Quantum constants of the motion for two-dimensional systems

E. G. Kalnins
Department of Mathematics, University of Waikato,
Hamilton, New Zealand,
e.kalnins@waikato.ac.nz
W. Miller, Jr.
School of Mathematics, University of Minnesota,
Minneapolis, Minnesota, 55455, U.S.A.,
miller@ima.umn.edu
and G. Pogosyan
Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
Dubna, Moscow Region, 14980, Russia,
pogosyan@thsun1.jinr.dubna.su

March 5, 2003

Abstract

Consider a non-relativistic Hamiltonian operator H in 2 dimensions consisting of a kinetic energy term plus a potential. We show that if the associated Schrödinger eigenvalue equation admits an orthogonal separation of variables then it is possible to generate algorithmically a canonical basis $\mathbf{Q}, \mathbf{P}$ where $P_1 = H$, P_2, are the other 2nd-order constants of the motion associated with the separable coordinates, and $[Q_i, Q_j] = [P_i, P_j] = 0$, $[Q_i, P_j] = \delta_{ij}$. The 3 operators Q_2, P_1, P_2 form a basis for the invariants. In general these are infinite-order differential operators. We shed some light on the general question of exactly when the Hamiltonian admits a constant of the motion that is polynomial in the momenta. We go further and consider all cases where the Hamilton-Jacobi equation admits a second-order constant of the motion, not necessarily associated with orthogonal separable coordinates, or even separable coordinates at all. In each of these cases we construct an additional constant of the motion.

1 Introduction

In the paper [1] the authors considered a classical Hamiltonian $H = H(\mathbf{x}, \mathbf{p}) = \sum_j g^{jj} p_j^2 + V(\mathbf{x})$ where $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{p} = (p_1, ..., p_n)$, such that the associated Hamilton-Jacobi equation $H(\mathbf{x}, \frac{\partial S}{\partial \mathbf{x}}) = E$ is additively separable in the orthogonal variables $\mathbf{x}$. In that case there is an explicit canonical change of coordinates from the variables $\mathbf{x}, \mathbf{p}$ with $\{x_i, p_j\} = \delta_{ij}$ to variables $\mathbf{Q}, \mathbf{P}$ where $P_1 = H$, $P_2, \cdots, P_n$ are the other second-order constants of the motion associated with the orthogonal separable x-coordinates, and $\{Q_i, Q_j\} = \{P_i, P_j\} = 0$, $\{Q_i, P_j\} = \delta_{ij}$. Thus the $2n-1$ functions $Q_2, \cdots, Q_n, P_1, \cdots, P_n$ form a basis for the invariants. Each invariant Q_j can be expressed as a sum of the form

$$Q_j = \sum_{k=1}^{n} M_k(x_k, \mathbf{P}), \tag{1}$$

see [2]. Through our new constructive approach we could say in advance for exactly which separable metrics and potentials Q_j is a polynomial in the momenta. We gave, in principle, a complete solution to this problem. Moreover, we showed how to characterize each term M_k in (1) by the Poisson brackets $\{M_k, P_j\}$. In this paper we extend our method to the quantum regime in two dimensions.

We have a Schrödinger operator

$$H \equiv L_1 = \Delta_2 + V(\mathbf{x}) \tag{2}$$

where Δ_2 is the Laplace-Beltrami operator on a real or complex two-dimensional Riemannian space and $\mathbf{x} = (x_1, x_2)$ are orthogonal coordinates on that space such that the Schrödinger equation

$$H\Psi = E\Psi \tag{3}$$

separates multiplicatively in these coordinates. We will show, in a formal but explicit sense, that it is possible to generate algorithmically a canonical basis $\mathbf{Q}, \mathbf{P}$ where $P_1 = H$, P_2 are the second-order constants of the motion associated with the separable coordinates, and $[Q_i, Q_j] = [P_i, P_j] = 0$, $[Q_i, P_j] = \delta_{ij}$, $1 \leq i, j \leq 2$. In general, the operators $\mathbf{Q}$ are of infinite order. (See [3] where we introduced infinite-order conformal symmetry operators for the time-dependent Schrödinger equation in one space variable.) We pay particular attention to the issue of when this method yields finite order differential symmetry operators. These questions of when a system with two second-order constants of the motion, classical or quantum, (generated by an orthogonal separation of variables) admits additional polynomial constants of the motion are closely related to the concept of superintegrability, [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

This first paper is concerned only with the two-dimensional quantum case, where no non-trivial R-separation occurs. However, we here also consider all cases where the Schrödinger equation admits a second-order constant of the motion, not necessarily associated with orthogonal separable coordinates, or even separable coordinates at all. In each instance we construct an additional constant of the motion. The second paper in this series will consider the three-dimensional case where R-separation becomes relevant.

2 Two-dimensional separable systems

Consider the case of orthogonal separable coordinates in a general Riemannian space, for which the Schrödinger operator has the form

$$H = L_1 = \frac{1}{f_1(x) + f_2(y)} \left(\partial_x^2 + \partial_y^2 + v_1(x) + v_2(y) \right). \tag{4}$$

and, due to the separability, there is the invariant

$$L_2 = \frac{f_2(y)}{f_1(x) + f_2(y)} \left(\partial_x^2 + v_1(x) \right) - \frac{f_1(x)}{f_1(x) + f_2(y)} \left(\partial_y^2 + v_2(y) \right),$$

i.e.,

$$[L_2, H] = 0,$$

[15, 16, 17, 18, 19]. We have the operator identities

$$f_1(x)H + L_2 = \partial_x^2 + v_1(x), \qquad f_2(y)H - L_2 = \partial_y^2 + v_2(y). \tag{5}$$

We look for a partial differential operator $M(H, L_2, x, \partial_x)$ that satisfies

$$[H, M] = \frac{1}{f_1(x) + f_2(y)} P(H, L_2). \tag{6}$$

where P is a given nonzero polynomial,

$$P(H, L_2) = \sum_{j,k} \alpha_{j,k} H^j L_2^k \tag{7}$$

and write it in the standard form

$$M(H, L_2, x, \partial_x) = \sum_{j,k} \left(X_{j,k}(x)\partial_x + \tilde{X}_{j,k}(x) \right) H^j L_2^k. \tag{8}$$

Comment 1 *If $M(H, L_2, x, \partial_x)$ contained partial dervitives in x of order ≥ 2 we could use the first identity $\partial_x^2 = f_1(x)H + L_2 - v_1(x)$, recursively, and rearrange terms to achieve the unique standard form (8).*

Note that we have the operator identity

$$[H, X(x)\partial_x + \tilde{X}(x)] = \tag{9}$$

$$\frac{1}{f_1(x) + f_2(y)}\Big((X'' + 2\tilde{X}')\partial_x + (\tilde{X}'' - v_1'X - 2v_1X') \\ +(2f_1X' + f_1'X)H + 2X'L_2\Big).$$

Thus the condition (6) is equivalent to the system of equations

$$X''_{j,k} + 2\tilde{X}'_{j,k} = 0,$$

$$\tilde{X}''_{j,k} - v_1'X_{j,k} - 2v_1X'_{j,k} + 2f_1X'_{j-1,k} + f_1'X_{j-1,k} + 2X'_{j,k-1} = \alpha_{j,k}.$$

Example 1 *We look for an M of first-order: $M = X(x)\partial_x + \tilde{X}(x)$ and take $P = \alpha_{0,0} + \alpha_{1,0}H + \alpha_{0,1}L_2$. Then equations (6) reduce to*

$$f_1'(x) = \frac{2\alpha_{1,0}}{c + \alpha_{0,1}x}, \qquad (c + \alpha_{0,1}x)v_1'(x) + 2\alpha_{0,1}v_1(x) + 2\alpha_{1,0} = 0.$$

Condition (8) makes sense, at least formally, for infinite order differential equations. Indeed, one can consider H, L_2 as parameters in these equations. Then once $M(H, L_2, x, \partial_x)$ is expanded as a power series in these parameters, the terms are reordered so that the powers of the parameters are on the right, before they are replaced by explicit differential operators. Alternatively one can consider the operator M as acting on a simultaneous eigenbasis of the commuting operators H and L_2, in which case the parameters are the eigenvalues. In this view we can write

$$M(H, L_2, x, \partial_x) = X(H, L_2, x)\partial_x + \tilde{X}(H, L_2, x) \tag{10}$$

and consider M as a first order ordinary differential operator in x that is analytic in the parameters H, L_2. Then the above system of equations can be written in the more compact form

$$X''' + 4(v_1 - f_1H - L_2)X' + 2(v_1' - f_1'H)X = -2P(H, L_2), \qquad \tilde{X} = -\frac{1}{2}X'. \tag{11}$$

The first equation (11) always has solutions for any f_1, v_1, say continuously differentiable. Thus we can always construct M and it will be analytic in the parameters H, L_2. (Of course, a basic question is for what choices of f_1, v_1, P do solutions M exist that are polynomials in the parameters H, L_2?) Further we have the result

Lemma 1 *Let* $\Psi_1(H, L_2, x), \Psi_2(H, L_2, x)$ *be a basis of solutions for the equation*

$$\left(\frac{d^2}{dx^2} + v_1(x) - f_1(x)H - L_2\right)\Psi(x) = 0. \tag{12}$$

Then $S_1(x) = \Psi_1^2, S_2(x) = \Psi_1\Psi_2, S_3(x) = \Psi_2^2$ *is a basis of solutions for the homogeneous equation*

$$S''' + 4(v_1 - f_1H - L_2)S' + 2(v_1' - f_1'H)S = 0. \tag{13}$$

Thus the general solution of the first equation (11) is a particular solution of this equation plus an arbitrary linear combination of S_1, S_2, S_3

