HUMBERT'S FUNCTIONS OF MATRIX ARGUMENTS –II

Lalit Mohan Upadhyaya* & H. S. Dhami**
Department of Mathematics,
University of Kumaun,
Almora Campus,
Almora (Uttaranchal),
India –263601.

2000 AMS Mathematics Subject Classification:
Primary: 33C65, 33C99.
Secondary: 60E, 62H, 44A05.
Key Words: Humbert's functions, matrix arguments, matrix-transform.

ABSTRACT

Four results concerning the Humbert's functions $\Phi_1, \Phi_3, \Xi_1, \Xi_2$ of matrix arguments are discussed along with a case of reducibility and transformation relations for the functions Φ_1 and Φ_2.

INTRODUCTION

In this paper we continue our study of Humbert's functions of matrix arguments. The Mathai's definitions for these functions have already been given in our previous papers [10, 11, 12], from where we shall draw upon the relevant results necessary for establishing the results here. Besides we have also given two results in the first section below, from Mathai [4], which shall be needed in deducing the transformation relations for the functions Φ_1 and Φ_2 of matrix arguments. All the matrices appearing in this paper are (pxp) real symmetric positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [3, 4].

* Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun (Uttaranchal), India-248179.
** To whom all the correspondence may be addressed.

1. Preliminary Results

THEOREM 1.1:

$$\Phi_1(a,b;c;-X,-Y)=\frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}\int_0^I |U|^{a-(p+1)/2}|I-U|^{c-a-(p+1)/2}\times$$

$$\left|I+U^{1/2}XU^{1/2}\right|^{-b} e^{-\mathrm{tr}(UY)}dU \qquad \cdots\cdots(1.1)$$

for $\mathrm{Re}(a,c-a)>(p-1)/2$.

THEOREM 1.2:

$$\Phi_2(b,b';c;-X,-Y)=\frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-b-b')}\iint |U_1|^{b-(p+1)/2}|U_2|^{b'-(p+1)/2}$$

$$\times|I-U_1-U_2|^{c-b-b'-(p+1)/2}e^{-\mathrm{tr}(XU_1+YU_2)}dU_1dU_2 \qquad \cdots\cdots(1.2)$$

for $\mathrm{Re}(b,b',c-b-b')>(p-1)/2$.

2. Results

THEOREM 2.1:

$$|P|^{-\alpha}\Xi_2(\alpha,\beta;\gamma;-P^{-1/2}XP^{-1/2},-Y)$$

$$=\frac{1}{\Gamma_p(\alpha)}\int_{T>0} e^{-\mathrm{tr}(PT)}|T|^{\alpha-(p+1)/2}\Phi_3(\beta;\gamma;-T^{1/2}XT^{1/2},-Y)dT \qquad \cdots\cdots(2.1)$$

for $\mathrm{Re}(\alpha)>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(2.1) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\Phi_3(\beta;\gamma;-T^{1/2}XT^{1/2},-Y)dXdY \qquad \cdots\cdots(2.2)$$

Applying the transformations

$X_1 = T^{1/2} X T^{1/2}$ (with $dX_1 = |T|^{(p+1)/2} dX$, and $|X_1| = |T||X|$) and then using eq.(1.1) of the authors' paper [12], the last expression yields,

$$|T|^{-\rho_1} \frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.3)$$

Substituting this expression on the right side of eq.(2.1) and then integrating out T in the resulting expression by using a Gamma integral produces,

$$|P|^{-(\alpha-\rho_1)} \frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.4)$$

Now, taking the M-transform of the left side of eq.(2.1) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we have,

$$\int_{X>0} \int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|P|^{-\alpha} \Xi_2(\alpha,\beta;\gamma;-P^{-1/2} X P^{-1/2},-Y) dX dY \qquad \cdots\cdots(2.5)$$

which, under the transformation

$X_2 = P^{-1/2} X P^{-1/2}$ (with $dX_2 = |P|^{-(p+1)/2} dX$, and $|X_2| = |P|^{-1}|X|$) and then using eq.(1.8) of the authors' paper [10] yields the same result as in eq.(2.4) above.

THEOREM 2.2:

$$|P|^{-\gamma} \left| I + P^{-1/2} X P^{-1/2} \right|^{-\beta} e^{-\mathrm{tr}(P^{-1}Y)}$$

$$= \frac{1}{\Gamma_p(\gamma)} \int_{T>0} e^{-\mathrm{tr}(PT)} |T|^{\gamma-(p+1)/2} \Phi_3(\beta;\gamma;-T^{1/2} X T^{1/2},-T^{1/2} Y T^{1/2}) dT \cdots (2.6)$$

for $\mathrm{Re}(\gamma) > (p-1)/2$.

