Designing Airplane Struts Using Minimal Surfaces

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Abstract

A model for minimizing the effects of skin drag and pressure drag is constructed. We show that a simple scaling technique can be used to transform a dual, constrained minimization problem into a volume constrained surface area minimization. We discuss some successes and failures with implementing numerical methods for the problem.

1 Introduction

There are several factors in the design of an airplane strut that impact the aerodynamic performance of the aircraft. The strut, which connects the wing of the aircraft to the engine, must be large enough to contain structural and mechanical components, but, at the same time, aerodynamic so as to minimize fluid drag.

There are at least four types of drag which are of concern with regard to the design of the strut [4]. The first is skin drag (D_s), which is a direct result of fluid viscosity. The second type is pressure drag (D_p) which is due to the pressure difference between the leading and trailing edge of the strut. The third is interference drag which decreases the performance of the aircraft. It is caused by the disruption of the flow of air around the wing due to the presence of the strut. The fourth is the induced drag, which is caused by vortex shedding.

In this report we discuss procedures for minimizing the combined skin drag and pressure drag. This model is formulated in §2. We review some theory for unconstrained and constrained area minimization, which is applicable for minimizing skin drag. In §3 we discuss the numerical methods that were used to perform the simulations, and we discuss the results. Finally, in §4 we give some conclusions and suggest future work.

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2 Model

The first type of fluid drag that we consider is the skin drag. A common approach to the problem of minimizing skin drag is to fix certain elements of the aerodynamic design and then to minimize the surface area, under these constraints. In this example we will fix the curves that define the intersections of the strut with the wing and the engine, and we require the trailing edge to be straight line (Fig. 1). Also, since the strut conceals structural members and electrical-mechanical plumbing, the smallest cross-section of the strut must have an area above a certain limit.

The phenomenon of pressure drag has a much more complicated theory than that of skin drag [6]. However, a good rule of thumb is that reducing the ratio of the thickness of the strut, t to its chord length, c, reduces pressure drag [6]. Consider a cross-section of the strut, shown in Fig. 2, taken parallel to the direction of flight and perpendicular to the axis of the strut. The chord length is the distance from the leading edge of the strut to its trailing edge, measured in the cross-section. The thickness is the maximum thickness of the cross-section, measured perpendicular to the chord. Both c and t are functions of the cross-section. We take as a simplifying assumption that pressure drag is proportional to the average of t/c, $\overline{t/c}$.

The minimization problem that we must solve is

Problem 2.0.1. *Find the minimum drag*

$$D = D_s + D_p = \eta_1 A + \eta_2 \overline{t/c}, \tag{2.1}$$

where η_1 and η_2 are constants, subject to a minimal cross-sectional area $\bar{A}$.

This problem is, as posed, very difficult. We can simplify it by replacing the minimal cross-section constraint with a fixed volume constraint, $V = V_o$, where V_o is the strut volume. The second way is to eliminate the $\overline{t/c}$ calculation by re-scaling the problem.

Simulations have shown that minimizing the area subject only to a volume constraint tends to collapse the leading edge curve towards the trailing edge, decreasing c. A bulge forms in the midsection of the strut, increasing t. The effect is demonstrated in fig. 3. The bulging effect can be diminished by the following process. The y-coordinates of the boundary curves are scaled by α, where $\alpha > 1$. The area is then minimized while constraining the volume to αV_o. The final step is to unscale. This process reduces the average of $\overline{t/c}$.

We say that a surface S_α is the α-distortion of S if its coordinates are $(x, \alpha y, z)$, where (x, y, z) are the coordinates of S, with distorted curves defined similarly. The transformed problem, which is what will be investigated in this report, is

Problem 2.0.2. *Find the surface, S, whose α-distortion, S_α, is the least area surface, given boundaries C_{w_α}, C_{t_α}, C_{e_α}, and the volume constraint αV_o.*

2.1 Some Theory of Unconstrained Minimal Surfaces

In this section we give some theory of unconstrained minimal surfaces, which will serve to introduce some important concepts. We begin by examining the following

