A NOTE ON BOUNDARY BLOW-UP PROBLEM OF $\Delta u = u^p$

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Abstract. Assume that Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. We study positive solutions to the problem, $\Delta u = u^p$ in Ω, $u(x) \to \infty$ as $x \to \partial\Omega$, where $p > 1$. Such solutions are called boundary blow-up solutions of $\Delta u = u^p$. We show that a boundary blow-up solution exists in any bounded domain if $1 < p < \frac{n}{n-2}$. In particular, when $n = 2$, there exists a boundary blow-up solution to $\Delta u = u^p$ for all $p \in (1, \infty)$. We also prove the uniqueness under the additional assumption that the domain satisfies the condition $\partial\Omega = \partial\overline{\Omega}$.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with $n \geq 2$ and let $\partial\Omega$ denote its boundary. In this article we study the problem

$$\Delta u(x) = f(u(x)) \quad \text{for} \quad x \in \Omega, \tag{1}$$

$$u(x) \to +\infty \quad \text{as} \quad d(x) := \operatorname{dist}(x, \partial\Omega) \to 0, \tag{2}$$

where $f(t) = t_+^p := \{\max(t, 0)\}^p$ with $p > 1$. Solutions to the problem (1), (2) are called *boundary blow-up solutions*.

In 1957, Keller [5] and Osserman [11] proved existence of solutions to problem (1), (2) for a rather general class of functions f; i.e., $f : \mathbb{R} \to [0, \infty)$ is a locally Lipschitz continuous function which is increasing and satisfies the following growth condition called Keller-Osserman condition:

$$\int_{t_0}^{\infty} \left\{ \int_0^t f(s)\, ds \right\}^{-1/2} dt < +\infty \quad \text{for all} \quad t_0 > 0. \tag{3}$$

It is easy to check that $f(t) = t_+^p$ with $p > 1$ satisfies (3). They showed that (3) is a necessary condition for the existence of blow-up solutions. Indeed, if the domain Ω is regular enough, say Lipschitz, then the existence of a classical solution to the problem (1), (2) is established by the method of supersolutions and subsolutions together with the uniform estimates of Keller [5]. We will briefly review existence results in the next section.

The case $f(t) = t_+^p$ with $p > 1$ is of special interest, and in this article only this case will be treated. Loewner and Nirenberg [7] studied the case when $p = \frac{n+2}{n-2}$ with $n > 2$, which is related to a problem in differential geometry. The problem (1), (2) is also related to probability theory. The equation $\Delta u = u_+^p$, $1 < p \leq 2$, appears in the analytical theory of a Markov process called superdiffusion; see [2]. By means of a probabilistic representation, a uniqueness result in domains with non-smooth boundary was established by Le Gall [6] in the case when $p = 2$. Later, Marcus and Véron [8, 9] extended the uniqueness in very general domains for all

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$p > 1$, using purely analytical method; they proved uniqueness in a domain whose boundary is locally represented as a graph of a continuous function. However, it is not clear whether a boundary blow-up solution exists or not in such a general domain. In [10], Matero constructed a boundary blow-up solution of $\Delta u = u_+^p$ with $1 < p < \infty$, in a two-dimensional domain with fractal boundary called the von Koch snowflake domain. His approach is based on the comparison with boundary blow-up solutions in a cut-off open cone.

We treat a special case when $p \in (1, \frac{n}{n-2})$ for $n \geq 3$ and $p \in (1, \infty)$ for $n = 2$. Some interesting results are obtained in that case. We will prove that a boundary blow-up solution exists in every bounded domain. As a consequence, it will imply a result of Matero [10] mentioned above. We will also show the uniqueness if the domain satisfies an additional assumption, $\partial\Omega = \partial\overline{\Omega}$. For example, if $\partial\Omega$ can be locally represented as a graph of a continuous function, then it satisfies the above condition. In this case, uniqueness was earlier proved by Marcus and Véron [8, 9].

2. Preliminaries

In this section, we briefly discuss the existence results of Keller [5], Loewner and Nirenberg [7]. We also introduce some terminology which will be used in the later parts of the paper. We begin with a simple lemma.

