EXTON'S QUADRUPLE HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENTS –I

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ABSTRACT

Exton [1] in 1972 defined and studied some properties of the hypergeometric functions of four variables. Prior to him no specific study of the quadruple hypergeometric functions had been made, except the four Lauricella functions $F_A^{(4)}, F_B^{(4)}, F_C^{(4)}$ and $F_D^{(4)}$ and some of their limiting cases. In the present paper we have defined the Exton's quadruple hypergeometric functions $K_5, K_{12}, K_{16}, K_{20}$ and K_{21} for the matrix arguments case and have proved a result for each of these functions besides a result for the function K_{11} and a case of reducibility for the function K_{12}.

INTRODUCTION

We have [5,6] earlier defined and studied the Exton's quadruple hypergeometric functions K_3, K_6 and K_{11} for the matrix arguments case. In continuation of these studies we are now defining and studying five more of the Exton's four variable hypergeometric functions with matrix

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arguments. All the matrices appearing in this paper are $(p\times p)$ real symmetric positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [2,3].

1. Preliminary Definitions

DEFINITION 1.1: The K_5 - function of matrix arguments

$$K_5 = K_5(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;-X,-Y,-Z,-T)$$

is defined as that class of functions for which the matrix transform (M-transform) is the following:

$$M(K_5) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_5(a,a,a,a;b_1,b_1,b_2,b_2;$$
$$c_1,c_2,c_3,c_4;-X,-Y,-Z,-T)dXdYdZdT$$
$$=\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b_1-\rho_1-\rho_2)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_3-\rho_4)}{\Gamma_p(b_2)}$$
$$\times\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)}\frac{\Gamma_p(c_2)}{\Gamma_p(c_2-\rho_2)}\frac{\Gamma_p(c_3)}{\Gamma_p(c_3-\rho_3)}\frac{\Gamma_p(c_4)}{\Gamma_p(c_4-\rho_4)}\times$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$
$$\text{for }\ \mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4,b_1-\rho_1-\rho_2,b_2-\rho_3-\rho_4,c_1-\rho_1,$$
$$c_2-\rho_2,c_3-\rho_3,c_4-\rho_4,\rho_1,\rho_2,\rho_3,\rho_4)>(p-1)/2. \qquad (1.1)$$

DEFINITION 1.2:

$$K_{12} = K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;-X,-Y,-Z,-T)$$
$$M(K_{12}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;$$
$$c_1,c_1,c_2,c_2;-X,-Y,-Z,-T)dXdYdZdT$$

$$
= \frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)} \frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)} \times
$$

$$
\frac{\Gamma_p(b_3-\rho_3)}{\Gamma_p(b_3)} \frac{\Gamma_p(b_4-\rho_4)}{\Gamma_p(b_4)} \frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1-\rho_2)} \frac{\Gamma_p(c_2)}{\Gamma_p(c_2-\rho_3-\rho_4)} \times
$$

$$
\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4) \qquad (1.2)
$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b_1-\rho_1, b_2-\rho_2, b_3-\rho_3, b_4-\rho_4,$
$c_1-\rho_1-\rho_2, c_2-\rho_3-\rho_4, \rho_1, \rho_2, \rho_3, \rho_4) > (p-1)/2.$

DEFINITION 1.3:

$$
K_{16} = K_{16}(a_1, a_2, a_3, a_4; b; -X, -Y, -Z, -T)
$$

$$
M(K_{16}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times
$$

$$
|Z|^{\rho_3-(p+1)/2} |T|^{\rho_4-(p+1)/2} K_{16}(a_1, a_2, a_3, a_4; b;
$$

$$
-X, -Y, -Z, -T)\,dX\,dY\,dZ\,dT
$$

$$
= \frac{\Gamma_p(a_1-\rho_1-\rho_2)}{\Gamma_p(a_1)} \frac{\Gamma_p(a_2-\rho_1-\rho_3)}{\Gamma_p(a_2)} \frac{\Gamma_p(a_3-\rho_2-\rho_4)}{\Gamma_p(a_3)} \times
$$

$$
\frac{\Gamma_p(a_4-\rho_3-\rho_4)}{\Gamma_p(a_4)} \frac{\Gamma_p(b)}{\Gamma_p(b-\rho_1-\rho_2-\rho_3-\rho_4)} \times
$$

$$
\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)
$$

for Re $(a_1-\rho_1-\rho_2, a_2-\rho_1-\rho_3, a_3-\rho_2-\rho_4, a_4-\rho_3-\rho_4,$ (1.3)
$b-\rho_1-\rho_2-\rho_3-\rho_4, \rho_1, \rho_2, \rho_3, \rho_4) > (p-1)/2.$

