1+\cdots+U_n)}\times
$$

$$
|U_1|^{b_1-(p-1)/2}\cdots|U_n|^{b_n-(p-1)/2}\,\Phi_2(a,a';c;-U_1^{1/2}X_1U_1^{1/2} \quad (7.22)
$$

$$
-\cdots-U_k^{1/2}X_kU_k^{1/2},-U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}-\cdots-U_n^{1/2}X_nU_n^{1/2})\times
$$

$$
dU_1\cdots dU_n
$$

for $\text{Re}(b_1,\cdots,b_n)>(p-1)/2.$

Proof: This theorem can be proved in a similar manner as the theorem 7.3.3.

Theorem 7.3.8:

$$
{}_{(1)}^{(k)}E_C^{(n)}(a,a',b;c_1,\cdots,c_n;-X_1,\cdots,-X_n)
$$

$$
=\frac{1}{\Gamma_p(a)\Gamma_p(a')}\int_{U>0}\int_{V>0}e^{-tr(U+V)}|U|^{a-(p+1)/2}\times
$$

$$
|V|^{a'-(p+1)/2}\,\Psi_2^{(n)}(b;c_1,\cdots,c_n;-U^{1/2}X_1U^{1/2},\cdots,-U^{1/2}X_kU^{1/2}, \quad (7.23)
$$

$$
-V^{1/2}X_{k+1}V^{1/2},\cdots,-V^{1/2}X_nV^{1/2})dUdV
$$

for $\text{Re}(a,a')>(p-1)/2.$

Proof: The theorem can be proved like the theorem 7.3.5 with the aid of definition 6.5 page 79 of Mathai [62].

Theorem 7.3.9:

$$
{}_{(1)}^{(k)}E_C^{(n)}(a,a',b;c_1,\cdots,c_n;-X_1,\cdots,-X_n)
$$

Continued to the next page … … … … …

$$
\begin{aligned}
&= \frac{1}{\Gamma_p(a)\Gamma_p(a')\Gamma_p(b)} \int_{U>0}\int_{V>0}\int_{W>0} e^{-tr(U+V+W)} \times \\
&|U|^{a-(p+1)/2} |V|^{a'-(p+1)/2} |W|^{b-(p+1)/2} \times \\
&{}_0F_1(\,;c_1;-W^{1/2}U^{1/2}X_1U^{1/2}W^{1/2}) \cdots {}_0F_1(\,;c_k; \\
&-W^{1/2}U^{1/2}X_kU^{1/2}W^{1/2})\, {}_0F_1(\,;c_{k+1};-W^{1/2}V^{1/2}X_{k+1}V^{1/2}W^{1/2}) \\
&\times \cdots {}_0F_1(\,;c_n;-W^{1/2}V^{1/2}X_nV^{1/2}W^{1/2})dUdVdW
\end{aligned} \tag{7.24}
$$

for $\mathrm{Re}(a,a',b) > (p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(7.24) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we obtain

$$
\begin{aligned}
&\int_{X_1>0}\cdots\int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times \\
&{}_0F_1(\,;c_1;-W^{1/2}U^{1/2}X_1U^{1/2}W^{1/2}) \cdots {}_0F_1(\,;c_k; \\
&-W^{1/2}U^{1/2}X_kU^{1/2}W^{1/2})\, {}_0F_1(\,;c_{k+1};-W^{1/2}V^{1/2}X_{k+1}V^{1/2}W^{1/2}) \\
&\times \cdots {}_0F_1(\,;c_n;-W^{1/2}V^{1/2}X_nV^{1/2}W^{1/2})dX_1\cdots dX_n
\end{aligned} \tag{7.25}
$$

Applying the transformations

$$Y_i = W^{1/2}U^{1/2}X_iU^{1/2}W^{1/2},\ Y_j = W^{1/2}V^{1/2}X_jV^{1/2}W^{1/2};$$

for $i = 1,\cdots k;\ j = k+1,\cdots,n;$

to the last expression and writing the M-transforms of the ${}_0F_1$ functions we are led to

$$|U|^{-\rho_1-\cdots-\rho_k}\,|V|^{-\rho_{k+1}-\cdots-\rho_n}\,|W|^{-\rho_1-\cdots-\rho_n}\times$$
$$\frac{\Gamma_p(c_1)\cdots\Gamma_p(c_n)}{\Gamma_p(c_1-\rho_1)\cdots\Gamma_p(c_n-\rho_n)}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n) \tag{7.26}$$

On substituting this expression on the right side of eq.(7.24) and integrating out U, V and W by using a Gamma integral produces $M\left[{}^{(k)}_{(1)}E^{(n)}_{C}\right]$ as given by eq.(7.3).

7.4 I discuss some transformation relations and cases of reducibility in this section. The assumptions of symmetry of a function of matrix arguments in its arguments, as assumed by Herz [22] and Mathai [60,62], are also assumed to be true here.

Theorem 7.4.1:

$$\lim_{\alpha\to\infty} {}^{(k)}_{(1)}E^{(n)}_{D}(\alpha, b_1,\cdots,b_n; c, c'; \frac{-X_1}{\alpha},\cdots,\frac{-X_n}{\alpha})$$
$$=\frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-b_1-\cdots-b_k)\Gamma_p(c'-b_{k+1}-\cdots-b_n)}\times$$
$$\int\cdots(n)\cdots\int|U_1|^{b_1-(p-1)/2}\cdots|U_n|^{b_n-(p-1)/2}\times$$
$$|I-U_1-\cdots-U_k|^{c-b_1-\cdots-b_k-(p+1)/2}\times$$
$$|I-U_{k+1}-\cdots-U_n|^{c'-b_{k+1}-\cdots-b_n-(p+1)/2}\times$$
$$e^{-tr(U_1X_1+\cdots+U_nX_n)}dU_1\cdots dU_n$$
$$\text{for } U_i>0, 0<U_1+\cdots+U_k<I, 0<U_{k+1}+\cdots+U_n<I, \text{ and} \tag{7.27}$$
$$\text{for } \mathrm{Re}(b_i, c-b_1-\cdots-b_k, c'-b_{k+1}-\cdots-b_n)>(p-1)/2; i=1,\cdots,n.$$

Proof: This result is a limiting case of the theorem 7.3.1 in which use of eq.(1.23) has been made.

Theorem 7.4.2: A case of reducibility:

$$\lim_{\alpha\to\infty} {}^{(k)}_{(1)}E^{(n)}_{D}[\alpha, b_1, \cdots, b_n; c, c'; \frac{-X}{\alpha}, \cdots(k)\cdots\frac{-X}{\alpha}, \frac{-Y}{\alpha}, \cdots(n-k)\cdots, \frac{-Y}{\alpha}] \tag{7.28}$$

$$= {}_1F_1(b_1+\cdots+b_k; c; -X)\times {}_1F_1(b_{k+1}+\cdots+b_n; c'; -Y)$$

Proof: We put $X_1 = \cdots = X_k = X$ and $X_{k+1} = \cdots = X_n = Y$ in eq.(7.27) and apply the following sets of transformations on its right side,

$V_1 = U_1, V_2 = U_1 + U_2, \cdots, V_k = U_1 + \cdots + U_k$; and,

$W_1 = U_{k+1}, W_2 = U_{k+1} + U_{k+2}, \cdots, W_{n-k} = U_{k+1} + \cdots + U_n$;

and integrate out the variables $V_1, \cdots, V_{k-1}$ and $W_1, \cdots, W_{n-k-1}$ one-by-one and in order, in the consequent expression and use eq.(2.3) to see this result.

Theorem 7.4.3:

$$\lim_{\alpha\to\infty} {}^{(k)}_{(2)}E^{(n)}_{D}(\alpha, \alpha, b_1, \cdots, b_n; c; \frac{-X_1}{\alpha}, \cdots, \frac{-X_n}{\alpha})$$

$$= \frac{\Gamma_p(c)}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-b_1-\cdots-b_n)}\int\cdots(n)\cdots\int\times$$

$$|U_1|^{b_1-(p+1)/2}\cdots|U_n|^{b_n-(p+1)/2}\times$$

(i) $$|I-U_1-\cdots-U_n|^{c-b_1-\cdots-b_n-(p+1)/2}\times \tag{7.29}$$

$$e^{-\mathrm{tr}(U_1X_1+\cdots+U_nX_n)}dU_1\cdots dU_n$$

where, $U_i' = U_i > 0$ and $0 < U_1 + \cdots + U_n < I$ and for $\mathrm{Re}(b_i, c-b_1-\cdots-b_n) > (p-1)/2; i = 1, \cdots, n$.

(ii)
$$\lim_{\alpha\to\infty} {}^{(k)}_{(2)}E_D^{(n)}(\alpha,\alpha,b_1,\cdots,b_n;c;\frac{-X}{\alpha},\cdots(n)\cdots,\frac{-X}{\alpha}) = {}_1F_1(b_1+\cdots+b_n;c;-X) \tag{7.30}$$

Proof: (i) This result is a limiting case of the theorem 7.3.4 in which use has been made of eq.(1.23).

(ii) By putting $X_1=\cdots=X_n=X$ in eq.(7.29) and applying the transformations, $V_1=U_1, V_2=U_1+U_2,\cdots,V_n=U_1+\cdots+U_n$; to it and integrating out $V_1,\cdots,V_{n-1}$ one-by-one and in order, in the resulting expression by employing a type-1 Beta integral and using eq.(2.3) this result can be had.

Theorem 7.4.4:

(i)
$$\lim_{\alpha\to\infty} {}^{(k)}_{(2)}E_D^{(n)}(a,a',\alpha,\cdots,\alpha;c;\frac{-X_1}{\alpha},\cdots,\frac{-X_n}{\alpha}) = \Phi_2(a,a';c;-X_1-\cdots-X_k,-X_{k+1}-\cdots-X_n) \tag{7.31}$$

(ii)
$$\lim_{\alpha\to\infty} {}^{(k)}_{(2)}E_D^{(n)}(a,a',\alpha,\cdots,\alpha;c;\frac{-X_1}{\alpha},\cdots,\frac{-X_n}{\alpha}) = \lim_{\beta\to\infty} F_1[\beta,a,a';c;\frac{-(X_1+\cdots+X_k)}{\beta},\frac{-(X_{k+1}+\cdots+X_n)}{\beta}] \tag{7.32}$$

Proof: This theorem is a limiting case of the theorem 7.3.6 and the result in eq.(7.31) then follows by the use of eq.(3.17) while, that in eq.(7.32) follows by the use of the theorem 4.8 page 65 of Mathai [62].