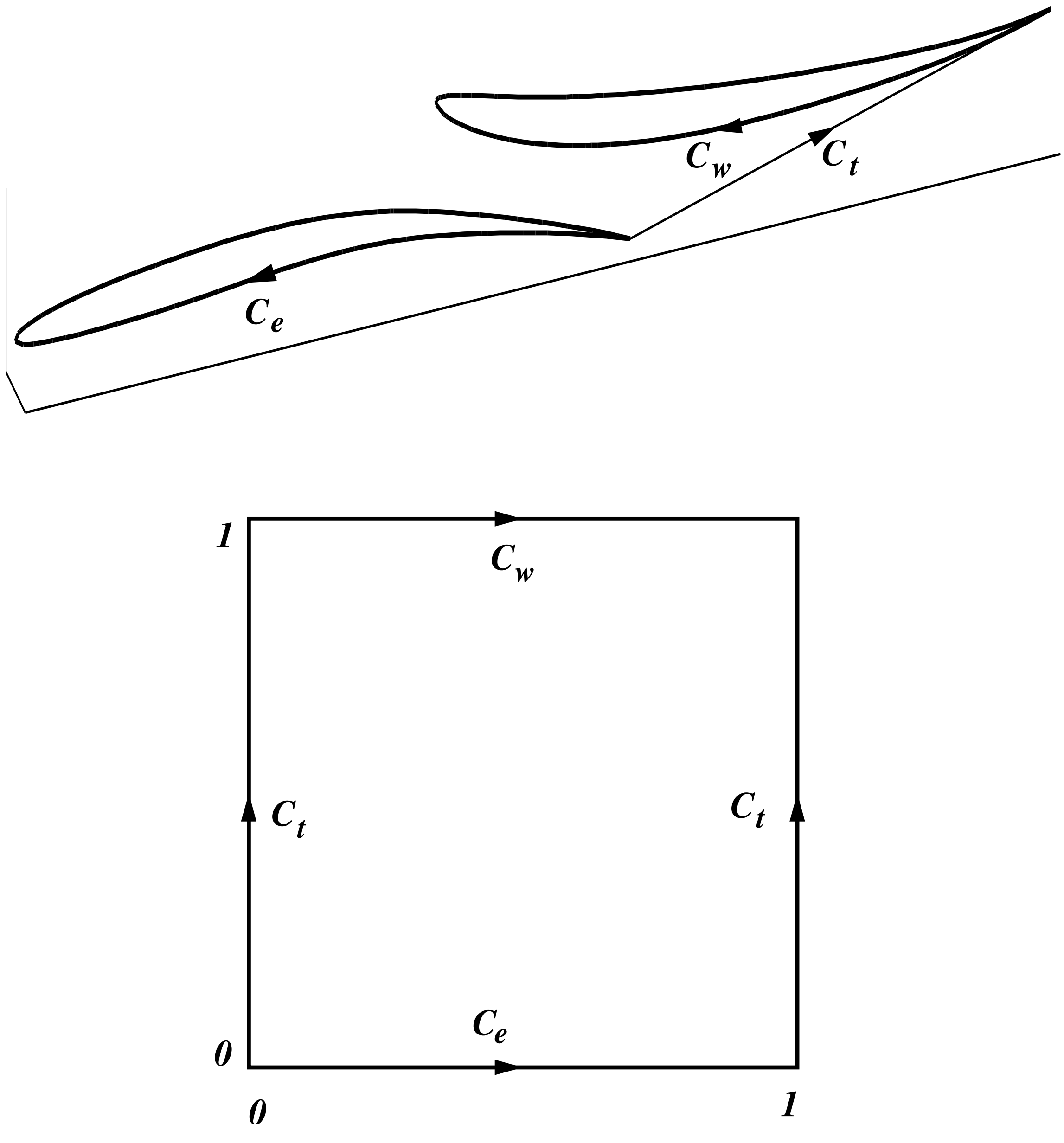

Figure 1: The bounding curves of the strut (top) with corresponding curves in the unit square (bottom). The upper curve is C_w, the lower, C_e. The straight connecting curve defines the trailing edge, and is labeled C_t.

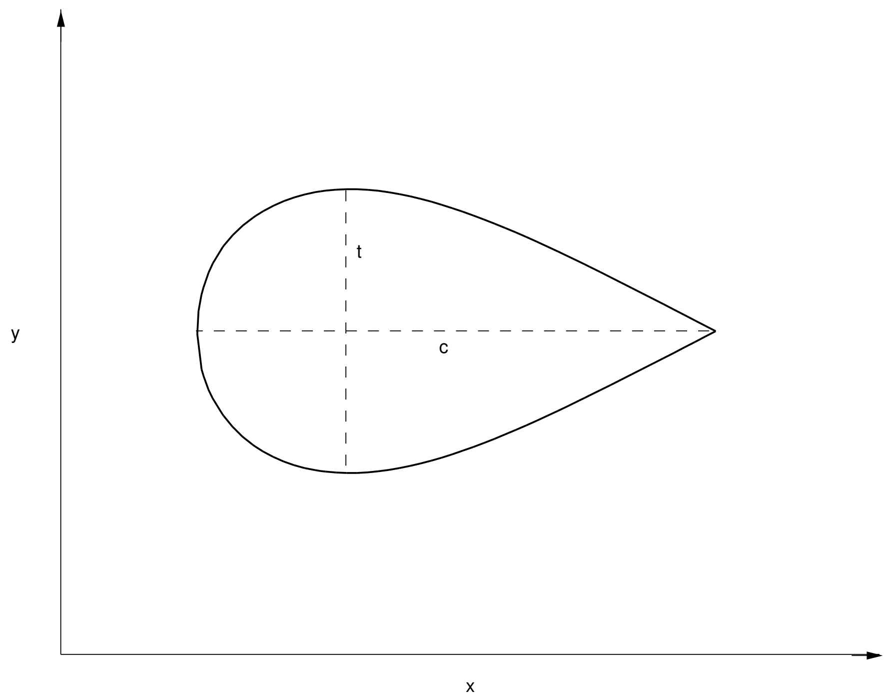

Figure 2: A cross-section of the strut showing the chord length, c and the thickness, t. Note the orientation of the axes; z is out of the page.