Lemma 2.1 (Comparison principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that f is increasing. Let $u, v \in C^2(\Omega)$ be solutions of $\Delta u \geq f(u)$ and $\Delta v \leq f(v)$ respectively. If $\liminf_{x \to \partial\Omega}(v - u)(x) \geq 0$, then $v \geq u$ in Ω.*

Proof. Suppose, to the contrary, that there exists $x_0 \in \Omega$ such that $u(x_0) > v(x_0)$. Then for sufficiently small $\epsilon > 0$, $\Omega_\epsilon := \{u - v > \epsilon\} \neq \emptyset$ and $\overline{\Omega}_\epsilon \subset \Omega$. Let $w := u - v - \epsilon$. Then $w = 0$ on $\partial\Omega_\epsilon$. Since f is increasing,

$$Lw \geq f(u) - f(v) \geq f(u) - f(v + \epsilon) \geq 0 \quad \text{in} \quad \Omega_\epsilon.$$

Then the maximum principle implies $w \leq 0$ in Ω_ϵ. This contradiction proves the lemma. □

Remark 2.2. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be bounded domains such that $\Omega_1 \Subset \Omega_2$, i.e., $\overline{\Omega}_1 \subset \Omega_2$. Suppose u_i $(i = 1, 2)$ are solutions to (1), (2) in Ω_i. Then, it follows from Lemma 2.1 that $u_1 \geq u_2$ in Ω_1.

The next theorem is quoted from [5]; see also [11].

Theorem 2.3 (Keller [5, pp. 505–507])**.** *Let u be a solution of* (1) *in a bounded domain Ω. There exist a continuous, decreasing function $g : (0, \infty) \to \mathbb{R}$ determined by f such that $\lim_{t\to 0} g(t) = +\infty$ and*

$$u(x) \leq g(d(x)), \quad \textit{where} \quad d(x) := \operatorname{dist}(x, \partial\Omega). \tag{4}$$

Using the above estimate (4), Keller proved the existence of a boundary blow-up solution. Although he claimed the existence in arbitrary domains, his argument seems to require certain smoothness assumption on Ω. Let Ω be a regular domain, say a Lipschitz domain. By the method of supersolutions and subsolutions (see e.g. [3, pp. 507-511]), one can show that, for each $m \geq 1$, there exists a unique solution $u_m \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ of (1) such that $u_m = \alpha_m$ on $\partial\Omega$, where $\alpha_m < \alpha_{m+1}$ and $\alpha_m \to \infty$ as $m \to \infty$. Then by the comparison principle, $\{u_m\}_{m=1}^\infty$ is an increasing sequence of functions. By (4), $u_m(x) \leq g(d(x))$ uniformly for $m \geq 1$. Denote by $u(x)$ the pointwise limit of $\{u_m(x)\}_{m=1}^\infty$. Then by the standard elliptic theory (see e.g. [4]), $u \in C^2(\Omega)$ and u is a solution of (1). As x approaches $\partial\Omega$, $u(x)$ increases indefinitely since $u_m = \alpha_m$ becomes infinite on $\partial\Omega$; thus u is a solution of the problem (1), (2).

The solution u constructed above is called a *minimal boundary blow-up solution*. Indeed, if v is a boundary blow-up solution, then by the comparison principle, $u_m \leq v$ in Ω for all $m \geq 1$ and thus, $u = \lim_{m\to\infty} u_m \leq v$ follows.

Loewner and Nirenberg [7] introduced another important solution of (1) called a *maximal solution* which is not necessarily a blow-up solution but can be constructed in any bounded domain Ω. Let $\{\Omega_m\}_{m=1}^{\infty}$ be an exhausting sequence of smooth subdomains of Ω; i.e., $\Omega_m \Subset \Omega_{m+1} \Subset \Omega$ and $\bigcup_{m=1}^{\infty} \Omega_m = \Omega$. Let u_m be the minimal blow-up solution in Ω_m for each $m \geq 1$, and let v be the minimal blow-up solution in a ball containing $\overline{\Omega}$. By Remark 2.2, $\{u_m\}_{m=1}^{\infty}$ is decreasing and bounded below by v. Hence, the limit function u exists and by the standard elliptic theory, it is a solution to (1). This solution u is maximal since if v is a solution of (1) in Ω, then by the comparison principle, we see $u_m \geq v$ for all $m \geq 1$. In next section, we will provide an example of maximal solution which is not a boundary blow-up solution; see Remark 3.3 below.

3. Main Results

We consider the problem (1), (2) with $f(t) = t_+^p$. Note that in this case, a solution to the problem (1), (2) must be positive, which is a simple consequence of the maximum principle. Indeed, more generally, let $t_0 := \sup\{t : f(t) = 0\}$. If $t_0 \neq -\infty$, then by continuity, $f(t_0) = 0$ and thus, $u \equiv t_0$ is a solution to (1). By Lemma 2.1, we find that any blow-up solution of (1) is bounded below by t_0.