Theorem 7.4.5: A transformation theorem:

$$\begin{aligned}&{}^{(k)}_{(1)}E_D^{(n)}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)\\ &=\left|I+X_1\right|^{-a}\,{}^{(k)}_{(1)}E_D^{(n)}[a,c-b_1-\cdots-b_k,b_2,\cdots,b_n;c,c';\\ &(I+X_1)^{-1/2}X_1(I+X_1)^{-1/2},-(I+X_1)^{-1/2}(X_2-X_1)(I+X_1)^{-1/2},\end{aligned}$$

Continued to the next page… … … … …

$$\cdots,-(I+X_1)^{-1/2}(X_k-X_1)(I+X_1)^{-1/2},-(I+X_1)^{-1/2}X_{k+1}\times$$
$$(I+X_1)^{-1/2},\cdots,-(I+X_1)^{-1/2}X_n(I+X_1)^{-1/2}] \tag{7.33}$$

where, $X_i-X_1>0$ for $i=2,\cdots,k$.

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$ {}^{(k)}_{(1)}E_D^{(n)}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)$$
$$=\left|I+X_k\right|^{-a}\ {}^{(k)}_{(1)}E_D^{(n)}[a,b_1,\cdots,b_{k-1},c-b_1-\cdots-b_k,b_{k+1},\cdots,$$
$$b_n;c,c';-(I+X_k)^{-1/2}(X_1-X_k)(I+X_k)^{-1/2},\cdots,-(I+X_k)^{-1/2}\times \tag{7.34}$$
$$(X_{k-1}-X_k)(I+X_k)^{-1/2},(I+X_k)^{-1/2}X_k(I+X_k)^{-1/2},$$
$$-(I+X_k)^{-1/2}X_{k+1}(I+X_k)^{-1/2},\cdots,-(I+X_k)^{-1/2}X_n(I+X_k)^{-1/2}]$$

where, $X_i-X_k>0$ for $i=1,\cdots,k-1$.

$$ {}^{(k)}_{(1)}E_D^{(n)}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)$$
$$=\left|I+X_{k+1}\right|^{-a}\ {}^{(k)}_{(1)}E_D^{(n)}[a,b_1,\cdots,b_k,c'-b_{k+1}-\cdots-b_n,b_{k+2},\cdots,$$
$$b_n;c,c';-(I+X_{k+1})^{-1/2}X_1(I+X_{k+1})^{-1/2},\cdots,-(I+X_{k+1})^{-1/2}\times$$
$$X_k(I+X_{k+1})^{-1/2},(I+X_{k+1})^{-1/2}X_{k+1}(I+X_{k+1})^{-1/2}, \tag{7.35}$$
$$-(I+X_{k+1})^{-1/2}(X_{k+2}-X_{k+1})(I+X_{k+1})^{-1/2},\cdots,$$
$$-(I+X_{k+1})^{-1/2}(X_n-X_{k+1})(I+X_{k+1})^{-1/2}]$$

where, $X_j-X_{k+1}>0$ for $j=k+2,\cdots,n$.

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$
\begin{aligned}
&{}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)\\
&=\left|I+X_n\right|^{-a}\,{}^{(k)}_{(1)}E^{(n)}_{D}[a,b_1,\cdots,b_{n-1},c'-b_{k+1}-\cdots-b_n;c,c';\\
&-(I+X_n)^{-1/2}X_1(I+X_n)^{-1/2},\cdots,-(I+X_n)^{-1/2}\times\\
&X_k(I+X_n)^{-1/2},-(I+X_n)^{-1/2}(X_{k+1}-X_n)(I+X_n)^{-1/2},\cdots,\\
&-(I+X_n)^{-1/2}(X_{n-1}-X_n)(I+X_n)^{-1/2},(I+X_n)^{-1/2}X_n\times\\
&(I+X_n)^{-1/2}]
\end{aligned}
\tag{7.36}
$$

where, $X_j-X_n>0$ for $j=k+1,\cdots,n-1$.

$$
\begin{aligned}
&{}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)\\
&=\left|I+X_i+X_{k+j}\right|^{-a}\,{}^{(k)}_{(1)}E^{(n)}_{D}[a,b_1,\cdots,b_{i-1},c-b_1-\cdots-b_k,b_{i+1},\\
&\cdots,b_k,b_{k+1},\cdots,b_{k+j-1},c'-b_{k+1}-\cdots-b_n,b_{k+j+1},\cdots,b_n;\\
&c,c';-(I+X_i+X_{k+j})^{-1/2}(X_1-X_i)(I+X_i+X_{k+j})^{-1/2},\cdots,\\
&-(I+X_i+X_{k+j})^{-1/2}(X_{i-1}-X_i)(I+X_i+X_{k+j})^{-1/2},\\
&(I+X_i+X_{k+j})^{-1/2}X_i(I+X_i+X_{k+j})^{-1/2},-(I+X_i+X_{k+j})^{-1/2}\\
&\times(X_{i+1}-X_i)(I+X_i+X_{k+j})^{-1/2},\cdots,-(I+X_i+X_{k+j})^{-1/2}\times\\
&(X_k-X_i)(I+X_i+X_{k+j})^{-1/2},-(I+X_i+X_{k+j})^{-1/2}\times
\end{aligned}
$$

Continued to the next page … … … … …

$$(X_{k+1}-X_{k+j})(I+X_i+X_{k+j})^{-1/2},\cdots,-(I+X_i+X_{k+j})^{-1/2}\times$$

$$(X_{k+j-1}-X_{k+j})(I+X_i+X_{k+j})^{-1/2},(I+X_i+X_{k+j})^{-1/2}\times$$

$$X_{k+j}(I+X_i+X_{k+j})^{-1/2},-(I+X_i+X_{k+j})^{-1/2}\times$$

$$(X_{k+j+1}-X_{k+j})(I+X_i+X_{k+j})^{-1/2},\cdots,$$

$$-(I+X_i+X_{k+j})^{-1/2}(X_n-X_{k+j})(I+X_i+X_{k+j})^{-1/2}]$$

where, $X_q-X_i>0,$ for $q=1,\cdots,i-1; X_m-X_i>0,$ for $m=i+1,$ $\cdots,k; X_r-X_{k+j}>0,$ for $r=k+1,\cdots,k+j-1; X_s-X_{k+j}>0,$ for $s=k+j+1,\cdots,n;$ and for $1\le i\le k$ and $1\le j\le n-k.$ (7.37)

Proof: To prove this theorem we first define the ${}^{(k)}_{(1)}E^{(n)}_D$ function through an integral representation:

$$ {}^{(k)}_{(1)}E^{(n)}_D(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)$$

$$=\frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-b_1-\cdots-b_k)\Gamma_p(c'-b_{k+1}-\cdots-b_n)}\times$$

$$\int\cdots(n)\cdots\int\left|U_1\right|^{b_1-(p-1)/2}\cdots\left|U_n\right|^{b_n-(p-1)/2}\times$$

$$\left|I-U_1-\cdots-U_k\right|^{c-b_1-\cdots-b_k-(p+1)/2}\times$$

$$\left|I-U_{k+1}-\cdots-U_n\right|^{c'-b_{k+1}-\cdots-b_n-(p+1)/2}\times$$

Continued to the next page … … … … …

$$\left|I + X_1^{1/2} U_1 X_1^{1/2} + \cdots + X_n^{1/2} U_n X_n^{1/2}\right|^{-a} dU_1 \cdots dU_k dU_{k+1} \cdots dU_n$$

$$\text{for } U_i > 0, 0 < U_1 + \cdots + U_k < I, 0 < U_{k+1} + \cdots + U_n < I, \text{ and} \tag{7.38}$$

$$\text{for } \mathrm{Re}(b_i, c - b_1 - \cdots - b_k, c' - b_{k+1} - \cdots - b_n) > (p-1)/2; i = 1, \cdots, n.$$

To obtain the result in eq.(7.33) we apply the transformations

$$U_1 = I - V_1 - \cdots - V_k, U_2 = V_2, \cdots, U_n = V_n;$$

to eq.(7.38) and by observing that,

$$\left|I + X_1^{1/2}(I - V_1 - \cdots - V_k) X_1^{1/2} + X_2^{1/2} V_2 X_2^{1/2} + \cdots + X_n^{1/2} V_n X_n^{1/2}\right|$$

$$= \left|I + X_1\right| \Big| I - (I + X_1)^{-1/2} X_1^{1/2} V_1 X_1^{1/2} (I + X_1)^{-1/2} + (I + X_1)^{-1/2} \times$$

$$(X_2 - X_1)^{1/2} V_2 (X_2 - X_1)^{1/2} (I + X_1)^{-1/2} + \cdots + (I + X_1)^{-1/2} \times$$

$$(X_k - X_1)^{1/2} V_k (X_k - X_1)^{1/2} (I + X_1)^{-1/2} + (I + X_1)^{-1/2} X_{k+1}^{1/2} \times$$

$$V_{k+1} X_{k+1}^{1/2} (I + X_1)^{-1/2} + \cdots + (I + X_1)^{-1/2} X_n^{1/2} V_n X_n^{1/2} (I + X_1)^{-1/2} \Big|$$

where, $X_i - X_1 > 0$, for $i = 2, \cdots, k$.

and then suitably interpreting the resulting expression in the light of eq.(7.38).

