VEHICLE NETWORKS: ACHIEVING REGULAR FORMATION

MADALENA CHAVES, ROBERT DAY, LUCIA GOMEZ-RAMOS, PARTHASARATHI NAG, ANCA WILLIAMS, WEI ZHANG

MENTOR: SONJA GLAVAŠKI

ABSTRACT. In this paper we will consider a network of vehicles exchanging information among themselves with the intention of achieving a specified polygonal formation. The network achieves the formation through decentralized feedback control, which is constructed from the available information. Several information flow laws are considered in order to improve the performance of the vehicle network. A stochastic model for information flow is also considered, allowing for the randomly breaking of the communication links among the vehicles.

1. Introduction

As we move into an era of autonomous vehicles, the study of control theory serves to design and analyze the performance of automated systems.

The cooperative use of unmanned vehicles, for example, requires some assurance of proper performance, especially when conditions are unfavorable. When centralized coordination is either disabled, impractical or tactically inadvisable, it is unclear whether individual vehicles will be able to properly use the information available to them, or how much information they may need to perform a desired task. One natural question is whether automated vehicles will be able to arrange themselves into a prespecified formation, assuming that one or all of the vehicles has incomplete information as to the whereabouts of other vehicles.

Applications include satellites required to arrange themselves in formation when only a portion of the group is in each one's line-of-sight, or unmanned aerial vehicles (UAVs) which may be summoned from patrol to survey or pursue a moving object below, when distance or other obstacles prevent communication. Deep exploration of space will inevitably require unmanned probes to position themselves relative to one another in order to observe various anomalies, under conditions in which signal delays make remote human control impossible, so that the vehicles must fend for themselves.

In recent papers [4, 5], the convergence of simple automated vehicles into formation using directed graphs to describe the vehicles' abilities to detect one another, has been explored. In [4, 5] graphical conditions are developed, which allow us to predict when the dynamical systems describing these autonomous vehicle formations will be stable in the traditional sense of Lyapunov. Assuming an underlying network structure for communication, they also demonstrate methods in which transmission of information amongst the vehicles facilitates a more efficient convergence to their predetermined relative positions. The formations considered are achieved when the vehicles in question assemble in proper position relative to a stationary central point which is not determined in advance, but will be arrived at through formation consensus.

In this work we extend these ideas of vehicle formation to include several new features. Since Lyapunov stability (a typical characterization of well-behaved control systems) requires asymptotic convergence of each of the variables used to describe a given vehicles behavior, we first provide a new definition of what is meant by a vehicle formation, a definition which includes dynamic or moving formations in which the positions of vehicles relative to a fixed frame of reference is not required. This definition will include convergence into stopped positions oriented about an undetermined final center (which will be Lyapunov stable, as in the studies of Fax and Murray), and also convergence of moving formations of vehicles which maintain fixed positions relative to one another without coming to a halt (as is desirable in satellite and surveillance formations for example.)

Further, we consider systems in which transmission and perception amongst the vehicles may be faulty, intermittent, or otherwise randomly varying. We introduce a stochastic element to the underlying graph in which arcs representing lines of communication may randomly disappear. After devising a well-behaved control system for individual vehicles facing the confusing conditions of intermittent sensory data, we also study the possibility of improved performance when vehicles transmit and receive information amongst themselves, again under the conditions of random loss.

2. Problem Formulation

We shall first give a brief overview of some algebraic graph theory concepts that will be used in modeling the information exchange among various vehicles in the network.

Definition 2.1. A directed graph G consists of a vertex set $V(G)$ and an edge set $A(G)$, where an arc or a directed edge is an ordered pair of distinct vertices $a = (u, v)$.The vertex u is called the tail of a and the vertex v is called the head of a.

A directed graph arises naturally in studies of sequential machines and in control theory. In this paper the information exchange between vehicles is represented by directed graphs.

Definition 2.2. Let u and v be two vertices of a directed graph G. An $u - v$ walk of G is a finite, alternating sequence of vertices and arcs

$$u = u_0, a_1, u_1, a_2, \dots , u_{n-1}, a_n, u_n = v$$

beginning with u and ending with v, and such that $a_i = (u_{i-1}, u_i)$, for $i = 1, \dots , n$.

Definition 2.3. Let $u, v \in V(G)$. Vertex v is said to be reachable from vertex u if the directed graph G contains an $u - v$ walk.

Definition 2.4. A directed graph G is strongly connected if, for every two distinct vertices of G, each vertex is reachable from the other.

Definition 2.5. The adjacency matrix of a graph G, denoted $A_d(G)$, is a square matrix of size $|V(G)| \times |V(G)|$ defined as follows:

$$A_d(i,j) \;=\; \begin{cases} 1 & \text{for } (u_i, u_j) \;\in\; A(G) \\ 0 & \text{elsewhere.} \end{cases}$$

where $u_i, u_j \in V(G)$.

Definition 2.6. The number of arcs incident into a vertex u_i is the in-degree of u_i. We denote by D the diagonal matrix with the in-degree of vertex u_i as the (i, i)-th entry.

Definition 2.7. The Laplacian matrix of a graph G is defined as

$$L = I - D^{-1}A_d.$$

An equivalent characterization of a strongly connected graph can be given in terms of the adjacency matrix. A directed graph G is strongly connected if there exists an integer $m > 0$ such that the matrix $(I + A_d)^m$ has *all entries strictly positive*.

Example 2.8. Here is an example of how the Laplacian is computed for a graph representing a network of vehicles.