Theorem 1 *The general solution of the third order non-homogeneous ordinary differential equation (11) can be obtained by quadratures from a basis* Ψ_1, Ψ_2 *of solutions of the second order homogeneous ordinary differential equation (12).*

PROOF: By the lemma, the general solution of (11) takes the form

$$X(H, L_2, x) = \sum_{k=1}^{3} c_k S_k(H, L_2, x) + X_0(H, L_2, x)$$

where X_0 is any particular solution of this equation. We review the standard construction of a particular solution of the non-homogeneous equation from a general solution of the homogeneous equation (13). This equation can be written as a first order system if we set $T = S', U = T'$. Then (13) is equivalent to the system

$$\frac{d}{dx}\begin{pmatrix} S \\ T \\ U \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2(v_1' - f_1'H) & -4(v_1 - f_1H - L_2) & 0 \end{pmatrix}\begin{pmatrix} S \\ T \\ U \end{pmatrix}, \tag{14}$$

or

$$\frac{d}{dx}\mathbf{X} = \mathcal{A}\mathbf{X}.$$

Then the invertible matrix

$$\mathcal{B} = \begin{pmatrix} S_1 & S_2 & S_3 \\ S_1' & S_2' & S_3' \\ S_1'' & S_2'' & S_3'' \end{pmatrix}$$

is a fundamental solution matrix for (14):

$$\frac{d}{dx}\mathcal{B} = \mathcal{A}\mathcal{B}.$$

(Note that $\det \mathcal{B}$ is the cube of the Wronskian of Ψ_1 and Ψ_2, hence a nonzero constant.) We want a fundamental solution matrix $\mathcal{S}$ for the non-homogeneous system

$$\frac{d}{dx}\begin{pmatrix} S \\ T \\ U \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2(v_1' - f_1'H) & -4(v_1 - f_1 H - L_2) & 0 \end{pmatrix}\begin{pmatrix} S \\ T \\ U \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2P(H, L_2) \end{pmatrix}, \tag{15}$$

or

$$\frac{d}{dx}\mathbf{X} = \mathcal{A}\mathbf{X} + \mathbf{A_0}.$$

Thus we want a solution of

$$\frac{d}{dx}\mathcal{S} = \mathcal{A}\mathcal{S} + \mathcal{A}_0, \qquad \mathcal{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2P(H, L_2) & -2P(H, L_2) & -2P(H, L_2) \end{pmatrix}.$$

Set $\mathcal{S}(x) = \mathcal{B}(x)\mathcal{C}(x)$, so that

$$\mathcal{S}' = \mathcal{B}'\mathcal{C} + \mathcal{B}\mathcal{C}' = \mathcal{A}\mathcal{B}\mathcal{C} + \mathcal{A}_0.$$

Since $\mathcal{B}' = \mathcal{A}\mathcal{B}$ we have $\mathcal{B}\mathcal{C}' = \mathcal{A}_0$ or

$$\mathcal{C}(x) = \int_{x_0}^{x} \mathcal{B}^{-1}(x)\mathcal{A}_0 dx + \mathcal{C}_0$$

where the constant x_0 and constant matrix $\mathcal{C}_0$ so that $\mathcal{C}$ is invertible. Then $\mathcal{S} = \mathcal{B}\mathcal{C}$ is the fundamental solution matrix. Q.E.D.

Theorem 2 *Let $\Psi_1(H, L_2 + \epsilon, x), \Psi_2(H, L_2 + \epsilon, x)$ be a basis of solutions for the equation*

$$\left(\frac{d^2}{dx^2} + v_1(x) - f_1(x)H - L_2 - \epsilon\right)\Psi(x) = 0, \tag{16}$$

such that the Wronskian is normalized by

$$\Psi_1\Psi_2' - \Psi_2\Psi' \equiv 1.$$

Set

$$w_j(H, L_2, x) = \frac{d}{d\epsilon}\Psi_j(, H, L_2{+}\epsilon, x)\,|_{\epsilon=0}, \qquad u_j(H, L_2, x) = \Psi_j(H, L_2{+}\epsilon, x)\,|_{\epsilon=0}, \qquad j = 1, 2.$$

Then

$$X_0(H, L_2, x) = u_1 w_2 - u_2 w_1 = u_1^2 \partial_{L_2}\frac{u_2}{u_1}$$

is a solution of the nonhomogeneous equation

$$X''' + 4(v_1 - f_1 H - L_2)X' + 2(v_1' - f_1'H)X = -2. \tag{17}$$

Recall that a basis for the solution space of the homogeneous equation is $(u_1)^2, (u_2)^2, u_1u_2$.

Similarly, we look for a finite order partial differential operator $N(H, L_2, y, \partial_y)$ that satisfies

$$[H, N] = \frac{-1}{f_1(x) + f_2(y)} P(H, L_2). \tag{18}$$

and write it in the standard form

$$N(H, L_2, y, \partial_y) = \sum_{j,k} \left(Y_{j,k}(y)\partial_y + \tilde{Y}_{j,k}(y) \right) H^j L_2^k. \tag{19}$$

Note the operator identity

$$[H, Y(y)\partial_y + \tilde{Y}(y)] = \tag{20}$$

$$\frac{1}{f_1(x) + f_2(y)} \Big((Y'' + 2\tilde{Y}')\partial_y + (\tilde{Y}'' - 2v_2Y' - v_2'Y)$$
$$+(2f_2Y' + f_2'Y)H - 2Y'L_2 \Big).$$

Thus, (18) is equivalent to the system of equations

$$Y''_{j,k} + 2\tilde{Y}'_{j,k} = 0,$$

$$\tilde{Y}''_{j,k} - 2v_2Y'_{j,k} - v_2'Y_{j,k} + 2f_2Y'_{j-1,k} + f_2'Y_{j-1,k} - 2Y'_{j,k-1} = -\alpha_{j,k}.$$

Then we write

$$N(H, L_2, y, \partial_y) = Y(H, L_2, y)\partial_y + \tilde{Y}(H, L_2, y) \tag{21}$$

and consider N as a first order ordinary differential operator in y that is analytic in the parameters H, L_2. The defining equations are

$$Y''' + 4(v_2 - f_2H + L_2)Y' + 2(v_2' - f_2'H)Y = 2P(H, L_2), \qquad \tilde{Y} = -\frac{1}{2}Y'. \tag{22}$$

Once we have obtained M and N, then we see that the operator $L_3 = M + N$ commutes with H:

$$[H, L_3] = [H, M] + [H, N] = \frac{1}{f_1 + f_2} P(H, L_2) - \frac{1}{f_1 + f_2} P(H, L_2) = 0.$$

Thus we can view L_3 as an infinite order differential symmetry operator for H. In special cases this will be a finite order operator.

It is important to note that L_3 is not just a function of H and L_2. Indeed, a straightforward computation yields

$$[L_2, N] = \frac{f_2}{f_1 + f_2} P(H, L_2), \qquad [L_2, N] = \frac{f_1}{f_1 + f_2} P(H, L_2),$$

so $[L_2, L_3] = P(H, L_2) \neq 0$.

Example 2 *Let us consider the quantum Hamiltonian*

$$H = \partial_x^2 + \partial_y^2 + x.$$

It is known to be associated with several symmetries, such as

$$\ell_0 = \partial_y, \quad \ell_1 = \{\partial_y, x\partial_y - y\partial_x\}_+ - \frac{1}{2}y^2,$$

$$\ell_2 = \partial_x\partial_y + \frac{1}{2}y,$$

where $\{A, B\}_+ = AB + BA$ *is the anticommutator of two operators. The occurence of* ℓ_0 *is obvious, because* y *is an ignorable variable for the Hamiltonian. How can we obtain* ℓ_1 *and* ℓ_2*, which are associated with the separation of the Schrödinger equation in parabolic and shifted parabolic coordinates, from our cartesian coordinate construction? The obvious separation in cartesian coordinates yields the additional second order symmetry*

$$L_2 = \frac{1}{2}(\partial_x^2 - \partial_y^2 + x).$$

Let us now consider the defining equations for a symmetry in the following form:

$$X''' + 4(x - \frac{1}{2}H - L_2)X' + 2X = (\frac{1}{2}H - L_2),$$

$$Y''' - 4(\frac{1}{2}H - L_2)Y' = -(\frac{1}{2}H - L_2).$$

These equations have the solutions

$$X = \frac{1}{2}(\frac{1}{2}H - L_2), \qquad Y = \frac{y}{4} - \frac{1}{8}.$$

The corresponding symmetry is thus finite and given by

$$L_3 = \frac{1}{2}(\partial_y^2\partial_x + \frac{1}{2}y\partial_y) - \frac{1}{4}\partial_y^2 = \{\ell_2, \partial_y\}_+ - \frac{1}{4}\partial_y^2 - \frac{1}{2}$$

We see that our construction yields reasonably easily the existence of ℓ_2 *and thereby* ℓ_1*. Note also that* $[\partial_y, \ell_1] = 2\ell_2$*.*

Example 3 *We consider cartesian coordinates in flat space and assume that the potential is separable in these coordinates. Thus we have* $f_1(x) = f_2(y) = \frac{1}{2}$ *and*

$$H = \partial_x^2 + \partial_y^2 + v_1(x) + v_2(y),$$

$$L_2 = \frac{1}{2}(\partial_x^2 + v_1(x)) - \frac{1}{2}(\partial_y^2 + v_2(y)).$$

We look for a 3rd order constant of the motion L_3*. Thus we have*

$$P(H, L_2) = \sum_{j+k\le 2} \alpha_{jk} H^j L_2^k$$

and M *must take the form*

$$M = (X_{10}\partial_x + \tilde{X}_{10})H + (X_{01}\partial_x + \tilde{X}_{01})L_2 + (X_{00}\partial_x + \tilde{X}_{00}, \quad \tilde{X}_{jk} = -\frac{1}{2}X'_{jk}.$$