PROOF: We take the M-transform of the right side of eq.(2.6) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, to get,

$$\int_{X>0} \int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$\Phi_3(\beta;\gamma;-T^{1/2} X T^{1/2},-T^{1/2} Y T^{1/2}) dX dY \qquad \cdots\cdots(2.7)$$

Applying the same transformation as for eq.(2.2)and a similar transformation for the Y variable and then using eq.(1.1) of the authors' paper [12], the expression (2.7) yields,

$$|T|^{-\rho_1-\rho_2}\,\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.8)$$

Substituting this expression on the right side of eq.(2.6) and then integrating out T in the resulting expression by using a Gamma integral gives,

$$|P|^{-(\gamma-\rho_1-\rho_2)}\,\frac{\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)} \qquad \cdots\cdots(2.9)$$

Now, taking the M-transform of the left side of eq.(2.6) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|P|^{-\gamma}\left|I+P^{-1/2}XP^{-1/2}\right|^{-\beta}e^{-\mathrm{tr}(P^{-1}Y)}dXdY \qquad \cdots\cdots(2.10)$$

Applying the same transformation as for eq.(2.5) and then integrating out X_2 by using a type-2 Beta integral and Y by using a Gamma integral, the expression (2.10) leads to the same result as in eq.(2.9).

THEOREM 2.3:

$$|P|^{-\alpha}\Phi_1(\alpha,\beta;\gamma;-P^{-1/2}XP^{-1/2},-P^{-1/2}YP^{-1/2})$$

$$=\frac{1}{\Gamma_p(\alpha)}\int_{T>0}e^{-\mathrm{tr}(PT)}|T|^{\alpha-(p+1)/2}\times$$

$$\Phi_3(\beta;\gamma;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dT \qquad \cdots\cdots(2.11)$$

for $\mathrm{Re}(\alpha)>(p-1)/2$.

PROOF: Following the same lines as in the proof of the theorem (2.2), the M-transform of the right side of eq.(2.11) gives,

$$|P|^{-(\alpha-\rho_1-\rho_2)}\,\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.12)$$

Now taking the M-transform of the left side of eq.(2.11) as in theorem (2.2) and applying the same transformation as in eq.(2.5) for the X variable and a similar transformation for the Y variable and then using eq.(1.5) of the authors' paper [10] produces the same result as in eq.(2.12).

THEOREM 2.4:

$$|P|^{-\alpha'} \Xi_1(\alpha,\alpha',\beta;\gamma;-X,-P^{-1/2}YP^{-1/2})$$

$$=\frac{1}{\Gamma_p(\alpha')}\int_{T>0} e^{-\mathrm{tr}(PT)}|T|^{\alpha'-(p+1)/2}\,\Xi_2(\alpha,\beta;\gamma;-X,-T^{1/2}YT^{1/2})\,dT \quad \cdots(2.13)$$

for $\mathrm{Re}(\alpha')>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(2.13) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$\Xi_2(\alpha,\beta;\gamma;-X,-T^{1/2}YT^{1/2})\,dXdY \quad \cdots\cdots(2.14)$$

Applying the transformation

$Y_1=T^{1/2}YT^{1/2}$ (with $dY_1=|T|^{(p+1)/2}dY$, and $|Y_1|=|T||Y|$) and then using eq.(1.8) of the authors' paper [10], the above expression gives,

$$|T|^{-\rho_2}\,\frac{\Gamma_p(\gamma)\Gamma_p(\alpha-\rho_1)\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \quad \cdots\cdots(2.15)$$

which, on substitution on the right side of eq.(2.13) and then integrating out T in the resulting expression by using a Gamma integral produces,

$$|P|^{-(\alpha'-\rho_2)}\,\frac{\Gamma_p(\gamma)\Gamma_p(\alpha-\rho_1)\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha'-\rho_2)}{\Gamma_p(\alpha')\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \quad \cdots\cdots(2.16)$$

Now, taking the M-transform of the left side of eq.(2.13) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we have,

$$\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|P|^{-\alpha'}\,\Xi_1(\alpha,\alpha',\beta;\gamma;-X,-P^{-1/2}YP^{-1/2})\,dXdY \quad \cdots\cdots(2.17)$$

which, under the transformation,

$Y_2 = P^{-1/2} Y P^{-1/2}$ (with $dY_2 = |P|^{-(p+1)/2} dY$, and $|Y_2| = |P|^{-1}|Y|$) and then using eq.(1.2) of the authors' paper [11] produces the same result as in eq.(2.16) above.