Consider a network of vehicles as follows, where the arc (i, j) implies that i sees j:

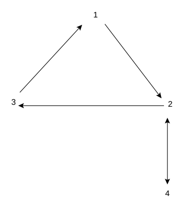

FIGURE 1. 4-Vehicle Network

The adjacency matrix, A_d, for this graph is given by:

$$A_d = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The in-degree matrix D for the graph is:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence, the Laplacian of the graph is

$$L = I - D^{-1}A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

In this work, we consider a network of N vehicles, among which there are some means of communication. We assume that each vehicle is a node in a strongly connected graph, G_1, with the corresponding Laplacian L_1. The adjacency matrix of this graph represents the communication links among the N vehicles: if $(A_d)_{ij} = 1$ then vehicle i can sense information from vehicle j (we also say that "i can see j"). Note that the links need not be reversible, so i may receive information from j but j doesn't receive information from i. The individual dynamics of each vehicle will be modeled by a discrete-time linear control system:

$$\begin{aligned} x^i_{k+1} &= A_1 x^i_k + B_1 u^i_k \\ z^i_k &= \frac{1}{|J_i|} \sum_{j \in J_i} ((x^i_k - h^i_0) - (x^j_k - h^j_0)) \end{aligned}$$

where, for each time k, x^i_k represents the state variable vector for vehicle i, u^i_k represents the control vector and z^i_k is the output. $J_i \subset [1, N] \backslash \{i\}$ represents the set of vehicles which vehicle i can sense. The output can be interpreted as the measurement of the positions of the neighbors $j \in J_i$ relative to the position of vehicle i. The vectors h^i_0 represent the desired relative positions of each vehicle with respect to the center of the formation.

The equations for the entire system can be written in a compact form, where the output is given in a natural way as follows:

$$\begin{aligned} x_{k+1} &= A x_k + B u_k \\ z_k &= L(x_k - h_0) \qquad (\Sigma) \end{aligned}$$

where $A = I_N \otimes A_1$, $B = I_N \otimes B_1$, $L = L_1 \otimes I_4$ (since, in our models, each state vector will have four coordinates) and L_1 is the Laplacian of the graph G_1.

We wish to investigate the problem of controlling this network of vehicles into a specified dynamic formation, where each vehicle will occupy a fixed position relative to the other vehicles (see Figure 2, for an example of a dynamic formation, and section 5 for more simulation results).

In section 3 we will introduce and discuss several notions of stability, and consider the communication among the vehicles to be in the form of sensory information (vehicle i receives information from the vehicles it can see, according to the graph). In section 4, we consider another type of communication among the vehicles: both sensory and transmitted information (vehicle i also sends some kind of information to the vehicles that can see him). In both sections we also consider the possibility of random loss of information, how this affects the stability of the network formation, and how to improve the controls. Finally, in section 5, a model is chosen for the vehicles' dynamics and several results of simulation tests are shown. Some remarks and conclusions end our paper.

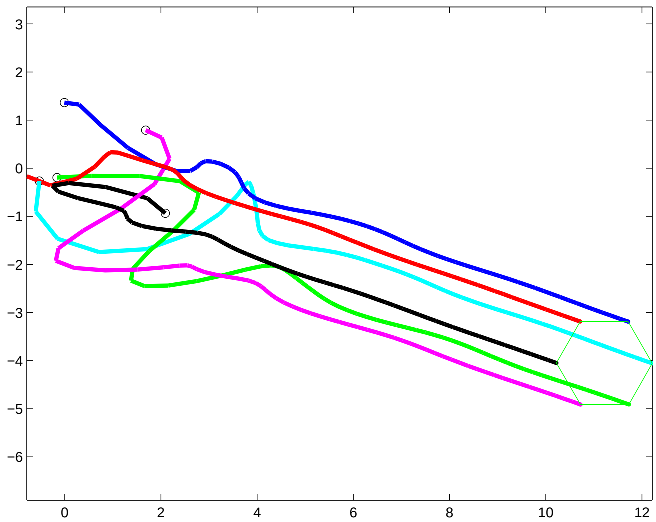

FIGURE 2. Achieving hexagonal formation, starting from random initial positions.

3. Sensed Information and Stochastic loss

Following approach from [4] we consider a set of N vehicles whose individual dynamics are given by the following discrete-time linear model:

$$\begin{aligned} x_{k+1}^i &= A_1 x_k^i + B_1 u_k^i \\ z_k^i &= \frac{1}{|J_i|}\sum_{j\in J_i}((x_k^i - h_0^i) - (x_k^j - h_0^j)) \qquad (\Sigma_1) \end{aligned}$$

where $x_k^i \in \mathbb{R}^4$ is the state variable vector, $u_k^i \in \mathbb{R}^2$ is the control variable vector and $z_k^i \in \mathbb{R}^4$ is the measurement signal error. The vector $h_0^i \in \mathbb{R}^4$ denotes the offset of vehicle i in the formation, i.e., the desired relative position of vehicle i with respect to the center of the formation. The set $J_i \subset [1, N] \setminus \{i\}$ represents the set of vehicles which vehicle i can sense. The sensory information exchange can be rewritten in terms of a matrix which is precisely the Laplacian of the graph, L_1 (formed by each vehicle and their sensing indices J_i).

Using the measurement signal z_k^i, we intend to construct a control:

$$u_k^i = F_1 z_k^i$$

that regulates the system Σ_1 into its position in a dynamic formation, in the sense defined below.

We now consider the entire system of N vehicles, with state vector $x_k = (x_k^1, \dots, x_k^N)' \in \mathbb{R}^{4N}$, control vector $u_k = (u_k^1, \dots, u_k^N)' \in \mathbb{R}^{2N}$, output vector $z_k = (z_k^1, \dots, z_k^N)' \in \mathbb{R}^{4N}$ and offset vector $h_0 = (h_0^1, \dots, h_0^N)' \in \mathbb{R}^{4N}$. We let $A = I_N \otimes A_1$ represent the matrix A_1 repeated N times along the diagonal of I_N. Similarly, suitable dimensional adjustments are made for the B_1 and F_1 matrices below. Using this notation we rewrite the entire system dynamics as follows:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ z_k &= L(x_k - h_0) \qquad (\Sigma) \end{aligned}$$

where $A = I_N \otimes A_1$, $B = I_N \otimes B_1$ and $L = L_1 \otimes I_4$ is the augmented Laplacian of the graph. The control law for the network is again of the form $u_k = Fz_k$.

3.1. **Notions of stability.** We next introduce some concepts that will be used throughout the paper. For any square matrix M, let $\Lambda(M)$ denote the set of eigenvalues of M.

Definition 3.1. The system Σ is stabilizable if there exists a matrix F such that $\Lambda(A+BFL)$ is in the interior of the unit circle. In particular, Σ is stabilizable with decentralized control if F can be written as $I_N \otimes F_1$.