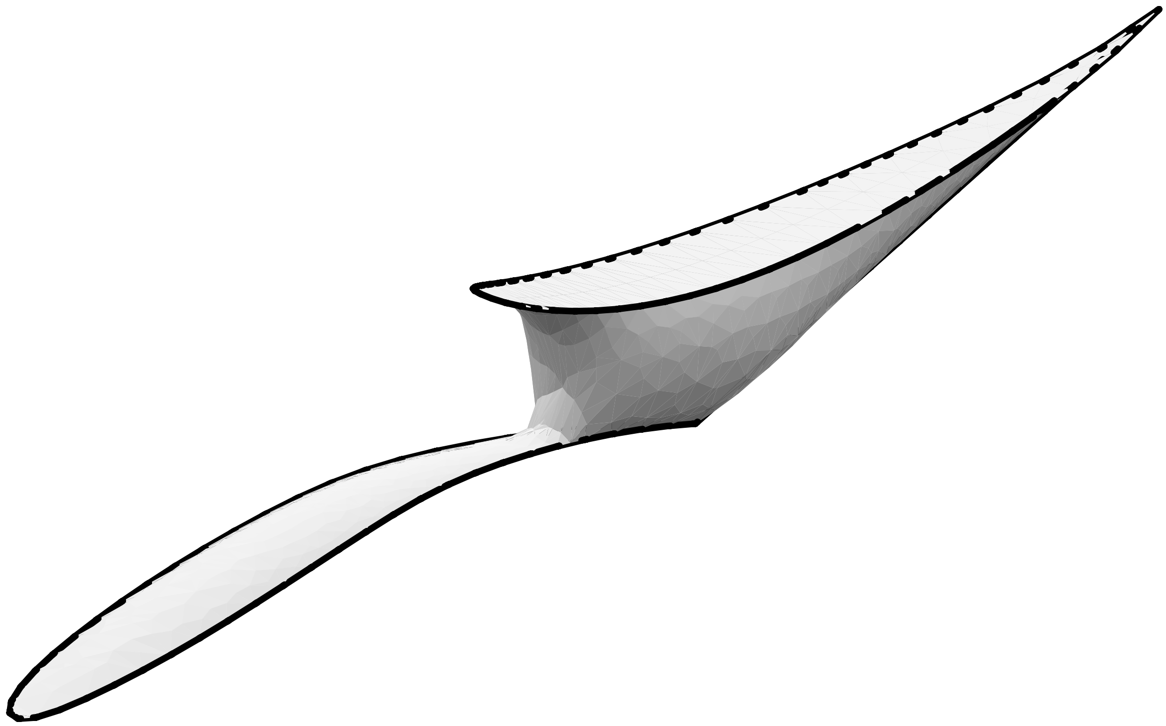

Figure 3: Area minimization with constrained volume. This snapshot, from a simulation with $V_o = 3 \times 10^{-5}$, shows that $\overline{t/c}$ can become large when pressure drag is neglected.

Problem 2.1.1. *Let C_w be the closed curve, shown in Fig. 1, that defines the intersection of the wing with the strut, and C_e be the closed curve that defines the intersection of the strut to the engine. The curve that defines the trailing edge of the strut is labeled C_t. C_w and C_e are assumed to be C^1 everywhere but where they meet the trailing edge (c.f. Fig. 1). Find the minimal surface whose boundaries coincide with these curves.*

Let $\mathbf{X}$ be a mapping of the the domain $D = (0,1) \times (0,1)$ into $\mathbf{R}^3$ (c.f Fig. 1). We take the coordinates of $D \subset \mathbf{R}^2$ to be (u, v) and assume $\mathbf{X} = (x(u,v), y(u,v), z(u,v))^T$. $\mathbf{X}$ is one of many parameterizations of the surface, $\mathbf{S}$, which is defined by the image of $\mathbf{X}$. In this discussion we consider only regular surfaces, in the sense of [2], for which one of the requirements is that partial derivatives of $\mathbf{X}$ of all orders are continuous in D. For such a surface we can define the surface normal by

$$\mathbf{N} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}. \tag{2.2}$$

It shall be convenient to have the definitions of the fundamental forms

$$E = \mathbf{X}_u \cdot \mathbf{X}_u, \quad F = \mathbf{X}_u \cdot \mathbf{X}_v, \quad G = \mathbf{X}_v \cdot \mathbf{X}_v, \tag{2.3}$$

and

$$e = \mathbf{N} \cdot \mathbf{X}_{uu}, \quad f = \mathbf{N} \cdot \mathbf{X}_{uv}, \quad g = \mathbf{N} \cdot \mathbf{X}_{vv}. \tag{2.4}$$

