Robust Strictly Positive Real Synthesis for Convex Combination of Sixth-Order Polynomials[1]

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Abstract For the two sixth-order polynomials $a(s)$ and $b(s)$, Hurwitz stability of their convex combination is necessary and sufficient for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both strictly positive real. Our reasoning method is constructive, and is insightful and helpful in solving the general robust strictly positive real synthesis problem.

Keywords Robust Stability, Strict Positive Realness, Robust Analysis and Synthesis

The strict positive realness (SPR) of transfer functions is an important performance specification, and plays a critical role in various fields such as absolute stability/hyperstability theory [12, 17], passivity analysis [9], quadratic optimal control [2] and adaptive system theory [13]. In recent years, stimulated by the parametrization method in robust stability analysis [3, 5], the study of robust strictly positive real systems has received much attention, and great progress has beem made [1, 4, 6, 7, 8, 10, 11, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. However, most results belong to the category of robust SPR analysis. Valuable results in robust SPR synthesis are rare. The following fundamental problem is still open [1, 4, 8, 10, 11, 14, 15, 16, 18, 22, 24, 26, 27, 28, 29]:

Suppose $a(s)$ and $b(s)$ are two n-th order Hurwitz polynomials, does there exist, and how to find a (fixed) polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR?

By the definition of SPR, it is easy to know that the Hurwitz stability of the convex combination of $a(s)$ and $b(s)$ is necessary for the existence of polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR. In [10, 11, 16], it was proved that, if $a(s)$ and $b(s)$ have the same even (or odd) parts, such a polynomial $c(s)$ always exists; In [1, 10, 11, 14, 19, 20, 22, 23, 26, 27], it was proved that, if $n \leq 4$ and $a(s), b(s) \in K$ (K is a stable interval polynomial set), such a polynomial $c(s)$ always exists; Recent results show that [19, 20, 22, 23, 24, 26, 28, 29], if $n \leq 5$ and $a(s)$ and $b(s)$ are the two endpoints of the convex combination of stable polynomials, such a polynomial $c(s)$ always exists. Some sufficient condition for robust SPR synthesis are presented in [1, 4, 8, 14, 19, 20, 23], especially, the design method in [19, 20] is numerically efficient for high-order polynomial segments and interval polynomials, and the derived conditions are necessary and sufficient for low-order polynomial segments and interval polynomials.

This paper shows that, for the two sixth-order polynomials $a(s)$ and $b(s)$, Hurwitz stability of their convex combination is necessary and sufficient for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both SPR. This also shows that the conditions given in [19, 20] are also

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necessary and sufficient, and the open problem above has a positive answer for the case of sixth-order polynomial segment. Our reasoning method is constructive, and is useful in solving the general robust SPR synthesis problem.

In this paper, P^n stands for the set of n-th order polynomials of s with real coefficients, R stands for the field of real numbers, $\partial(p)$ stands for the order of polynomial $p(\cdot)$, and $H^n \subset P^n$ stands for the set of n-th order Hurwitz stable polynomials with real coefficients.

In the following definition, $p(\cdot) \in P^m, q(\cdot) \in P^n, f(s) = p(s)/q(s)$ is a rational function.

Definition 1 [25] $f(s)$ is said to be strictly positive real(SPR), if

(i) $\partial(p) = \partial(q)$;

(ii) $f(s)$ is analytic in $\mathrm{Re}[s] \geq 0$, (namely, $q(\cdot) \in H^n$);

(iii) $\mathrm{Re}[f(j\omega)] > 0, \quad \forall \omega \in R$.

If $f(s) = p(s)/q(s)$ is proper, it is easy to get the following property:

Property 1 [7] If $f(s) = p(s)/q(s)$ is a proper rational function, $q(s) \in H^n$, and $\forall \omega \in R, \mathrm{Re}[f(j\omega)] > 0$, then $p(s) \in H^n \cup H^{n-1}$.