Definition 3.2. N vehicles are said to be in a dynamic formation if at all times they are ordered as the vertices of a pre-specified 2-complex.

In our paper we will consider N vehicles to achieve "formation" if they are ordered on the vertices of a regular polygon. Notice that this formation can be dynamic, i.e. the positions of the vehicles may not be constant with respect to some global reference point. This is not the case in previous work presented in [4].

Definition 3.3. A system Σ is said to be F-stable (Formation-stable) if

$$\lim_{k\to\infty} \|L(x_k - h_0)\|_{\mathbb{R}^{4N}} = 0$$

where $h_0 \in \mathbb{R}^4$ is the offset vector relative to formation center.

Definition 3.4. A system Σ is said to be L-stable if there exists an element $x_{ss} \in \mathbb{R}^{4N}$ such that

$$\lim_{k\to\infty} \|x_k - x_{ss}\|_{\mathbb{R}^{4N}} = 0.$$

In this case, x_{ss} is called the steady-state of the system Σ.

Note that a system is L-stable if it converges asymptotically to a fixed state x_{ss}. For a network of vehicles, L-stability corresponds to all the vehicles in the network coming to a full stop at the point x_{ss} (this point will necessarily have the velocity coordinates equal to zero).

Given any vector $h_0 \in \mathbb{R}^{4N}$, define $H = \{\alpha h_0 : \alpha \in \mathbb{R}\}$. The next Proposition gives a necessary condition for the network Σ to be both F-stable and L-stable. In other words, the network not only achieves the formation, but the vehicles also asymptotically converge to a fixed position in the plane.

This is, however, a very strong condition for F-stability, since the vehicles in the network may achieve (and remain in) formation without coming to a full stop (dynamic formation).
Both situations (F-stability with and without L-stability) have been observed in simulations, for two different models of vehicles.

Proposition 3.5. *Assume that system Σ is stabilizable, i.e., there exists a matrix F such that $\Lambda(A + BFL)$ is in the interior of the unit circle. Then the system Σ is L-stable. Furthermore, assume that the matrix F also satisfies*

$$(I - \tilde{A})^{-1}BFL + I : H \to Null(L)$$

where $\tilde{A} = A + BFL$. Then the system is simultaneously L-stable and F-stable.

Proof: Assume that the system is stable. Note that

$$\begin{aligned} x_{k+1} &= Ax_k + BFL(x_k - h_0) \\ &= (A + BFL)x_k - BFLh_0 \end{aligned}$$

Let us denote

$$BFLh_0 = q,$$

then

$$\begin{aligned} x_{k+1} &= \tilde{A}x_k - q \\ &= \tilde{A}^{k+1}x_0 - \sum_{i=0}^{k} \tilde{A}^i q \end{aligned}$$

Taking limits we obtain:

$$\begin{aligned} \lim_{k\to\infty} x_{k+1} &= \lim_{k\to\infty} (\tilde{A}^{k+1}x_0 - \sum_{i=0}^{k} \tilde{A}^i q) \\ &= -(I - \tilde{A})^{-1}BFLh_0 \\ &= x_{ss} \end{aligned}$$

Therefore the system is L-stable with steady state x_{ss}. Next, let us consider the error term.

$$\begin{aligned} z_k &= L(x_k - h_0) \\ \lim_{k\to\infty} z_k &= \lim_{k\to\infty} L(x_k - h_0) \\ &= L \lim_{k\to\infty} (x_k - h_0) \\ &= -L\left((I - \tilde{A})^{-1}BFLh_0 + h_0\right) \\ &= -L\left((I - \tilde{A})^{-1}BFL + I\right) h_0 \\ &= 0 \text{ (by hypothesis)} \end{aligned}$$

Therefore the system is F-stable. Hence the system Σ is both F and L-stable.

3.2. **Stochastic loss of information.** Following approach from [7], we next consider the case when sensed information is received with some probability. We will try to incorporate some results from [7] into the previous framework, and thus extend the formation stability analysis.

At each time step, we associate to each arc (i, j) (with $(A_d)_{ij} \neq 0$), a number θ_k^{ij} which is zero if the sensed information that i would receive from j is lost — in this case, we say that the arc was broken or invisible at time k. After [7], we define

$$\theta_k^{ij} = \begin{cases} 1, & \text{if } i \text{ sees } j \\ 0, & \text{if } i \text{ doesn't see } j. \end{cases} \tag{1}$$

For each pair (i, j), θ_k^{ij} is a Bernoulli process with $\mathbf{P}(\theta_k^{ij} = 1) = p$. We assume that every arc (i, j) has the same probability p of not breaking at time k, and that the random variables θ_k^{ij}, $k = 1, 2, \ldots$ are independent. We also assume that each vehicle has access to θ_k^{ij}, in other words, each vehicle knows when an arc is broken.

Whenever an arc is invisible, vehicle i does not receive the new position of its neighbor $j \in J_i$. To overcome this loss of information, we allow vehicle i to use the *last available information* received from vehicle j. So, as discussed in [7], a *sensory information vector* can be defined as follows:

$$w_k^{ij} = w_{k-1}^{ij} + \theta_k^{ij}(x_k^j - w_{k-1}^{ij}) \tag{2}$$

which represents the information that i receives from j at time k. If the arc (i, j) was not broken, then i receives the new position information, x_k^j, but if the arc was broken, then i receives the latest position information available, w_{k-1}^{ij}. (Note that there is a different vector w_k^{ij} associated with each arc (i, j): if, for instance, $j_1, j_2 \in J_i$, then vehicle i may receive old information from neighbor j_1 and new information from another neighbor j_2.)