The area of the surface is defined by

$$A(\mathbf{S}) = \int_D \|\mathbf{X}_u \times \mathbf{X}_v\| \; dudv = \int_D \sqrt{EG - F^2} \; dudv. \tag{2.5}$$

Let $\mathbf{S}$ be a surface of least area, with the specified boundaries. Standard variational calculations [3, 5] tells us that $\mathbf{X}$ solves the Euler equations [6]:

$$\frac{\partial}{\partial u}\left(\frac{G\mathbf{X}_u - F\mathbf{X}_v}{\sqrt{EG - F^2}}\right) + \frac{\partial}{\partial v}\left(\frac{E\mathbf{X}_v - F\mathbf{X}_u}{\sqrt{EG - F^2}}\right) = 0. \tag{2.6}$$

The equations of (2.6) are degenerate, because, while variations are allowed tangent and normal to the surface, only normal variations can change the area functional. Variations tangent to the surface change the parameterizations of the surface, but leave the area functional invariant. Indeed, in general, the area is invariant under re-parameterization [6]. We expect, therefore, to find as a necessary condition for least area a single equation, rather than three.

A more appropriate variation to consider is the normal variation

$$\mathbf{\Phi}(u, v, t) = \mathbf{X}(u, v) + th(u, v)\mathbf{N}(u, v), \tag{2.7}$$

where $h(u, v)$ is a smooth mapping of D to $\mathbf{R}$. Assuming that the surface is of least area, we have [2]

$$\frac{d}{dt} A(\mathbf{X} + th\mathbf{N})|_{t=0} = -2\int_D H\sqrt{EG - F^2}h \; dudv = 0, \tag{2.8}$$

where H, the mean curvature of $\mathbf{S}$, is defined by

$$H = \frac{1}{2}\frac{Eg - 2Ff - Ge}{EG - F^2}. \tag{2.9}$$

Thus $H = 0$ if $\mathbf{S}$ is of least area, i.e., $H = 0$ is a necessary condition for least area. By definition a surface is *minimal* if and only if $H = 0$ [2]. This definition is a bit misleading since $H = 0$ is only a critical point condition, i.e., if $H = 0$ the surface is a critical point of the area functional, not necessarily the true surface of least area.

Suppose we wish to calculate the parameterization which gives the surface of least area (or a stationary point of the area functional in general). The equation $H = 0$ is underdetermined as a system. To overcome this difficulty we can specify a parameterization of $\mathbf{S}$ with two additional equations.

We first consider the conformality conditions, which are the analogs to curve parameterizations by arc length, as a way of specifying the parameterization type. The mapping $\mathbf{X}$ will be conformal if and only if

$$\mathbf{X}_u \cdot \mathbf{X}_u = \mathbf{X}_v \cdot \mathbf{X}_v, \quad \mathbf{X}_u \cdot \mathbf{X}_v = 0. \tag{2.10}$$

Hence, we have the following [6]

Proposition 2.1.2. *Let* $\mathbf{S}$ *be a regular parameterized surface, whose parameterization is* $\mathbf{X} : D \mapsto \mathbf{R}^3$*, and assume that* $\mathbf{X}$ *is conformal. Then*

$$\mathbf{X}_{uu} + \mathbf{X}_{vv} = 2\lambda^2 H\mathbf{N}, \tag{2.11}$$

where $\lambda = \mathbf{X}_u \cdot \mathbf{X}_u = \mathbf{X}_v \cdot \mathbf{X}_v$.

Proof. This property follows from the calculations

$$
\begin{aligned}
\mathbf{X}_u \cdot (\mathbf{X}_{uu} + \mathbf{X}_{vv}) &= 0, && (2.12)\\
\mathbf{X}_v \cdot (\mathbf{X}_{uu} + \mathbf{X}_{vv}) &= 0, && (2.13)\\
\mathbf{N} \cdot (\mathbf{X}_{uu} + \mathbf{X}_{vv}) &= 2\lambda^2 H. && (2.14)
\end{aligned}
$$

□

Thus, assuming $\mathbf{X}$ is conformal, $\mathbf{X}$ is minimal ($H = 0$) if and only if the coordinates of $\mathbf{X}$ are harmonic functions of (u, v), i.e.,

$$\Delta \mathbf{X} = \mathbf{0}. \tag{2.15}$$

Suppose that we solve $\Delta \mathbf{X} = \mathbf{0}$ on D with the boundary conditions in a fixed parametric form. What have we found? If $\mathbf{X}$ is conformal then $H = 0$, but, in general, this cannot be expected. In fact the nonlinearity did not disappear; it went to the boundary. When we parameterize the boundary conditions we are pinning down the solution to $\Delta \mathbf{X} = \mathbf{0}$. But it is unknown a priori which parameterization (of the boundary) makes $\mathbf{X}$ conformal.