Similarly the result in eq.(7.34) follows from eq.(7.38) by the use of the transformations

$$U_1 = V_1, U_2 = V_2, \cdots, U_k = I - V_1 - \cdots - V_k, U_{k+1} = V_{k+1}, \cdots, U_n = V_n.$$

To obtain the result in eq.(7.35) the transformations are,

$$U_1 = V_1, \cdots, U_k = V_k, U_{k+1} = I - V_{k+1} - \cdots - V_n,$$

$$U_{k+2} = V_{k+2}, \cdots, U_n = V_n;$$

while, those for the result in eq.(7.36) are

$$U_1 = V_1, \cdots, U_{n-1} = V_{n-1}, U_n = I - V_{k+1} - \cdots - V_n.$$

The result in eq.(7.37) is a combination of the above two categories of results. It is obtained from eq.(7.38) by the application of the transformations

$$U_1 = V_1, \cdots, U_{i-1} = V_{i-1}, U_i = I - V_1 - \cdots - V_k, U_{i+1} = V_{i+1}, \cdots,$$

$$U_k = V_k, U_{k+1} = V_{k+1}, \cdots, U_{k+j-1} = V_{k+j-1}, U_{k+j} = I - V_{k+1} - \cdots$$

$-V_n, U_{k+j+1} = V_{k+j+1}, \cdots, U_n = V_n;$ where $1 \le i \le k$ and $1 \le j \le n-k$; and by observing that

$$\Big| I + X_1^{1/2} V_1 X_1^{1/2} + \cdots + X_{i-1}^{1/2} V_{i-1} X_{i-1}^{1/2} + X_i^{1/2} (I - V_1 - \cdots - V_k) X_i^{1/2} +$$

$$+ X_{i+1}^{1/2} V_{i+1} X_{i+1}^{1/2} + \cdots + X_k^{1/2} V_k X_k^{1/2} + X_{k+1}^{1/2} V_{k+1} X_{k+1}^{1/2} + \cdots +$$

$$X_{k+j-1}^{1/2} V_{k+j-1} X_{k+j-1}^{1/2} + X_{k+j}^{1/2} (I - V_{k+1} - \cdots - V_n) X_{k+j}^{1/2} + X_{k+j+1}^{1/2} \times$$

$$V_{k+j+1} X_{k+j+1}^{1/2} + \cdots + X_n^{1/2} V_n X_n^{1/2} \Big|$$

$$= \left| I + X_i + X_{k+j} \right| \Big| I + (I + X_i + X_{k+j})^{-1/2} (X_1 - X_i)^{1/2} V_1 (X_1 - X_i)^{1/2} \times$$

$$(I + X_i + X_{k+j})^{-1/2} + \cdots + (I + X_i + X_{k+j})^{-1/2} (X_{i-1} - X_i)^{1/2} V_{i-1} \times$$

$$(X_{i-1} - X_i)^{1/2} (I + X_i + X_{k+j})^{-1/2} - (I + X_i + X_{k+j})^{-1/2} X_i^{1/2} V_i X_i^{1/2} \times$$

$$(I + X_i + X_{k+j})^{-1/2} + (I + X_i + X_{k+j})^{-1/2} (X_{i+1} - X_i)^{1/2} V_{i+1} \times$$

$$(X_{i+1} - X_i)^{1/2} (I + X_i + X_{k+j})^{-1/2} + \cdots + (I + X_i + X_{k+j})^{-1/2} (X_k - X_i)^{1/2} \times$$

Continued to the next page … … … … …

$$
\begin{aligned}
& V_k (X_k - X_i)^{1/2} (I + X_i + X_{k+j})^{-1/2} + (I + X_i + X_{k+j})^{-1/2} \times \\
& (X_{k+1} - X_{k+j})^{1/2} V_{k+1} (X_{k+1} - X_{k+j})^{1/2} (I + X_i + X_{k+j})^{-1/2} + \cdots \\
& + (I + X_i + X_{k+j})^{-1/2} (X_{k+j-1} - X_{k+j})^{1/2} V_{k+j-1} \times \\
& (X_{k+j-1} - X_{k+j})^{1/2} (I + X_i + X_{k+j})^{-1/2} - (I + X_i + X_{k+j})^{-1/2} X_{k+j}^{1/2} \times \\
& V_{k+j} X_{k+j}^{1/2} (I + X_i + X_{k+j})^{-1/2} + (I + X_i + X_{k+j})^{-1/2} \times \\
& (X_{k+j+1} - X_{k+j})^{1/2} V_{k+j+1} (X_{k+j-1} - X_{k+j})^{1/2} \times \\
& (I + X_i + X_{k+j})^{-1/2} + \cdots + (I + X_i + X_{k+j})^{-1/2} (X_n - X_{k+j})^{1/2} V_n \times \\
& (X_n - X_{k+j})^{1/2} (I + X_i + X_{k+j})^{-1/2} \Bigg|
\end{aligned}
$$

where, $X_q - X_i > 0$, for $q = 1, \cdots, i-1$; $X_m - X_i > 0$, for $m = i+1, \cdots, k$; $X_r - X_{k+j} > 0$, for $r = k+1, \cdots, k+j-1$; $X_s - X_{k+j} > 0$, for $s = k+j+1, \cdots, n$; and for $1 \le i \le k$ and $1 \le j \le n-k$.

Theorem 7.4.6: Special cases:

(i)
$$
{}^{(0)}_{(1)}E_D^{(n)}(a, b_1, \cdots, b_n; c; -X_1, \cdots, -X_n) = {}^{(0)}_{(2)}E_D^{(n)}(a, b_1, \cdots, b_n; c; -X_1, \cdots, -X_n) = F_D^{(n)}(a, b_1, \cdots, b_n; c; -X_1, \cdots, -X_n) \qquad (7.39)
$$

(ii)
$$
{}^{(1)}_{(1)}E_D^{(2)}(a, b_1, b_2; c, c'; -X_1, -X_2) = F_2(a, b_1, b_2; c, c'; -X_1, -X_2) \qquad (7.40)
$$

(iii)
$$ {}^{(1)}_{(1)}E^{(3)}_{D}(a,b_1,b_2,b_3;c,c';-X_1,-X_2,-X_3) = F_G(a,a,a,b_1,b_2,b_3;c,c',c';-X_1,-X_2,-X_3) \tag{7.41} $$

(iv)
$$ {}^{(3)}_{(1)}E^{(4)}_{D}(a,b_1,b_2,b_3,b_4;c,c';-X,-Y,-Z,-T) = K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,c';-X,-Y,-Z,-T) \tag{7.42} $$

(v)
$$ {}^{(0)}_{(1)}E^{(n)}_{C}(a,b;c_1,\cdots,c_n;-X_1,\cdots,-X_n) = F^{(n)}_{C}(a,b;c_1,\cdots,c_n;-X_1,\cdots,-X_n) \tag{7.43} $$

(vi)
$$ {}^{(1)}_{(1)}E^{(2)}_{C}(a,a',b;c_1,c_2;-X_1,-X_2) = F_2(b,a,a';c_1,c_2;-X_1,-X_2) \tag{7.44} $$

(vii)
$$ {}^{(1)}_{(1)}E^{(3)}_{C}(a,a',b;c_1,c_2,c_3;-X_1,-X_2,-X_3) = F_E(b,b,b,a,a',a';c_1,c_2,c_3;-X_1,-X_2,-X_3) \tag{7.45} $$

Proof: (i) This result follows by putting $k=0$ in eqs.(7.1) and (7.2) and then comparing the result with eq.(4.5).

(ii) This result is obtained by putting $k=1$ and $n=2$ in eq.(7.1) and then comparing the result with eq.(3.2).

(iii) To obtain this result we put $k=1$ and $n=3$ in eq.(7.1) and compare the result with eq.(5.3).

(iv) The result in eq.(7.42) can be had by putting $k=3$ and $n=4$ in eq.(7.1) and comparing the consequent expression with eq.(6.11).

(v) On putting $k=0$ in eq.(7.3) and then comparing the outcome with eq.(4.4), this result can be inferred.

(vi) By putting $k=1$ and $n=2$ in eq.(7.3) and comparing the outcome with eq.(3.2), we achieve this result.

(vii) Putting $k=1$ and $n=3$ in eq.(7.3) and comparing the consequent result with eq.(5.1) we obtain this result.

CHAPTER VIII

THE GENERALIZED HORN'S FUNCTIONS OF MATRIX ARGUMENTS

In this chapter I shall define the two Horn's functions ${}^{(k)}H_3^{(n)}$ and ${}^{(k)}H_4^{(n)}$ for the matrix arguments case and shall also establish some results for both of these functions.

8.1 Definitions

8.1.1 The Horn's function ${}^{(k)}H_3^{(n)}$ of matrix arguments

$${}^{(k)}H_3^{(n)} = {}^{(k)}H_3^{(n)}(a, b_{k+1}, \cdots, b_n; c; -X_1, \cdots, -X_n)$$

is defined as that class of functions which has the following matrix-transform (M-transform):

$$M[{}^{(k)}H_3^{(n)}] = \int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2}$$

$$\times {}^{(k)}H_3^{(n)}(a, b_{k+1}, \cdots, b_n; c; -X_1, \cdots, -X_n) dX_1 \cdots dX_n$$

$$= \frac{\Gamma_p(a - 2\rho_1 - \cdots - 2\rho_k - \rho_{k+1} - \cdots - \rho_n)}{\Gamma_p(a)\Gamma_p(b_{k+1}) \cdots \Gamma_p(b_n)\Gamma_p(c - \rho_1 - \cdots - \rho_n)} \times \tag{8.1}$$

$$\Gamma_p(b_{k+1} - \rho_{k+1}) \cdots \Gamma_p(b_n - \rho_n)\Gamma_p(c)\Gamma_p(\rho_1) \cdots \Gamma_p(\rho_n)$$

for $\text{Re}(a - 2\rho_1 - \cdots - 2\rho_k - \rho_{k+1} - \cdots - \rho_n, b_{k+1} - \rho_{k+1}, \cdots,$
$b_n - \rho_n, c - \rho_1 - \cdots - \rho_n, \rho_i) > (p-1)/2, \quad i = 1, \cdots, n.$

8.1.2 ${}^{(k)}H_4^{(n)} = {}^{(k)}H_4^{(n)}(a, b_{k+1}, \cdots, b_n; c_1, \cdots, c_n; -X_1, \cdots, -X_n)$

$$M[{}^{(k)}H_4^{(n)}] = \int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2}$$

$$\times {}^{(k)}H_4^{(n)}(a, b_{k+1}, \cdots, b_n; c_1, \cdots, c_n; -X_1, \cdots, -X_n) dX_1 \cdots dX_n$$

Continued to the next page … … … … …

$$= \frac{\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n)}{\Gamma_p(a)\Gamma_p(b_{k+1})\cdots\Gamma_p(b_n)\Gamma_p(c_1-\rho_1)\cdots\Gamma_p(c_n-\rho_n)}\times$$

$$\Gamma_p(b_{k+1}-\rho_{k+1})\cdots\Gamma_p(b_n-\rho_n)\Gamma_p(c_1)\cdots\Gamma_p(c_n)\times$$

$$\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n) \tag{8.2}$$

for $\mathrm{Re}(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n, b_{k+1}-\rho_{k+1},\cdots,$
$b_n-\rho_n, c_i-\rho_i, \rho_i) > (p-1)/2, \quad i=1,\cdots,n.$

8.2 In this section we prove three theorems- two for the function ${}^{(k)}H_3^{(n)}$ and one for the function ${}^{(k)}H_4^{(n)}$. It is interesting to note that all the theorems of this section hold good for the case p=2 only and all these results are different from the corresponding results in the scalar case.