Vehicle i will now use w_k^{ij} to compute the control u_k^i. In the case of a broken arc, we improve the sensory information vector by *estimating the current position of vehicle* j. Using the old information w_{k-1}^{ij}, and assuming that j would follow its drift direction, the vector w_k^{ij} becomes

$$w_k^{ij} = A_1 w_{k-1}^{ij} + \theta_k^{ij}(x_k^j - A_1 w_{k-1}^{ij}). \tag{3}$$

Then each vehicle, $i = 1, \ldots, N$, in the network evolves according to

$$x_{k+1}^i = A_1 x_k^i + B_1 F_1 \frac{1}{|J_i|} \sum_{j \in J_i} (x_k^i - h_0^i - (w_k^{ij} - h_0^j)). \tag{4}$$

For the rest of this section, we will analyze the simple case of a 2 vehicle strongly connected network (see Figure 3). The Laplacian for

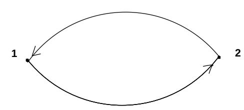

FIGURE 3. Information Exchange for 2 Vehicle Network

this graph is

$$L_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We have the following system of equations for the closed loop system

$$X_{k+1} = \hat{A}_k X_k + BFLh_0, \tag{5}$$

where X_{k+1} is the augmented vector of the state vectors together with the estimated state vectors:

$$X_{k+1} = \begin{pmatrix} x_{k+1}^1 \\ x_{k+1}^2 \\ w_{k+1}^{21} \\ w_{k+1}^{12} \end{pmatrix}$$

and

$$\hat{A}_k = \begin{pmatrix} A_1 + B_1 F_1 & 0 & 0 & -B_1 F_1 \\ 0 & A_1 + B_1 F_1 & -B_1 F_1 & 0 \\ \theta_k^{21} I & 0 & (1 - \theta_k^{21}) A_1 & 0 \\ 0 & \theta_k^{12} I & 0 & (1 - \theta_k^{12}) A_1 \end{pmatrix}.$$

For each k, $\hat{A}_k \in \{\hat{A}^0, \hat{A}^{01}, \hat{A}^{10}, \hat{A}^1\}$ corresponding to the 4 possible combinations of information loss across the network. ($\hat{A}^0$ corresponds to $\theta_k^{21} = \theta_k^{12} = 0$; $\hat{A}^{01}$ corresponds to $\theta_k^{21} = 0$, $\theta_k^{12} = 1$; etc.)

A equivalent way of writing system (5) is given next, which may provide some insight into the problem.

Lemma 3.6. *An equivalent equation for system (5) is*

$$\tilde{X}_{k+1} = \hat{A}_k \tilde{X}_k + \hat{B}_k h_0 \tag{6}$$

where $\tilde{X}_k = (x_k^1 - h_0^1, x_k^2 - h_0^2, w_k^{21} - h_0^1, w_k^{12} - h_0^2)'$ *is a translation of the augmented vector* X_k*,* $\hat{A}_k \in \mathbb{R}^{16\times 16}$ *is as defined above and* $\hat{B}_k \in \mathbb{R}^{16\times 8}$ *is given by*

$$\hat{B}_k = \begin{pmatrix} A_1 - I & 0 \\ 0 & A_1 - I \\ (1-\theta_k^{21})(A_1 - I) & 0 \\ 0 & (1-\theta_k^{12})(A_1 - I) \end{pmatrix}.$$

Proof. Note that, for each vehicle $i = 1, 2$, the individual dynamics may be rewritten

$$\begin{aligned} x_{k+1}^i - h_0^i &= (A_1 + B_1 F_1)(x_k^i - h_0^i) - B_1 F_1 \frac{1}{|J_i|} \sum_{j \in J_i} (w_k^{ij} - h_0^j) \\ &\quad + (A_1 - I) h_0^i, \end{aligned}$$

by adding and subtracting the necessary terms in h_0^i. Likewise for the sensory information vector

$$\begin{aligned} w_{k+1}^{ij} - h_0^j &= A_1(w_k^{ij} - h_0^j) + \theta_k^{ij}(x_k^j - h_0^j - A_1(w_k^{ij} - h_0^j)) \\ &\quad + (1-\theta_k^{ij})(A_1 - I) h_0^j \end{aligned}$$

Putting these together we obtain the desired equation for the translated state variables $\tilde{X}_k$.

For our next result, we will consider the simplified case

$$\theta_k^{12} = \theta_k^{21} = \theta_k,$$

that is, we will assume that the losses of information in the 2 vehicle network are characterized by a single Bernoulli process. Thus, at each step k, the matrix $\hat{A}_k$ will take values in $\{\hat{A}^0, \hat{A}^1\}$, corresponding to the situations: "all information is lost" or "all information is passed on".

This is certainly a much simpler version of the general problem we are studying, but in some sense we may look at this simplified model as a "worst case" analysis of our problem. In the simulations we will consider independent random variables θ_k^{ij} and thus allowing for partial loss of information.

Under these simplified conditions several useful results from [7] may be applied, namely, Definition 1, and Theorems 1, 2 in that paper. For reference, we state these next.

Definition 3.7. (see Definition 1 from [7]) The system given by (6) with $\hat{B}_k h_0 \equiv 0$ is mean-square stable (MSS) if for every initial state (x_0, θ_0)

$$\lim_{k \to +\infty} E[\|x_k\|^2] = 0.$$

Theorem 3.8. *(see Theorem 2 from [7]) System (6) with $\hat{B}_k h_0 \equiv 0$ is mean-square stable (MSS) iff there exists a matrix $M > 0$ such that*

$$M - \sum_{j=0}^{1} p_j (\hat{A}^j)' M \hat{A}^j > 0$$

where p_j is the probability $\mathbf{P}(\theta_k = j)$, $j = 0, 1$.

Proposition 3.9. *Assume that*

$$(A_1 - I) h_0^i = 0, \quad i = 1, 2.$$

Assume also that $\theta_k^{12} = \theta_k^{21} = \theta_k$. Then, system (6) is mean-square stable if and only if there exists a symmetric, positive definite matrix M such that the following LMI is satisfied:

$$\begin{pmatrix} Z & \sqrt{p} Z' (\hat{A}^0)' & \sqrt{1-p} Z' (\hat{A}^1)' \\ \sqrt{p} \hat{A}^0 Z & Z & 0 \\ \sqrt{1-p} \hat{A}^1 Z & 0 & Z \end{pmatrix} > 0,$$

where $Z = M^{-1}$.

Proof. We know from [7] (Theorem 2) that the system (6) is mean-square stable iff there exists a positive definite symmetric matrix M such that the following holds

$$M \; - \; p(\hat{A}^0)' M \hat{A}^0 \; - \; (1-p)(\hat{A}^1)' M \hat{A}^1 \; > \; 0. \tag{7}$$

Let us consider the following positive definite block matrix

$$A \; = \; \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{12} \; = \; (A_{21})'$ and A_{22} is invertible.