Hereafter, we always assume $p \in (1, \infty)$ when $n = 2$, and $p \in (1, \frac{n}{n-2})$ when $n \geq 3$. We will show that in that case, a boundary blow-up solution exists in any bounded domain, which obviously include the domain considered by Matero in [10]. Also, by using Safonov's iteration technique in [12], we prove uniqueness provided that Ω satisfies the condition $\partial\Omega = \partial\overline{\Omega}$. For example, if $\partial\Omega$ can be locally represented as a graph of a continuous function, then it satisfies the above condition.

3.1. **Construction of a barrier in $\mathbb{R}^n \setminus \{0\}$.** We will construct a solution of $\Delta u = u_+^p$ in $\mathbb{R}^n \setminus \{0\}$ which blows up at the origin. We look for a solution of the form $v(x) = c_p\,|x|^{-\gamma}$, where $c_p, \gamma > 0$. Since v is positive and radially symmetric, $v(r) = c_p r^{-\gamma}$, where $r = |x|$, must solve the following ODE:

$$v''(r) + \frac{n-1}{r}v'(r) = v^p(r) \quad \text{in} \quad (0,\infty). \tag{5}$$

Hence, the unknown constants c_p, γ should satisfy

$$c_p\gamma(\gamma + 2 - n)r^{-\gamma-2} = c_p^p r^{-\gamma p}. \tag{6}$$

Set $\gamma = 2/(p-1)$ so that $\gamma + 2 = \gamma p$. The assumption $c_p > 0$ requires a restriction on p, namely $2/(p-1) > n-2$. It is satisfied for all $p > 1$ when $n = 2$ and for $p \in (1, \frac{n}{n-2})$ when $n \geq 3$. If we choose

$$c_p = \{\gamma(\gamma + 2 - n)\}^{\gamma/2} = \left\{\frac{2n - 2(n-2)p}{(p-1)^2}\right\}^{1/(p-1)}, \tag{7}$$

it follows $c_p^p = c_p\gamma(\gamma + 2 - n)$.

Then, $v(x) = c_p\,|x|^{-\gamma}$ is a solution of $\Delta v = v_+^p$ on $\mathbb{R}^n \setminus \{0\}$ such that $v(x) \to +\infty$ as $|x| \to 0$. We summarize the above result as a lemma.

Lemma 3.1. *Let $p > 1$ when $n = 2$, and let $p \in (1, \frac{n}{n-2})$ when $n \geq 3$. Then, $v(x) := c_p\,|x|^{-\gamma}$ is a solution of $\Delta v = v_+^p$ in $\mathbb{R}^n \setminus \{0\}$ such that $v(x) \to +\infty$ as $|x| \to 0$. Here, $\gamma = 2/(p-1)$ and $c_p = \left\{\dfrac{2n - 2(n-2)p}{(p-1)^2}\right\}^{1/(p-1)}$.*

3.2. **Existence and uniqueness of boundary blow-up solution.**

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, there exists a solution u to the problem* (1), (2).

Proof. Let $\{\Omega_m\}_{m=1}^{\infty}$ be an exhausting sequence of smooth subdomains of Ω, and let u_m be the minimal blow-up solution of (1) in Ω_m. Then, the limit $u := \lim_{m\to\infty} u_m$ is a maximal solution; see Section 2.

We need to show that u is indeed a boundary blow-up solution. For any $y \in \Omega$, choose a point $y_0 \in \partial\Omega$ such that $d(y) = |y - y_0|$. Let $v(x) := c_p\,|x - y_0|^{-\gamma}$ with c_p, γ defined as in Lemma 3.1. Since $y_0 \notin \Omega_m$ for each $m \geq 1$, wee find $v(x) < +\infty$ for all $x \in \Omega_m$. Hence, we conclude by Lemma 2.1 that

$$u_m(y) \geq v(y) = c_p d^{-\gamma}(y) \tag{8}$$

provided m is large enough so that $y \in \Omega_m$.

Therefore, by passing to the limit, we find $u(y) \geq c_p d^{-\gamma}(y)$ for any $y \in \Omega$. Clearly, $u(y) \to +\infty$ as $d(y) \to 0$, and thus, u is a desired solution. □

Remark 3.3. In Theorem 3.2, the restriction that $p < \frac{n}{n-2}$ when $n > 2$ is essential. Let $\Omega := \{x \in \mathbb{R}^n : 0 < |x| < 1\}$, where $n > 2$. Brézis and Véron [1] showed that if $p \geq \frac{n}{n-2}$, then any positive solution u of $\Delta u = u^p$ in Ω satisfies $\overline{\lim}_{x\to 0} u(x) < +\infty$. Consequently, there is no solution of the problem (1), (2) in Ω. This also shows that in general, a maximal solution is not necessarily a boundary blow-up solution.