Theorem 8.2.1:

$${}^{(k)}H_3^{(n)}(a,b_{k+1},\cdots,b_n;c;-X_1,\cdots,-X_n)$$

$$= \frac{\Gamma_p(c)}{\Gamma_p[(a+1)/2]\Gamma_p(b_{k+1})\cdots\Gamma_p(b_n)}\times$$

$$\frac{1}{\Gamma_p[c-b_{k+1}-\cdots-b_n-(a+1)/2]}\int\cdots(n-k+1)\cdots\times$$

$$\int\left|U_{k+1}\right|^{b_{k+1}-(p+1)/2}\cdots\left|U_n\right|^{b_n-(p+1)/2}\times$$

$$\left|I-U_{k+1}-\cdots-U_n\right|^{c-b_{k+1}-\cdots-b_n-(p+1)/2}\times$$

$$\left|V\right|^{(a+1)/2-(p+1)/2}\left|I-V\right|^{c-b_{k+1}-\cdots-b_n-(a+1)/2-(p+1)/2}$$

$$\times\left|I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2}\right|^{-a}\times$$

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$$\left| I + 4(I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots + U_n^{1/2} X_n U_n^{1/2})^{-1} \times \right.$$

$$(I - U_{k+1} - \cdots - U_n)^{1/2} V^{1/2} (X_1 + \cdots + X_k) V^{1/2} (I - U_{k+1} -$$

$$\cdots - U_n)^{1/2} (I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots \tag{8.3}$$

$$\left. + U_n^{1/2} X_n U_n^{1/2})^{-1} \right|^{-(2a+1)/4} dU_{k+1} \cdots dU_n dV$$

for $\mathrm{Re}[(a+1)/2, b_{k+1}, \cdots, b_n, c - b_{k+1} - \cdots - b_n - (a+1)/2]$

$> (p-1)/2$ and for $p = 2$, where $U'_{k+1} = U_{k+1} > 0, \cdots, U'_n = U_n$

> 0 and $0 < U_{k+1} + \cdots + U_n < I.$

Proof: Taking the M-transform of the function involving $X_1, \cdots, X_n$ on the right side of eq.(8.3) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we have,

$$\int_{X_1 > 0} \cdots \int_{X_n > 0} |X_1|^{\rho_1 - (p+1)/2} \cdots |X_k|^{\rho_k - (p+1)/2} \times$$

$$|X_{k+1}|^{\rho_{k+1} - (p+1)/2} \cdots |X_n|^{\rho_n - (p+1)/2} \times$$

$$\left| I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots + U_n^{1/2} X_n U_n^{1/2} \right|^{-a} \times$$

$$\left| I + 4(I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots + U_n^{1/2} X_n U_n^{1/2})^{-1} \times \right.$$

$$(I - U_{k+1} - \cdots - U_n)^{1/2} V^{1/2} (X_1 + \cdots + X_k) V^{1/2} \times$$

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$$(I-U_{k+1}-\cdots-U_n)^{1/2}(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots$$
$$+U_n^{1/2}X_nU_n^{1/2})^{-1}\Big|^{-(2a+1)/4} dX_1\cdots dX_k dX_{k+1}\cdots dX_n \qquad (8.4)$$

Making use of the transformations

$Y_1=X_1, Y_2=X_1+X_2,\cdots,Y_k=X_1+\cdots+X_k; Y_j=U_j^{1/2}X_jU_j^{1/2}$, for $j=k+1,\cdots,n$; with $dY_1\cdots dY_k=dX_1\cdots dX_k$ and $dY_j=|U_j|^{(p+1)/2}dX_j$ for $j=k+1,\cdots,n$; (on using eq.(6.6) page 95 of Mathai [57]) and $|X_1|=|Y_1|, |X_2|=|Y_2-Y_1|,\cdots,$ $|X_k|=|Y_k-Y_{k-1}|; |Y_j|=|U_j||X_j|$, for $j=k+1,\cdots,n$; where $0<Y_1<Y_2<\cdots<Y_k$ and $Y_j>0$ for $j=k+1,\cdots,n$;

we can have

$$|U_{k+1}|^{-\rho_{k+1}}\cdots|U_n|^{-\rho_n}\int\cdots\int|Y_1|^{\rho_1-(p+1)/2}\times$$
$$|Y_2-Y_1|^{\rho_2-(p+1)/2}\cdots|Y_k-Y_{k-1}|^{\rho_k-(p+1)/2}\times$$
$$|Y_{k+1}|^{\rho_{k+1}-(p+1)/2}\cdots|Y_n|^{\rho_n-(p+1)/2}\times$$
$$|I+Y_{k+1}+\cdots+Y_n|^{-a}\,\Big|I+4(I+Y_{k+1}+\cdots+Y_n)^{-1}\times \qquad (8.5)$$
$$(I-U_{k+1}-\cdots-U_n)^{1/2}V^{1/2}Y_kV^{1/2}(I-U_{k+1}-\cdots-U_n)^{1/2}\times$$
$$(I+Y_{k+1}+\cdots+Y_n)^{-1}\Big|^{-(2a+1)/4}dY_1\cdots dY_k dY_{k+1}\cdots dY_n$$

Integrating out the variables $Y_1,\cdots,Y_{k-1}$ by using a type-1 Beta integral and then making use of another transformation

$$Z_k = 4(I+Y_{k+1}+\cdots+Y_n)^{-1}(I-U_{k+1}-\cdots-U_n)^{1/2}V^{1/2}Y_k \times$$
$$V^{1/2}(I-U_{k+1}-\cdots-U_n)^{1/2}(I+Y_{k+1}+\cdots+Y_n)^{-1}, \text{ with}$$
$$dZ_k = 4^{p(p+1)/2}\left|I+Y_{k+1}+\cdots+Y_n\right|^{-(p+1)} \times$$
$$\left|I-U_{k+1}-\cdots-U_n\right|^{(p+1)/2}\left|V\right|^{(p+1)/2} dY_k \text{ and}$$
$$\left|Z_k\right| = 4^p\left|I+Y_{k+1}+\cdots+Y_n\right|^{-2}\left|I-U_{k+1}-\cdots-U_n\right|\left|V\right|\left|Y_k\right|$$

the expression (8.5) yields,

$$4^{-p(\rho_1+\cdots+\rho_k)}\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_k)}{\Gamma_p(\rho_1+\cdots+\rho_k)}\left|U_{k+1}\right|^{-\rho_{k+1}}\cdots\left|U_n\right|^{-\rho_n} \times$$
$$\left|I-U_{k+1}-\cdots-U_n\right|^{-(\rho_1+\cdots+\rho_k)}\left|V\right|^{-(\rho_1+\cdots+\rho_k)} \times$$
$$\int_{Z_k>0}\int_{Y_{k+1}>0}\cdots(n-k)\cdots\int_{Y_n>0}\left|Z_k\right|^{\rho_1+\cdots+\rho_k-(p+1)/2} \tag{8.6}$$
$$\times\left|Y_{k+1}\right|^{\rho_{k+1}-(p+1)/2}\cdots\left|Y_n\right|^{\rho_n-(p+1)/2}\left|I+Z_k\right|^{-(2a+1)/4}$$
$$\times\left|I+Y_{k+1}+\cdots+Y_n\right|^{-(a-2\rho_1-\cdots-2\rho_k)} dZ_k dY_{k+1}\cdots dY_n$$

Integrating out the variable Z_k by using a type-2 Beta integral and the variables $Y_{k+1},\cdots,Y_n$ by using a type-2 Dirichlet integral, the above expression produces

$$4^{-p(\rho_1+\cdots+\rho_k)}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\left|U_{k+1}\right|^{-\rho_{k+1}}\cdots\left|U_n\right|^{-\rho_n} \times$$
$$\left|I-U_{k+1}-\cdots-U_n\right|^{-(\rho_1+\cdots+\rho_k)}\left|V\right|^{-(\rho_1+\cdots+\rho_k)} \times \tag{8.7}$$
$$\Gamma_p[(2a+1)/4-\rho_1-\cdots-\rho_k]\frac{\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n)}{\Gamma_p[(2a+1)/4]\Gamma_p(a-2\rho_1-\cdots-2\rho_k)}$$

Substitution of this expression on the right side of eq.(8.3) gives the following expression,

$$4^{-p(\rho_1+\cdots+\rho_k)}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\frac{\Gamma_p[(2a+1)/4-\rho_1-\cdots-\rho_k]}{\Gamma_p[(2a+1)/4]}$$

$$\times\frac{\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n)}{\Gamma_p(a-2\rho_1-\cdots-2\rho_k)\Gamma_p[(a+1)/2]\Gamma_p(b_{k+1})\cdots\Gamma_p(b_n)}\times$$

$$\frac{\Gamma_p(c)}{\Gamma_p[c-b_{k+1}-\cdots-b_n-(a+1)/2]}\int\cdots(n-k+1)\cdots\int\times$$

$$|U_{k+1}|^{b_{k+1}-\rho_{k+1}-(p+1)/2}\cdots|U_n|^{b_n-\rho_n-(p+1)/2}\times \tag{8.8}$$

$$|I-U_{k+1}-\cdots-U_n|^{c-b_{k+1}-\cdots-b_n-(\rho_1+\cdots+\rho_k)-(p+1)/2}\times$$

$$|V|^{(a+1)/2-(\rho_1+\cdots+\rho_k)-(p+1)/2}\times$$

$$|I-V|^{c-b_{k+1}-\cdots-b_n-(a+1)/2-(p+1)/2}\,dU_{k+1}\cdots dU_n dV$$

Integrating out the variables $U_{k+1},\cdots,U_n$ and V by using type-1 Dirichlet integral and a type-1 Beta integral respectively and observing that

$$4^{-p(\rho_1+\cdots+\rho_k)}\frac{\Gamma_p[(a+1)/2-\rho_1-\cdots-\rho_k]}{\Gamma_p[(a+1)/2]\Gamma_p[(2a+1)/4]}\times$$

$$\frac{\Gamma_p[(2a+1)/4-\rho_1-\cdots-\rho_k]}{\Gamma_p(a-2\rho_1-\cdots-2\rho_k)}=\frac{1}{\Gamma_p(a)},\ \text{for } p=2; \tag{8.9}$$

by using eq.(6.13) page 84 of Mathai [62], simplification of the expression (8.8) finally yields, $M[{}^{(k)}H_3^{(n)}]$ as given by eq.(8.1).