DEFINITION 1.4:

$$
K_{20} = K_{20}(a_1, a_1, b_3, b_4; b_1, b_2, a_2, a_2; c, c, c, c; -X, -Y, -Z, -T)
$$

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$$M(K_{20}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_{20}(a_1,a_1,b_3,b_4;b_1,b_2,a_2,a_2;$$
$$c,c,c,c;-X,-Y,-Z,-T)dXdYdZdT$$
$$=\frac{\Gamma_p(a_1-\rho_1-\rho_2)}{\Gamma_p(a_1)}\frac{\Gamma_p(a_2-\rho_3-\rho_4)}{\Gamma_p(a_2)}\frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)}\times$$
$$\frac{\Gamma_p(b_3-\rho_3)}{\Gamma_p(b_3)}\frac{\Gamma_p(b_4-\rho_4)}{\Gamma_p(b_4)}\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3-\rho_4)}\times$$
$$\times\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$
for $\mathrm{Re}(a_1-\rho_1-\rho_2, a_2-\rho_3-\rho_4, b_i-\rho_i, \rho_i,$
$c-\rho_1-\rho_2-\rho_3-\rho_4) > (p-1)/2$, where $i=1,\cdots,4$. (1.4)

DEFINITION 1.5:

$$K_{21} = K_{21}(a,a,b_6,b_5;b_1,b_2,b_3,b_4;c,c,c,c;-X,-Y,-Z,-T)$$
$$M(K_{21}) = \int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_{21}(a,a,b_6,b_5;b_1,b_2,b_3,b_4;$$
$$c,c,c,c;-X,-Y,-Z,-T)dXdYdZdT$$
$$=\frac{\Gamma_p(a-\rho_1-\rho_2)}{\Gamma_p(a)}\frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)}\frac{\Gamma_p(b_3-\rho_3)}{\Gamma_p(b_3)}\times$$
$$\frac{\Gamma_p(b_4-\rho_4)}{\Gamma_p(b_4)}\frac{\Gamma_p(b_6-\rho_3)}{\Gamma_p(b_6)}\frac{\Gamma_p(b_5-\rho_4)}{\Gamma_p(b_5)}\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3-\rho_4)}$$
$$\times\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$
for $\mathrm{Re}(a-\rho_1-\rho_2, b_6-\rho_3, b_5-\rho_4, b_i-\rho_i, \rho_i,$ (1.5)
$c-\rho_1-\rho_2-\rho_3-\rho_4) > (p-1)/2$, where $i=1,\cdots,4$.

2. **Results**

THEOREM 2.1:

$$K_5(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;-X,-Y,-Z,-T)$$

$$= \frac{1}{\Gamma_p(a)} \int_{S>0} e^{-\mathrm{tr}(S)} |S|^{a-(p+1)/2} \Psi_2(b_1;c_1,c_2;-S^{1/2}XS^{1/2},$$
$$-S^{1/2}YS^{1/2})\Psi_2(b_2;c_3,c_4;-S^{1/2}ZS^{1/2},-S^{1/2}TS^{1/2})dS \tag{2.1}$$

for $\mathrm{Re}(a) > (p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(2.1) with respect to the variables X,Y,Z,T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively, we get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} |T|^{\rho_4-(p+1)/2} \Psi_2(b_1;c_1,c_2;-S^{1/2}XS^{1/2}, \tag{2.2}$$
$$-S^{1/2}YS^{1/2})\Psi_2(b_2;c_3,c_4;-S^{1/2}ZS^{1/2},-S^{1/2}TS^{1/2})dXdYdZdT$$