By standard Gaussian Elimination i.e. multiplying the second row by $A_{12} A_{22}^{-1}$ and subtracting the resultant from the first row we obtain

$$\begin{pmatrix} A_{11} \; - \; A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11} \; - \; A_{12}A_{22}^{-1}A_{21} \; > \; 0$ is the *Schur Complement*.

Pre and postmultiplying (7) by $Z \; = \; M^{-1}$ and acknowledging the fact that $Z \; = \; Z'$ we obtain the following inequality:

$$(8) \qquad Z \; - \; \sqrt{p}(\hat{A}^0 Z)' Z^{-1} \sqrt{p}\hat{A}^0 Z \; - \; \sqrt{1-p}(\hat{A}^1 Z)' Z^{-1} \hat{A}^1 Z \; > \; 0.$$

This in turn is equivalent to

(9)

$$Z \; - \; \begin{pmatrix} \sqrt{p}(\hat{A}^0 Z)' & \sqrt{1-p}(\hat{A}^1 Z)' \end{pmatrix} \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}^{-1} \begin{pmatrix} \sqrt{p}\hat{A}^0 Z \\ \sqrt{1-p}\hat{A}^1 Z \end{pmatrix} > 0$$

Setting

$$A_{11} = Z, \quad A_{22} = \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}, \quad A_{21} = \begin{pmatrix} \sqrt{p}\hat{A}^0 Z \\ \sqrt{1-p}\hat{A}^1 Z \end{pmatrix}$$

and comparing (9) with the Schur complement mentioned above we obtain the desired LMI.

From section 5 we can see that our examples do indeed satisfy the condition $(A_1 - I)h_0^i = 0$ for $i = 1, 2$, so Proposition 3.9 applies to our examples.

4. Information Flow Laws and Stochastic Loss

In this section we consider that the vehicles that can sense each other may also be able to exchange information. We then introduce stochastic loss of data for a network of vehicles that can sense and communicate simultaneously. We again assume that vehicles know when data is received, or if the data was lost, i.e. they have access to θ_k. In particular, we suppose that whenever vehicle i can sense j, he also receives a relay of the positions of the vehicles that j sees.

As in [4] in the case when no information is lost we consider the following simple information law for each vehicle i:

$$p_{k+1}^i = z_k^i + \frac{1}{|J_i|} \sum_{j \in J_i} p_k^j$$

In other words, vehicle i senses information and also receives information at time k. The information he will have at time $k+1$ is the sum of the information sensed and the average of the information he received from his neighbors at the previous step. Globally, the information law for the system becomes:

$$p_k = \left(\sum_{l=0}^{k} G^l \right) L(x_k - h_0)$$

where $G = I - L$. The dynamics for the system are:

$$x_{k+1} = Ax_k + BFp_k.$$

In the case when vehicle i receives information from j with some probability p, we assume that every arc (i, j) has the same probability p of not breaking at time k (i.e. information can be transmitted), and that the random variables θ_k^{ij}, $k = 1, 2, \ldots$ are independent.

One question investigated by the authors was whether the incorporation of an information flow law as above into the model (3) would improve the convergence of the vehicles to the formation. Since in this random loss system there is "old" and not very accurate information on the vehicles' positions being passed around, it is conceivable that an information flow law would only increase the level of inaccuracy in the system and thus hinder, rather than improve, the convergence. Indeed the results from our simulations in section 5 confirm this.

To avoid increasing the level of inaccuracy we develop an alternative information flow law. We will see in the next section that this particular information flow law leads to some improvements. For each pair (i, j), such that $j \in J_i$, the information that vehicle i receives from j is given by:

$$
\begin{aligned}
p_k^{i,j} &= w_k^{i,j}, \quad \text{for } \theta_k^{i,j} = 1 \\
p_k^{i,j} &= \frac{\sum_{l \in J_i} \theta_k^{i,l} \theta_k^{l,j} w_k^{l,j}}{\sum_{l \in J_i} \theta_k^{i,l} \theta_k^{l,j}}, \quad \text{for } \theta_k^{i,j} = 0 \\
p_k^{i,j} &= 0, \quad \text{for } \sum_{l \in J_i} \theta_k^{i,l} \theta_k^{l,j} = 0.
\end{aligned} \tag{10}
$$

For $j = i$ we set $p_k^{i,i} = x_k^i$.

Note that this law does not involve any "old" information:

- if the arc (i, j) is not broken, then $\theta_k^{i,l} = 1$ and therefore $p_k^{i,j} = w_k^{i,j} \equiv x_k^j$;
- if the arc (i, j) is broken, then $\theta_k^{i,l} = 0$, but any other broken links ($\theta_k^{i,l} = 0$ or $\theta_k^{l,j} = 0$), which would bring in some old information, do not contribute to $p_k^{i,j}$.

The main idea behind this flow law is that, if the arc (i, j) is broken (so "i does not see j") at time k, then vehicle i may still receive vehicle j's new position information, in the following situation (see Figure 4).

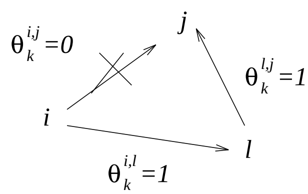

FIGURE 4. Vehicle i may receive j's position through l.

Suppose there exists some other vehicle l , such that

$$l \in J_i \quad \text{and} \quad j \in J_l$$

and at time k both

$$\theta_k^{i,l} = 1 \quad \text{and} \quad \theta_k^{l,j} = 1$$

(that is, both arcs (i, l) and (l, j) are not broken). Then, information flow law (10) states that vehicle i receives the new position information from j through its neighbor l. If several vehicles satisfy these two conditions, an average is computed. Information flow law (10) may be expected to improve the communication in the network, especially in the case of high connectivity graphs.

The state transition for each vehicle is given by

$$
\begin{aligned}
z_k^i &= \frac{1}{|J_i|} \sum_{j \in J_i} ((x_k^i - h_0^i) - (p_k^{i,j} - h_0^j)) \\
x_k^i &= A_1 x_k^i + B_1 F_1 z_k^i.
\end{aligned}
$$

5. Simulations

To test and explore our results, a linear discrete-time system was implemented and simulated for several different situations, as described below. A simple statistical analysis, based on Monte Carlo methods, is also presented, that provides a measure of our system's performance.