The following theorem is the main result of this paper:

Theorem 1 Suppose $a(s) = s^6+a_1s^5+a_2s^4+a_3s^3+a_4s^2+a_5s+a_6 \in H^6, b(s) = s^6+b_1s^5+b_2s^4+b_3s^3+b_4s^2+b_5s+b_6 \in H^6$, the necessary and sufficient condition for the existence of a polynomial $c(s)$ such that $c(s)/a(s)$ and $c(s)/b(s)$ are both Strictly Positive Real is

$$\lambda b(s) + (1-\lambda)a(s) \in H^6, \lambda \in [0,1].$$

Since SPR transfer functions enjoy convexity property, by Property 1, we can easily get the necessary part of the theorem.

To prove sufficiency, we must first introduce some lemmas.

Lemma 1 Suppose $s^6+a_1s^5+a_2s^4+a_3s^3+a_4s^2+a_5s+a_6 \in H^6$, then the following quadratic curve is an ellipse in the first quadrant of the x-y-z-p space:

$$\begin{cases} (a_2x+z-a_1y-a_3)^2 - 4(a_1-x)(a_5+a_3y+a_1p-a_2z-a_4x) = 0 \\ a_6x+a_4z-a_3p-a_5y = 0 \\ a_5p-a_6z = 0 \end{cases}$$

and this ellipse is tangent with the line

$$\begin{cases} a_6x+a_4z-a_3p-a_5y = 0 \\ a_5p-a_6z = 0 \\ a_1-x = 0 \end{cases}$$

at

$$\begin{cases} x = a_1, \\ y = \frac{a_5a_6a_1-a_5a_4a_2a_1+a_5a_4a_3+a_3a_6a_2a_1-a_3^2a_6}{-a_5a_4a_1+a_5^2+a_3a_6a_1}, \\ z = -a_5\frac{a_5a_2a_1-a_6a_1^2-a_5a_3}{-a_5a_4a_1+a_5^2+a_3a_6a_1}, \\ p = -a_6\frac{a_5a_2a_1-a_6a_1^2-a_5a_3}{-a_5a_4a_1+a_5^2+a_3a_6a_1}, \end{cases}$$

tangent with the line

$$\begin{cases} a_6x+a_4z-a_3p-a_5y = 0 \\ a_5p-a_6z = 0 \\ a_5+a_3y+a_1p-a_2z-a_4x = 0 \end{cases}$$

at

$$\begin{cases} x = \frac{-a_5^3+a_5^2a_4a_1+a_3^3a_6+a_5^2a_2a_3-a_5a_3^2a_4-2a_5a_3a_6a_1}{a_5^2a_2^2-a_5^2a_4+a_5a_4^2a_1-2a_5a_6a_1a_2-a_3a_6a_1a_4+a_6^2a_1^2+a_5a_3a_6-a_5a_3a_4a_2+a_3^2a_6a_2}, \\ y = \frac{a_6a_4a_3^2-a_3a_6^2a_1-a_3a_5a_4^2-a_5^2a_6+a_5^2a_4a_2}{a_5^2a_2^2-a_5^2a_4+a_5a_4^2a_1-2a_5a_6a_1a_2-a_3a_6a_1a_4+a_6^2a_1^2+a_5a_3a_6-a_5a_3a_4a_2+a_3^2a_6a_2}, \\ z = a_5\frac{a_2a_5^2-a_5a_6a_1-a_5a_4a_3+a_3^2a_6}{a_5^2a_2^2-a_5^2a_4+a_5a_4^2a_1-2a_5a_6a_1a_2-a_3a_6a_1a_4+a_6^2a_1^2+a_5a_3a_6-a_5a_3a_4a_2+a_3^2a_6a_2}, \\ p = a_6\frac{a_2a_5^2-a_5a_6a_1-a_5a_4a_3+a_3^2a_6}{a_5^2a_2^2-a_5^2a_4+a_5a_4^2a_1-2a_5a_6a_1a_2-a_3a_6a_1a_4+a_6^2a_1^2+a_5a_3a_6-a_5a_3a_4a_2+a_3^2a_6a_2}. \end{cases}$$

Proof Since $a(s)$ is Hurwitz stable, Lemma 1 is proved by a direct calculation.

