TIME AVERAGING AND TURBULENCE TERMS IN METEOROLOGY

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Abstract. We discuss averaging in time for the planetary boundary layer in the Boussinesq approximation. We introduce a notion of instantaneous turbulent kinetic energy (ITKE) and then derive a balance equation for ITKE.

1. Introduction

It is important to have a rigorous treatment of averaging (or smoothing) processes applied to the Euler and Navier-Stokes Equations, especially since averaged values are registered by meteorological devices, they are also used for numerical simulations (ARPS, etc.).

2. Various Notions of Time Averaging

Any (scalar) flow $u(x,t)$ may be represented as (cf. [1, 2])

$$u(x,t) = \sum_{k=1}^{K} u_k(x,t), \tag{2.1}$$

where each $u_k(x,t)$ $(k = 1, 2 \ldots, K)$ describes a specific feature of the flow, where x is a spatial variable and t is time

In atmospheric science it is customary to view (2.1) as

$$u(x,t) = \overline{u(x,t)} + u'(x,t), \tag{2.2}$$

where $\overline{u(x,t)}$ is a **slowly varying average** component and $u'(x,t)$ is a **rapidly fluctuating** or **turbulent** component.

Date: February 25, 2003.

Standard (Reynolds) Time Averaging. We assume that $u(x,t)$ is a C^1 function of its variables. We assume that there is a typical averaging period T that includes a "large" number of oscillations in $u'(x,t)$, then $\overline{u(x,t)}$ is independent of time and is given by

$$\overline{u(x,t)} \equiv \frac{1}{T}\int_0^T u(x,s)\,ds, \tag{2.3}$$

so in this case $\overline{u(x,t)}$ is not slowly varying but a constant (in time) function for $t \in [0,T]$. It is easy to show that

$$\overline{u'(x,t)} = 0, \quad \overline{\overline{u(x,t)}v(x,t)} = \overline{u(x,t)}\,\overline{v(x,t)}, \quad \overline{\overline{u(x,t)}v'(x,t)} = 0 \tag{2.4}$$

and

$$\overline{\overline{u(x,t)}} = \overline{u(x,t)} \tag{2.5}$$

Using (2.4), we have

$$\overline{u(x,t)v(x,t)} = \overline{u(x,t)}\,\overline{v(x,t)} + \overline{u'(x,t)v'(x,t)} \tag{2.6}$$

and

$$\begin{aligned}&\overline{u(x,t)v(x,t)w(x,t)} = \overline{u(x,t)}\,\overline{v(x,t)}\,\overline{w(x,t)} + \overline{u(x,t)}\,\overline{v'(x,t)w'(x,t)}\\ &+ \overline{v(x,t)}\,\overline{u'(x,t)w'(x,t)} + \overline{w(x,t)}\,\overline{v'(x,t)u'(x,t)}\\ &+ \overline{u'(x,t)v'(x,t)w'(x,t)}\end{aligned} \tag{2.7}$$

Also for spatial derivatives under appropriate smoothness assumptions we have

$$\overline{\frac{\partial}{\partial x}u(x,t)} = \frac{\partial}{\partial x}\overline{u(x,t)}, \quad \overline{\frac{\partial}{\partial x}u'(x,t)} = \frac{\partial}{\partial x}\overline{u'(x,t)} = 0 \tag{2.8}$$

However, temporal derivatives and time averages do not commute, i.e.

$$\begin{aligned}&\overline{\frac{\partial}{\partial t}u(x,t)} = \overline{\frac{\partial}{\partial t}u'(x,t)}\\ =&\frac{1}{T}(u(x,T) - u(x,0)) = \frac{1}{T}(u'(x,T) - u'(x,0))\\ \neq&0 = \frac{\partial}{\partial t}\overline{u(x,t)} = \frac{\partial}{\partial t}\overline{u'(x,t)}.\end{aligned} \tag{2.9}$$

The problem with (2.9) might be helped by assuming that there exists an appropriate averaging period T so that

$$\overline{u(x,t)} = u(x,0) = u(x,T), \quad u'(x,0) = u'(x,T) = 0. \tag{2.10}$$

However, this is a superficial remedy since $\overline{u(x,t)}$ is a constant (in time) function and temporal derivative is zero.