We haven't addressed applicability of conformality conditions to our problem. Conformal maps preserve angles. Since D is a square domain the corners into which the vertices of the square are mapped should have angle $\pi/2$. This is not the case for the most natural mappings of D. We consider parameterizing over a domain which has four distinct vertices (corners), the angles at which agree with those of the corners of the surface. This, however, complicates the boundary conditions.

A different approach is to adopt less restrictive conditions on the parameterization. Consider the arc length parameterization conditions

$$\mathbf{X}_u \cdot \mathbf{X}_{uu} = 0, \quad \mathbf{X}_v \cdot \mathbf{X}_{vv} = 0. \tag{2.16}$$

These conditions do not require that angles are preserved under the mapping. If we require our boundary curves to be parameterized by arclength then equations (2.16) and equation (2.9) form the following square, second-order, nonlinear system of PDE's

$$
\begin{aligned}
0 &= \mathbf{X}_u \cdot \mathbf{X}_{uu},\\
0 &= \mathbf{X}_v \cdot \mathbf{X}_{vv},\\
0 &= (\mathbf{X}_u \cdot \mathbf{X}_u)(\mathbf{X}_{vv} \cdot \mathbf{X}_u \times \mathbf{X}_v) - 2(\mathbf{X}_u \cdot \mathbf{X}_v)(\mathbf{X}_{uv} \cdot \mathbf{X}_u \times \mathbf{X}_v)\\
&+ (\mathbf{X}_v \cdot \mathbf{X}_v)(\mathbf{X}_{uu} \cdot \mathbf{X}_v \times \mathbf{X}_v).
\end{aligned} \tag{2.17}
$$

2.2 Constrained Volume Least Area Surfaces

Adding a volume constraint to a regular surface is straightforward. Let S be the boundary of body B. Then the volume of B, V_0, is calculated as follows.

$$
\begin{aligned}
V_o &= \int_{\mathbf{B}} dxdydz \\
&= \int_{\mathbf{B}} \nabla \cdot (x, y, x)^T \; dxdydz \\
&= \oint_{\mathbf{S}} \mathbf{N} \cdot (x, y, z)^T d\mathbf{S} \\
&= \int_{\mathbf{D}} \mathbf{N} \cdot \mathbf{X} ||\mathbf{X}_u \times \mathbf{X}_v|| \; dudv. \qquad (2.18)
\end{aligned}
$$

This equation can be added to Eq. (2.5), using a Lagrange multiplier, to form a new objective function that can be minimized directly.

3 Numerical Methods For Finding Least Area Surfaces

Here we present two numerical methods for approximating minimal surface. In the first approach we solve system (2.17) using the finite difference method (FDM). The second is a direct approach that minimizes the surface energy independent of parameterizations. As this method is used in a code called *Surface Evolver* [1], we call it the Surface Evolver Method (SEM).

3.1 Finite Difference Method

For the FDM, we first generate a regular N_u by N_v mesh of $\bar{D}$ with spatial step sizes $h_u = N_u^{-1}$ in the u direction and $h_v = N_v^{-1}$ in the v direction. Let the unknown values of $\mathbf{X}$ at the interior mesh points $(u_i = ih_u, v_j = jh_v)$ be labeled $\mathbf{X}_{ij}$ where $1 \leq i \leq N_u - 1$, $1 \leq j \leq N_v - 1$. We replace the derivatives of (2.17) with the second-order accurate finite differences

$$
\begin{aligned}
\mathbf{X}_u &\approx \frac{\mathbf{X}_{i+1,j} - \mathbf{X}_{i-1,j}}{2h_u}, \qquad (3.1) \\
\mathbf{X}_v &\approx \frac{\mathbf{X}_{i,j+1} - \mathbf{X}_{i,j-1}}{2h_v}, \qquad (3.2) \\
\mathbf{X}_{uu} &\approx \frac{\mathbf{X}_{i+1,j} - 2\mathbf{X}_{i,j} + \mathbf{X}_{i-1,j}}{h_u^2}, \qquad (3.3) \\
\mathbf{X}_{vv} &\approx \frac{\mathbf{X}_{i,j+1} - 2\mathbf{X}_{i,j} + \mathbf{X}_{i,j-1}}{h_v^2}, \qquad (3.4) \\
\mathbf{X}_{uv} &\approx \frac{\mathbf{X}_{i+1,j+1} - \mathbf{X}_{i-1,j+1} - \mathbf{X}_{i+1,j-1} + \mathbf{X}_{i-1,j-1}}{4h_u h_v}. \qquad (3.5)
\end{aligned}
$$