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Theorem 8.2.2:

$$ {}^{(k)}H_3^{(n)}(a, b_{k+1}, \cdots, b_n; c; -X_1, \cdots, -X_n) $$

$$ = \frac{\Gamma_p(c)}{\Gamma_p(b_{k+1}) \cdots \Gamma_p(b_n) \Gamma_p(c - b_{k+1} - \cdots - b_n)} \times $$

$$ \int \cdots (n-k+1) \cdots \int \left|U_{k+1}\right|^{b_{k+1}-(p+1)/2} \cdots \left|U_n\right|^{b_n-(p+1)/2} \times $$

$$ \left|I - U_{k+1} - \cdots - U_n\right|^{c-b_{k+1}-\cdots-b_n-(p+1)/2} \times $$

$$ \left|I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots + U_n^{1/2} X_n U_n^{1/2}\right|^{-a} \times $$

$$ {}_2F_1[(a+1)/2, (2a+1)/4; c - b_{k+1} - \cdots - b_n; $$

$$ -4(I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots + U_n^{1/2} X_n U_n^{1/2})^{-1} (I - U_{k+1} - $$

$$ \cdots - U_n)^{1/2} (X_1 + \cdots + X_k)(I - U_{k+1} - \cdots - U_n)^{1/2} \times $$

$$ (I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots + U_n^{1/2} X_n U_n^{1/2})^{-1}] dU_{k+1} \cdots dU_n \qquad (8.10) $$

where $\mathrm{Re}(b_{k+1}, \cdots, b_n, c - b_{k+1} - \cdots - b_n) > (p-1)/2$, $U'_{k+1} = U_{k+1} > 0, \cdots, U'_n = U_n > 0$ and $0 < U_{k+1} + \cdots + U_n < I$ and $p = 2$.

Proof: Taking the M-transform of the right side of eq.(8.10) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we have,

$$ \int_{X_1>0} \cdots \int_{X_n>0} \left|X_1\right|^{\rho_1-(p+1)/2} \cdots \left|X_k\right|^{\rho_k-(p+1)/2} \times $$

$$ \left|X_{k+1}\right|^{\rho_{k+1}-(p+1)/2} \cdots \left|X_n\right|^{\rho_n-(p+1)/2} \left|I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \right. $$

$$\cdots + U_n^{1/2} X_n U_n^{1/2}\Big|^{-a} {}_2F_1[(a+1)/2, (2a+1)/4; c - b_{k+1} -$$

$$\cdots - b_n; -4(I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots + U_n^{1/2} X_n U_n^{1/2})^{-1} \times$$

$$(I - U_{k+1} - \cdots - U_n)^{1/2}(X_1 + \cdots + X_k)(I - U_{k+1} - \qquad (8.11)$$

$$\cdots - U_n)^{1/2}(I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots + U_n^{1/2} X_n U_n^{1/2})^{-1}] \times$$

$$dX_1 \cdots dX_k dX_{k+1} \cdots dX_n$$

Applying the transformations

$$Y_1 = X_1, Y_2 = X_1 + X_2, \cdots, Y_k = X_1 + \cdots + X_k; Y_j = U_j^{1/2} X_j U_j^{1/2},$$

for $j = k+1, \cdots, n;$ then from eq.(6.7) page 95 of Mathai [57], we have,

$$dY_1 \cdots dY_k = dX_1 \cdots dX_k, \text{also, } dY_j = \left|U_j\right|^{(p+1)/2} dX_j \text{ for } j =$$

$$k+1, \cdots, n \text{ and } 0 < Y_1 < Y_2 < \cdots < Y_k \text{ and } Y_j > 0 \; (j = k+1, \cdots, n).$$

Using these transformations in the expression (8.11) and then integrating out the variables $Y_1, \cdots, Y_{k-1}$ one-by-one and in order by using a type-1 Beta integral, we are led to

$$\left|U_{k+1}\right|^{-\rho_{k+1}} \cdots \left|U_n\right|^{-\rho_n} \frac{\Gamma_p(\rho_1) \cdots \Gamma_p(\rho_k)}{\Gamma_p(\rho_1 + \cdots + \rho_k)} \int_{Y_k > 0} \cdots (n-k+1) \cdots$$

$$\times \int_{Y_n > 0} \left|Y_k\right|^{\rho_1 + \cdots + \rho_k - (p+1)/2} \left|Y_{k+1}\right|^{\rho_{k+1} - (p+1)/2} \cdots \times$$

$$\left|Y_n\right|^{\rho_n - (p+1)/2} \left|I + Y_{k+1} + \cdots + Y_n\right|^{-a} \times$$

$${}_2F_1[(a+1)/2, (2a+1)/4; c - b_{k+1} - \cdots - b_n;$$

$$-4(I + Y_{k+1} + \cdots + Y_n)^{-1}(I - U_{k+1} - \cdots - U_n)^{1/2} Y_k \times$$

Continued to the next page … … … … …

$$(I-U_{k+1}-\cdots-U_n)^{1/2}(I+Y_{k+1}+\cdots+Y_n)^{-1}]dY_k dY_{k+1}\cdots dY_n \quad (8.12)$$

Now using the following transformation in the expression (8.12),

$$Z_k = 4(I+Y_{k+1}+\cdots+Y_n)^{-1}(I-U_{k+1}-\cdots-U_n)^{1/2}Y_k \times$$

$$(I-U_{k+1}-\cdots-U_n)^{1/2}(I+Y_{k+1}+\cdots+Y_n)^{-1}, \text{with } dZ_k =$$

$$4^{p(p+1)/2}\left|I+Y_{k+1}+\cdots+Y_n\right|^{-(p+1)} \times$$

$$\left|I-U_{k+1}-\cdots-U_n\right|^{(p+1)/2} dY_k \ \text{ and } \ \left|Z_k\right| = 4^p \times$$

$$\left|I+Y_{k+1}+\cdots+Y_n\right|^{-2}\left|I-U_{k+1}-\cdots-U_n\right|\left|Y_k\right|$$

and then writing the M-transform of the ${}_2F_1$- function and integrating out the variables $Y_{k+1},\cdots,Y_n$ by using a type-2 Dirichlet integral, we have

$$4^{-p(\rho_1+\cdots+\rho_k)}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\left|U_{k+1}\right|^{-\rho_{k+1}}\ldots\left|U_n\right|^{-\rho_n} \times$$

$$\left|I-U_{k+1}-\cdots-U_n\right|^{-(\rho_1+\cdots+\rho_k)}\Gamma_p[(2a+1)/4-\rho_1-\cdots-\rho_k]\times$$

$$\frac{\Gamma_p[(a+1)/2-\rho_1-\cdots-\rho_k]}{\Gamma_p[(2a+1)/4]\Gamma_p(a-2\rho_1-\cdots-2\rho_k)\Gamma_p[(a+1)/2]}\times \quad (8.13)$$

$$\frac{\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n)\Gamma_p(c-b_{k+1}-\cdots-b_n)}{\Gamma_p(c-b_{k+1}-\cdots-b_n-\rho_1-\cdots-\rho_k)}$$

Substituting this expression on the right side of eq.(8.10) and integrating out the variables $U_{k+1},\cdots,U_n$ in the resulting expression by using a type-1 Dirichlet integral and using eq.(8.9) we have $M[{}^{(k)}H_3^{(n)}]$ as given by eq.(8.1).

Theorem 8.2.3:

$$
\begin{aligned}
&{}^{(k)}H_4^{(n)}(a, b_{k+1}, \cdots, b_n; c_1, \cdots, c_n; -X_1, \cdots, -X_n) \\
&= \frac{\Gamma_p(c_{k+1}) \cdots \Gamma_p(c_n)}{\Gamma_p(b_{k+1}) \cdots \Gamma_p(b_n) \Gamma_p(c_{k+1} - b_{k+1}) \cdots \Gamma_p(c_n - b_n)} \times \\
&\int \cdots (n-k) \cdots \int |U_{k+1}|^{b_{k+1} - (p+1)/2} \cdots |U_n|^{b_n - (p+1)/2} \times \\
&|I - U_{k+1}|^{c_{k+1} - b_{k+1} - (p+1)/2} \cdots |I - U_n|^{c_n - b_n - (p+1)/2} \times \\
&\left| I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots + U_n^{1/2} X_n U_n^{1/2} \right|^{-a} \times \\
&F_C^{(k)}[(a+1)/2, (2a+1)/4; c_1, \cdots, c_k; -4(I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \\
&\cdots + U_n^{1/2} X_n U_n^{1/2})^{-1} X_1 (I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots \\
&+ U_n^{1/2} X_n U_n^{1/2})^{-1}, \cdots, -4(I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots \\
&+ U_n^{1/2} X_n U_n^{1/2})^{-1} X_k (I + U_{k+1}^{1/2} X_{k+1} U_{k+1}^{1/2} + \cdots \\
&+ U_n^{1/2} X_n U_n^{1/2})^{-1}] dU_{k+1} \cdots dU_n
\end{aligned}
\tag{8.14}
$$

where $p = 2$, and $0 < U_j < I$ for $j = k+1, \cdots n$ and for $\mathrm{Re}(b_{k+1}, \cdots, b_n, c_{k+1} - b_{k+1}, \cdots, c_n - b_n) > (p-1)/2$.