The system is derived from the double integrator model, $\ddot{x} = u$, for each vehicle $i = 1, \dots, N$, by discretization:

$$x_{k+1}^i = A_1 x_k^i + B_1 u_k^i \tag{11}$$

$$z_k^i = \frac{1}{|J_i|} \sum_{l \in J_i} ((x_k^i - h_0^i) - (x_k^l - h_0^l)) \tag{12}$$

$$J_i = \{l \in \{1, 2, \dots, N\} : a_{il} = 1\},$$

with matrices

$$A_1 = \begin{pmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} (\Delta t)^2/2 & 0 \\ \Delta t & 0 \\ 0 & (\Delta t)^2/2 \\ 0 & \Delta t \end{pmatrix},$$

where i represents the i-th vehicle of the network, Δt is the discretization time interval ($\Delta t = 0.1$) and $u^i = (u_1^i, u_2^i)'$ represents the vehicle acceleration along the x and y directions in the plane, respectively. The set J_i is the family of all vehicles that can be seen by vehicle i. The quantity z_k^i represents the information directly available to vehicle i, which consists of the differences between its position and the position of its neighbours in J_i. The offset vectors h_0^i represent the position of the vehicle i relative to the center of the formation.

A variant of the double integrator was also tested in our simulations, consisting of adding a friction term proportional to the velocity:

$$\ddot{x} + c\dot{x} = u.$$

For this variant, the corresponding discrete-time system is of the form (11) with matrices A_1, B_1 given by

$$A_1 = \begin{pmatrix} 1 & \frac{1}{c}(1 - e^{-c\Delta t}) & 0 & 0 \\ 0 & e^{-c\Delta t} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{c}(1 - e^{-c\Delta t}) \\ 0 & 0 & 0 & e^{-c\Delta t} \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} \frac{\Delta t}{c} - \frac{1}{c^2}(1 - e^{-c\Delta t}) & 0 \\ \frac{1}{c}(1 - e^{-c\Delta t}) & 0 \\ 0 & \frac{\Delta t}{c} - \frac{1}{c^2}(1 - e^{-c\Delta t}) \\ 0 & \frac{1}{c}(1 - e^{-c\Delta t}) \end{pmatrix}.$$

5.1. **Decentralized control.** In this paper, we restrict our examples to deal with linear feedback control of the form

$$u_k = FLx_k - FLh_0, \tag{13}$$

where $h_0 \in \mathbb{R}^{4N \times 4N}$ is a constant vector, containing information about the desired formation. In our examples, we set h_0 to be the positions at the vertices of a regular N-gon:

$$h_0^i = \begin{pmatrix} \cos(\frac{2\pi i}{N}) \\ 0 \\ \sin(\frac{2\pi i}{N}) \\ 0 \end{pmatrix}, \quad i = 1, \dots, N. \tag{14}$$

The matrix $F \in \mathbb{R}^{2N \times 4N}$ is to be suitably chosen so that the network will converge to the desired F-formation.

The general problem of finding a matrix K that stabilizes the linear system $x_{k+1} = Ax_k + Bu_k$ with output feedback $u_k = KLx_k$, has an optimal solution K provided by the LQ regulator method. Then the system $x_{k+1} = (A + BKL)x_k$ is stable and the solution K is optimal in the sense that some cost function of u_k and z_k is minimized.

However, a matrix F obtained by the LQ method would generally result in a centralized control. For practical purposes, designing a *decentralized control*, is of greater interest and is one of the goals of this paper. With a decentralized control each vehicle in the network is able to independently construct its own control u_k^i from its sensed and communicated information. Based on the structure of the matrices B_1 and L a F_1 is designed as follows:

$$F_1 = \begin{pmatrix} -0.8 & -0.8 & 0 & 0 \\ 0 & 0 & -0.8 & -0.8 \end{pmatrix}, \tag{15}$$

and let $F = \operatorname{diag}(F_1, \dots, F_1)$. Then, it is clear that each u_k^i picks up only the information that vehicle i receives form its neighbors, $j \in J_i$ (see Figure 5).

Another advantage of choosing this matrix F is that it is *independent of* L (as opposed to what happens in the LQ regulator case), and thus

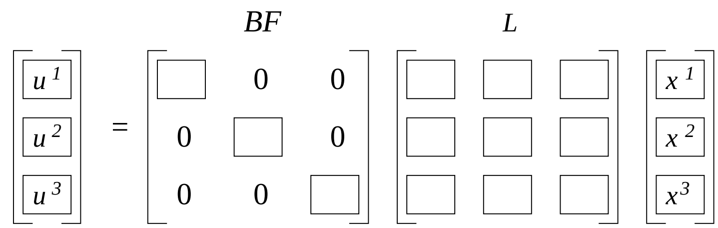

FIGURE 5. Block structure for decentralized control.

it is not necessary to evaluate F for each graph. In all simulations described below we use this constant feedback matrix.

5.2. **Information flow laws.** As a first step towards understanding the role of an information flow law such as described in [4], we compare the evolution of our network of vehicles and its convergence to F-stable formation, in the two cases:

(i) each vehicle has access only to its sensed information;
(ii) each vehicle has access to its sensed information as well as to extra communicated information from its neighbors, in the form of an *information flow law*.

As described in the previous section for our simulations we first implement the following flow law:

$$p_k = \left(\sum_{q=0}^{k} G^q \right) L(x_k - h_0) \tag{16}$$

and compare the evolution of the two systems

$$\begin{aligned} x_{k+1} &= Ax_k + BFL(x_k - h_0) \\ x_{k+1} &= Ax_k + BFp_k. \end{aligned}$$

From Figure 6, it is clear that the network with information flow law achieves the desired formation in a more efficient way.

5.3. **Stochastic loss of information.** Next we consider the effect that random losses of sensed information have on the vehicles in the network, and address the question of how to control vehicles to the desired formation under these adverse conditions.