Now condition (6) leads to the system of equations (labeled by the powers (j,k) *of* $H^jL_2^k$*)*

$$\begin{array}{ll} (2,0) & X'_{10} = \alpha_{20}, \\ (1,1) & X'_{01} + 2X'_{10} = \alpha_{11}, \\ (0,2) & 2X'_{01} = \alpha_{02}, \\ (1,0) & -\frac{1}{2}X'''_{10} - v_1X_{10} - 2v_1X'_{10} + X'_{00} = \alpha_{10}, \\ (0,1) & -\frac{1}{2}X'''_{01} - v_1X_{01} - 2v_1X'_{01} + 2X'_{00} = \alpha_{01}, \\ (0,0) & -\frac{1}{2}X'''_{00} - v_1X_{00} - 2v_1X'_{00} = \alpha_{00}. \end{array} \tag{23}$$

These equations are equivalent to

$$X_{10} = \alpha_{20}x + c_{10}, \quad X_{01} = \frac{1}{2}\alpha_{02}x + c_{01}, \quad \alpha_{02} + 4\alpha_{20} = 2\alpha_{11}, \tag{24}$$

$$X'_{00} = (\alpha_{20}x + c_{10})v'_1 + 2\alpha_{20}v_1 + \alpha_{10}, \tag{25}$$

$$2X'_{00} = (\frac{1}{2}\alpha_{02}x + c_{01})v'_1 + \alpha_{02}v_1 + \alpha_{01}, \tag{26}$$

$$-\frac{1}{2}X'''_{00} - v_1X_{00} - 2v_1X'_{00} = \alpha_{00}. \tag{27}$$

In the following computation we rule out the trivial case v_1 *a constant. Then we have the following possibilities:*

CASE 1: $(\alpha_{02} - 4\alpha_{20})x + (2c_{01} - 4c_{10}) \not\equiv 0$.

Then we can eliminate X'_{00} *from (25), (26), and solve the first order differential equation for* v_1 *to obtain*

$$v_1(x) = \frac{\alpha_{01} - 2\alpha_{10}}{\alpha_{02} - 4\alpha_{20}}\left[-1 + \frac{\xi}{([\frac{1}{2}\alpha_{02} - 2\alpha_{20}]x + [c_{01} - 2c_{10}])^2}\right].$$

where ξ *is a constant. ¿From this we can solve (25) to get*

$$X_{00} = [\alpha_{10} - \alpha_{20}(\alpha_{01} - 2\alpha_{10}]x - \frac{(\alpha_{01} - 2\alpha_{10})\xi}{\alpha_{02} - 4\alpha_{20}}\frac{[c_{10}(\frac{1}{2}\alpha_{02} - 2\alpha_{20}) - \alpha_{20}(c_{01} - 2c_{10})]}{([\frac{1}{2}\alpha_{02} - 2\alpha_{20}]x + [c_{01} - 2c_{10}])^2} + d_1.$$

Note that we must have $\alpha_{02} - 4\alpha_{20} \neq 0$ for otherwise v_1 would be a constant. Now we normalize our equations so that $\alpha_{02} - 4\alpha_{20} = 2$ and translate in the x coordinate to achieve $c_{01} - 2c_{10} = 0$. Substituting these expressions for X_{00} and v_1 into (27) and equating coefficients of powers of x we find the condition $c_{10} = 0$ Thus our solution is

$$\alpha_{02}-4\alpha_{20} = 2,\ c_{01} = c_{10} = 0,\ \alpha_{00} = (\alpha_{01}-2\alpha_{10})\left(\alpha_{10} - 2\alpha_{20}(\alpha_{01} - 2\alpha_{10})\right),\ \alpha_{11} = 1+4\alpha_{20},$$

$$v_1(x) = \frac{\alpha_{01} - 2\alpha_{10}}{2}\left(-1 + \frac{\xi}{x^2}\right), \quad X_{00} = \left[\alpha_{10} - \alpha_{20}(\alpha_{01} - 2\alpha_{10})\right] x + d_1,$$

$$X_{10} = \alpha_{20}x, \qquad X_{01} = \frac{1}{2}\alpha_{02}x.$$

CASE 2: $(\alpha_{02} - 4\alpha_{20})x + (2c_{01} - 4c_{10}) \equiv 0.$

Then we have

$$2\alpha_{10} - \alpha_{01} = 0,\ X_{10} = \alpha_{20}x + c_{10},\ X_{01} = 2X_{10},\ 4\alpha_{20} = \alpha_{11},$$

and the remaining equations are

$$X'_{00} = (\alpha_{20}x + c_{10})v'_1 + 2\alpha_{20}v_1 + \alpha_{10} \tag{28}$$

and (27). Assume first that $\alpha_{20} \neq 0$ Then we can normalize so $\alpha_{20} = 1$ and translate in x to achieve $c_{10} = 0$. Then we substitute (28) into (27) and solve for X_{00} to get

$$X_{00} = \frac{-1}{v'_1}\left[\alpha_{00} + \frac{1}{2}v'''_1 x + v''_1 + v'''_1 + 2v_1v'_1x + 4v_1^2 + 2\alpha_{10}v_1\right].$$

Computing X'_{00} from this expression and substituting back into (28) we obtain a fourth-order ordinary differential equation for the potential:

$$-v_1^{(4)}v'_1(1 + \frac{x}{2}) + v'''_1(\frac{1}{2}v''_1x + v''_1 - \frac{3}{2}v'_1)+ \tag{29}$$

$$v''_1(\alpha_{00} + v''_1 + 4v_1^2 + 2\alpha_{10}v_1 - 8v_1v'_1) - {v'_1}^2(3v'_1x + 4v_1 + 3\alpha_{10}) = 0.$$

If $\alpha_{20} = 0$ we have

$$c_{01} = \alpha_{02} = \alpha_{01} = \alpha_{11} = 0,\ X_{01} = 0,$$

$$X_{00} = c_{10}v_1 + \alpha_{10}x + d_0,$$

as well as equation (27). Substituting the expression for X_{00} into (27) we obtain a third-order ordinary differential equation for the potential:

$$\frac{1}{2}c_{10}v'''_1 + 3c_{10}v'_1v_1 + \alpha_{10}xv'_1 + d_0v'_1 + 2\alpha_{10}v_1 = -\alpha_{00}.$$

If $c_{10} \neq 0$ then we can normalize so $c_{10} = 1$ and add a constant to v_1 so that $d_0 = 0$. Then the equation for the potential simplifies to:

$$\frac{1}{2}v_1''' + 3v_1'v_1 + \alpha_{10}xv_1' + 2\alpha_{10}v_1 + \alpha_{00} = 0. \tag{30}$$

Finally, if $c_{10} = 0, \alpha_{10} \neq 0$ we can normalize so $\alpha_{10} = 1$ and translate in x to achieve $d_0 = 0$. Then we find the potential

$$v_1(x) = -\frac{\alpha_{00}}{2} + \frac{d_1}{x^2}. \tag{31}$$

A very similar computation, with the same polynomial $P(H, L_2)$ yields the possibilities for $v_2(y)$, and the construction of the operator N. Then $L_3 = M + N$ is a third-order quantum constant of the motion.

Example 4 *We look for third-order invariants where the manifold is a space of revolution, with x as an ignorable coordinate. Then we can take $f_1(x) \equiv 0$. The solutions for $v_1(x)$ are essentially reparametrizations of the solutions of the preceding example. Note, however, that $\alpha_{20} = 0$.*

Indeed, for any constant δ, $f_1(x), f_2(y), v_1(x), v_2(y)$ are solutions of (11), (22) for H, L_2 and some functions X, Y if and only if

$$\tilde{f}_1(x) = f_1(x) - \delta,\ \tilde{f}_2(y) = f_2(y) + \delta,\ \tilde{v}_1(x) = v_1(x),\ \tilde{v}_2(y) = v_2(y),$$

$$\tilde{H} = H,\ \tilde{L}_2 = L_2 + \delta H$$

also satisfy (11), (22). Thus we can set $\delta = \frac{1}{2}$ and read off the solution for $\tilde{f}_1(x) \equiv 0$ from the previous example with $f_1(x) = \frac{1}{2}$.

Example 5 *We look for third-order invariants where $v_1(x) \equiv 0$. Then the equations for $f_1(x)$ reduce to two:*

$$(\frac{1}{2}\alpha_{02}x + c_{01})f_1''' + 2\alpha_{02}f_1'' - 2(\alpha_{01}x + 2c_{00})f' - 4\alpha_{01}f = -4\alpha_{10}, \tag{32}$$

$$3\alpha_{02}f + \frac{3}{2}(\frac{1}{2}\alpha_{02}x + c_{01})f' - [\alpha_{20} - \alpha_{11}f + \alpha_{02}f^2]\frac{f''}{f'^2} = \frac{3}{2}\alpha_{11}. \tag{33}$$

These coupled equations are difficult to solve in general. Two obvious solutions are

$$f_1(x) = Ax, \qquad f_1(x) = \frac{A}{x^2} + B$$

where the relation between A, B and the other parameters can be determined by substitution into the equations.

Comment 2 *For any constant δ, $f_1(x), f_2(y), v_1(x), v_2(y)$ are solutions of (11), (22) for H, L_2 and some functions X, Y if and only if*

$$\tilde{f}_1(x) = f_1(x),\ \tilde{f}_2(y) = f_2(y),\ \tilde{v}_1(x) = v_1(x) + \delta f_1(x),\ \tilde{v}_2(y) = v_2(y) + \delta f_2(y),$$

$$\tilde{H} = H + \delta,\ \tilde{L}_2 = L_2$$

also satisfy (11), (22).