Lemma 2 Suppose $s^6+a_1s^5+a_2s^4+a_3s^3+a_4s^2+a_5s+a_6 \in H^6$, then the following quadratic curve is an ellipse in the first quadrant of the x-y-z-p space:

$$\begin{cases} (a_5+a_3y+a_1p-a_2z-a_4x)^2-4(a_2x+z-a_1y-a_3)(a_6x+a_4z-a_3p-a_5y)=0 \\ a_1-x=0 \\ a_5p-a_6z=0 \end{cases}$$

and this ellipse is tangent with the line

$$\begin{cases} a_1-x=0 \\ a_5p-a_6z=0 \\ a_2x+z-a_1y-a_3=0 \end{cases}$$

at

$$\begin{cases} x=a_1, \\ y=\frac{-a_5^2+a_5a_4a_1-a_5a_2^2a_1+a_5a_2a_3+a_6a_1^2a_2-a_1a_6a_3}{-a_2a_1a_5+a_1^2a_6+a_3a_5}, \\ z=a_5\frac{a_3^2-a_3a_2a_1+a_4a_1^2-a_1a_5}{-a_2a_1a_5+a_1^2a_6+a_3a_5}, \\ p=a_6\frac{a_3^2-a_3a_2a_1+a_4a_1^2-a_1a_5}{-a_2a_1a_5+a_1^2a_6+a_3a_5}, \end{cases}$$

tangent with the line

$$\begin{cases} a_1-x=0 \\ a_5p-a_6z=0 \\ a_6x+a_4z-a_3p-a_5y=0 \end{cases}$$

at

$$\begin{cases} x=a_1, \\ y=\frac{a_4^2a_5a_1-a_3a_4a_1a_6-a_5^2a_4+a_3a_5a_6-a_2a_1a_5a_6+a_1^2a_6^2}{-a_5^2a_2+a_5a_4a_3-a_3^2a_6+a_5a_6a_1}, \\ z=a_5\frac{a_5a_4a_1-a_1a_6a_3-a_5^2}{-a_5^2a_2+a_5a_4a_3-a_3^2a_6+a_5a_6a_1}, \\ p=a_6\frac{a_5a_4a_1-a_1a_6a_3-a_5^2}{-a_5^2a_2+a_5a_4a_3-a_3^2a_6+a_5a_6a_1}.. \end{cases}$$

Proof Since $a(s)$ is Hurwitz stable, Lemma 2 is proved by a direct calculation.

Lemma 3 Suppose $s^6+a_1s^5+a_2s^4+a_3s^3+a_4s^2+a_5s+a_6 \in H^6$, then the following quadratic curve is an ellipse in the first quadrant of the x-y-z-p space:

$$\begin{cases} (a_6x+a_4z-a_3p-a_5y)^2-4(a_5+a_3y+a_1p-a_2z-a_4x)(a_5p-a_6z)=0 \\ a_1-x=0 \\ a_2x+z-a_1y-a_3=0 \end{cases}$$

and this ellipse is tangent with the line

$$
\begin{cases}
a_1 - x = 0 \\
a_2 x + z - a_1 y - a_3 = 0 \\
a_5 + a_3 y + a_1 p - a_2 z - a_4 x = 0
\end{cases}
$$

at

$$
\begin{cases}
x = a_1, \\
y = -\frac{a_3 a_5 + a_3 a_2^2 a_1 + a_1^2 a_6 - a_2 a_3^2 - a_4 a_2 a_1^2}{a_3^2 - a_3 a_2 a_1 + a_4 a_1^2 - a_1 a_5}, \\
z = -\frac{a_2 a_1 a_3^2 - a_2 a_1^2 a_5 + 2 a_1 a_3 a_5 + a_1^3 a_6 - a_3^3 - a_3 a_4 a_1^2}{a_3^2 - a_3 a_2 a_1 + a_4 a_1^2 - a_1 a_5}, \\
p = -\frac{2 a_5 a_4 a_1 + a_4 a_1 a_2 a_3 - a_5^2 - a_5 a_2^2 a_1 + a_5 a_2 a_3 - a_1 a_6 a_3 + a_6 a_1^2 a_2 - a_4 a_3^2 - a_4^2 a_1^2}{a_3^2 - a_3 a_2 a_1 + a_4 a_1^2 - a_1 a_5},
\end{cases}
$$

tangent with the line

$$
\begin{cases}
a_1 - x = 0 \\
a_2 x + z - a_1 y - a_3 = 0 \\
a_5 p - a_6 z = 0
\end{cases}
$$