Theorem 3.4. *In addition, assume that Ω satisfies $\partial\Omega = \partial\overline{\Omega}$. Then, the solution of the problem* (1), (2) *is unique.*

Proof. Let u_1, u_2 be two boundary blow-up solutions in Ω. We claim that the following estimate holds:

$$N_1 d^{-\gamma}(x) \leq u_i(x) \leq N_2 d^{-\gamma}(x), \quad \text{for all } x \in \Omega; \quad i = 1, 2, \tag{9}$$

where $N_1, N_2 > 0$ are constants depending only on n and p.

Fix $x_0 \in \Omega$ and denote $r := d(x_0)$. Choose $z_0 \in \partial\Omega$ such that $|x_0 - z_0| = r$. From the assumption that $\partial\Omega = \partial\overline{\Omega}$, there exists a point $y_0 \in B_r(z_0) \setminus \overline{\Omega}$. Note that $r \leq |x_0 - y_0| \leq 2r$. Let $v(x) := c_p\,|x - y_0|^{-\gamma}$. Since Ω is bounded and $y_0 \notin \overline{\Omega}$, we find, by see Lemma 2.1, that $u_i(x) \geq v(x)$, where $i = 1, 2$. In particular,

$$u_i(x_0) \geq c_p\,|x_0 - y_0|^{-\gamma} \geq c_p 2^{-\gamma} d^{-\gamma}(x_0); \quad i = 1, 2. \tag{10}$$

Also, by considering a ball $B_r(x_0)$ and the minimal boundary blow-up solution in that ball as a comparison function, it is not hard to see $u_i(x_0) \leq N_2 d(x_0)^{-\gamma}$, $i = 1, 2$, for some constant $N_2 > 0$ depending only on n and p; see e.g. [5].

Therefore, we conclude that the estimate (9) holds. Once we obtain the estimate (9), $u_1 \equiv u_2$ will follow from the iteration technique of Safonov in [12]. For the reader's convenience, we will reproduce his technique here.

Assume, to the contrary, that $u_2(x_1)/u_1(x_1) > k > 1$ for some $x_1 \in \Omega$. Let $\Omega_0 := \{u_2 > ku_1\} \cap B_r(x_1)$, where $r = \frac{1}{2}\, d(x_1)$. Then, we find

$$\Delta(u_2 - ku_1) = u_2^p - ku_1^p > (k^p - k)u_1^p \geq c_1 k r^{-\gamma p},$$

where $c_1 = 2^{-\gamma p} N_1^p (k^{p-1} - 1)$. Therefore, $\Delta(u_2 - ku_1 + w) \geq 0$ in Ω_0, where $w = \dfrac{c_1}{2n}\, k r^{-\gamma p}\,(r^2 - |x - x_1|^2)$. By the maximum principle

$$w(x_1) < (u_2 - ku_1 + w)\,(x_1) \leq \sup_{\partial\Omega_0}(u_2 - ku_1 + w).$$

Note that the maximum must be achieved on $\partial B_r(x_1) \cap \overline{\Omega}_0 \subset \partial\Omega_0$; otherwise, it is achieved on $\{u_2 = ku_1\} \cap \overline{B_r(x_1)}$, where we have $u_2 - ku_1 + w \leq w(x_1)$. Hence, $w(x_1) < (u_2 - ku_1)(x_2)$, where $x_2 \in \partial B_r(x_1) \cap \partial\Omega_0 \subset \Omega$. On the other hand, by (9), we find (recall $x_2 \in \partial B_r(x_1)$ so that $d(x_2) \geq r$)

$$w(x_1) = \frac{c_1}{2n}\, k r^{-\gamma p} r^2 = \frac{c_1}{2n}\, k r^{-\gamma} \geq c_2 k u_1(x_2),$$

where $c_2 = \dfrac{c_1}{2nN_2} = \dfrac{N_1^p}{2^{\gamma p+1} n N_2}(k^{p-1} - 1)$. Therefore, we conclude $u_2(x_2)/u_1(x_2) > (1 + c_2)k$. By iterating the above process (both c_1 and c_2 are monotone increasing in k), we obtain a sequence of points $\{x_j\}_{j=1}^{\infty}$ in Ω satisfying $u_2(x_j)/u_1(x_j) > (1 + c_2)^j k$, which tends to infinity as $j \to \infty$. On the other hand, by (9),

$$\frac{u_2(x)}{u_1(x)} < \frac{N_2 d^{-\gamma}(x)}{N_1 d^{-\gamma}(x)} = \frac{N_2}{N_1} \quad \forall x \in \Omega.$$

This contradiction proves the uniqueness. □

Remark 3.5. If $\Omega = \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$, then $\partial\Omega \neq \partial\overline{\Omega}$. The uniqueness fails in this case; see [13].

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