Running Averages. We define

$$\overline{u(x,t)} \equiv \frac{1}{2T}\int_{t-T}^{t+T} u(x,s)\,ds. \tag{2.11}$$

Then, assuming (2.2), we see that (2.4)–(2.7) fail, i.e.,

$$\overline{u'(x,t)} \neq 0, \quad \overline{\overline{u(x,t)}v(x,t)} \neq \overline{u(x,t)}\,\overline{v(x,t)}, \quad \overline{\overline{u(x,t)}v'(x,t)} \neq 0, \tag{2.12}$$

$$\overline{u(x,t)v(x,t)} \neq \overline{u(x,t)}\,\overline{v(x,t)} + \overline{u'(x,t)v'(x,t)}, \tag{2.13}$$

However, (2.8) holds with some adjustments:

$$\overline{\frac{\partial}{\partial x}u(x,t)} = \frac{\partial}{\partial x}\overline{u(x,t)}, \quad \overline{\frac{\partial}{\partial x}u'(x,t)} = \frac{\partial}{\partial x}\overline{u'(x,t)} \neq 0 \tag{2.14}$$

and we see that (2.11) repairs the problem in (2.9) since now

$$\overline{\frac{\partial}{\partial t}u(x,t)} = \frac{1}{2T}(u(x,t+T) - u(x,t-T)) = \frac{\partial}{\partial t}\overline{u(x,t)}. \tag{2.15}$$

3. Turbulent Kinetic Energy

In the Boussinesq Approximation for incompressible media we have

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + F_{rx} \tag{3.1}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + F_{ry} \tag{3.2}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_0} + F_{rz} \tag{3.3}$$

$$\frac{D\theta}{Dt} = -w\frac{d\theta_0}{dz} \tag{3.4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3.5}$$

where u, v, w are the velocity components, p is pressure, θ is potential temperature and $\rho_0(z)$, $\theta_0(z)$ are given functions. Note that by (3.5)

$$\begin{aligned} \frac{Du}{Dt} &\equiv \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + u\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}_{0} \\ &= \frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z}. \end{aligned} \tag{3.6}$$

Also note that by (2.8) and (3.5) we have

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0. \tag{3.7}$$

We set (cf. [1])

$$\frac{\overline{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z}. \tag{3.8}$$

Definition 3.1. Instantaneous turbulent kinetic energy ITKE (x, y, z, t) is a function defined by

$$\text{ITKE}\,(x, y, z, t) :\equiv \frac{1}{2}\left((u')^2 + (v')^2 + (w')^2\right). \tag{3.9}$$

A standard notion of turbulent kinetic energy

$$TKE\,(x, y, z) \equiv \frac{1}{2}\overline{((u')^2 + (v')^2 + (w')^2)} \tag{3.10}$$

and various balance equations are available for TKE[1] Note that

$$TKE\,(x, y, z) = \overline{\text{ITKE}\,(x, y, z, t)} \tag{3.12}$$

[1]The balance equation

$$\frac{\overline{D}}{Dt}(TKE) = MP + BPL + TR - \varepsilon \tag{3.11}$$

is known but differs from our balance.

Theorem 3.1. *Let u, v, w, p, θ be the functions of (x, y, z, t) satisfying the Boussinesq approximation system of partial differential equations* (3.1)–(3.5)*. Then*