3 Comparison with the classical case

We review the corresponding classical case as treated in [1]. The Hamiltonian has the form

$$H = L_1 = \frac{p_x^2 + p_y^2 + v_1(x) + v_2(y)}{f_1(x) + f_2(y)}. \tag{34}$$

and, due to the separability, there is the invariant

$$L_2 = \frac{f_2(y)(p_x^2 + v_1(x)) - f_1(x)(p_y^2 + v_2(y))}{f_1(x) + f_2(y)}.$$

We look for a function $M(H, L_2, x, p_x)$ which satisfies

$$\{H, M\} = \frac{1}{f_1(x) + f_2(y)}. \tag{35}$$

The condition has the form

$$(-v_1'(x) + f_1'(x)H)\frac{\partial M}{\partial p_x} + 2p_x\frac{\partial M}{\partial x} = 1. \tag{36}$$

Note that

$$p_x^2 = L_2 + f_1(x)H - v_1(x). \tag{37}$$

We consider x to be a function of the independent variable $U = p_x^2$ and write (36) in the form

$$\frac{dM}{dp_x} = \frac{1}{-v_1'(x) + f_1'(x)H} = \frac{dx}{dU}.$$

The solution is

$$M(H, L_2, p_x) = \int\ {U'}^{-1} dp_x$$

where

$$U(x) = p_x^2 = -v_1(x) + f_1(x)H + L_2,$$

and we consider $U'^{-1} = \frac{dx}{dU}$ to be a function of $U = p_x^2$. As shown in [1] the possible functions M that are polynomials in p_x are exactly those that arise for systems in which x can be expressed as a function of p_x, L_2, H in the form

$$x = Q\left(\frac{U - L_2 + \alpha H + \beta}{\gamma H + \delta}\right)$$

where Q is any nonconstant polynomial and $\alpha, \beta, \gamma, \delta$ are constants with $|\alpha| + |\beta| \neq 0$. If Q has order m then M will be a rational function of the momenta p_x, p_y; it is the function $\tilde{M} = (\gamma H + \delta)^m M$ that is a polynomial. Note that $\tilde{M}$ satisfies

$$\{H, \tilde{M}\} = \frac{(\gamma H + \delta)^m}{f_1(x) + f_2(y)}.$$

It is hard to directly compare this p_x-based formulation of the classical system with our quantum formulation, because in the quantum case it is unclear how to express the operator of multiplication by x as a function of the operator ∂_x. However, it does make sense to formally expand $\tilde{M}$ in the form

$$\tilde{M}(x, P_x, H, L_2) = \sum_{i,j,k=0}^{\infty} M_{ijk}(x) p_x^i H^j L_2^k$$

and then use (37), recursively, to reduce this expression to the standard form

$$\tilde{M}(x, p_x, H, L_2) = \sum_{j,k=0}^{\infty} \left(X_{jk}(x) p_x + \tilde{X}_{jk}\right) H^j L_2^k = X(x, H, L_2) p_x + \tilde{X}(x, H, L_2) \tag{38}$$

We determine $\tilde{M} = X p_x + \tilde{X}$ by requiring that

$$\{H, \tilde{M}\} = \frac{1}{f_1(x) + f_2(y)} P(H, L_2). \tag{39}$$

where P is a given nonzero polynomial,

$$P(H, L_2) = \sum_{j,k} \alpha_{j,k} H^j L_2^k. \tag{40}$$

This is equivalent to the system of equations

$$\tilde{X}'_{j,k} = 0,$$

$$-v_1' X_{j,k} - 2 v_1 X'_{j,k} + 2 f_1 X'_{j-1,k} + f_1' X_{j-1,k} + 2 X'_{j,k-1} = \alpha_{j,k},$$

or, in terms of the functions $X(x, H, L_2), \tilde{X}(x, H, L_2)$,

$$2(v_1 - f_1 H - L_2) X' + (v_1' - f_1' H) X = -P(H, L_2), \qquad \tilde{X}' = 0. \tag{41}$$

Comparing equations (11) and (41), we see that our classical construction of constants of the motion carries over directly to the quantum case only for quantum systems such that $X''' = 0$. This quantization condition is seldom satisfied. To be precise, let us consider solutions $X(x, H, L_2)$ of (41) that are analytic in the variables (H, L_2) in a neighborhood of $(0, 0)$.

Theorem 3 *Formal solutions $X(x, H, L_2)$ of the classical equations (41) also satisfy the quantum equations (11) for the following metric components and potentials:*

$$v_1(x) = \frac{\beta}{(x+\alpha)^2}, \qquad f_1(x) = \frac{\gamma}{(x+\alpha)^2}, \tag{42}$$

$$v_1(x) = \frac{\gamma_1 x + \delta_1}{(x+\beta)^2} - \beta_1, \qquad f_1(x) = \frac{\gamma_2 x + \delta_2}{(x+\beta)^2} - \beta_2, \tag{43}$$

$$v_1(x) = \beta_1, \qquad f_1(x) = \beta_2, \tag{44}$$

$$v_1(x) = \alpha_1 x + \beta_1, \qquad f_1(x) = \alpha_2 x + \beta_2. \tag{45}$$

PROOF: Suppose $X(x, H, L_2) \not\equiv 0$ satisfies the classical equations corresponding to the metric and potential functions $f_1(x), v_1(x)$, and set

$$F(x, H, L_2) = v_1(x) - f_1(x)H - L_2.$$

Then X will also satisfy the quantum conditions (11) provided $X''' = 0$, i.e.,

$$X(x, H, L_2) = A(H, L_2)x^2 + B(H, L_2)x + C(H, L_2),$$

where A, B, C are analytic functions of the variables (H, L_2) in a neighborhood of $(0, 0)$. Then the condition (41)

$$2(2Ax + B)F + (Ax^2 + Bx + c)F' = -P(H, L_2), \tag{46}$$

where P is a nonzero polynomial, has the solution

$$F(x, H, L_2) = \frac{-(\frac{1}{3}Ax^3 + \frac{1}{2}Bx^2 + Cx) + D}{(Ax^2 + Bx + C)^2} = v_1(x) - f_1(x)H - L_2, \tag{47}$$

where $D = D(H, L_2)$.

If we choose $(H, L_2) = (H_0, L_0), (H_0', L_0')$ where $H_0 \neq H_)'$ in the common domain of definition of A, B, C and such that $|A| + |B| + |C| > 0$ at these points, then we can solve the equations

$$F(x, H_0, L_0) = v_1(x) - f_1(x)H - L_0, \qquad F(x, H_0', L_0') = v_1(x) - f_1(x)H - L_0'$$

for $v_1(x), f_1(x)$ to get expressions of the form

$$v_1(x) = \frac{P_1(x)}{Q(x)}, \qquad f_1(x) = \frac{P_2(x)}{Q(x)},$$

where P_1, P_2, Q are polynomials in x without a common factor. From (47) we have

$$Q(x)\left[-\frac{AP}{3}x^3 - \frac{BP}{2}x^2 - CPx + D\right] = [P_1(x) - P_2(x)H - Q(x)L_2] \quad (48)$$
$$\times \left[A^2x^4 + 2ABx^3 + (B^2 + 2AC)x^2 + 2BCx + C^2\right].$$

Let $m =$ order Q. Then the order of the polynomial dependence in x on the left-hand side of (48) is $m+3$, whereas the order in x on the right-hand side is $\geq m+4$, but the coefficient of this term is proportional to A^2. We have a contradiction unless $A \equiv 0$. Thus (48) becomes

$$Q(x)\left[-\frac{BP}{2}x^2 - CPx + D\right] = [P_1(x) - P_2(x)H - Q(x)L_2]\,(Bx+C)^2. \quad (49)$$

Case 1: order $Q >$ max order (P_1, P_2)

This implies $BP = 2B^2L_2$. If $B = 0$ then (49) implies $CP = 0$, hence $C = 0$. This is impossible, since $X \not\equiv 0$. Hence $B \neq 0$ and $P = 2L_2B$. Thus, we can write F in the form

$$F(x, H, L_2) = 4L_2^2\frac{L_2C^2P + D}{(Px + 2L_2C)^2} - L_2 = v_1(x) - F_1(x)H - L_2.$$

This will be in the desired form

$$F(x, H, L_2) = \frac{\beta - \gamma H}{(x+\alpha)^2} - L_2$$

provided $C = \alpha/2, P = L_2, D = \frac{1}{4}\beta - \frac{1}{4}\gamma H - \frac{1}{2}\alpha^2 L_2$, where α, β, γ are constants.

Case 2: order $Q =$ max order (P_1, P_2)

Then there are constants $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_3 \neq 0$, $|\alpha_1| + |\alpha_2| > 0$, such that

$$(\alpha_1 - \alpha_2 H - \alpha_3 L_2)B^2 = -\frac{\alpha_3}{2}BP.$$

If $B = 0$ then $CP = 0$, so $C = 0$, which is impossible. Hence $B \neq 0$ and

$$B = \frac{P}{R}, \quad \text{where } R = -\frac{2}{\alpha_3}(\alpha_1 - \alpha_2 H - \alpha_3 L_2).$$

Thus condition (49) becomes

$$-\frac{1}{2}Q(x)R\left[P^2x^2+2RPCx-2RD\right]=\left[P_1(x)-P_2(x)H-Q(x)L_2\right](Px+RC)^2. \tag{50}$$

There are two possibilities:

$$\text{i) } RC=\beta P \qquad \text{ii) } -2DR=R^2C^2.$$

For possibility i) we have

$$-\frac{1}{2}Q(x)R\left[P^2x^2+2\beta Px-2RD\right]=\left[P_1(x)-P_2(x)H-Q(x)L_2\right]P^2(x+\beta)^2,$$

so we must have $R^2D=\tilde{P}$, a polynomial in H, L_2 and $-2\frac{\beta R}{P}x+\frac{\tilde{P}}{P^2}=\tilde{R}x+\tilde{S}$, where $\tilde{R},\tilde{S}$ are linear in H, L_2. Thus we obtain

$$v_1(x)-f_1(x)H-L_2=\left[\frac{\gamma_1x+\delta_1}{(x+\beta)^2}-\beta_1\right]-\left[\frac{\gamma_2x+\delta_2}{(x+\beta)^2}-\beta_2\right]H-L_2.$$

For possibility ii) we have simply $v_1(x)=\beta_1, f_1(x)=\beta_2$.