Applying the transformations,

$$X_1 = S^{1/2}XS^{1/2}, Y_1 = S^{1/2}YS^{1/2}, Z_1 = S^{1/2}ZS^{1/2}, T_1 = S^{1/2}TS^{1/2};$$

with, $dX_1 = |S|^{(p+1)/2}dX, dY_1 = |S|^{(p+1)/2}dY, dZ_1 = |S|^{(p+1)/2}dZ,$

$dT_1 = |S|^{(p+1)/2}dT;$ and, $|X_1| = |S||X|, |Y_1| = |S||Y|, |Z_1| = |S||Z|,$

$|T_1| = |S||T|;$ to the above expression and then writing the M-transforms of the two Ψ_2- functions involved gives,

$$|S|^{-\rho_1-\rho_2-\rho_3-\rho_4} \frac{\Gamma_p(b_1-\rho_1-\rho_2)}{\Gamma_p(b_1)} \frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)} \frac{\Gamma_p(c_2)}{\Gamma_p(c_2-\rho_2)} \times$$

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$$\frac{\Gamma_p(c_3)}{\Gamma_p(c_3-\rho_3)}\frac{\Gamma_p(c_4)}{\Gamma_p(c_4-\rho_4)}\frac{\Gamma_p(b_2-\rho_3-\rho_4)}{\Gamma_p(b_2)}\times \tag{2.3}$$
$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

Substituting this expression on the right side of eq.(2.1) and then integrating out S in the resulting expression by using a Gamma integral gives $M(K_5)$ as given by eq.(1.1).

THEOREM 2.2:

$$K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;-X,-Y,-Z,-T)$$
$$=\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(c_1-b_1-b_2)\Gamma_p(c_2-b_3-b_4)}$$
$$\times\int\int\int\int|U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}\times$$
$$|S|^{b_4-(p+1)/2}|I-U-V|^{c_1-b_1-b_2-(p+1)/2}\times$$
$$|I-W-S|^{c_2-b_3-b_4-(p+1)/2}\Big|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}$$
$$+W^{1/2}ZW^{1/2}+S^{1/2}TS^{1/2}\Big|^{-a}\,dUdVdWdS \tag{2.4}$$

for $\text{Re}(b_i,c_1-b_1-b_2,c_2-b_3-b_4)>(p-1)/2,$ where $i=1,\cdots,4.$

PROOF: Taking the M-transform of the right side of eq.(2.4) with respect to the variables X,Y,Z,T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively, we obtain,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\Big|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}$$

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$$+W^{1/2}ZW^{1/2}+S^{1/2}TS^{1/2}\Big|^{-a} dXdYdZdT \qquad (2.5)$$

On making use of the transformations,

$$X_1 = U^{1/2}XU^{1/2}, Y_1 = V^{1/2}YV^{1/2}, Z_1 = W^{1/2}ZW^{1/2}, T_1 = S^{1/2}TS^{1/2};$$

in the above expression and then integrating out X_1, Y_1, Z_1, T_1 by using a type-2 Dirichlet integral yields,

$$|U|^{-\rho_1}|V|^{-\rho_2}|W|^{-\rho_3}|S|^{-\rho_4}\frac{\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)}{\Gamma_p(a)}\times \qquad (2.6)$$

$$\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)$$

Substituting this expression on the right side of eq.(2.4) and then integrating out the variables U,V and W,S in the resulting expression by using a type-1 Dirichlet integral generates $M(K_{12})$ as given by eq.(1.2).

THEOREM 2.3: A case of reducibility:-

(i) $$\lim_{\alpha\to\infty} K_{12}(\alpha,\alpha,\alpha,\alpha;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;$$

$$\frac{-X}{\alpha},\frac{-Y}{\alpha},\frac{-Z}{\alpha},\frac{-T}{\alpha}) \qquad (2.7)$$

$$=\Phi_2(b_1,b_2;c_1;-X,-Y)\Phi_2(b_3,b_4;c_2;-Z,-T)$$

(ii) $$\lim_{\alpha\to\infty} K_{12}(\alpha,\alpha,\alpha,\alpha;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;$$

$$\frac{-X}{\alpha},\frac{-X}{\alpha},\frac{-Z}{\alpha},\frac{-Z}{\alpha}) \qquad (2.8)$$

$$={}_1F_1(b_1+b_2;c_1;-X)\,{}_1F_1(b_3+b_4;c_2;-Z)$$

PROOF: (i). This result follows by putting $a=\alpha$ in eq.(2.4) and replacing X by X/α, etc. and then proceeding to the limit as $\alpha\to\infty$ keeping in mind the lemma (2.3.1) page 40 of Mathai [3], and finally using the theorem 4.3 page 63 of Mathai [3] in the resulting expression.