$$\overline{\frac{D}{Dt}(ITKE)} = MP + BPL + TR - \varepsilon, \tag{3.13}$$

where the mechanical production

$$\begin{aligned} MP \equiv & -\overline{u'u'}\frac{\partial \overline{u}}{\partial x} - \overline{u'v'}\frac{\partial \overline{u}}{\partial y} - \overline{u'w'}\frac{\partial \overline{u}}{\partial z} \\ & -\overline{v'u'}\frac{\partial \overline{v}}{\partial x} - \overline{v'v'}\frac{\partial \overline{v}}{\partial y} - \overline{v'w'}\frac{\partial \overline{v}}{\partial z} \\ & -\overline{w'u'}\frac{\partial \overline{w}}{\partial x} - \overline{w'v'}\frac{\partial \overline{w}}{\partial y} - \overline{w'w'}\frac{\partial \overline{w}}{\partial z} \\ & -\overline{u'u'\frac{\partial u'}{\partial x}} - \overline{u'v'\frac{\partial u'}{\partial y}} - \overline{u'w'\frac{\partial u'}{\partial z}} \\ & -\overline{v'u'\frac{\partial v'}{\partial x}} - \overline{v'v'\frac{\partial v'}{\partial y}} - \overline{v'w'\frac{\partial v'}{\partial z}} \\ & -\overline{w'u'\frac{\partial w'}{\partial x}} - \overline{w'v'\frac{\partial w'}{\partial y}} - \overline{w'w'\frac{\partial w'}{\partial z}}, \end{aligned} \tag{3.14}$$

the buyoncy production or loss

$$BPL \equiv \overline{w'\theta'}\frac{g}{\theta_0}, \tag{3.15}$$

the transport redistribution

$$TR \equiv -\frac{1}{\rho_0}\left(\overline{u'\frac{\partial p'}{\partial x} + v'\frac{\partial p'}{\partial y} + w'\frac{\partial p'}{\partial z}}\right) = -\frac{1}{\rho_0}\overline{\mathbf{u}'\cdot\nabla p'}, \tag{3.16}$$

and the frictional dissipation

$$\varepsilon \equiv -\left(\overline{u'F_{rx}{}' + v'F_{ry}{}' + w'F_{rz}{}'}\right) = -\overline{\mathbf{u}'\cdot\mathbf{F}_r{}'}. \tag{3.17}$$

Proof. We average (3.6) with use of (2.6) and (3.7) to obtain

$$\begin{aligned} \overline{\frac{Du}{Dt}} &= \overline{\frac{\partial u}{\partial t}} + \frac{\partial}{\partial x}(\overline{u}\,\overline{u} + \overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u}\,\overline{v} + \overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u}\,\overline{w} + \overline{u'w'}) \\ &= \overline{\frac{\partial u}{\partial t}} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} + \frac{\partial}{\partial x}(\overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'}) \\ &= \overline{\frac{\partial u'}{\partial t}} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} + \frac{\partial}{\partial x}(\overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'}). \end{aligned} \tag{3.18}$$

Then the mean equations of (3.1)–(3.5) will be (3.7) and

$$\overline{\frac{\partial u}{\partial t}}+\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}+\overline{w}\frac{\partial \overline{u}}{\partial z}=-\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x}+f\overline{v}-\left[\frac{\partial}{\partial x}(\overline{u'u'})+\frac{\partial}{\partial y}(\overline{u'v'})+\frac{\partial}{\partial z}(\overline{u'w'})\right]+\overline{F_{rx}}, \tag{3.19}$$

$$\overline{\frac{\partial v}{\partial t}}+\overline{u}\frac{\partial \overline{v}}{\partial x}+\overline{v}\frac{\partial \overline{v}}{\partial y}+\overline{w}\frac{\partial \overline{v}}{\partial z}=-\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial y}-f\overline{u}-\left[\frac{\partial}{\partial x}(\overline{v'u'})+\frac{\partial}{\partial y}(\overline{v'v'})+\frac{\partial}{\partial z}(\overline{v'w'})\right]+\overline{F_{ry}}, \tag{3.20}$$

$$\overline{\frac{\partial w}{\partial t}}+\overline{u}\frac{\partial \overline{w}}{\partial x}+\overline{v}\frac{\partial \overline{w}}{\partial y}+\overline{w}\frac{\partial \overline{w}}{\partial z}=-\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x}+g\frac{\overline{\theta}}{\theta_0}-\left[\frac{\partial}{\partial x}(\overline{w'u'})+\frac{\partial}{\partial y}(\overline{w'v'})+\frac{\partial}{\partial z}(\overline{w'w'})\right]+\overline{F_{rx}}, \tag{3.21}$$

$$\overline{\frac{\partial \theta}{\partial t}}+\overline{u}\frac{\partial \overline{\theta}}{\partial x}+\overline{v}\frac{\partial \overline{\theta}}{\partial y}+\overline{w}\frac{\partial \overline{\theta}}{\partial z}=-\overline{w}\frac{d\theta_0}{dz}-\left[\frac{\partial}{\partial x}(\overline{\theta'u'})+\frac{\partial}{\partial y}(\overline{\theta'v'})+\frac{\partial}{\partial z}(\overline{\theta'w'})\right]. \tag{3.22}$$