Recall from the previous sections, vehicle i receives from vehicle j at time k the sensory information vector (3), with an estimation of the current position of j

$$w_k^{ij} = A_1 w_{k-1}^{ij} + \theta_k^{ij}(x_k^j - A_1 w_{k-1}^{ij}).$$

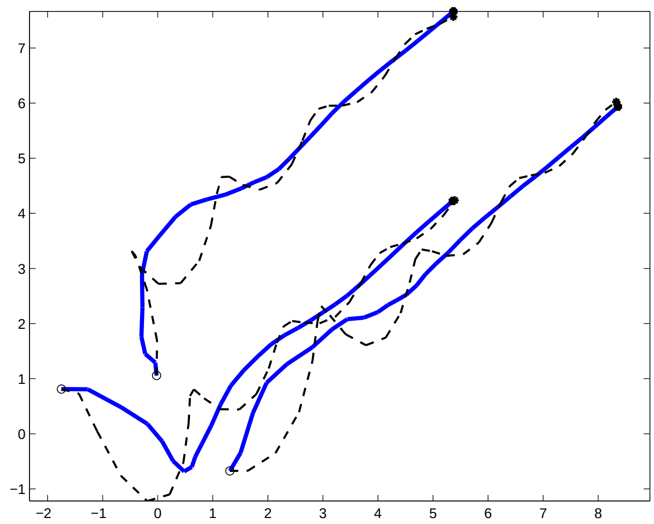

FIGURE 6. Triangle formation: no information flow (dashed line) and with information flow (solid line).

Then each vehicle in the network evolves according to (4).

The estimation of the current position as in (3) (as opposed to the more basic equation (2)) is observed to introduce an improvement in the convergence of system (4) to the formation. (see Figure 7.)

We argued in section 4 that incorporation of an information flow law into the model (4) would not necessarily improve the convergence of the vehicles to the formation. Indeed, for the double integrator model, information flow laws (including (16)) did not particularly improve the F-stability of the closed loop system (4).

However, the new information flow law (10) considerably improved the convergence for the alternative model with friction (see Figure 8).

5.4. **Monte Carlo methods and statistical analysis.** To evaluate the performance of our models under random losses of information, and to try to provide some estimates of rapidity of convergence and deviation from the formation, we focused on the following two questions:

(i) How fast do the vehicles converge to the formation (or come closer to F-stability), depending on the probability p of no information loss?
(ii) After a given fixed time has elapsed, what is the deviation of the network from the desired formation, as a function of the probability p?

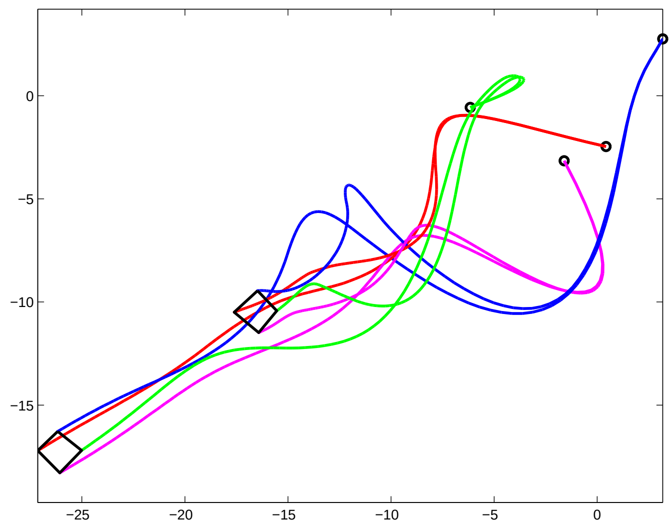

Figure 7. Square formation with $p = 0.4$ of information communication. Comparison between the two sensory information vectors: with estimation of the current position (left corner); with no estimation (center).

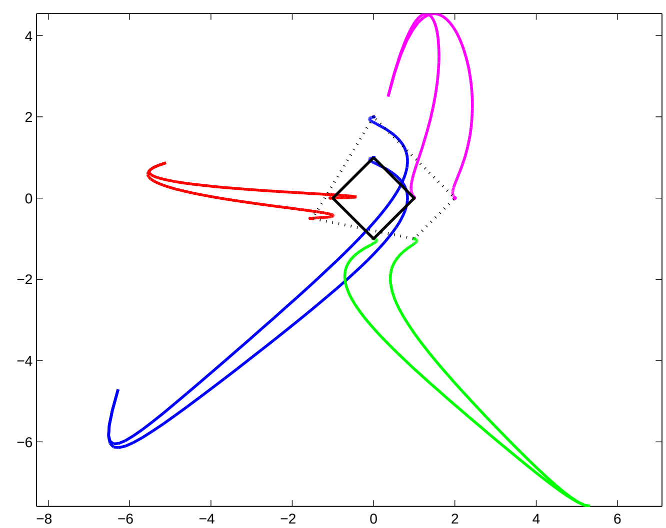

Figure 8. Square formation with random loss of information (friction model): no information flow (dashed line) and with information flow (solid line).

Using a Monte Carlo approach, for each probability $p \in \mathbb{P} := \{0.05s : s = 0, 1, \ldots, 20\}$, the system is simulated M times. For each of these simulations, the initial condition x_0 is a random vector, and the random variables θ_k^{ij} take the value 1 with probability p. For each of these simulations a certain quantity $f = f(p)$ (supposed to depend on the paramenter p) is recorded

$$f(r,p), \quad r = 1, \dots, M, \quad p \in \mathbb{P},$$

and then an estimate of the quantity $f = f(p)$ can be obtained by averaging

$$f(p) = \frac{1}{M} \sum_{r=1}^{M} f(r,p).$$

For question (i), it is necessary to establish the following measure: the network is said to be "within ε of the N-gon formation at time k" if the sides and the diagonals of the N-gon formed by the vehicles at time k do not differ from the ideal values for more than ε. Thus in case

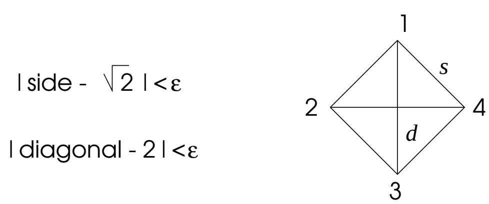

FIGURE 9. Estimating the square formation.