Proof: We take the M-transform of the right side of eq.(8.14) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively to get,

$$
\int_{X_1 > 0} \cdots \int_{X_n > 0} |X_1|^{\rho_1 - (p+1)/2} \cdots |X_k|^{\rho_k - (p+1)/2} \times
$$

Continued to the next page … … … … …

$$\left|X_{k+1}\right|^{\rho_{k+1}-(p+1)/2}\cdots\left|X_n\right|^{\rho_n-(p+1)/2}\times$$

$$\left|I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2}\right|^{-a}F_C^{(k)}[(a+1)/2,$$

$$(2a+1)/4;c_1,\cdots,c_k;-4(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots$$

$$+U_n^{1/2}X_nU_n^{1/2})^{-1}X_1(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots$$

$$+U_n^{1/2}X_nU_n^{1/2})^{-1},\cdots,-4(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+$$

$$U_n^{1/2}X_nU_n^{1/2})^{-1}X_k(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+$$

$$U_n^{1/2}X_nU_n^{1/2})^{-1}]dX_1\cdots dX_k dX_{k+1}\cdots dX_n \tag{8.15}$$

Making use of the transformations

$$Y_i=4(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}X_i\times$$

$$(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1},\ i=1,\cdots,k;$$

$$Y_j=U_j^{1/2}X_jU_j^{1/2},\ j=k+1,\cdots,n;\ \text{so that,}$$

$$Y_i=4(I+Y_{k+1}+\cdots+Y_n)^{-1}X_i(I+Y_{k+1}+\cdots+Y_n)^{-1};$$

$$dY_i=4^{p(p+1)/2}\left|I+Y_{k+1}+\cdots+Y_n\right|^{-(p+1)}dX_i,\ i=1,\cdots,k;$$

$$dY_j=\left|U_j\right|^{(p+1)/2}dX_j,\ j=k+1,\cdots,n;\ \text{and}$$

$$\left|Y_i\right|=4^p\left|I+Y_{k+1}+\cdots+Y_n\right|^{-2}\left|X_i\right|,\ i=1,\cdots,k;\ \left|Y_j\right|=\left|U_j\right|\left|X_j\right|,$$

$$j=k+1,\cdots,n;$$

in the expression (8.15) and then using the M-transform of an $F_C^{(n)}$-function in the resulting expression and integrating out the variables $Y_{k+1},\cdots,Y_n$ by using a type-2 Dirichlet integral, we have

$$4^{-p(\rho_1+\cdots+\rho_k)}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\left|U_{k+1}\right|^{-\rho_{k+1}}\ldots\left|U_n\right|^{-\rho_n}\times$$
$$\frac{\Gamma_p(c_1)\cdots\Gamma_p(c_k)\Gamma_p[(a+1)/2-\rho_1-\cdots-\rho_k]}{\Gamma_p(c_1-\rho_1)\cdots\Gamma_p(c_k-\rho_k)\Gamma_p[(a+1)/2]}\times \tag{8.16}$$
$$\frac{\Gamma_p[(2a+1)/4-\rho_1-\cdots-\rho_k]}{\Gamma_p[(2a+1)/4]\Gamma_p(a-2\rho_1-\cdots-2\rho_k)}\times$$
$$\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n)$$

Using this expression on the right side of eq.(8.14) and integrating out the variables $U_{k+1},\cdots,U_n$ in the resulting expression by using a type-1 Beta integral and using eq.(8.9) finally yields $M[{}^{(k)}H_4^{(n)}]$ as given by eq.(8.2).

8.3 Now we proceed to prove four more results for the generalized Horn's functions of matrix arguments. The results which are being established in this section are valid for all finite values of p.

Theorem 8.3.1:

$$\begin{aligned}&{}^{(k)}H_3^{(n)}(a,b_{k+1},\cdots,b_n;c;-X_1,\cdots,-X_n)\\&=\frac{1}{\Gamma_p(a)}\int_{S>0}e^{-\mathrm{tr}(S)}\left|S\right|^{a-(p+1)/2}\Phi_3^{(n-k+1)}[b_{k+1},\cdots,b_n;c;\\&-S(X_1+\cdots+X_k)S',-S^{1/2}X_{k+1}S^{1/2},\cdots,-S^{1/2}X_nS^{1/2}]dS\end{aligned} \tag{8.17}$$

for $S=S'>0$ and $\mathrm{Re}(a)>(p-1)/2$.

Proof: We take the M-transform of the right side of eq.(8.17) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively to get

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_k|^{\rho_k-(p+1)/2}\times$$

$$|X_{k+1}|^{\rho_{k+1}-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\,\Phi_3^{(n-k+1)}[b_{k+1},\cdots, \quad (8.18)$$

$$b_n;c;-S(X_1+\cdots+X_k)S',-S^{1/2}X_{k+1}S^{1/2},\cdots,-S^{1/2}X_nS^{1/2}]\times$$

$$dX_1\cdots dX_k dX_{k+1}\cdots dX_n$$

Applying the transformations

$$Y_i = SX_iS', \text{ and } Y_j = S^{1/2}X_jS^{1/2} \text{ with } dY_i = |S|^{(p+1)}\,dX_i; dY_j =$$

$$|S|^{(p+1)/2}\,dX_j \text{ and } |Y_i| = |S|^2|X_i|, |Y_j| = |S||X_j| \text{ for } i = 1,\cdots,k$$

and $j = k+1,\cdots,n$ the expression (8.18) yields

$$|S|^{-2(\rho_1+\cdots+\rho_k)-\rho_{k+1}-\cdots-\rho_n}\int_{Y_1>0}\cdots\int_{Y_n>0}|Y_1|^{\rho_1-(p+1)/2}\times$$

$$\times\cdots|Y_k|^{\rho_k-(p+1)/2}|Y_{k+1}|^{\rho_{k+1}-(p+1)/2}\cdots|Y_n|^{\rho_n-(p+1)/2}\times \quad (8.19)$$

$$\Phi_3^{(n-k+1)}[b_{k+1},\cdots,b_n;c;-(Y_1+\cdots+Y_k),-Y_{k+1},\cdots,-Y_n]\times$$

$$dY_1\cdots dY_k dY_{k+1}\cdots dY_n$$

Now, applying the transformations,

$$Z_1 = Y_1, Z_2 = Y_1+Y_2,\cdots,Z_k = Y_1+\cdots+Y_k; Z_j = Y_j,$$

for $j = k+1,\cdots,n$ then from eq.(6.7) page 95 of Mathai [57] we have

$$dY_1\cdots dY_k = dZ_1\cdots dZ_k, \text{ also, } dY_j = dZ_j \text{ for } j = k+1,\cdots,n \text{ and}$$

$$0 < Z_1 < Z_2 < \cdots < Z_k, Z_j > 0\ (j = k+1,\cdots,n)$$

which render the expression (8.19) as below

$$|S|^{-2(\rho_1+\cdots+\rho_k)-\rho_{k+1}-\cdots-\rho_n}\int\cdots(n)\cdots\int|Z_1|^{\rho_1-(p+1)/2}\times$$

$$|Z_2-Z_1|^{\rho_2-(p+1)/2}\cdots|Z_k-Z_{k-1}|^{\rho_k-(p+1)/2}\times$$

Continued to the next page … … … … …

$$\left|Z_{k+1}\right|^{\rho_{k+1}-(p+1)/2}\cdots\left|Z_n\right|^{\rho_n-(p+1)/2}\Phi_3^{(n-k+1)}[b_{k+1},\cdots, \qquad (8.20)$$
$$b_n;c;-Z_k,-Z_{k+1},\cdots,-Z_n]dZ_1\cdots dZ_k dZ_{k+1}\cdots dZ_n$$

Integrating out the variables $Z_1,\cdots,Z_{k-1}$ one-by-one and in order by using a type-1 Beta integral and then using the following definition of M-transform of a $\Phi_3^{(n)}$ - function

$$M(\Phi_3^{(n)})=\int_{X_1>0}\cdots\int_{X_n>0}\left|X_1\right|^{\rho_1-(p+1)/2}\cdots\left|X_n\right|^{\rho_n-(p+1)/2}\times$$
$$\Phi_3^{(n)}(a_2,\cdots,a_n;c;-X_1,\cdots,-X_n)dX_1\cdots dX_n \qquad (8.21)$$
$$=\frac{\Gamma_p(a_2-\rho_2)}{\Gamma_p(a_2)}\cdots\frac{\Gamma_p(a_n-\rho_n)}{\Gamma_p(a_n)}\frac{\Gamma_p(c)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)}{\Gamma_p(c-\rho_1-\cdots-\rho_n)}$$

for $\mathrm{Re}(a_2-\rho_2,\cdots,a_n-\rho_n,c-\rho_1-\cdots-\rho_n,\rho_1,\cdots,\rho_n)>(p-1)/2.$

the expression (8.20) yields,

$$|S|^{-2(\rho_1+\cdots+\rho_k)-\rho_{k+1}-\cdots-\rho_n}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\times$$
$$\frac{\Gamma_p(c)\Gamma_p(b_{k+1}-\rho_{k+1})\cdots\Gamma_p(b_n-\rho_n)}{\Gamma_p(b_{k+1})\cdots\Gamma_p(b_n)\Gamma_p(c-\rho_1-\cdots-\rho_n)} \qquad (8.22)$$

Substituting this expression on the right side of eq.(8.17) and then integrating out S in the resulting expression by using a Gamma integral produces $M[{}^{(k)}H_3^{(n)}]$ as given by eq.(8.1).