Now we first subtract the left-hand side (3.19) from the left-hand side (3.1) and multiply the result by u', i.e.,

$$\begin{aligned}&u'(\frac{\partial u}{\partial t}-\overline{\frac{\partial u}{\partial t}}+u'\frac{\partial u}{\partial x}+\overline{u}\frac{\partial u'}{\partial x}+v'\frac{\partial u}{\partial y}+\overline{v}\frac{\partial u'}{\partial y}+w'\frac{\partial u}{\partial z}+\overline{w}\frac{\partial u'}{\partial z})\\ =&u'(\frac{\partial u'}{\partial t}-\overline{\frac{\partial u}{\partial t}}+u'\frac{\partial u}{\partial x}+\overline{u}\frac{\partial u'}{\partial x}+v'\frac{\partial u}{\partial y}+\overline{v}\frac{\partial u'}{\partial y}+w'\frac{\partial u}{\partial z}+\overline{w}\frac{\partial u'}{\partial z}).\end{aligned} \tag{3.23}$$

Next, we subtract the left-hand side (3.20) from the left-hand side (3.2) and multiply the result by v', i.e.,

$$\begin{aligned}&v'(\frac{\partial v}{\partial t}-\overline{\frac{\partial v}{\partial t}}+u'\frac{\partial v}{\partial x}+\overline{u}\frac{\partial v'}{\partial x}+v'\frac{\partial v}{\partial y}+\overline{v}\frac{\partial v'}{\partial y}+w'\frac{\partial v}{\partial z}+\overline{w}\frac{\partial v'}{\partial z})\\ =&v'(\frac{\partial v'}{\partial t}-\overline{\frac{\partial v}{\partial t}}+u'\frac{\partial v}{\partial x}+\overline{u}\frac{\partial v'}{\partial x}+v'\frac{\partial v}{\partial y}+\overline{v}\frac{\partial v'}{\partial y}+w'\frac{\partial v}{\partial z}+\overline{w}\frac{\partial v'}{\partial z}).\end{aligned} \tag{3.24}$$

Finally, we subtract the left-hand side (3.21) from the left-hand side (3.3) and multiply the result by w', i.e.,

$$\begin{aligned}&w'(\frac{\partial w}{\partial t}-\overline{\frac{\partial w}{\partial t}}+u'\frac{\partial w}{\partial x}+\overline{u}\frac{\partial w'}{\partial x}+v'\frac{\partial w}{\partial y}+\overline{v}\frac{\partial w'}{\partial y}+w'\frac{\partial w}{\partial z}+\overline{w}\frac{\partial w'}{\partial z})\\ =&\,w'(\frac{\partial w'}{\partial t}-\overline{\frac{\partial w}{\partial t}}+u'\frac{\partial w}{\partial x}+\overline{u}\frac{\partial w'}{\partial x}+v'\frac{\partial w}{\partial y}+\overline{v}\frac{\partial w'}{\partial y}+w'\frac{\partial w}{\partial z}+\overline{w}\frac{\partial w'}{\partial z}).\end{aligned} \tag{3.25}$$

We subtract the right-hand side (3.19) from the right-hand side (3.1) and multiply the result by u', i.e.,

$$u'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}+fv'+\frac{\partial}{\partial x}(\overline{u'u'})+\frac{\partial}{\partial y}(\overline{u'v'})+\frac{\partial}{\partial z}(\overline{u'w'}).+F_{rx}{}'\right) \tag{3.26}$$