THEOREM 2.5:

$$\text{(i)}\ \Phi_1(a,b;c;I,-Y) = \frac{\Gamma_p(c)\Gamma_p(c-b-a)}{\Gamma_p(c-b)\Gamma_p(c-a)} \times {}_1F_1(a;c-b;-Y) \qquad \cdots\cdots(2.18)$$

$$\text{(ii)}\ \Phi_1(a,b;c;-X,-Y) = e^{-tr(Y)}|I+X|^{-b} \times$$

$$\Phi_1[c-a,b;c;(I+X)^{-1/2}X(I+X)^{-1/2},Y] \qquad \cdots\cdots(2.19)$$

$$\text{(iii)}\ \Phi_2(b,b';c;-X,-Y) = e^{-tr(X)}\Phi_2(c-b-b',b';c;X,X-Y) \qquad \cdots\cdots(2.20)$$

$$= e^{-tr(Y)}\Phi_2(b,c-b-b';c;Y-X,Y) \qquad \cdots\cdots(2.21)$$

PROOF: (i) The result in eq.(2.18) follows by making $X \to -I$ in eq.(1.1) and then using the theorem (2.3.4) page 42 of Mathai [4].

(ii)Consider eq.(1.1) along with the observation,

$$\left|I + U^{1/2}XU^{1/2}\right| = |I + XU| = \left|I + X^{1/2}UX^{1/2}\right| \qquad \cdots\cdots(2.22)$$

Now, applying the transformation $V = I - U$ along with the observation,

$$\left|I + X^{1/2}(I-V)X^{1/2}\right| = |I+X|\left|I-(I+X)^{-1/2}X^{1/2}VX^{1/2}(I+X)^{-1/2}\right|$$

the desired result follows immediately by a suitable interpretation of the resulting expression in the light of eq.(1.1) along with eq.(2.22).

(iii) The result in eq.(2.20) follows by applying the transformations $V_1 = I - U_1 - U_2, V_2 = U_2$ (so that, $dV_1 dV_2 = dU_1 dU_2$) to eq.(1.2) and suitably interpreting the resulting expression as per eq.(2.20). The result in eq.(2.21) also follows similarly from eq.(1.2).

References

1. Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G. (1954). Tables of Integral Transforms, Vol. I, McGraw Hill, New York, Toronto and London.

2. Exton H. (1976). Multiple Hypergeometric Functions and Applications, Ellis Horwood Limited, Publishers, Chichester.
3. Mathai A.M. (1992). Jacobians of Matrix Transformations I, Centre for Mathematical Sciences, Trivandrum, India.
4. Mathai A.M. (1993). Hypergeometric Functions of Several Matrix Arguments, Centre for Mathematical Sciences, Trivandrum, India.
5. Mathai A.M. (1993). Appell's and Humbert's Functions of Matrix Arguments, Linear Algebra and its Applications; 183, pp. 201-221.
6. Mathai A.M. (1995). Special Functions of Matrix Arguments-III; Proceedings of the National Academy of Sciences, India; LXV (IV) pp. 367-393.
7. Mathai A.M., Pederzoli G. (1996). Some Transformations for Functions of Matrix Arguments; Indian J. Pure Appl. Math. 27(3), pp. 277-284.
8. Srivastava H.M., Karlsson P.W. (1985). Multiple Gaussian Hypergeometric Series. Ellis Horwood Limited, Publishers, Chichester.
9. Upadhyaya Lalit Mohan, Dhami H.S.(Nov.2001). Matrix Generalizations of Multiple Hypergeometric Functions, # 1818 IMA Preprints Series, University of Minnesota, Minneapolis, U.S.A.
10. Upadhyaya Lalit Mohan, Dhami H.S.(Mar. 2002). Appell's and Humbert's Functions of Matrix Arguments-I. # 1848 IMA Preprints Series, University of Minnesota, Minneapolis, U.S.A.
11. Upadhyaya Lalit Mohan, Dhami H.S.(Apr. 2002). Appell's and Humbert's Functions of Matrix Arguments-II. # 1853 IMA Preprints Series, University of Minnesota, Minneapolis, U.S.A.
12. Upadhyaya Lalit Mohan, Dhami H.S.(May 2002). Humbert's Functions of Matrix Arguments-I. # 1856 IMA Preprints Series, University of Minnesota, Minneapolis, U.S.A.