(ii).Replacing the two Φ_2- functions in eq. (2.7) by their integral representations as given by the theorem 4.3 page 63 of Mathai [3], we get,

$$\lim_{\alpha\to\infty} K_{12}(\alpha,\alpha,\alpha,\alpha;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;$$
$$\frac{-X}{\alpha},\frac{-Y}{\alpha},\frac{-Z}{\alpha},\frac{-T}{\alpha})$$
$$=\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(c_1-b_1-b_2)\Gamma_p(c_2-b_3-b_4)}$$
$$\times\int\int\int\int|U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}\times \qquad (2.9)$$
$$|S|^{b_4-(p+1)/2}|I-U-V|^{c_1-b_1-b_2-(p+1)/2}\times$$
$$|I-W-S|^{c_2-b_3-b_4-(p+1)/2}e^{-\mathrm{tr}(UX+VY+WZ+ST)}\times$$
$$dUdVdWdS$$

The result in eq.(2.8) is obtained by putting Y=X and T=Z in eq. (2.9) and then applying the transformations,

$$U_1=U, V_1=U+V, W_1=W, S_1=W+S; \text{ with } dU_1dV_1=dUdV,$$

$$dW_1dS_1=dWdS, \text{ where, } 0< U_1<V_1<I, 0< W_1<S_1<I;$$

to it and then integrating out U_1 and W_1 in the resulting expression by using a type-1 Beta integral which leads to the desired result in the light of theorem 2.3.4 page 42 of Mathai [3].

THEOREM 2.4:

$$K_{16}(a_1,a_2,a_3,a_4;b;-X,-Y,-Z,-T)$$
$$=\frac{1}{\Gamma_p(a_2)\Gamma_p(a_3)}\int_{S_1>0}\int_{S_2>0}e^{-\mathrm{tr}(S_1+S_2)}|S_1|^{a_2-(p+1)/2}\times$$
$$|S_2|^{a_3-(p+1)/2}\Phi_2(a_1,a_4;b;-S_1^{1/2}XS_1^{1/2}-S_2^{1/2}YS_2^{1/2}, \qquad (2.10)$$
$$-S_1^{1/2}ZS_1^{1/2}-S_2^{1/2}TS_2^{1/2})dS_1dS_2$$
for $\mathrm{Re}(a_2,a_3)>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(2.10) with respect to the variables X,Y,Z,T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively, we have,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\Phi_2(a_1,a_4;b;-S_1^{1/2}XS_1^{1/2} \tag{2.11}$$
$$-S_2^{1/2}YS_2^{1/2},-S_1^{1/2}ZS_1^{1/2}-S_2^{1/2}TS_2^{1/2})dXdYdZdT$$

Making use of the transformations,

$$X_1=S_1^{1/2}XS_1^{1/2},Y_1=S_2^{1/2}YS_2^{1/2},Z_1=S_1^{1/2}ZS_1^{1/2},T_1=S_2^{1/2}TS_2^{1/2};$$

in the above equation and then applying the following transformations in the resulting expression so obtained,

$$X_2=X_1,Y_2=X_1+Y_1;Z_2=Z_1,T_2=Z_1+T_1;\text{ with, }dX_1dY_1= \tag{2.12}$$
$$dX_2dY_2,\text{ and, }dZ_1dT_1=dZ_2dT_2;\text{ where, }0<X_2<Y_2\text{ and}$$

$0<Z_2<T_2$, followed by first, integrating out of X_2 and Z_2 by utilizing a type-1 Beta integral, and afterwards writing the M-transform of a Φ_2 function in the consequent expression leads us to,

$$|S_1|^{-\rho_1-\rho_3}|S_2|^{-\rho_2-\rho_4}\frac{\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)}{\Gamma_p(a_1)\Gamma_p(a_4)}\times$$
$$\frac{\Gamma_p(b)\Gamma_p(a_1-\rho_1-\rho_2)\Gamma_p(a_4-\rho_3-\rho_4)}{\Gamma_p(b-\rho_1-\rho_2-\rho_3-\rho_4)} \tag{2.13}$$

Putting back this expression on the right side of eq.(2.10) and integrating out S_1 and S_2 by employing a Gamma integral generates $M(K_{16})$ as given by eq.(1.3) above.