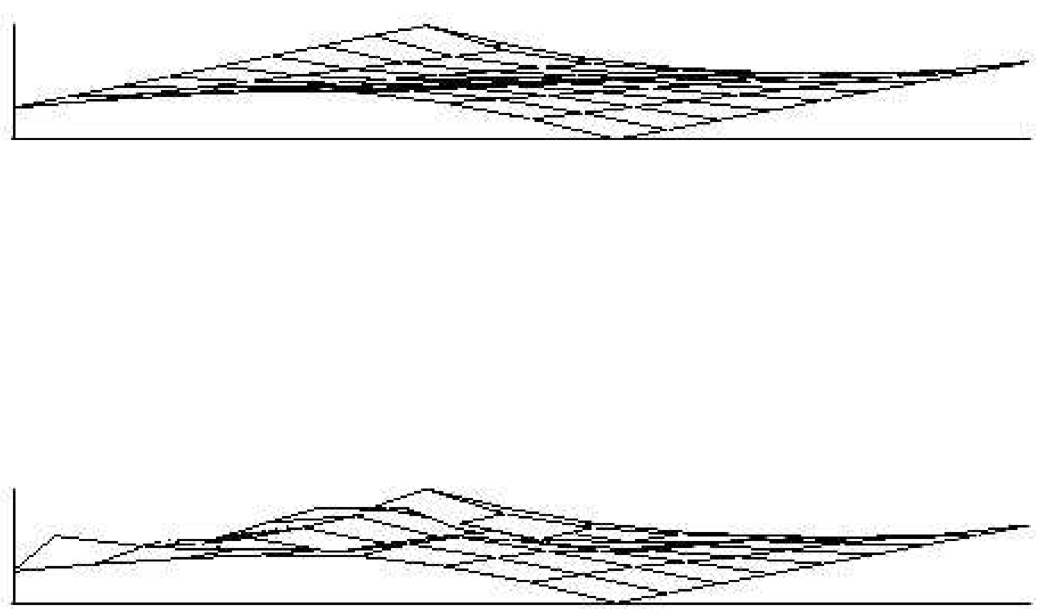

Figure 4: Results of running the FDM code to solve (2.17). The intial guess (top) is a ruled surface. The final result (bottom) is clearly not minimal.

Discretization of this form results in the formation of $3(N_u-1)(N_v-1)$ nonlinear equations in the same number of unknowns.

As a test run, we used code based on the above scheme to calculate the (simple) minimal surface with the following boundary conditions

$$\begin{aligned} \mathbf{X}|_{v=0} &= (u,0,0), \quad \mathbf{X}|_{v=1} = (u,1,u), \quad u \in [0,1], \\ \mathbf{X}|_{u=0} &= (0,v,0), \quad \mathbf{X}|_{u=1} = (1,v,v), \quad v \in [0,1]. \end{aligned} \tag{3.6}$$

(3.7)

The code converged rapidly to the solution provided the initial guess was reasonable.

We had no success, however, solving (2.17) with the initial conditions shown in Fig. 1. This was initially attributed to the degree of nonlinearity of the discretized system of equations and to the coarse mesh that was being used. We were able to rule out the latter. The former was addressed by a continuation algorithm to globalize the convergence zone. We used the homotopy map

$$(1-\alpha_i)\mathbf{X}_{vv} + \alpha_i \begin{pmatrix} \mathbf{X}_u \cdot \mathbf{X}_{uu} \\ \mathbf{X}_v \cdot \mathbf{X}_{vv} \\ \tilde{H} \end{pmatrix} = 0, \tag{3.8}$$

with continuation parameters α_i. Here $\tilde{H}$ is the right hand side of the third equation in (2.17). Still the code failed to find a minimal surface.

The results shown in Fig. 4 were typical for the output of the code. The lack of success of the code raised the question of well-posedness of (2.17). Surface Evolver confirmed that the system (2.17) was ill-posed with the given boundary conditions.

3.2 Surface Evolver Method

The SEM is a direct method that minimizes the surface energy of a faceted surface. An initial faceted surface with fixed boundary vertices and free interior vertices is required. This results in a finite dimensional minimization problem for the total energy. This problem is solved by a conjugate gradient method [1].

We ran Surface Evolver to find the unconstrained minimal surface for the boundary conditions shown in Fig. 1. The results are displayed in Fig. 5. It is clear why the FDM code did not converge to a minimal surface. The limiting surface is not regular. Further iterations would produce one surface on the top and one on the bottom, with the trailing edge providing a location for pinch-off.

3.3 Results Using Surface Evolver

An earlier observation noted that a volume constrained minimal surface had a bulge in the midsection. This is equivalent to solving the distorted problem with a parameter of $\alpha = 1$. Increasing α had the effect of reducing the mid section bulge and, at the same time, reducing the pressure drag term.