at

$$
\begin{cases}
x = a_1, \\
y = -\frac{a_5 a_6 a_1 - a_5 a_4 a_2 a_1 + a_5 a_4 a_3 + a_3 a_6 a_2 a_1 - a_3^2 a_6}{a_5 a_4 a_1 - a_1 a_6 a_3 - a_5^2}, \\
z = -a_5 \frac{-a_2 a_1 a_5 + a_1^2 a_6 + a_3 a_5}{a_5 a_4 a_1 - a_1 a_6 a_3 - a_5^2}, \\
p = -a_6 \frac{-a_2 a_1 a_5 + a_1^2 a_6 + a_3 a_5}{a_5 a_4 a_1 - a_1 a_6 a_3 - a_5^2}.
\end{cases}
$$

Proof Since $a(s)$ is Hurwitz stable, Lemma 3 is proved by a direct calculation.

For notational simplicity, denote

$$
\begin{aligned}
\Omega^a_{e1} := \{(x,y,z,p)| \quad & (a_2 x + z - a_1 y - a_3)^2 - 4(a_1 - x)(a_5 + a_3 y + a_1 p - a_2 z \\
& - a_4 x) < 0, a_6 x + a_4 z - a_3 p - a_5 y = 0, a_5 p - a_6 z = 0\}
\end{aligned}
$$

$$
\begin{aligned}
\Omega^a_{e2} := \{(x,y,z,p)| \quad & (a_5 + a_3 y + a_1 p - a_2 z - a_4 x)^2 - 4(a_2 x + z - a_1 y - a_3)(a_6 x \\
& + a_4 z - a_3 p - a_5 y) < 0, a_1 - x = 0, a_5 p - a_6 z = 0\}
\end{aligned}
$$

$$
\begin{aligned}
\Omega^a_{e3} := \{(x,y,z,p)| \quad & (a_6 x + a_4 z - a_3 p - a_5 y)^2 - 4(a_5 + a_3 y + a_1 p - a_2 z \\
& - a_4 x)(a_5 p - a_6 z) < 0, a_1 - x = 0, a_2 x + z - a_1 y - a_3 = 0\}
\end{aligned}
$$

$$
\begin{aligned}
\Omega^b_{e1} := \{(x,y,z,p)| \quad & (b_2 x + z - b_1 y - b_3)^2 - 4(b_1 - x)(b_5 + b_3 y + b_1 p - b_2 z \\
& - b_4 x) < 0, b_6 x + b_4 z - b_3 p - b_5 y = 0, b_5 p - b_6 z = 0\}
\end{aligned}
$$

$$
\begin{aligned}
\Omega^b_{e2} := \{(x,y,z,p)| \quad & (b_5 + b_3 y + b_1 p - b_2 z - b_4 x)^2 - 4(b_2 x + z - b_1 y - b_3)(b_6 x \\
& + b_4 z - b_3 p - b_5 y) < 0, b_1 - x = 0, b_5 p - b_6 z = 0\}
\end{aligned}
$$

$$
\begin{aligned}
\Omega^b_{e3} := \{(x,y,z,p)| \quad & (b_6 x + b_4 z - b_3 p - b_5 y)^2 - 4(b_5 + b_3 y + b_1 p - b_2 z \\
& - b_4 x)(b_5 p - b_6 z) < 0, b_1 - x = 0, b_2 x + z - b_1 y - b_3 = 0\}
\end{aligned}
$$

In what follows, (A, B) stands for the set of points in the line segment connecting the point A and the point B in the x-y-z-p space, not including the endpoints A and B. Denote

$$
\Omega^a := \{(x,y,z,p)|(x,y,z,p) \in (A,B) \cup (A,C) \cup (B,C), \forall A \in \Omega^a_{e1}, \forall B \in \Omega^a_{e2}, \forall C \in \Omega^a_{e3}\}
$$

$$
\Omega^b := \{(x,y,z,p)|(x,y,z,p) \in (A,B) \cup (A,C) \cup (B,C), \forall A \in \Omega^b_{e1}, \forall B \in \Omega^b_{e2}, \forall C \in \Omega^b_{e3}\}
$$