THEOREM 2.5:

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

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$$= \frac{1}{\Gamma_p(b_1)\cdots\Gamma_p(b_4)} \int_{S_1>0}\cdots\int_{S_4>0} e^{-\mathrm{tr}(S_1+\cdots+S_4)} \times$$

$$|S_1|^{b_1-(p+1)/2} \cdots |S_4|^{b_4-(p+1)/2}\, \Psi_2(a;c,d:-S_1^{1/2}XS_1^{1/2} \quad (2.14)$$

$$-S_2^{1/2}YS_2^{1/2} - S_3^{1/2}ZS_3^{1/2}, -S_4^{1/2}TS_4^{1/2})dS_1\cdots dS_4$$

for Re $(b_i) > (p-1)/2,\ i = 1,\cdots,4.$

PROOF: Taking the M-transform of the right side of eq.(2.14) with respect to the variables X, Y, Z, T and the parameters $\rho_1, \rho_2, \rho_3, \rho_4$ respectively, we achieve,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0} |X|^{\rho_1-(p+1)/2}\,|Y|^{\rho_2-(p+1)/2} \times$$

$$|Z|^{\rho_3-(p+1)/2}\,|T|^{\rho_4-(p+1)/2}\, \Psi_2(a;c,d:-S_1^{1/2}XS_1^{1/2} \qquad (2.15)$$

$$-S_2^{1/2}YS_2^{1/2} - S_3^{1/2}ZS_3^{1/2}, -S_4^{1/2}TS_4^{1/2})dXdYdZdT$$

The application of the transformations,

$$X_1 = S_1^{1/2}XS_1^{1/2}, Y_1 = S_2^{1/2}YS_2^{1/2}, Z_1 = S_3^{1/2}ZS_3^{1/2}, T_1 = S_4^{1/2}TS_4^{1/2}; \quad (2.16)$$

to the above expression followed by the use of another set of transformations,

$$X_2 = X_1, Y_2 = X_1 + Y_1, Z_2 = X_1 + Y_1 + Z_1;\ \text{with, } dX_1dY_1dZ_1 = dX_2dY_2dZ_2;\ \text{where, } 0 < X_2 < Y_2 < Z_2;$$

and then first integrating out X_2 and Y_2 one-by-one and in order by using a type-1 Beta integral, afterwards, invoking the M-transform of a Ψ_2 function gives,

$$|S_1|^{-\rho_1}\cdots|S_4|^{-\rho_4}\, \frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_4)\Gamma_p(a-\rho_1-\cdots-\rho_4)\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(a)\Gamma_p(d-\rho_4)\Gamma_p(c-\rho_1-\rho_2-\rho_3)} \quad (2.17)$$

Substituting this expression on the right side of eq.(2.14) and integrating out $S_1,\cdots,S_4$ by the help of a Gamma integral gives $M(K_{11})$ as given by eq.(2.3) of the authors' paper [6].

THEOREM 2.6:

$$K_{20}(a_1,a_1,b_3,b_4;b_1,b_2,a_2,a_2;c,c,c,c;-X,-Y,-Z,-T)$$

$$=\frac{1}{\Gamma_p(b_1)\cdots\Gamma_p(b_4)}\int_{S_1>0}\cdots\int_{S_4>0}e^{-tr(S_1+\cdots+S_4)}\times$$

$$|S_1|^{b_1-(p+1)/2}\cdots|S_4|^{b_4-(p+1)/2}\,\Phi_2(a_1,a_2;c:-S_1^{1/2}XS_1^{1/2} \quad (2.18)$$

$$-S_2^{1/2}YS_2^{1/2},-S_3^{1/2}ZS_3^{1/2}-S_4^{1/2}TS_4^{1/2})dS_1\cdots dS_4$$

for $\mathrm{Re}\,(b_i)>(p-1)/2,\ i=1,\cdots,4.$

PROOF: Taking the M-transform of the right side of eq.(2.18) with respect to the variables X, Y, Z, T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively, and then first applying the same set of transformations as in eq.(2.16), followed by the application of another set of transformations as in eq.(2.12), after that integrating out X_2 and Z_2 by employing a type-1 Beta integral and subsequently utilizing the M-transform of a Φ_2 function yields,

$$|S_1|^{-\rho_1}\cdots|S_4|^{-\rho_4}\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_4)\Gamma_p(a_1-\rho_1-\rho_2)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c-\rho_1-\cdots-\rho_4)}\times \quad (2.19)$$

$$\Gamma_p(a_2-\rho_3-\rho_4)\Gamma_p(c)$$

Substitution of this expression in eq.(2.18) followed by integration of $S_1,\cdots,S_4$ by the use of a Gamma integral ultimately generates $M(K_{20})$ as given by eq.(1.4).