We began with a volume estimate $V_o = 1\times10^{-5}$. The estimate must be sufficient ensure that the resulting strut can fit all the necessary structural and mechanical components.

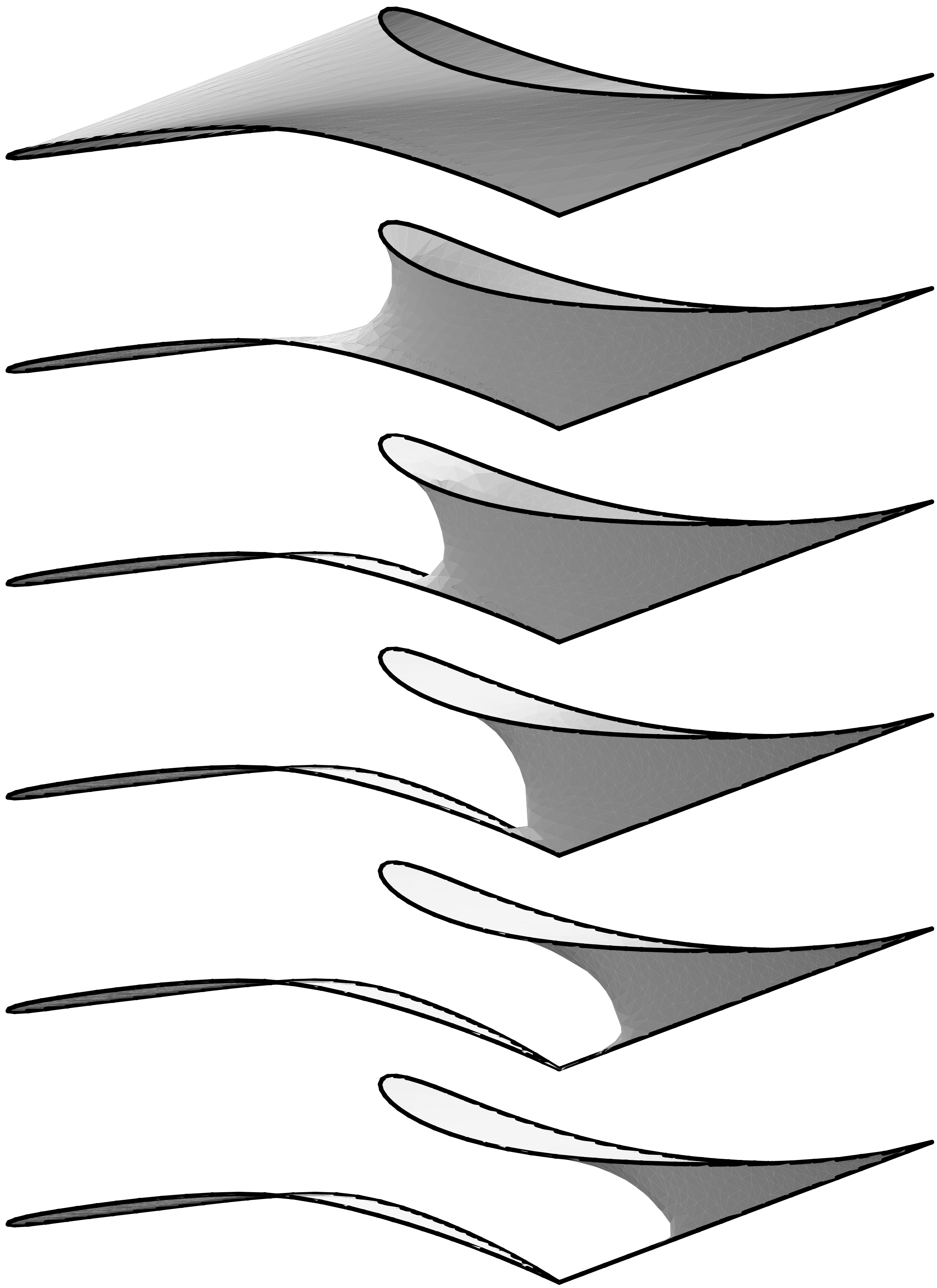

Figure 5: Evolution to the minimal surface. This series of images, created using Surface Evolver, shows why the system in (2.17) is ill-posed given the boundary conditions.