Theorem 8.3.2:

$${}^{(k)}H_3^{(n)}(a,b_{k+1},\cdots,b_n;c;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(a)\Gamma_p(b_{k+1})\cdots\Gamma_p(b_n)}\int_{S>0}\cdots(n-k+1)\cdots\times$$

Continued to the next page … … … … …

$$\int_{T_n>0} e^{-tr(S+T_{k+1}+\cdots+T_n)} |S|^{a-(p+1)/2} |T_{k+1}|^{b_{k+1}-(p+1)/2} \times$$
$$\cdots |T_n|^{b_n-(p+1)/2} {}_0F_1[\,;c;-S(X_1+\cdots+X_k)S' \tag{8.23}$$
$$-S^{1/2}(T_{k+1}^{1/2}X_{k+1}T_{k+1}^{1/2}+\cdots+T_n^{1/2}X_nT_n^{1/2})S^{1/2}]dSdT_{k+1}\cdots dT_n$$
for $\mathrm{Re}(a, b_{k+1}, \cdots, b_n) > (p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(8.23) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we get

$$\int_{X_1>0}\cdots\int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_k|^{\rho_k-(p+1)/2} \times$$
$$|X_{k+1}|^{\rho_{k+1}-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} {}_0F_1[\,;c;-S(X_1+\cdots+ \tag{8.24}$$
$$X_k)S' - S^{1/2}(T_{k+1}^{1/2}X_{k+1}T_{k+1}^{1/2}+\cdots+T_n^{1/2}X_nT_n^{1/2})S^{1/2}]\times$$
$$dX_1\cdots dX_k dX_{k+1}\cdots dX_n$$

Applying the transformations

$$Y_1 = X_1, Y_2 = X_1 + X_2, \cdots, Y_k = X_1+\cdots+X_k; Y_j = T_j^{1/2}X_jT_j^{1/2},$$

for $j = k+1, \cdots, n;$ with $dY_1\cdots dY_k = dX_1\cdots dX_k,$ also, $dY_j =$

$|T_j|^{(p+1)/2} dX_j$ for $j = k+1, \cdots, n$ and $0 < Y_1 < Y_2 < \cdots < Y_k$

and $Y_j > 0\ (j = k+1, \cdots, n),$ to the expression (8.24) and integrating out $Y_1, \cdots, Y_{k-1}$ as in theorem 8.3.1 by using a type-1 Beta integral we obtain

$$\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_k)}{\Gamma_p(\rho_1+\cdots+\rho_k)} |T_{k+1}|^{-\rho_{k+1}} \cdots |T_n|^{-\rho_n} \int_{Y_k>0}\cdots(n-k+1)\cdots$$
$$\int_{Y_n>0} |Y_k|^{\rho_1+\cdots+\rho_k-(p+1)/2} |Y_{k+1}|^{\rho_{k+1}-(p+1)/2} \cdots\times$$

Continued to the next page … … … … …

$$\left|Y_n\right|^{\rho_n-(p+1)/2}\ {}_0F_1[\ ;c;-SY_kS'-S^{1/2}(Y_{k+1}+\cdots+Y_n)S^{1/2}]\times \tag{8.25}$$
$$dY_k dY_{k+1}\cdots dY_n$$

Now applying another set of transformations,

$$Z_k=SY_kS';Z_j=S^{1/2}Y_jS^{1/2};\ \text{with}\ dZ_k=|S|^{p+1}dY_k;dZ_j=$$
$$|S|^{(p+1)/2}dY_j;\ \text{and}\ \left|Z_k\right|=|S|^2\left|Y_k\right|;\left|Z_j\right|=|S|\left|Y_j\right|;\ \text{for}\ j=k+1,\cdots,n;$$

to the expression (8.25) and using the theorem 2.2.3 we get

$$\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\left|T_{k+1}\right|^{-\rho_{k+1}}\ldots\left|T_n\right|^{-\rho_n}\times$$
$$|S|^{-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n}\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\cdots-\rho_n)} \tag{8.26}$$

Substituting this expression on the right side of eq.(8.23) and then integrating out the variables $S,T_{k+1},\cdots,T_n$ in the resulting expression by using a Gamma integral produces $M[{}^{(k)}H_3^{(n)}]$ as given by eq.(8.1).

Theorem 8.3.3:

$$^{(k)}H_4^{(n)}(a,b_{k+1},\cdots,b_n;c_1,\cdots,c_n;-X_1,\cdots,-X_n)$$
$$=\frac{1}{\Gamma_p(a)}\int_{S>0}e^{-tr(S)}|S|^{a-(p+1)/2}\ {}_0F_1(\ ;c_1;-SX_1S')\cdots$$
$${}_0F_1(\ ;c_k;-SX_kS')\,{}_1F_1(b_{k+1};c_{k+1};-S^{1/2}X_{k+1}S^{1/2})\cdots\times \tag{8.27}$$
$${}_1F_1(b_n;c_n;-S^{1/2}X_nS^{1/2})dS$$

for $\operatorname{Re}(a)>(p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(8.27) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_k|^{\rho_k-(p+1)/2}\times$$
$$|X_{k+1}|^{\rho_{k+1}-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\;{}_0F_1(\,;c_1;-SX_1S')\cdots$$
$${}_0F_1(\,;c_k;-SX_kS')\,{}_1F_1(b_{k+1};c_{k+1};-S^{1/2}X_{k+1}S^{1/2})\cdots \tag{8.28}$$
$${}_1F_1(b_n;c_n;-S^{1/2}X_nS^{1/2})dX_1\cdots dX_k dX_{k+1}\cdots dX_n$$

Applying the following transformations to the above expression

$$Y_i=SX_iS',\ Y_j=S^{1/2}X_jS^{1/2};\ \text{with } dY_i=|S|^{p+1}dX_i,\ dY_j=|S|^{(p+1)/2}$$
$$dX_j;\ \text{and}\ |Y_i|=|S|^2|X_i|,\ |Y_j|=|S||X_j|;\ \text{for } i=1,\cdots,k,\ j=k+1,\cdots,n;$$

and then writing the M-transforms of the ${}_0F_1$ and ${}_1F_1$ functions we obtain,

$$|S|^{-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n}\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)}\cdots\frac{\Gamma_p(c_n)}{\Gamma_p(c_n-\rho_n)}\times$$
$$\frac{\Gamma_p(b_{k+1}-\rho_{k+1})}{\Gamma_p(b_{k+1})}\cdots\frac{\Gamma_p(b_n-\rho_n)}{\Gamma_p(b_n)}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n) \tag{8.29}$$

On substituting this expression on the right side of eq.(8.27) and integrating out the variable S in the resulting expression by using a Gamma integral, the outcome is $M[{}^{(k)}H_4^{(n)}]$ as given by eq.(8.2).

Theorem 8.3.4: Special Cases-

(i)
$$\begin{aligned}&{}^{(0)}H_3^{(n)}(a,b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)\\&=F_D^{(n)}(a,b_1,\cdots,b_n;c;-X_1,\cdots,-X_n)\end{aligned} \tag{8.30}$$

(ii)
$$\begin{aligned}&{}^{(0)}H_4^{(n)}(a,b_1,\cdots,b_n;c_1,\cdots,c_n;-X_1,\cdots,-X_n)\\&=F_A^{(n)}(a,b_1,\cdots,b_n;c_1,\cdots,c_n;-X_1,\cdots,-X_n)\end{aligned} \tag{8.31}$$

Proof: (i) This result is obtained by putting $k = 0$ in eq.(8.1) and then comparing the result with eq.(4.5).
(ii) Similarly, this result can be had by putting $k = 0$ in eq.(8.2) and then comparing the outcome with eq.(4.2).

CHAPTER IX

THE GENERALIZED SRIVASTAVA $H_B^{(n)}$ AND $H_C^{(n)}$ FUNCTIONS OF MATRIX ARGUMENTS

9.1 I have already discussed the Srivastava's triple hypergeometric functions H_B and H_C of matrix arguments in chapter fifth of this thesis. This concluding chapter of the present thesis is aimed at describing the Srivastava functions $H_B^{(n)}$ and $H_C^{(n)}$, which are the generalizations of the H_B and H_C functions discussed earlier, for the case of matrix arguments. First I will define these functions for matrix arguments and then prove three results concerning these two functions.

9.2 Definitions:

9.2.1 The Srivastava function $H_B^{(n)}$ of matrix arguments,

$$H_B^{(n)} = H_B^{(n)}(\alpha_1,\cdots,\alpha_n;\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_n)$$

is defined as that class of functions which has the following matrix-transform:

$$M\left[H_B^{(n)}\right] = \int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$

$$H_B^{(n)}(\alpha_1,\cdots,\alpha_n;\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_n)dX_1\cdots dX_n$$

$$=\frac{\Gamma_p(\alpha_1-\rho_1-\rho_n)\Gamma_p(\alpha_2-\rho_1-\rho_2)\cdots\Gamma_p(\alpha_n-\rho_{n-1}-\rho_n)}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)\cdots\Gamma_p(\alpha_n)\Gamma_p(\gamma_1-\rho_1)\cdots\Gamma_p(\gamma_n-\rho_n)}\times \qquad (9.1)$$

$$\Gamma_p(\gamma_1)\cdots\Gamma_p(\gamma_n)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)$$

for $\mathrm{Re}(\alpha_1-\rho_1-\rho_n,\alpha_2-\rho_1-\rho_2,\cdots,\alpha_n-\rho_{n-1}-\rho_n,\gamma_i-\rho_i,\rho_i)$ $>(p-1)/2,$ where, $i=1,\cdots,n.$

9.2.2 $H_C^{(n)} = H_C^{(n)}(\alpha_1, \cdots, \alpha_n; \gamma; -X_1, \cdots, -X_n)$

$$M\left[H_C^{(n)}\right] = \int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$
$$H_C^{(n)}(\alpha_1, \cdots, \alpha_n; \gamma; -X_1, \cdots, -X_n) dX_1 \cdots dX_n$$
$$= \frac{\Gamma_p(\alpha_1 - \rho_1 - \rho_n)\Gamma_p(\alpha_2 - \rho_1 - \rho_2) \cdots \Gamma_p(\alpha_n - \rho_{n-1} - \rho_n)}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2) \cdots \Gamma_p(\alpha_n)\Gamma_p(\gamma - \rho_1 - \cdots - \rho_n)} \times \tag{9.2}$$
$$\Gamma_p(\gamma)\Gamma_p(\rho_1) \cdots \Gamma_p(\rho_n)$$

for $\mathrm{Re}(\alpha_1 - \rho_1 - \rho_n, \alpha_2 - \rho_1 - \rho_2, \cdots, \alpha_n - \rho_{n-1} - \rho_n, \gamma_1 - \rho_1 - \cdots - \rho_n, \rho_i) > (p-1)/2$, where, $i = 1, \cdots, n$.

9.3 In this concluding section of the thesis I prove three results- one for the function $H_B^{(n)}$ and two for the function $H_C^{(n)}$ of matrix arguments.