Case 3: order $Q <$ max order (P_1, P_2)

Then we must have $B=0$, so (49) becomes

$$Q(x)\left[-CPx+D\right]=\left[P_1(x)-P_2(x)H-Q(x)L_2\right]C^2.$$

This implies $P=(-\alpha_1-\alpha_2H)C$ where $|\alpha_1|+|\alpha_2|>0$. Thus

$$F(x,H,L_2)=\frac{-CPx+D}{c^2}=v_1(x)-f_1(x)H-L_2,$$

so $D=(\beta_1-\beta_2H-L_2)C^2$ and

$$v_1(x)=\alpha_1x+\beta_1, \qquad f_1(x)=\alpha_2x+\beta_2.$$

Q.E.D.

Operators M associated with (42,43,45) are first order, whereas case (43) is infinite order. These are the only pairs $(v_1(x), f_1(x))$ such that the classical functions $\tilde{M}=X(x,H,L_2)p_x$ correspond exactly to the quantum operators $M=X(x,H,L_2)\partial_x-\frac{1}{2}X'(x,H,L_2)$.

Comment 3 *Noncanonical versions of operators constructed in this paper can also provide insight into the structure of invariants for a given problem. For example, consider the classical harmonic oscillator* $\tilde{H}=p_x^2+p_y^2+x^2+y^2$*,*

$f_1 = f_2 = \frac{1}{2}$, $v_1 = x^2, v_2 = y^2$. The classical constant of the motion that follows from our construction can be written as [1]

$$\tilde{L} = \tilde{M} + \tilde{N} = \log(\frac{p_x + ix}{p_x - ix}) - \log(\frac{p_y + iy}{p_y - iy}).$$

This suggests that the corresponding quantum operators should be $H = -\partial_x^2 - \partial_y^2 + x^2 + y^2$ and

$$L = M + N = \log\left((-\partial_x + x)(-\partial_x - x)^{-1}\right) - \log\left((-\partial_y + y)(-\partial_y - y)^{-1}\right)$$
$$= \log\left((-\partial_x + x)^2(-\partial_y - y)^2(\partial_x^2 - x^2 - 1)^{-1}(\partial_y^2 - y^2 + 1)^{-1}\right)$$
$$= \log\left((\partial_{xy}^2 - xy - \{x\partial_y - y\partial_x\})^2(\partial_x^2 - x^2 - 1)^{-1}(\partial_y^2 + y^2 + 1)^{-1}\right),$$

where

$$[H, L] = 0, \qquad [H, M] = -[H, N] = -4.$$

This is, in fact, correct. Note that since $-\partial_x^2 + x^2$, $-\partial_y^2 + y^2$ commute with H, this suggests that $\partial_{xy}^2 - xy$, $x\partial_y - y\partial_x$ commute with H, also correct. Indeed, since $[a^2, H] = -4a^2$ for $a = -\partial_x + x$, we can use the formal identity

$$[F(S), H] = F'(S)[S, H]$$

with $S = (a^2(\partial_x^2 - x^2 - 1)^{-1})$, $F(S) = \log(S)$ to obtain

$$[M, H] = [F(S), H] = -4.$$

Similarly $[N, H] = 4$. Thus the operator $M = \log\left((-\partial_x + x)(-\partial_x - x)^{-1}\right)$ defines a canonical set, although it is not written in canonical form.

4 A new approach to the classical case

Consider again the classical separable Hamiltonian in two dimensions:

$$H = L_1 = \frac{p_x^2 + p_y^2 + v_1(x) + v_2(y)}{f_1(x) + f_2(y)}. \tag{51}$$

Due to the separability, there is the invariant

$$L_2 = \frac{f_2(y)(p_x^2 + v_1(x)) - f_1(x)(p_y^2 + v_2(y))}{f_1(x) + f_2(y)},$$

and there are the relations

$$p_x^2 = -v_1 + f_1 H + L_2, \qquad p_y^2 = -v_2 + f_2 H - L_2.$$

In our earlier work we constructed functions L_1', L_2' satisfying the canonical commutation relations

$$\{L_i, L_j\} = \{L_i', L_j'\} = 0, \qquad \{L_i, L_j'\} = \delta_{ij}.$$

However, due to the form of construction it is difficult to detect the presence of invariants that contain terms such as $p_x p_y$. One can obtain only powers of these invariants from the canonical invariants. Thus we search for these invariants directly, starting out with the classical case.

We construct a function

$$\tilde{L} = A(x, y, L_1, L_2) p_x p_y + D(x, y, L_1, L_2) \tag{52}$$

such that $\{H, \tilde{L}\} = 0$, but $\{L_2, \tilde{L}\} \neq 0$. The condition $\{H, \tilde{L}\} = 0$ is equivalent to the system of equations

$$\begin{aligned} D_x &= -\tfrac{1}{2} A f_2' H - A_y(-v_2 + f_2 H - L_2) + \tfrac{1}{2} A v_2' \\ &= -(-v_2 + f_2 H - L_2)^{\frac{1}{2}} \partial_y [(-v_2 + f_2 H - L_2)^{\frac{1}{2}} A] \\ D_y &= -\tfrac{1}{2} A f_1' H - A_x(-v_1 + f_1 H + L_2) + \tfrac{1}{2} A v_1' \\ &= -(-v_1 + f_1 H + L_2)^{\frac{1}{2}} \partial_x [(-v_1 + f_1 H + L_2)^{\frac{1}{2}} A] \end{aligned} \tag{53}$$

The integrability condition for this system is

$$\begin{aligned} A_{xx} \quad & (-v_1 + f_1 H + L_2) + \tfrac{3}{2} A_x(-v_1' + f_1' H) + \tfrac{1}{2} A(-v_1'' + f_1'' H) \\ = \quad & A_{yy}(-v_2 + f_2 H - L_2) + \tfrac{3}{2} A_y(-v_2' + f_2' H) + \tfrac{1}{2} A(-v_2'' + f_2'' H), \end{aligned} \tag{54}$$

or, equivalently,

$$\begin{aligned} & (-v_1 + f_1 H + L_2)^{\frac{1}{2}} \partial_x [(-v_1 + f_1 H + L_2)^{\frac{1}{2}} \partial_x D] \\ & = (-v_2 + f_2 H - L_2)^{\frac{1}{2}} \partial_y [(-v_2 + f_2 H - L_2)^{\frac{1}{2}} \partial_y D]. \end{aligned} \tag{55}$$

In terms of new coordinates $\tilde{x}, \tilde{y}$ such that

$$\frac{d\tilde{x}}{dx} = (-v_1 + f_1 H + L_2)^{-\frac{1}{2}}, \qquad \frac{d\tilde{y}}{dy} = (-v_2 + f_2 H - L_2)^{-\frac{1}{2}},$$

this last equation is

$$\partial^2_{\tilde{x}\tilde{x}} D = \partial^2_{\tilde{y}\tilde{y}} D,$$

with general solution $D = D^{(1)}(\tilde{x} + \tilde{y}) + D^{(2)}(\tilde{x} - \tilde{y})$, where $D^{(1)}, D^{(2)}$ are arbitrary functions. Then

$$(-v_1 + f_1 H + L_2)^{\frac{1}{2}} (-v_2 + f_2 H - L_2)^{\frac{1}{2}} A = D^{(1)}(\tilde{x} + \tilde{y}) - D^{(2)}(\tilde{x} - \tilde{y}).$$

Thus for any separable metric and potential we can always find invariants of the form (52). Moreover, a straightforward computation yields $\{L_2, \tilde{L}\} = 2(D_y p_y - D_x p_x)$. Thus any invariant $\tilde{L}$, such that D is nonconstant as a function of x, y, is independent of L_2. A major question here is when are such invariants polynomials in the momenta?

Example 6 *Consider the harmonic oscillator*

$$H = p_x^2 + p_y^2 + \omega^2(x^2 + y^2).$$

Here $v_1 = \omega^2 x^2, v_2 = \omega^2 y^2, f_1 = f_2 = \frac{1}{2}$. *A solution for* $\tilde{L}$ *is* $\tilde{L} = p_x p_y + \omega^2 xy$.