Lemma 4 Suppose $a(s) = s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6 \in H^6, b(s) = s^6 + b_1 s^5 + b_2 s^4 + b_3 s^3 + b_4 s^2 + b_5 s + b_6 \in H^6$, if $\Omega^a \cap \Omega^b \neq \phi$, take $(x,y,z,p) \in \Omega^a \cap \Omega^b$, and let $c(s) :=$

$s^5+(x-\varepsilon)s^4+ys^3+zs^2+ps+\varepsilon$ (ε is a sufficiently small positive number), then for $\dfrac{c(s)}{a(s)}$ and $\dfrac{c(s)}{b(s)}$, we have $\forall\omega\in R, \mathrm{Re}[\dfrac{c(j\omega)}{a(j\omega)}]>0$ and $\mathrm{Re}[\dfrac{c(j\omega)}{b(j\omega)}]>0$.

Proof Suppose $(x,y,z,p)\in\Omega^a\cap\Omega^b$, let $c(s):=s^5+(x-\varepsilon)s^4+ys^3+zs^2+ps+\varepsilon, \varepsilon>0, \varepsilon$ sufficiently small.

$\forall\omega\in R$, consider

$$\begin{aligned}\mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}]= & \frac{1}{\mid a(j\omega)\mid^2}[(a_1-x)\omega^{10}+(a_2x+z-a_1y-a_3)\omega^8+(a_5+a_3y+a_1p\\ & -a_2z-a_4x)\omega^6+(a_6x+a_4z-a_3p-a_5y)\omega^4+(a_5p-a_6z)\omega^2\\ & +\varepsilon(\omega^{10}-a_2\omega^8+(a_4-1)\omega^6+(-a_6+a_2)\omega^4-a_4\omega^2+a_6)]\end{aligned}$$

In order to prove that $\forall\omega\in R, \mathrm{Re}[\dfrac{c(j\omega)}{a(j\omega)}]>0$, let $t=\omega^2$, we only need to prove that, for any $\varepsilon>0, \varepsilon$ sufficiently small, the following polynomial $f_1(t)$ satisfies

$$\begin{aligned}f_1(t):= & t[(a_1-x)t^4+(a_2x+z-a_1y-a_3)t^3+(a_5+a_3y+a_1p\\ & -a_2z-a_4x)t^2+(a_6x+a_4z-a_3p-a_5y)t+(a_5p-a_6z)]\\ & +\varepsilon(t^5-a_2t^4+(a_4-1)t^3+(-a_6+a_2)t^2-a_4t+a_6)]>0, \quad \forall t\in[0,+\infty).\end{aligned}$$

Since $(x,y,z,p)\in\Omega^a$, by the definition of Ω^a, it is easy to know that

$$\begin{aligned}g_1(t):= & (a_1-x)t^4+(a_2x+z-a_1y-a_3)t^3+(a_5+a_3y+a_1p-a_2z-a_4x)t^2\\ & +(a_6x+a_4z-a_3p-a_5y)t+(a_5p-a_6z)>0, \quad \forall t\in[0,+\infty).\end{aligned}$$

Moreover, we obviously have $f_1(0)>0$, and for any $\varepsilon>0$, when t is a sufficiently large or sufficiently small positive number, we have $f_1(t)>0$, namely, there exist $0<t_1<t_2$ such that, for all $\varepsilon>0$, $t\in[0,t_1]\cup[t_2,+\infty)$, we have $f_1(t)>0$.

Denote

$$M_1=\inf_{t\in[t_1,t_2]} tg_1(t),$$

$$N_1=\sup_{t\in[t_1,t_2]}|t^5-a_2t^4+(a_4-1)t^3+(-a_6+a_2)t^2-a_4t+a_6|.$$

Then $M_1>0$ and $N_1>0$. Choosing $0<\varepsilon<\dfrac{M_1}{N_1}$, by a direct calculation, we have

$$\begin{aligned}f_1(t):= & t[(a_1-x)t^4+(a_2x+z-a_1y-a_3)t^3+(a_5+a_3y+a_1p\\ & -a_2z-a_4x)t^2+(a_6x+a_4z-a_3p-a_5y)t+(a_5p-a_6z)]\\ & +\varepsilon(t^5-a_2t^4+(a_4-1)t^3+(-a_6+a_2)t^2-a_4t+a_6)>0, \quad \forall t\in[0,+\infty).\end{aligned}$$