(i), the estimated quantity is $f(p) =$ "iteration k, when vehicles come within 0.05 of the square formation".

In Figure 10, for a network of four vehicles and $M = 10$, we can see the estimated number of iterations that are necessary for system (4) to be within $\varepsilon = 0.05$ of the square formation, as a function of p. As expected, for low p the probability of information loss is large $(1-p)$, and the vehicles need quite a long time to reach the desired formation. An interesting observation is that, as p increases, there seems to be an abrupt transition and, for $p \geq 0.3$, the number of iterations necessary for the network to come to formation does not differ very much, and lies in the range $120 - 160$.

In Figure 11, the error between the exact formation and the position of the vehicles after a fixed number of iterations (40) is plotted against the probability p. As expected, this error decreases as the probability of having no broken arcs increases.

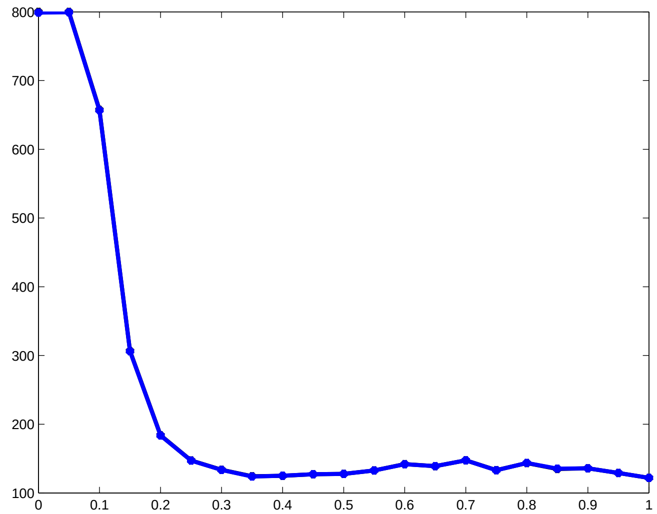

FIGURE 10. Estimated time to reach square formation, as a function of p.

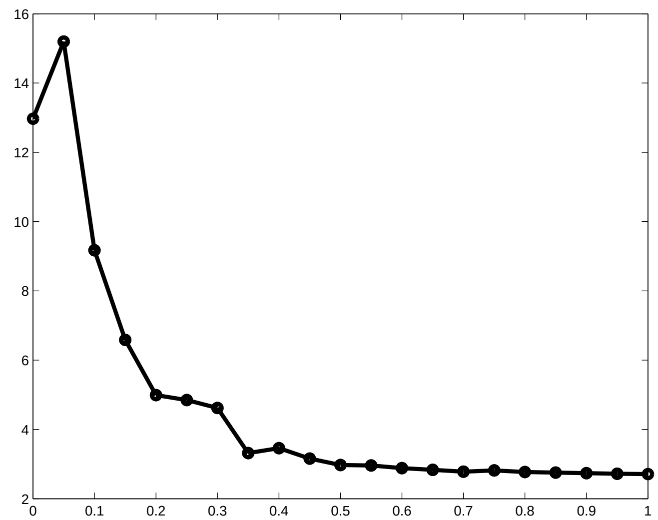

FIGURE 11. Deviation from formation, after 40 iterations, as a function of p.

6. Conclusion

Based on the formulation of new theoretical results together with several simulations of decentralized network control systems, we consider this work successful. Not only was our initial goal of smoothly inserting stochastic loss of communication, both passive and active, into a working model achieved, but as is often true in the best of scientific research, many of the results of our experiments were unexpected and pleasantly surprising. The stringency of the graph-theoretical sufficient conditions for system stability (proposed in the work of Fax and Murray) made it unclear whether or not our simulated vehicle networks could be expected to reach formation and stay in formation subject to more than a very modest stochastic disturbance of the graph.

We found to our astonishment that although the static network model required strong connectivity to ensure stability, vehicles in a stochastic model were able to achieve and maintain formation even when the instantaneous graph was not even expected to be connected for the majority of time iterations! It seems from experiment that we need only require strong connectivity in the *underlying* network structure, that is, the collection of all arcs available over the entire course of the simulation. Neglecting the pursuit of optimal control in our choice of vehicle design and feedback mechanism, we were amazed by the performance of our virtual vehicles when we placed them in even the toughest of virtual conditions. Statistical analysis shows that these vehicles were able to consistently converge into formation with even as much as two-thirds of all communication lost. Using only the simplest mechanism for estimating the position of their intermittently disappearing neighbors, they were able to collect themselves into position in a number of time steps which was impressively comparable to their non-stochastic cousins. Not one member of our team predicted such excellent performance with such simple tools. Initially considering our goal to be to describe how bad the stochastic system would get and how quickly, we are happy to say not very.

Despite the overall success of the stochastic model in general, the mixed results in the model of transmitted signal flow leave much room for improvement. Particularly, some confusing experimental results in which the system performed better without relayed information for very low probability of communication loss (suppressed from this work due to a lack of verification) deserves explanation and may be treated in later work. Although we could not properly insert the available (static network) information flow law into our stochastic model (due to its unwaveringly consistent circulation of information, obviously not suited for graphs which may not be connected for several instants), it is clear that a new information flow law, more sophisticated than our makeshift single-layer approach is to be desired.

Along with this project's apparent successes and obvious drawbacks, it is worth mentioning certain aspects of the general problem, beyond the scope of this work, if only to inspire future study. How do various graph-theoretical aspects of the underlying vehicle communication network affect overall performance in the stochastic system? For example, how much will graph periodicity interfere with smooth flow of transmitted information at various levels of arc loss? Can collision avoidance mechanisms be easily integrated, or could these precautions interfere with the seeming simplicity of our demonstrations? What happens when we extend the model to three-dimensional vehicle formations? Can these ideas developed for vehicle formations be extended to more general, higher-dimensional decentralized control problems?

Though the answers to these questions may not come quickly, we rest assured that the pursuit of control theory guides us into the technological future smoothly and safely. Through this work we hope to have confidently and scientifically advanced the premise that unmanned vehicles will serve mankind reliably, even under very harsh circumstances.

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