Consider the anharmonic oscillator

$$H = p_x^2 + p_y^2 + \omega^2(4x^2 + y^2).$$

Here $v_1 = 4\omega^2 x^2, v_2 = \omega^2 y^2, f_1 = f_2 = \frac{1}{2}$. *A solution for* $\tilde{L}$ *is*

$$\tilde{L} = p_y(xp_y - yp_x) - \omega^2 xy^2 = -yp_x p_y + \frac{x}{2}H - xL_2 - 2\omega^2 xy^2.$$

Another difficulty with the canonical functions that we have computed is that, due to the form of their construction, they can't directly express invariants with cross terms, such as $yp_x - xp_y$. The missing class of invariants is of the general form

$$\hat{L} = B(x, y, L_1, L_2)p_x + C(x, y, L_1, L_2)p_y \tag{56}$$

such that $\{H, \hat{L}\} = 0$, but $\{L_2, \hat{L}\} \neq 0$. The condition $\{H, \hat{L}\} = 0$ is equivalent to the system of equations

$$\begin{aligned} C_x + B_y = 0, \\ C_y(-v_2 + f_2 H - L_2) + \frac{1}{2}C(-v_2' + f_2' H) + \\ B_x(-v_1 + f_1 H + L_2) + \frac{1}{2}B(-v_1' + f_1' H) = 0 \end{aligned} \tag{57}$$

In terms of new coordinates $\tilde{x}, \tilde{y}$ such that

$$\frac{d\tilde{x}}{dx} = (-v_1 + f_1 H + L_2)^{-\frac{1}{2}}, \qquad \frac{d\tilde{y}}{dy} = (-v_2 + f_2 H - L_2)^{-\frac{1}{2}},$$

we have

$$(\partial_{\tilde{x}\tilde{x}} - \partial_{\tilde{y}\tilde{y}})\tilde{B} = 0, \qquad (\partial_{\tilde{x}\tilde{x}} - \partial_{\tilde{y}\tilde{y}})\tilde{C} = 0,$$
$$\tilde{B} = (-v_1 + f_1 + L_2)^{\frac{1}{2}}B, \qquad \tilde{C} = (-v_2 + f_2 - L_2)^{\frac{1}{2}}C,$$

with general solutions

$$\tilde{B} = B^{(1)}(\tilde{x} + \tilde{y}) + B^{(2)}(\tilde{x} - \tilde{y}), \tilde{C} = B^{(1)}(\tilde{x} + \tilde{y}) - B^{(2)}(\tilde{x} - \tilde{y}),$$

where $B^{(1)}, B^{(2)}$ are arbitrary functions. Thus for any separable metric and potential we can always find invariants of the form (56). Moreover, a straightforward computation yields $\{L_2, \hat{L}\} \neq 0$, for $\hat{L} \neq 0$. Again it is important to know when such invariants are polynomials in the momenta.

Example 7 *Consider the harmonic oscillator*

$$H = p_x^2 + p_y^2 + \omega^2(x^2 + y^2).$$

Here $v_1 = \omega^2 x^2, v_2 = \omega^2 y^2, f_1 = f_2 = \frac{1}{2}$. *A polynomial solution for* $\hat{L}$ *is* $\hat{L} = yp_x - xp_y$.

Example 8 *We set* $C = \partial_x f, B = -\partial_y f$ *and look for solutions of the form* $f = X(x, H, L_2)Y(y, H, L_2)$. *The necessary and sufficient conditions for this are*

$$(-v_1 + f_1 H + L_2)X'' + \frac{1}{2}(-v_1' + f_1' H)X' = CX, \tag{58}$$

$$(-v_2 + f_2 H - L_2)Y'' + \frac{1}{2}(-v_2' + f_2' H)X' = CY,$$

where $C = C(H, L_2)$. *Consider the case of flat space where* $f_1 = f_2 = \frac{1}{2}$. *Then the condition for the* x*-equation takes the form*

$$(-v_1 + \lambda)X'' + \frac{1}{2}(-v_1' + f_1' H)X' = CX, \ \lambda = \frac{1}{2}H + L_2,$$

with a similar condition for the y*-equation. If we look for a solution of the form* $X = X_1 + \lambda X_2$ *with* $\partial_\lambda C = 0$ *we find two cases:*

$$v_1 = \frac{1}{4}Cx^2 + \frac{B}{x^2}, \quad X = \frac{1}{2}Cx^2 + \lambda, \tag{59}$$

$$v_1 = 2g(x) - Cx^2 + 2K, \quad X = g(x) + \lambda x, \tag{60}$$

where

$$(-3g + Cx^2 - 2K)g' + Cxg' - Cg = 0.$$

5 Quantum version of the new approach

We consider orthogonal separable coordinates in a general Riemannian space, for which the Schrödinger operator has the form

$$H = L_1 = \frac{1}{f_1(x) + f_2(y)} \left(\partial_x^2 + \partial_y^2 + v_1(x) + v_2(y)\right). \tag{61}$$

and the separability invariant is

$$L_2 = \frac{f_2(y)}{f_1(x) + f_2(y)} \left(\partial_x^2 + v_1(x)\right) - \frac{f_1(x)}{f_1(x) + f_2(y)} \left(\partial_y^2 + v_2(y)\right).$$

We have the usual operator identities

$$f_1(x)H + L_2 = \partial_x^2 + v_1(x), \qquad f_2(y)H - L_2 = \partial_y^2 + v_2(y), \tag{62}$$

and look for a partial differential operator $\tilde{L}(H, L_2, x, y)$ that satisfies

$$[H, \tilde{L}] = 0. \tag{63}$$

and $[L_2, \tilde{L}] \neq 0$. We require that the invariant take the standard form

$$\tilde{L}(H, L_2, x, y) = \sum_{j,k} \left(A_{j,k}(x,y)\partial_{xy} + B_{j,k}(x,y)\partial_x + C_{j,k}(x,y)\partial_y + D_{j,k}(x,y)\right) H^j L_2^k. \tag{64}$$

Using the operator identities

$$\partial_x H = H\partial_x - \frac{f_1'}{f_1+f_2}H + \frac{v_1'}{f_1+f_2},$$

$$\partial_y H = H\partial_y - \frac{f_2'}{f_1+f_2}H + \frac{v_2'}{f_1+f_2},$$

$$\partial_x L_2 = L_2\partial_x - \frac{f_1' f_2}{f_1+f_2}H + \frac{f_2 v_1'}{f_1+f_2},$$

$$\partial_y L_2 = L_2\partial_y + \frac{f_1 f_2'}{f_1+f_2}H - \frac{f_1 v_2'}{f_1+f_2},$$

we see that

$$(f_1(x) + f_2(y))[H, A(x,y)\partial_{xy} + B(x,y)\partial_x + C(x,y)\partial_y + D(x,y)] =$$

$$(A_{xx} + A_{yy} + 2B_y + 2C_x)\partial_{xy} + (B_{xx} + B_{yy} - 2A_y v_2 + 2D_x - Av_2')\partial_x$$
$$+(2A_y f_2 + Af_2')\partial_x H - 2A_y\partial_x L_2 + (C_{xx} + C_{yy} - 2A_x v_1 + 2D_y - Av_1')\partial_y$$
$$+(2A_x f_1 + Af_1')\partial_y H + 2A_x\partial_y L_2 + (D_{xx} + D_{yy} - 2B_x v_1 - 2C_y v_2 - Bv_1' - Cv_2')$$
$$+(2B_x f_1 + 2C_y f_2 + Bf_1' + Cf_2')H + (2B_x - 2C_y)L_2.$$

Thus the condition (63) is equivalent to the system of equations

$$\partial_{xx}A_{j,k} + \partial_{yy}A_{j,k} + 2\partial_y B_{j,k} + 2\partial_x C_{j,k} = 0, \tag{65}$$

$$\begin{aligned}\partial_{xx}B_{j,k} + \partial_{yy}B_{j,k} - 2\partial_y A_{j,k}v_2 + 2\partial_x D_{j,k} - A_{j,k}v_2' +\\ (2\partial_y A_{j-1,k}f_2 + A_{j-1,k}f_2') - 2\partial_y A_{j,k-1} = 0,\end{aligned} \tag{66}$$

$$\partial_{xx}C_{j,k}+\partial_{yy}C_{j,k}-2\partial_x A_{j,k}v_1+2\partial_y D_{j,k}-A_{j,k}v_1'+ \\ (2\partial_x A_{j-1,k}f_1+A_{j-1,k}f_1')+2\partial_x A_{j,k-1}=0, \quad (67)$$

$$\partial_{xx}D_{j,k}+\partial_{yy}D_{j,k}-2\partial_x B_{j,k}v_1-2\partial_y C_{j,k}v_2-B_{j,k}v_1'-C_{j,k}v_2' \quad (68)$$
$$+(2\partial_x B_{j-1,k}f_1+2\partial_y C_{j-1,k}f_2+B_{j-1,k}f_1'+C_{j-1,k}f_2')+(2\partial_x B_{j,k-1}-2\partial_y C_{j,k-1})=0.$$

In terms of the general symbolic operator

$$\tilde{L}(H,L_2,x,y)=A(x,y,H,L_2)\partial_{xy}+B(x,y,H,L_2)\partial_x + C(x,y,H,L_2)\partial_y \\ +D(x,y,H,L_2), \quad (69)$$

the conditions are

$$A_{xx}+A_{yy}+2B_y+2C_x=0, \quad (70)$$
$$B_{xx}+B_{yy}-2A_y v_2+2D_x-Av_2'+(2A_y f_2+Af_2')H-2A_y L_2=0, \quad (71)$$
$$C_{xx}+C_{yy}-2A_x v_1+2D_y-Av_1'+(2A_x f_1+Af_1')H+2A_x L_2=0, \quad (72)$$
$$D_{xx}+D_{yy}-2B_x v_1-2C_y v_2-Bv_1'-Cv_2' \quad (73)$$
$$+(2B_x f_1+2C_y f_2+Bf_1'+Cf_2')H+(2B_x-2C_y)L_2=0.$$

Consider the special case of conditions (70,71,72,73) such that $A\equiv 0$. Then there is a function $G(x,y,H,L_2)$ such that

$$B=-\partial_x G, \qquad C=\partial_y G,$$

and the determining conditions simplify to

$$1) \qquad G_{xxxy}+G_{xyyy}=0,$$

$$2) \qquad \frac{1}{2}G_{xxxx}+2G_{xx}v_1+G_x v_1'-(2G_{xx}f_1+G_x f_1')H-2G_{xx}L_2= \\ \frac{1}{2}G_{yyyy}+2G_{yy}v_2+G_y v_2'-(2G_{yy}f_2+G_y f_2')H+2G_{yy}L_2.$$

The first determining equation means that

$$G(x,y)=K(x,y)+F(x)+J(y)$$

where F,J are arbitrary and K is harmonic: $K_{xx}+K_{yy}=0$. This representation is unique in K,F,J, up to the addition of the harmonic separable function $\tilde{K}(x,y)=\frac{a}{2}(x^2-y^2)+bx+cy+d$. Alternatively, we can write

$$G(x,y)=z_1(x+iy)+z_2(x-iy)+F(x)+J(y)$$

where z_1,z_2 are arbitrary analytic functions. Then only condition 2) remains to be satisfied.