Namely,

$$\forall\omega\in R, \mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}]>0.$$

Similarly, since $(x,y,z,p)\in\Omega^b$, there exist $0<t_3<t_4$ such that, for all $\varepsilon>0$, $t\in[0,t_3]\cup[t_4,+\infty)$, we have $f_2(t)>0$,where

$$\begin{aligned}f_2(t):= & t[(b_1-x)t^4+(b_2x+z-b_1y-b_3)t^3+(b_5+b_3y+b_1p\\ & -b_2z-b_4x)t^2+(b_6x+b_4z-b_3p-b_5y)t+(b_5p-b_6z)]\\ & +\varepsilon(t^5-b_2t^4+(b_4-1)t^3+(-b_6+b_2)t^2-b_4t+b_6)]\end{aligned}$$

Denote

$$\begin{aligned}g_2(t):= & (b_1-x)t^4+(b_2x+z-b_1y-b_3)t^3+(b_5+b_3y+b_1p-b_2z-b_4x)t^2\\ & +(b_6x+b_4z-b_3p-b_5y)t+(b_5p-b_6z),\end{aligned}$$

$$M_2 = \inf_{t\in[t_3,t_4]} tg_2(t),$$
$$N_2 = \sup_{t\in[t_3,t_4]} |t^5 - b_2t^4 + (b_4-1)t^3 + (-b_6+b_2)t^2 - b_4t + b_6|.$$

Then $M_2 > 0$ and $N_2 > 0$. Choosing $0 < \varepsilon < \frac{M_2}{N_2}$, we have

$$\forall \omega \in R, \mathrm{Re}[\frac{c(j\omega)}{b(j\omega)}] > 0.$$

Thus, by choosing $0 < \varepsilon < \min\{\frac{M_1}{N_1}, \frac{M_2}{N_2}\}$, Lemma 4 is proved.

Lemma 5 Suppose $a(s) = s^6 + a_1s^5 + a_2s^4 + a_3s^3 + a_4s^2 + a_5s + a_6 \in H^6, b(s) = s^6 + b_1s^5 + b_2s^4 + b_3s^3 + b_4s^2 + b_5s + b_6 \in H^6$, if $\lambda b(s) + (1-\lambda)a(s) \in H^6, \lambda \in [0,1]$, then $\Omega^a \cap \Omega^b \neq \phi$

Proof If $\forall \lambda \in [0,1], a_\lambda(s) := \lambda b(s) + (1-\lambda)a(s) \in H^6$, by Lemmas 1-3, for any $\lambda \in [0,1]$, $\Omega^{a_\lambda}_{e1}, \Omega^{a_\lambda}_{e2}$ and $\Omega^{a_\lambda}_{e3}$ are three ellipses in the first quadrant of the x-y-z-p space, denote

$$\Omega^{a_\lambda} := \{(x,y,z,p)|(x,y,z,p) \in (A,B) \cup (A,C) \cup (B,C), \forall A \in \Omega^{a_\lambda}_{e1}, \forall B \in \Omega^{a_\lambda}_{e2}, \forall C \in \Omega^{a_\lambda}_{e3}\}.$$

Apparently, when λ changes continuously from 0 to 1, Ω^{a_λ} will change continuously from Ω^a to Ω^b, $\Omega^{a_\lambda}_{e1}$ will change continuously from Ω^a_{e1} to Ω^b_{e1}, $\Omega^{a_\lambda}_{e2}$ will change continuously from Ω^a_{e2} to Ω^b_{e2}, and $\Omega^{a_\lambda}_{e3}$ will change continuously from Ω^a_{e3} to Ω^b_{e3}.

Now assume $\Omega^a \cap \Omega^b = \phi$, by the definitions of Ω^a and Ω^b, and Lemmas 1-3, $\exists u > 0$, $v > 0, u \neq a_1, u \neq b_1$, and $\exists k \in \{1,2,3\}$, such that the following plane in the x-y-z-p space

$$l: \quad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} + \frac{p}{r} = 1$$

separates Ω^a and Ω^b, meanwhile, l is tangent with $\Omega^a_{e1}, \Omega^a_{e2}, \Omega^a_{e3}$ and Ω^b_{ek} simultaneously (or tangent with $\Omega^b_{e1}, \Omega^b_{e2}, \Omega^b_{e3}$ and Ω^a_{ek} simultaneously).