Then we subtract the right-hand side (3.20) from the right-hand side (3.2) and multiply the result by v', i.e.,

$$v'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial y} - fu' + \frac{\partial}{\partial x}(\overline{v'u'}) + \frac{\partial}{\partial y}(\overline{v'v'}) + \frac{\partial}{\partial z}(\overline{v'w'}). + F_{ry}{}'\right) \tag{3.27}$$

Finally, we subtract the right-hand side (3.21) from the right-hand side (3.3) and multiply the result by w', i.e.,

$$w'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g\frac{\theta'}{\theta_0} + \frac{\partial}{\partial x}(\overline{w'u'}) + \frac{\partial}{\partial y}(\overline{w'v'}) + \frac{\partial}{\partial z}(\overline{w'w'}) + F_{rz}{}'\right). \tag{3.28}$$

The sum of (3.23)–(3.25) equals the sum of (3.26)–(3.28), so

$$\begin{aligned}
&u'(\frac{\partial u'}{\partial t} - \overline{\frac{\partial u}{\partial t}} + u'\frac{\partial u}{\partial x} + \overline{u}\frac{\partial u'}{\partial x} + v'\frac{\partial u}{\partial y} + \overline{v}\frac{\partial u'}{\partial y} + w'\frac{\partial u}{\partial z} + \overline{w}\frac{\partial u'}{\partial z})\\
&+v'(\frac{\partial v'}{\partial t} - \overline{\frac{\partial v}{\partial t}} + u'\frac{\partial v}{\partial x} + \overline{u}\frac{\partial v'}{\partial x} + v'\frac{\partial v}{\partial y} + \overline{v}\frac{\partial v'}{\partial y} + w'\frac{\partial v}{\partial z} + \overline{w}\frac{\partial v'}{\partial z})\\
&+w'(\frac{\partial w'}{\partial t} - \overline{\frac{\partial w}{\partial t}} + u'\frac{\partial w}{\partial x} + \overline{u}\frac{\partial w'}{\partial x} + v'\frac{\partial w}{\partial y} + \overline{v}\frac{\partial w'}{\partial y} + w'\frac{\partial w}{\partial z} + \overline{w}\frac{\partial w'}{\partial z})\\
&=u'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + fv' + \frac{\partial}{\partial x}(\overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'}) + F_{rx}{}'\right)\\
&\quad + v'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial y} - fu' + \frac{\partial}{\partial x}(\overline{v'u'}) + \frac{\partial}{\partial y}(\overline{v'v'}) + \frac{\partial}{\partial z}(\overline{v'w'}) + F_{ry}{}'\right)\\
&\quad + w'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g\frac{\theta'}{\theta_0} + \frac{\partial}{\partial x}(\overline{w'u'}) + \frac{\partial}{\partial y}(\overline{w'v'}) + \frac{\partial}{\partial z}(\overline{w'w'}) + F_{rz}{}'\right)
\end{aligned} \tag{3.29}$$

or

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{1}{2}\left((u')^2+(v')^2+(w')^2\right)\right]-\left[u'\overline{\frac{\partial u}{\partial t}}+v'\overline{\frac{\partial v}{\partial t}}+w'\overline{\frac{\partial w}{\partial t}}\right]\\
&+u'u'\frac{\partial u}{\partial x}+u'\overline{u}\frac{\partial u'}{\partial x}+u'v'\frac{\partial u}{\partial y}+u'\overline{v}\frac{\partial u'}{\partial y}+u'w'\frac{\partial u}{\partial z}+u'\overline{w}\frac{\partial u'}{\partial z}\\
&+v'u'\frac{\partial v}{\partial x}+v'\overline{u}\frac{\partial v'}{\partial x}+v'v'\frac{\partial v}{\partial y}+v'\overline{v}\frac{\partial v'}{\partial y}+v'w'\frac{\partial v}{\partial z}+v'\overline{w}\frac{\partial v'}{\partial z}\\
&+w'u'\frac{\partial w}{\partial x}+w'\overline{u}\frac{\partial w'}{\partial x}+w'v'\frac{\partial w}{\partial y}+w'\overline{v}\frac{\partial w'}{\partial y}+w'w'\frac{\partial w}{\partial z}+w'\overline{w}\frac{\partial w'}{\partial z}\\
&=u'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}+fv'+\frac{\partial}{\partial x}(\overline{u'u'})+\frac{\partial}{\partial y}(\overline{u'v'})+\frac{\partial}{\partial z}(\overline{u'w'})+F_{rx}{}'\right)\\
&\quad+v'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial y}-fu'+\frac{\partial}{\partial x}(\overline{v'u'})+\frac{\partial}{\partial y}(\overline{v'v'})+\frac{\partial}{\partial z}(\overline{v'w'})+F_{ry}{}'\right)\\
&\quad+w'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial z}+g\frac{\theta'}{\theta_0}+\frac{\partial}{\partial x}(\overline{w'u'})+\frac{\partial}{\partial y}(\overline{w'v'})+\frac{\partial}{\partial z}(\overline{w'w'})+F_{rz}{}'\right)
\end{aligned}
\tag{3.30}
$$