Example 9 *If we make the ansatz $G = X(x,H,L_2)Y(y,H,L_2)$ then, in addition to the well known angular momentum invariant given earlier, we find the following polynomial invariants:*

$$X = (\frac{1}{4} + L_2)\cos x + s(1+\beta H),\ Y = (\frac{1}{4} + L_2)\cosh y + t(1+\xi H), \quad (74)$$

$$v_1(x) = 2s\frac{\sin x}{\cos^2 x} + \frac{a_1}{\cos^2 x},\ f_1(x) = -2s\beta\frac{\sin x}{\cos^2 x} + \frac{a_2}{\cos^2 x},$$
$$v_2(y) = 2t\frac{\sinh y}{\cosh^2 y} + \frac{b_1}{\cosh^2 y},\ f_2(y) = -2t\xi\frac{\sinh y}{\cosh^2 y} + \frac{b_2}{\cosh^2 y},$$
$$D = -\frac{1}{2}(\frac{1}{4} + L_2)\left(t\cos x(1+\xi H) + s\cosh y(1+\beta H)\right).$$
$$\tilde{L} = -2x(y^2 + 4L_2)\partial_x + 2y(x^2 - 4L_2)\partial_y + x^2 - y^2, \quad (75)$$
$$v_1(x) = \frac{1}{8}x^2 + \frac{a_1}{x^2},\ f_1(x) = \frac{a_2}{x^2},\ v_2(y) = \frac{1}{8}y^2 + \frac{b_1}{y^2},\ f_2(y) = \frac{b_2}{y^2}.$$

Example 10 *Again we consider the special case of conditions (70,71,72,73) such that $A \equiv 0$ where now we require*

$$G(x,y) = -2\log(X(x) + Y(y)) + \mathcal{F}(x) + \mathcal{J}(y) = K(x,y) + F(x) + J(y)$$

where F, J are arbitrary and K is harmonic. Then the harmonic requirement on K implies that

$$K = -2\log(X+Y) + \tilde{F}(x) + \tilde{J}(y)$$

where

$$(X')^2 = \frac{\alpha}{12}X^4 + \frac{\beta}{3}X^3 + \gamma X^2 + 2\delta X + \phi,$$
$$(Y')^2 = -\frac{\alpha}{12}Y^4 + \frac{\beta}{3}Y^3 - \gamma Y^2 + 2\delta Y - \phi,$$
$$X'' = \frac{\alpha}{6}X^3 + \frac{\beta}{2}X^2 + \gamma X + \delta, \quad Y'' = -\frac{\alpha}{6}Y^3 + \frac{\beta}{2}Y^2 - \gamma Y + \delta.$$

Further,

$$\tilde{F}(x) = \frac{1}{3}\frac{X'''}{X'}, \qquad \tilde{J}(y) = \frac{1}{3}\frac{Y'''}{Y'},$$

and the metric and potential terms have the solution

$$v_1 - f_1 H = \frac{-\frac{a}{12}X^4 - \frac{b}{3}X^3 + \frac{b_1}{2}X^2 + \eta_1 X + \eta_2}{24(X')^2},$$

$$v_2 - f_2 H = \frac{\frac{a}{12}Y^4 - \frac{b}{3}Y^3 - \frac{b_1}{2}Y^2 + \eta_1 Y - \eta_2}{24(Y')^2}.$$

Here, $\alpha, \beta, \gamma, \delta, \phi$ *and*

$a = a^{(1)}+a^{(2)}H, b = b^{(1)}+b^{(2)}H, b_1 = b_1^{(1)}+b_1^{(2)}H, \eta_1 = \eta_1^{(1)}+\eta_1^{(2)}H, \eta_2 = \eta_2^{(1)}+\eta_2^{(2)}H$

are parameters. The remaining condition is

$$\frac{1}{2}F''''+2F''(v_1-f_1H-L_2)+F'(v_1-f_1'H)-\frac{1}{2}J''''-2J''(v_2-f_2H-L_2)-J'(v_2'-f_2'H) =$$

$$\frac{1}{36}\left(\frac{a}{2}X^2 + bX - \frac{a}{2}Y^2 + bY\right) + \frac{2}{3}\left(\frac{X'''}{X'}(v_1 - f_1H) - \frac{Y'''}{Y'}(v_2 - f_2H)\right)$$
$$+\tilde{F}'(v_1' - f_1'H) - \tilde{J}'(v_2' - f_2'H).$$

The simplest family of solutions is obtained by setting $F \equiv \tilde{F}, J \equiv \tilde{J}$ *and* $\alpha = \beta = a = b = 0$.

6 The general case

Now we consider the general case of conditions (70,71,72,73). Then there are two functions $F(x, y, H, L_2), G(x, y, H, L_2)$ such that

$$A = \partial_{xy}F, \qquad B = -\frac{1}{2}\partial_{xyy}F - \partial_x G, \qquad C = -\frac{1}{2}\partial_{xxy}F + \partial_y G,$$

and the determining conditions simplify to

$$1) \qquad 2G_{xyyy}+\frac{1}{2}F_{xyyyyy}+2F_{xyyy}(v_2-f_2H+L_2)+3F_{xyy}(v_2'-f_2H)+F_{xy}(v_2''-f_2''H) =$$

$$-2G_{xxxy}+\frac{1}{2}F_{xxxxxy}+2F_{xxxy}(v_1-f_1H-L_2)+3F_{xxy}(v_1'-f_1'H)+F_{xy}(v_1''-f_1''H),$$

$$\begin{aligned}2) \qquad \frac{1}{2}F_{xxxxyy} + 2F_{xxyy}(v_1 - f_1H) + F_{xxy}(v_2' - f_2'H) + \frac{1}{2}G_{xxxx} &\quad + \\ 2G_{xx}(v_1 - f_1H - L_2) + G_x(v_1' - f_1'H) &\quad = \\ -\frac{1}{2}F_{xxyyyy} - 2F_{xxyy}(v_2 - f_2H) - F_{xyy}(v_1' - f_1'H) + \frac{1}{2}G_{yyyy} &\quad + \\ 2G_{yy}(v_2 - f_2H + L_2) + G_y(v_2' - f_2'H).&\end{aligned}$$

Now we show that there are always solutions for these equations in which $A \not\equiv 0$, for any v_1, v_2, f_1, f_2. Set $G \equiv 0$ and make the ansatz $F = \mathcal{X}(x, H, L_2)\mathcal{Y}(y, H, L_2)$ where $\mathcal{X}'\mathcal{Y}' \neq 0$. Then equations 1) and 2) become

$$1) \qquad \frac{1}{2}\mathcal{X}^{(1)}\mathcal{Y}^{(5)}+2\mathcal{X}^{(1)}\mathcal{Y}^{(3)}(v_2-f_2H+L_2)+3\mathcal{X}^{(1)}\mathcal{Y}^{(2)}(v_2'-f_2'H)+\mathcal{X}^{(1)}\mathcal{Y}^{(1)}(v_2''-f_2''H) =$$

$$\frac{1}{2}\mathcal{X}^{(5)}\mathcal{Y}^{(1)}+2\mathcal{X}^{(3)}\mathcal{Y}^{(1)}(v_1-f_1H-L_2)+3\mathcal{X}^{(2)}\mathcal{Y}^{(1)}(v_1'-f_1'H)+\mathcal{X}^{(1)}\mathcal{Y}^{(1)}(v_1''-f_1''H),$$

$$2)\qquad \frac{1}{2}\mathcal{X}^{(4)}\mathcal{Y}^{(2)}+\frac{1}{2}\mathcal{X}^{(2)}\mathcal{Y}^{(4)}+2\mathcal{X}^{(2)}\mathcal{Y}^{(2)}(v_1+v_2-f_1H-f_2H)+$$
$$\mathcal{X}^{(2)}\mathcal{Y}^{(1)}(v_2'-f_2'H)+\mathcal{X}^{(1)}\mathcal{Y}^{(2)}(v_1'-f_1'H)=0.$$

Variables separate in these equations and we have

$$1a)\qquad \frac{1}{2}\mathcal{X}^{(5)}+2\mathcal{X}^{(3)}(v_1-f_1H-L_2)+3\mathcal{X}^{(2)}(v_1'-f_1'H)+\mathcal{X}^{(1)}(v_1''-f_1''H)=\alpha\mathcal{X}^{(1)},$$

$$1b)\qquad \frac{1}{2}\mathcal{Y}^{(5)}+2\mathcal{Y}^{(3)}(v_2-f_2H+L_2)+3\mathcal{Y}^{(2)}(v_2'-f_2'H)+\mathcal{Y}^{(1)}(v_2''-f_2''H)=\alpha\mathcal{Y}^{(1)},$$

where α is a constant. Similarly, if $\mathcal{X}^{(2)}\mathcal{Y}^{(2)}\neq 0$ we have the separation equations

$$2a)\qquad \frac{1}{2}\mathcal{X}^{(4)}+2\mathcal{X}^{(2)}(v_1-f_1H-L_2)+\mathcal{X}^{(1)}(v_1'-f_1'H)=\beta\mathcal{X}^{(2)},$$

$$2b)\qquad \frac{1}{2}\mathcal{Y}^{(4)}+2\mathcal{Y}^{(2)}(v_2-f_2H+L_2)+\mathcal{Y}^{(1)}(v_2'-f_2'H)=-\beta\mathcal{Y}^{(2)},$$

These equations are consistent if $\alpha=\beta=0$. Now set $X=\mathcal{X}', Y=\mathcal{Y}'$. We have a solution of equations (70,71,72,73) whenever $X'Y'\neq 0$ and X and Y satisfy the ordinary differential equations

$$X'''+4X'(v_1-f_1H-L_2)+2X(v_1'-f_1'H) = 0 \tag{76}$$
$$Y'''+4Y'(v_2-f_2H+L_2)+2Y(v_2'-f_2'H) = 0, \tag{77}$$

essentially the same as the third order homogeneous ordinary differential equation (13).