Without loss of generality, suppose that l is tangent with $\Omega^a_{e1}, \Omega^a_{e2}, \Omega^a_{e3}$ and Ω^b_{ek} simultaneously.

Since l is tangent with $\Omega^a_{e1}, \Omega^a_{e2}$ and Ω^a_{e3} simultaneously, $a(s)$ is Hurwitz stable and $u > 0, u \neq a_1, v > 0$, by a lengthy calculation, we get that the necessary and sufficient condition for l being tangent with $\Omega^a_{e1}, \Omega^a_{e2}$ and Ω^a_{e3} simultaneously is

$$uv^3 - a_1v^3 - a_2uv^2 + a_3v^2 + a_4uv - a_5v - a_6u = 0, w = -uv, r = -v^2 \quad (1)$$

Since $w = -uv, r = -v^2, l$ is tangent with Ω^b_{ek}, by a direct calculation, we have

$$uv^3 - b_1v^3 - b_2uv^2 + b_3v^2 + b_4uv - b_5v - b_6u = 0 \quad (2)$$

¿From (1) and (2), we obviously have $\forall \lambda \in [0,1]$,

$$uv^3 - a_{\lambda 1}v^3 - a_{\lambda 2}uv^2 + a_{\lambda 3}v^2 + a_{\lambda 4}uv - a_{\lambda 5}v - a_{\lambda 6}u = 0, w = -uv, r = -v^2 \quad (3)$$

where $a_{\lambda i} := a_i + \lambda(b_i - a_i), i = 1,2,3,4,5,6$. (3) shows that l is also tangent with $\Omega^{a_\lambda}_{ek} (\forall \lambda \in [0,1])$. But l separates Ω^a_{ek} and Ω^b_{ek}, and when λ changes continuously from 0 to 1, $\Omega^{a_\lambda}_{ek}$ will change continuously from Ω^a_{ek} to Ω^b_{ek}, which is obviously impossible. This completes the proof.

From Theorem 2.4 in [20], or the proof of Lemma 5 in [24], we have

Lemma 6 Suppose $a(s) = s^6 + a_1s^5 + a_2s^4 + a_3s^3 + a_4s^2 + a_5s + a_6 \in H^6, b(s) = s^6 + b_1s^5 + b_2s^4 + b_3s^3 + b_4s^2 + b_5s + b_6 \in H^6, c(s) = s^5 + xs^4 + ys^3 + zs^2 + ps + q$, if $\forall \omega \in R, \mathrm{Re}[\frac{c(j\omega)}{a(j\omega)}] > 0$ and $\mathrm{Re}[\frac{c(j\omega)}{b(j\omega)}] > 0$, take

$$\widetilde{c}\,(s) := c(s) + \delta \cdot d(s), \quad \delta > 0,\ \delta \text{ sufficiently small}$$

(where $d(s)$ is an arbitrarily given monic sixth-order polynomial), then $\dfrac{\widetilde{c}\,(s)}{a(s)}$ and $\dfrac{\widetilde{c}\,(s)}{b(s)}$ are both strictly positive real.

The sufficiency of Theorem 1 is now proved by combining Lemmas 1-6.

Remark 1 From the proof of Theorem 1, we can see that this paper not only proves the existence, but also provides a design method.

Remark 2 The method provided in this paper is constructive, and is insightful and helpful in solving the general robust SPR synthesis problem. This subject is currently under investigation.

Remark 3 Our results can easily be generalized to discrete-time case.

Remark 4 If $\dfrac{c(s)}{a(s)}$ and $\dfrac{c(s)}{b(s)}$ are both SPR, it is easy to know that $\forall \lambda \in [0,1]$, $\dfrac{c(s)}{\lambda a(s) + (1-\lambda) b(s)}$ is also SPR.

Remark 5 The stability of polynomial segment can be checked by many efficient methods, e.g., eigenvalue method, root locus method, value set method, etc. [3, 5].

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