We can rewrite (3.30) in the following way:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{1}{2}\left((u')^2+(v')^2+(w')^2\right)\right]+\overline{u}\frac{\partial}{\partial x}\left[\frac{1}{2}\left((u')^2+(v')^2+(w')^2\right)\right]\\
&+\overline{v}\frac{\partial}{\partial y}\left[\frac{1}{2}\left((u')^2+(v')^2+(w')^2\right)\right]+\overline{w}\frac{\partial}{\partial z}\left[\frac{1}{2}\left((u')^2+(v')^2+(w')^2\right)\right]\\
&-\left[u'\overline{\frac{\partial u}{\partial t}}+v'\overline{\frac{\partial v}{\partial t}}+w'\overline{\frac{\partial w}{\partial t}}\right]\\
&+u'u'\frac{\partial u}{\partial x}+u'v'\frac{\partial u}{\partial y}+u'w'\frac{\partial u}{\partial z}+v'u'\frac{\partial v}{\partial x}+v'v'\frac{\partial v}{\partial y}+v'w'\frac{\partial v}{\partial z}\\
&+w'u'\frac{\partial w}{\partial x}+w'v'\frac{\partial w}{\partial y}+w'w'\frac{\partial w}{\partial z}\\
&=u'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}+fv'+\frac{\partial}{\partial x}(\overline{u'u'})+\frac{\partial}{\partial y}(\overline{u'v'})+\frac{\partial}{\partial z}(\overline{u'w'})+F_{rx}{}'\right)\\
&\quad+v'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial y}-fu'+\frac{\partial}{\partial x}(\overline{v'u'})+\frac{\partial}{\partial y}(\overline{v'v'})+\frac{\partial}{\partial z}(\overline{v'w'})+F_{ry}{}'\right)\\
&\quad+w'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial z}+g\frac{\theta'}{\theta_0}+\frac{\partial}{\partial x}(\overline{w'u'})+\frac{\partial}{\partial y}(\overline{w'v'})+\frac{\partial}{\partial z}(\overline{w'w'})+F_{rz}{}'\right)
\end{aligned}
\tag{3.31}
$$

or

$$
\begin{aligned}
&\overline{\frac{D}{Dt}}\left[\frac{1}{2}\left((u')^2+(v')^2+(w')^2\right)\right]\\
&-\left[u'\overline{\frac{\partial u}{\partial t}}+v'\overline{\frac{\partial v}{\partial t}}+w'\overline{\frac{\partial w}{\partial t}}\right]\\
&+u'u'\frac{\partial u}{\partial x}+u'v'\frac{\partial u}{\partial y}+u'w'\frac{\partial u}{\partial z}+v'u'\frac{\partial v}{\partial x}+v'v'\frac{\partial v}{\partial y}+v'w'\frac{\partial v}{\partial z}\\
&+w'u'\frac{\partial w}{\partial x}+w'v'\frac{\partial w}{\partial y}+w'w'\frac{\partial w}{\partial z}\\
&=u'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}+fv'+\frac{\partial}{\partial x}(\overline{u'u'})+\frac{\partial}{\partial y}(\overline{u'v'})+\frac{\partial}{\partial z}(\overline{u'w'})+F_{rx}{}'\right)\\
&\quad+v'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial y}-fu'+\frac{\partial}{\partial x}(\overline{v'u'})+\frac{\partial}{\partial y}(\overline{v'v'})+\frac{\partial}{\partial z}(\overline{v'w'})+F_{ry}{}'\right)\\
&\quad+w'\left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial z}+g\frac{\theta'}{\theta_0}+\frac{\partial}{\partial x}(\overline{w'u'})+\frac{\partial}{\partial y}(\overline{w'v'})+\frac{\partial}{\partial z}(\overline{w'w'})+F_{rz}{}'\right)
\end{aligned}
\tag{3.32}
$$