Theorem 9.3.1:

$$H_B^{(n)}(\alpha_1, \cdots, \alpha_n; \gamma_1, \cdots, \gamma_n; -X_1, \cdots, -X_n)$$
$$= \frac{1}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2) \cdots \Gamma_p(\alpha_n)} \int_{T_1>0} \cdots \int_{T_n>0} e^{-\mathrm{tr}(T_1 + \cdots + T_n)}$$
$$\times |T_1|^{\alpha_1-(p+1)/2} |T_2|^{\alpha_2-(p+1)/2} \cdots |T_n|^{\alpha_n-(p+1)/2} \times \tag{9.3}$$
$${}_0F_1(\,;\gamma_1; -T_2^{1/2} T_1^{1/2} X_1 T_1^{1/2} T_2^{1/2})\, {}_0F_1(\,;\gamma_2; -T_3^{1/2} T_2^{1/2} X_2 T_2^{1/2} \times$$
$$T_3^{1/2}) \cdots {}_0F_1(\,;\gamma_n; -T_1^{1/2} T_n^{1/2} X_n T_n^{1/2} T_1^{1/2}) dT_1 \cdots dT_n$$

for $\mathrm{Re}(\alpha_i) > (p-1)/2$; where, $i = 1, \cdots, n$.

Proof: Taking the M-transform of the right side of eq.(9.3) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ respectively, we have,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$

$${}_0F_1(\,;\gamma_1;-T_2^{1/2}T_1^{1/2}X_1T_1^{1/2}T_2^{1/2})\,{}_0F_1(\,;\gamma_2;-T_3^{1/2}T_2^{1/2}X_2T_2^{1/2} \tag{9.4}$$

$$\times T_3^{1/2})\cdots\,{}_0F_1(\,;\gamma_n;-T_1^{1/2}T_n^{1/2}X_nT_n^{1/2}T_1^{1/2})dX_1\cdots dX_n$$

Making use of the transformations

$$Y_1=T_2^{1/2}T_1^{1/2}X_1T_1^{1/2}T_2^{1/2},\ Y_2=T_3^{1/2}T_2^{1/2}X_2T_2^{1/2}T_3^{1/2},\cdots,$$

$$Y_n=T_1^{1/2}T_n^{1/2}X_nT_n^{1/2}T_1^{1/2};$$

in the last expression and using the M-transform of a ${}_0F_1$ function, we get

$$|T_1|^{-\rho_1-\rho_n}|T_2|^{-\rho_1-\rho_2}\cdots|T_n|^{-\rho_{n-1}-\rho_n}\times$$

$$\frac{\Gamma_p(\gamma_1)\cdots\Gamma_p(\gamma_n)}{\Gamma_p(\gamma_1-\rho_1)\cdots\Gamma_p(\gamma_n-\rho_n)}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n) \tag{9.5}$$

which is to be substituted on the right side of eq.(9.3), followed by integrating out of $T_1,\cdots,T_n$ by using a Gamma integral to achieve $M\left[H_B^{(n)}\right]$ as given by eq.(9.1).

Theorem 9.3.2:

$$H_C^{(n)}(\alpha_1,\cdots,\alpha_n;\gamma;-X_1,\cdots,-X_n)$$

$$=\frac{1}{\Gamma_p(\alpha_1)\cdots\Gamma_p(\alpha_n)}\int_{T_1>0}\cdots\int_{T_n>0}e^{-tr(T_1+\cdots+T_n)}\times$$

$$|T_1|^{\alpha_1-(p+1)/2}\cdots|T_n|^{\alpha_n-(p+1)/2}\,{}_0F_1(\,;\gamma;-T_2^{1/2}T_1^{1/2}X_1\times$$

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$$T_1^{1/2} T_2^{1/2} - T_3^{1/2} T_2^{1/2} X_2 T_2^{1/2} T_3^{1/2} - \cdots - T_1^{1/2} T_n^{1/2} X_n \times$$

$$T_n^{1/2} T_1^{1/2}) dT_1 \cdots dT_n \qquad (9.6)$$

for $\mathrm{Re}(\alpha_i) > (p-1)/2;$ where, $i = 1, \cdots, n.$

Proof: This theorem follows in the same manner as the previous theorem, except that the use of eq.(2.4) is to be made here.

Theorem 9.3.3:

$$H_C^{(2m)}(\alpha_1, \cdots, \alpha_{2m}; \gamma; -X_1, \cdots, -X_{2m})$$

$$= \frac{1}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_3)\cdots\Gamma_p(\alpha_{2m-1})} \int_{T_1>0} \cdots (m) \cdots \int_{T_m>0} \times$$

$$e^{-\mathrm{tr}(T_1+\cdots+T_m)} |T_1|^{\alpha_1-(p+1)/2} |T_2|^{\alpha_3-(p+1)/2} \cdots \times$$

$$|T_{m-1}|^{\alpha_{2m-3}-(p+1)/2} |T_m|^{\alpha_{2m-1}-(p+1)/2} \Phi_2^{(m)}(\alpha_2, \alpha_4,$$

$$\cdots, \alpha_{2m}; \gamma; -T_1^{1/2} X_1 T_1^{1/2} - T_2^{1/2} X_2 T_2^{1/2}, -T_2^{1/2} X_3 T_2^{1/2} - T_3^{1/2} \times$$

$$X_4 T_3^{1/2}, \cdots, -T_{m-1}^{1/2} X_{2m-3} T_{m-1}^{1/2} - T_m^{1/2} X_{2m-2} T_m^{1/2},$$

$$-T_m^{1/2} X_{2m-1} T_m^{1/2} - T_1^{1/2} X_{2m} T_1^{1/2}) dT_1 \cdots dT_m \qquad (9.7)$$

for $\mathrm{Re}(\alpha_1, \alpha_3, \cdots, \alpha_{2m-1}) > (p-1)/2.$

Proof: Taking the M-transform of the right side of eq.(9.7) with respect to the variables $X_1, \cdots, X_{2m}$ and the parameters $\rho_1, \cdots, \rho_{2m}$ respectively, we obtain

$$\int_{X_1>0} \cdots (2m) \cdots \int_{X_{2m}>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_{2m}|^{\rho_{2m}-(p+1)/2} \times$$

Continued to the next page … … … … …

$$\Phi_2^{(m)}(\alpha_2,\alpha_4,\cdots,\alpha_{2m};\gamma;-T_1^{1/2}X_1T_1^{1/2}-T_2^{1/2}X_2T_2^{1/2},$$
$$-T_2^{1/2}X_3T_2^{1/2}-T_3^{1/2}X_4T_3^{1/2},\cdots,-T_{m-1}^{1/2}X_{2m-3}T_{m-1}^{1/2} \tag{9.8}$$
$$-T_m^{1/2}X_{2m-2}T_m^{1/2},-T_m^{1/2}X_{2m-1}T_m^{1/2}-T_1^{1/2}X_{2m}T_1^{1/2})\times$$
$$dX_1\cdots dX_{2m}$$

Applying the transformations

$$Z_1=T_1^{1/2}X_1T_1^{1/2}, Z_2=T_2^{1/2}X_2T_2^{1/2}, Z_3=T_2^{1/2}X_3T_2^{1/2}, Z_4=T_3^{1/2}\times$$
$$X_4T_3^{1/2},\cdots,Z_{2m-3}=T_{m-1}^{1/2}X_{2m-3}T_{m-1}^{1/2}, Z_{2m-2}=T_m^{1/2}X_{2m-2}T_m^{1/2},$$
$$Z_{2m-1}=T_m^{1/2}X_{2m-1}T_m^{1/2}, Z_{2m}=T_1^{1/2}X_{2m}T_1^{1/2};$$

to the last expression followed by the use of another set of transformations in it,

$U_1=Z_1, U_2=Z_1+Z_2; U_3=Z_3, U_4=Z_3+Z_4;\cdots;U_{2m-3}=Z_{2m-3},$
$U_{2m-2}=Z_{2m-3}+Z_{2m-2}; U_{2m-1}=Z_{2m-1}, U_{2m}=Z_{2m-1}+Z_{2m};$
with, $dU_1dU_2=dZ_1dZ_2, dU_3dU_4=dZ_3dZ_4,\cdots,dU_{2m-3}dU_{2m-2}=$
$dZ_{2m-3}dZ_{2m-2}, dU_{2m-1}dU_{2m}=dZ_{2m-1}dZ_{2m};$ where, $0<U_1<U_2,$
$0<U_3<U_4,\cdots,0<U_{2m-3}<U_{2m-2}, 0<U_{2m-1}<U_{2m};$

then integrating out the m variables $U_1,U_3,\cdots,U_{2m-3},U_{2m-1}$ in the ensuing expression by employing a type-1 Beta integral and afterwards using the eq.(4.6) leads us to

$$|T_1|^{-\rho_1-\rho_{2m}}|T_2|^{-\rho_2-\rho_3}\ldots|T_m|^{-\rho_{2m-2}-\rho_{2m-1}}\times$$
$$\frac{\Gamma_p(\alpha_2-\rho_1-\rho_2)\Gamma_p(\alpha_4-\rho_3-\rho_4)\cdots\Gamma_p(\alpha_{2m}-\rho_{2m-1}-\rho_{2m})}{\Gamma_p(\alpha_2)\Gamma_p(\alpha_4)\cdots\Gamma_p(\alpha_{2m})\Gamma_p(\gamma-\rho_1-\cdots-\rho_{2m})} \tag{9.9}$$
$$\times\Gamma_p(\gamma)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_{2m})$$

Substituting this expression on the right side of eq.(9.7) and then integrating out the variables $T_1, \cdots, T_m$ in the resulting expression by using a Gamma integral produces

$$\frac{\Gamma_p(\alpha_2-\rho_1-\rho_2)\Gamma_p(\alpha_4-\rho_3-\rho_4)\cdots\Gamma_p(\alpha_{2m}-\rho_{2m-1}-\rho_{2m})}{\Gamma_p(\alpha_2)\Gamma_p(\alpha_4)\cdots\Gamma_p(\alpha_{2m})\Gamma_p(\gamma-\rho_1-\cdots-\rho_{2m})}$$

$$\times\frac{\Gamma_p(\gamma)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_{2m})}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_3)\cdots\Gamma_p(\alpha_{2m-1})}\Gamma_p(\alpha_1-\rho_1-\rho_{2m})\times \tag{9.10}$$

$$\Gamma_p(\alpha_3-\rho_2-\rho_3)\cdots\Gamma_p(\alpha_{2m-1}-\rho_{2m-2}-\rho_{2m-1})$$

which is $M\left[H_C^{(2m)}\right]$ as can be seen from eq.(9.2) when interpreted for the case $n = 2m$.

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