THEOREM 2.7:

$$K_{21}(a,a,b_6,b_5;b_1,b_2,b_3,b_4;c,c,c,c;-X,-Y,-Z,-T)$$

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$$=\frac{1}{\Gamma_p(a)\left\{\prod_{i=1}^{6}\Gamma_p(b_i)\right\}}\int_{R>0}\int_{S_1>0}\cdots\int_{S_6>0}e^{-\mathrm{tr}(R+\sum_{i=1}^{6}S_i)}\times$$

$$|R|^{a-(p+1)/2}\left\{\prod_{i=1}^{6}|S_i|^{b_i-(p+1)/2}\right\}{}_0F_1(\,;c;-S_1^{1/2}R^{1/2}XR^{1/2}S_1^{1/2}$$

$$-S_2^{1/2}R^{1/2}YR^{1/2}S_2^{1/2}-S_6^{1/2}S_3^{1/2}ZS_3^{1/2}S_6^{1/2}-S_5^{1/2}S_4^{1/2}TS_4^{1/2}S_5^{1/2})\times \quad (2.20)$$

$$dR\left\{\prod_{i=1}^{6}dS_i\right\}$$

for $\mathrm{Re}(a,b_i)>(p-1)/2,\ i=1,\cdots,6.$

PROOF: Taking the M-transform of the right side of eq.(2.20) with respect to the variables X, Y, Z, T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively, we obtain,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\,{}_0F_1(\,;c;-S_1^{1/2}R^{1/2}XR^{1/2}S_1^{1/2} \quad (2.21)$$

$$-S_2^{1/2}R^{1/2}YR^{1/2}S_2^{1/2}-S_6^{1/2}S_3^{1/2}ZS_3^{1/2}S_6^{1/2}-S_5^{1/2}S_4^{1/2}TS_4^{1/2}S_5^{1/2})$$

$$\times dXdYdZdT$$

The application of the transformations,

$$X_1=S_1^{1/2}R^{1/2}XR^{1/2}S_1^{1/2},\ Y_1=S_2^{1/2}R^{1/2}YR^{1/2}S_2^{1/2},$$

$$Z_1=S_6^{1/2}S_3^{1/2}ZS_3^{1/2}S_6^{1/2},\ T_1=S_5^{1/2}S_4^{1/2}TS_4^{1/2}S_5^{1/2};\ \text{with,}$$

$$dX_1=|S_1|^{(p+1)/2}|R|^{(p+1)/2}dX,\ dY_1=|S_2|^{(p+1)/2}|R|^{(p+1)/2}dY,$$

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$$dZ_1 = \left|S_6\right|^{(p+1)/2}\left|S_3\right|^{(p+1)/2} dZ, dT_1 = \left|S_5\right|^{(p+1)/2}\left|S_4\right|^{(p+1)/2} dT;$$

and, $\left|X_1\right| = \left|S_1\right|\left|R\right|\left|X\right|, \left|Y_1\right| = \left|S_2\right|\left|R\right|\left|Y\right|, \left|Z_1\right| = \left|S_6\right|\left|S_3\right|\left|Z\right|, \left|T_1\right| = \left|S_5\right|\left|S_4\right|\left|T\right|;$

to the last equation followed by the use of the theorem (3.3) page 55 of Mathai [3], produces,

$$\left\{\prod_{i=1}^{4}\left|S_i\right|^{-\rho_i}\right\}\left|S_5\right|^{-\rho_4}\left|S_6\right|^{-\rho_3}\left|R\right|^{-\rho_1-\rho_2}\frac{\left\{\prod_{i=1}^{4}\Gamma_p(\rho_i)\right\}\Gamma_p(c)}{\Gamma_p(c-\rho_1-\cdots-\rho_4)} \quad (2.22)$$

Replacing this expression on the right side of eq.(2.20), subsequently integrating out the variables of integration by using a Gamma integral generates $M(K_{21})$ in agreement with eq.(1.5).

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