We average (3.32) and obtain using (2.4)

$$
\begin{aligned}
&\overline{\overline{\frac{D}{Dt}}\left[\frac{1}{2}\left((u')^2+(v')^2+(w')^2\right)\right]}+\overline{u'u'\frac{\partial u}{\partial x}}+\overline{u'v'\frac{\partial u}{\partial y}}+\overline{u'w'\frac{\partial u}{\partial z}}\\
&+\overline{v'u'\frac{\partial v}{\partial x}}+\overline{v'v'\frac{\partial v}{\partial y}}+\overline{v'w'\frac{\partial v}{\partial z}}+\overline{w'u'\frac{\partial w}{\partial x}}+\overline{w'v'\frac{\partial w}{\partial y}}+\overline{w'w'\frac{\partial w}{\partial z}}\\
&=-\frac{1}{\rho_0}\overline{\left(u'\frac{\partial p'}{\partial x}+v'\frac{\partial p'}{\partial y}+w'\frac{\partial p'}{\partial z}\right)}+\overline{w'\theta'}\frac{g}{\theta_0}+\overline{u'F_{rx}{}'+v'F_{ry}{}'+w'F_{rz}{}'}
\end{aligned}
\tag{3.33}
$$

or using also (2.4), (2.7), (2.8)

$$
\begin{aligned}
&\overline{\overline{\frac{D}{Dt}}\left[\frac{1}{2}\left((u')^2+(v')^2+(w')^2\right)\right]}+\overline{u'u'}\,\frac{\partial\overline{u}}{\partial x}+\overline{u'v'}\,\frac{\partial\overline{u}}{\partial y}+\overline{u'w'}\,\frac{\partial\overline{u}}{\partial z}\\
&+\overline{v'u'}\,\frac{\partial\overline{v}}{\partial x}+\overline{v'v'}\,\frac{\partial\overline{v}}{\partial y}+\overline{v'w'}\,\frac{\partial\overline{v}}{\partial z}+\overline{w'u'}\,\frac{\partial\overline{w}}{\partial x}+\overline{w'v'}\,\frac{\partial\overline{w}}{\partial y}+\overline{w'w'}\frac{\partial\overline{w}}{\partial z}\\
&+\overline{u'u'\frac{\partial u'}{\partial x}}+\overline{u'v'\frac{\partial u'}{\partial y}}+\overline{u'w'\frac{\partial u'}{\partial z}}+\overline{v'u'\frac{\partial v'}{\partial x}}+\overline{v'v'\frac{\partial v'}{\partial y}}+\overline{v'w'\frac{\partial v'}{\partial z}}\\
&+\overline{w'u'\frac{\partial w'}{\partial x}}+\overline{w'v'\frac{\partial w'}{\partial y}}+\overline{w'w'\frac{\partial w'}{\partial z}}\\
&=-\frac{1}{\rho_0}\overline{\left(u'\frac{\partial p'}{\partial x}+v'\frac{\partial p'}{\partial y}+w'\frac{\partial p'}{\partial z}\right)}+\overline{w'\theta'}\frac{g}{\theta_0}+\overline{u'F_{rx}{}'+v'F_{ry}{}'+w'F_{rz}{}'}
\end{aligned}
\tag{3.34}
$$

□

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