Now suppose that $X'Y'\equiv 0$, but $X'\neq 0$. Then we have $v_2'-f_2'H=0$ so v_2', f_2' are constants. Further, X satisfies the ordinary differential equation

$$\frac{1}{2}X''''+2X''(v_1-f_1L-L_2)+3X'(v_1'-f_1'H)+X'(v_1''-f_1''H)=0.$$

Finally, if $X=Y=1$ we have $v_2''-f_2''H=v_1''-f_1''H=\alpha(H)$, corresponding to oscillator potentials.

7 Lie form and nonorthogonal separation in two dimensions

We know that if a Hamiltonian

$$H = \sum_{i,j=1}^{2} g^{ij} p_i p_j$$

admits a constant of the motion L that is quadratic in the momenta

$$L = \sum_{i,j=1}^{2} a^{ij} p_i p_j, \qquad \{H, L\} = 0 \tag{78}$$

and if the roots of the determinant $|a^{ij} - \lambda g^{ij}|$ are distinct, then the eigenforms define new (separable) variables x_1, x_2 and the associated Schrödinger operator can be written in Liouville form

$$H = \frac{1}{f_1(x_1) + f_2(x_2)} \left(\partial_{x_1 x_1} + \partial_{x_2 x_2} + v_1(x_1) + v_2(x_2) \right).$$

However, it may be that the roots of this determinant are equal. In this case H cannot be put into Liouville form, but rather Lie form, which for a suitable choice of variables (non-separable) is

$$H = \frac{1}{x + B(y)} \left(\partial_{xy} + \frac{1}{2} K(y) \right) + \frac{1}{2} U'(y). \tag{79}$$

The associated quantum constant is

$$L = \partial_{xx} - 2yH + U(y). \tag{80}$$

How can we calculate a third invariant? We look for a quantum constant of the form

$$L' = M(H, L, y)\partial_y + N(H, L, y).$$

Applying the condition $[H, L'] = 0$, we see that the functions M and N are of the form

$$\begin{aligned} M &= \frac{1}{2H - U'(y)}, \qquad (81) \\ N &= -\frac{1}{2} \int \frac{U''K - 2K'H + K'U' + 4B'H^2 + 4B'U'H + B'U'^2}{\sqrt{L + 2yH - U}(2H - U')^2} dy. \end{aligned}$$

Here we have used the formal relation

$$\partial_x = \sqrt{L + 2yH - U}.$$

According to our operational calculus, these relations make sense if we expand in an (ordered) power series in H, L or when the operators are applied to a simultaneous eigenspace of H and L, as explained earlier. Note in particular that if we consider the free Hamiltonian then L' has the particularly simple form

$$L' = H \int \frac{B'(y)}{\sqrt{L + 2yH}} dy - \partial_y. \tag{82}$$

While this appears to be only a formal expression it is useful for obtaining concrete results.

Example 11 . *Consider the zero potential case $K = U \equiv 0$, $B(y) = y^2$. We can formally evaluate the integral in the expression for L' by integrating by parts, and then multiply by $3H$ to obtain*

$$L'' = 2(L + 2yH)^{1/2}(-L + yH) - 3H\partial_y \tag{83}$$

This expression can immediately be interpreted as the differential operator

$$\hat{L} = 2\partial_x(-L + yH) - 3\partial_y H = -2\partial_x^3 + \frac{1}{x + y^2}(6y\partial_x^2\partial_y - 3\partial_y^2\partial_x)$$

which can be verified to commute with H. Indeed, if $B(y)$ is a polynomial then, through integration by parts, we can always uncover a symmetry operator of finite order. In this particular example the Hamiltonian also admits a second order symmetry operator

$$\hat{N} = x\partial_x^2 + \frac{3}{4}\partial_y^2 - \frac{3xy + \frac{1}{3}y^3}{x + y^2}\partial_x\partial_y,$$

and $[\hat{N}, L] = \hat{L}$. However, for general polynomial $B(y)$ the corresponding invarian $\hat{L}$ cannot be obtained as a commutator of other finite differential invariants.

It is clear from the method of the example that if one takes $U(y)$ as a constant and $B(y)$ and $K(y)$ as polynomials then, as before, we can generate explicit finite order differental operators that commute with H. In any case it is always possible to give a formal series in H and L which can be taken to represent an infinite differential operator.

There is one remaining possibility for a quadratic constant of the motion (78) in two dimensions: the constant may be associated with *nonorthogonal* separation of variables. In two dimensions there is only one case: separation in light cone (null) coordinates, [19]. For this case the Hamiltonian takes the form

$$H = \partial_z\partial_{\bar{z}} + f(\bar{z})$$

and there is a first-order symmetry operator ∂_z, so ∂_{zz} is a second-order constant of the motion. In addition there is a quadratic constant

$$L = M\partial_z + \frac{i}{2}\int \bar{z}\frac{df}{d\bar{z}}d\bar{z}.$$

where $M = i(z\partial_z - \bar{z}\partial_{\bar{z}})$.

References

[1] E.G.Kalnins, J. Kress, W.Miller Jr. and G.S.Pogosyan. Complete sets of invariants for dynamical systems that admit a separation of variables; *J. Math. Phys.* **43**, 3592–3609, (2002).

[2] V.I. Arnold. Mathematical Methods of Classical Mechanics. (translated by K. Vogtmann and A. Weinstein) *Graduate Texts in Mathematics, 60, Springer-Verlag*, New York, 1978.

[3] E.G.Kalnins, R.D.Levine and W.Miller Jr. Conformal symmetries and generalized recurrences for heat and Schrödinger equations in one spatial dimension; pp. 237-256, in the volume "Mechanics, Analysis and Geometry: 200 years after Lagrange", M. M. Francaviglia, ed., North Holland, Amsterdam 1991.

[4] L.P.Eisenhart. Enumeration of Potentials for Which One-Particle Schrödinger Equations Are Separable; *Phys.Rev.* **74**, 87 (1948).

[5] J.Friš, V.Mandrosov, Ya.A.Smorodinsky, M.Uhlir and P.Winternitz. On Higher Symmetries in Quantum Mechanics; *Phys.Lett.* **16**, 354 (1965).

[6] J.Friš, Ya.A.Smorodinskii, M.Uhlír and P.Winternitz. Symmetry Groups in Classical and Quantum Mechanics; *Sov.J.Nucl.Phys.* **4**, 444 (1967).

[7] A.A.Makarov, Ya.A.Smorodinsky, Kh.Valiev and P.Winternitz. A Systematic Search for Nonrelativistic Systems with Dynamical Symmetries; *Nuovo Cimento* **A 52**, 1061 (1967).

[8] D.Bonatos, C.Daskaloyannis and K.Kokkotas. Deformed Oscillator Algebras for Two-Dimensional Quantum Superintegrable Systems; *Phys. Rev.* **A 50**, 3700 (1994).

[9] F.Calogero. Solution of a Three-Body Problem in One Dimension; *J.Math.Phys.* **10**, 2191 (1969).

[10] L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian and V.M.Ter-Antonyan. Elliptic Basis for a Circular Oscillator. *Nuovo Cimento*, **B 88**, 43 (1985), Two-Dimensional Hydrogen Atom: I. Elliptic Bases; *Theor.Math.Phys.* **61**, 1021 (1984); Hidden symmetry, Separation of Variables and Interbasis Expansions in the Two-Dimensional Hydrogen Atom. *J.Phys.*, **A 18**, 455 (1985).

[11] J.Hietarinta. Direct methods for the search of the second invariant. *Physics Report*, **147**, 87-154, (1987).

[12] E.G.Kalnins, W.Miller Jr. and G.S.Pogosyan. Superintegrability and associated polynomial solutions. Euclidean space and the sphere in two dimensions; *J.Math.Phys.* **37**, 6439, (1996).

[13] E.G.Kalnins, J. Kress, W.Miller Jr. and G.S.Pogosyan. Completeness of superintegrability in two-dimensional constant curvature spaces; *J. Phys. A: Math Gen.* **34**, 4705–4720, (2001).

[14] S.Gravel and P.Winternitz. Superintegrability with third-order integrals in quantum and classical mechanics; *CRM Preprint* , (2002).

[15] P. Stäckel. Habilitationsschrift, Halle, 1891

[16] L.P. Eisenhart. Separable Systems of Stäckel; *Annals of Math.* (2) **35**, 284-305 (1934).

[17] E.G. Kalnins and W. Miller. Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations; *SIAM J. Math. Anal.* **11**, 1011-1026 (1980).

[18] W.Miller, Jr. Symmetry and Separation of Variables. *Addison-Wesley Publishing Company*, Providence, Rhode Island, 1977.

[19] E.G. Kalnins, Separation of Variables for Riemannian Spaces of Constant Curvature, *Pitman*, Monographs and Surveys in Pure and Applied Mathematics 28, Longman, Essex, England, 1986.