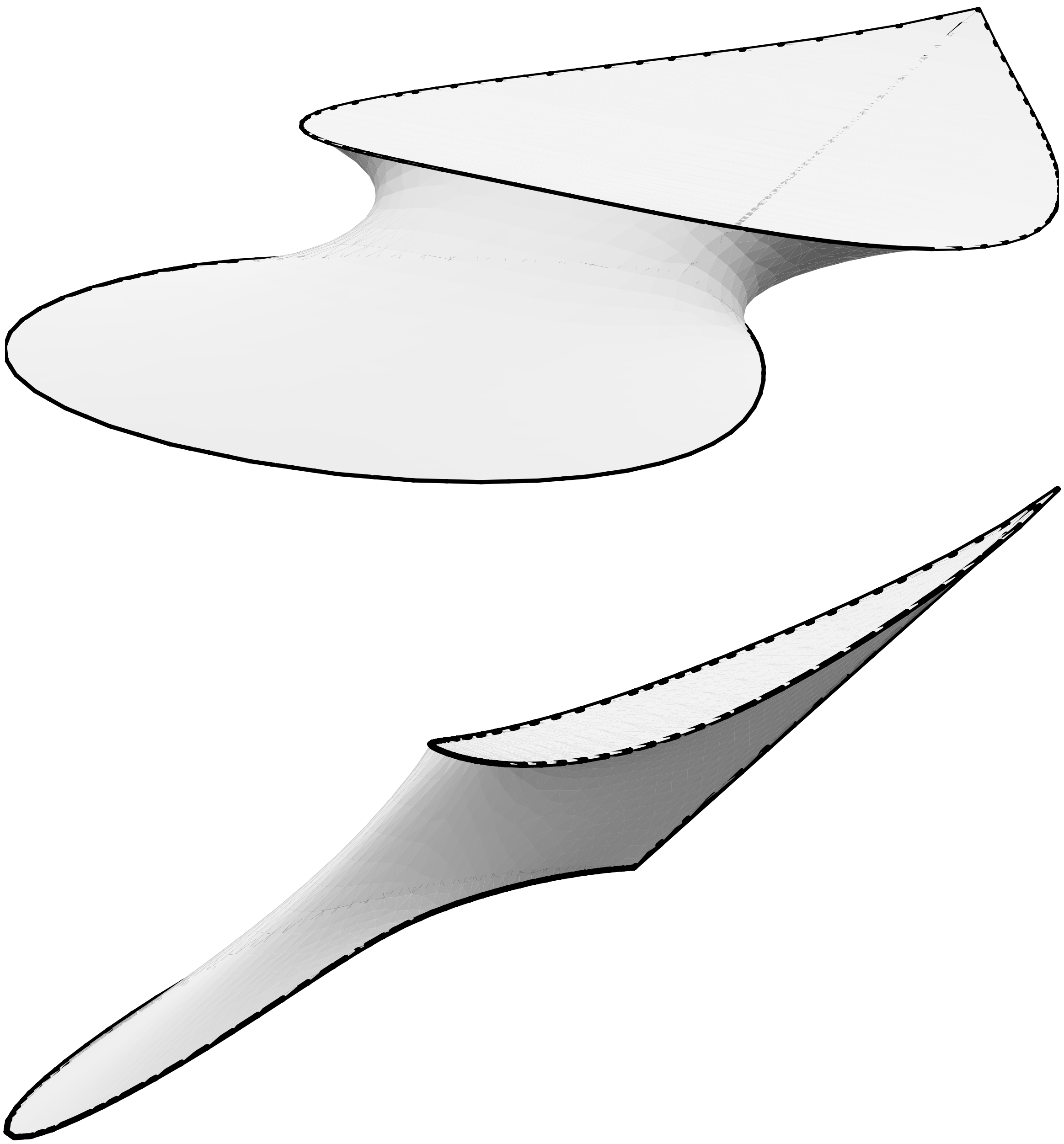

Figure 6: The α-distorted surface, near its volume-constrained minimum, shown before (top) and after (bottom) unscaling. These figures are from a simulation with $V = 3\times10^{-5}$ and $\alpha = 10$. This shows that the minimal surface obtained via the distortion method has its leading edge in tact.

We then experimented with values of α to determine which gave the right trade-off between minimizing surface area (skin drag) and minimizing $\overline{t/c}$ (pressure drag).

Results of the method are depicted in 6. For comparison we note that for $\alpha = 1$ the leading edge has deteriorated significantly. When $\alpha = 10$, however, the leading edge is in tact. The trade-off is that the surface area has increased for $\alpha = 10$ over $\alpha = 1$. However, the surface shown at the bottom of Fig. 6, the unscaled surface, has 10% less surface area than the ruled surface (the initial guess), a significant decrease. Since a good rule of thumb is that skin drag and pressure drag contribute about equally to the total drag, the trade is probably a good one.

4 Conclusions

The general method that we describe in this report is quite simple. Its utility depends upon some further investigation of the Physics of the problem and the true trade-off between minimizing area and $\overline{t/c}$. This depends quite a lot on the unknowns η_1 and η_2, the values for which we have neglected to include. We have shown some evidence of a simple tool to balance surface area reduction with having an acceptable value for $\overline{t/c}$.

The method that we defined using parameterizations is not useless. In the general case, however, H is not zero, so a more general variational approach will be needed. Using Surface Evolver helps to explore what problems are solvable with parameterizations and which are not. What the results of using the α distortion method have shown is that under reasonable conditions there is no pinch-off effect, making the problem amenable to parameterization methods.

References

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