On the Significance of the Titius-Bode Law for the Distribution of the Planets[*]

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Abstract

The radii of the planetary and satellite orbits are in approximate agreement with geometric progressions. The question of whether the observed patterns have some physical basis or are due to chance may be addressed using a Monte Carlo approach. We find that the estimated probability of chance occurrence depends sensitively on the restrictions imposed on the population of orbits. We argue that it is not possible to conclude unequivocally that laws of Titius-Bode type are, or are not, significant. Therefore, the possibility of a physical explanation for the observed distributions remains open.

Key Words: Celestial Mechanics, Orbits, Planetary Dynamics

1 Introduction

The approximate regularity in the sequence of distances from the Sun of the planets of our solar system, which is described by the empirical relationship known as the Titius-Bode Law, has been a subject of interest and controversy for centuries. The law played a significant rôle during the search for new planetary bodies. The discovery of Uranus by Herschel in 1781 and of the largest asteroid Ceres by Piazzi in 1801 appeared to confirm the accuracy of the law. Both Adams and Leverrier used the Titius-Bode law in their calculations for a new planet (Neptune). However, there is a substantial deviation between the observed orbital radius of Neptune and the value indicated by the empirical law. For Pluto, the connection breaks down completely.

The Titius-Bode Law, or Bode's Law for short, states that the orbital radii of the planets are given, in astronomical units, by the formula

$$r_n = 0.4 + 0.3 \times 2^n \,, \quad n = -\infty, 0, 1, 2, 3, \ldots \,. \tag{1}$$

The values produced by this formula and the observed values are compared in Table 1. It is clear that, with the exceptions noted above, the agreement is remarkable. However, Newman, *et al.*, (1994) have considered the psychological tendency to find pattern where none exists, and have also discussed how inappropriate inferences regarding astronomical phenomena have been drawn from statistical analyses. Interest in the Titius-Bode law has been heightened by the discovery of extra-solar planets, although it may be many years before its relevance in this context can be tested.

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Table 1: Planetary radii given by the Titius-Bode Law compared to the Observed values.

n	Planetary Body	Radius (AU) from Bode's Law	Observed mean Radius (AU)
$-\infty$	Mercury	0.4	0.39
0	Venus	0.7	0.72
1	Earth	1.0	1.00
2	Mars	1.6	1.52
3	(Ceres)	2.8	2.77
4	Jupiter	5.2	5.20
5	Saturn	10.0	9.54
6	Uranus	19.6	19.18
7	Neptune	38.8	30.06
8	Pluto	77.2	39.44

Nieto (1972) traces the history of the Titius-Bode Law up to about 1970, and reviews many attempts to explain it in physical terms. Several references to more recent work may be found in Hayes and Tremaine (1998). Despite the distinguished part the law has played in the evolution of our knowledge of planetary dynamics, no theoretical explanation of it has been advanced which has found general acceptance. Indeed, the view has frequently been expressed that the putative relationship between the orbital radii is coincidental, and that the observed pattern is due to chance. It is this question which we wish to address.

2 A Probabilistic Paradox

The decision as to whether a given event is the result of chance, or is so unlikely as to suggest a definite causative origin, is fraught with difficulty. The measure of probability of the observed event is not normally definable in a unique manner, so that different conclusions may result from different methods of measurement. A simple example illustrates the problem. Let us consider the question: *What is the probability that a randomly chosen chord intersecting a circle will have a radius greater than the length of the side of an inscribed equilateral triangle?* (For a unit circle, the length is $\sqrt{3}$.) We consider three alternative methods of defining the chord:

1. Choose an arbitrary point within the circle as the mid-point of the chord.
2. Specify randomly the two points at which the chord intersects the circle.
3. Choose a random point on an arbitrary radius as the mid-point of the chord.

Elementary reasoning shows that the probability of the event is $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{2}$ respectively for the three methods. The paradox is resolved by recognizing that the question is not well posed: the answer depends on the method by which the chord is chosen. To have a unique answer, we must specify the manner of choice.

The same problem arises in deciding whether Bode's Law is mearly a coincidence or something deeper. We may ask if the observed planetary pattern could have occurred by chance, but the estimated probability may depend strongly upon the manner of its estimation.

3 The Uranian Satellite System

To study the significance of patterns of the Titius-Bode type, Murray and Dermott (1999) considered a geometric progression of orbital periods

$$T_n = T_0 A^n \,, \quad n = 1, 2, 3, \ldots \tag{2}$$

and compared the values produced by this formula with the observed periods of the five principal satellites of Uranus. The parameters T_0 and A were obtained by considering the logarithm of (2) and mimimizing the mean square deviation

$$\chi^2 = \frac{1}{5} \sum_{n=1}^{5} [\log T_n^{\mathrm{obs}} - (\log T_0 + n \log A)]^2 \,, \tag{3}$$

where T_n^{obs} are the observed periods, given in Table 2. The resulting values are $T_0 = 0.7919$ and $A = 1.777$. For these parameters the discrepancy is $\chi = 0.0247$ and the periods given by (2) are in close agreement with the observed values (Table 2). The question is whether this agreement is statistically significant.

Murray and Dermott (1999) used a Monte Carlo technique to address this question. Using a method similar to that of Dermott (1973), they generated a series of 10^5 sets of periods for the five satellites, random but for certain restrictions on their distribution. The period of the innermost satellite was fixed in agreement with the observed period of Miranda ($T_1 = 1.413$). The other four periods were generated by the formula

$$\frac{T_{n+1}}{T_n} = L + x_n (U - L) \,, \quad n = 1, 2, 3, 4 \,, \tag{4}$$

where L and U are fixed lower and upper limits on the ratio of successive periods and x_n are randomly chosen in the interval $[0, 1]$. For the observed system, $L = 1.546$ and $U = 2.101$. For each system, the parameters T_0 and A which mimimized the deviation χ were determined. The number of systems having root mean square deviation χ less that the deviation ($\chi_0 = 0.0247$) for the observed system was calculated and thus the probability $P(\chi < \chi_0)$ of this event was estimated to be 0.79. Murray and Dermott concluded that the probability that the observed configuration of satellites has arisen by chance is about 80%. It is this conclusion which we believe is open to question.

Table 2: Periods of the Uranian satellites given by the Murray-Dermott formula compared to the observed values.

n	Satellite	Murray-Dermott Fitted Period	Observed Period
1	Miranda	1.407	1.413
2	Ariel	2.500	2.520
3	Umbriel	4.442	4.144
4	Titania	7.893	8.706
5	Oberon	14.02	13.46

In Fig. 1, a sample of the population of 10^5 sets of orbital periods of the five satellites in the population chosen by Murray and Dermott is illustrated in the upper left panel (for clarity, only 50 cases are shown). The limiting cases permitted under the imposed restrictions are indicated by the dashed lines. We see that the satellite periods fall within a triangular region on the plot. In Fig. 1(b)

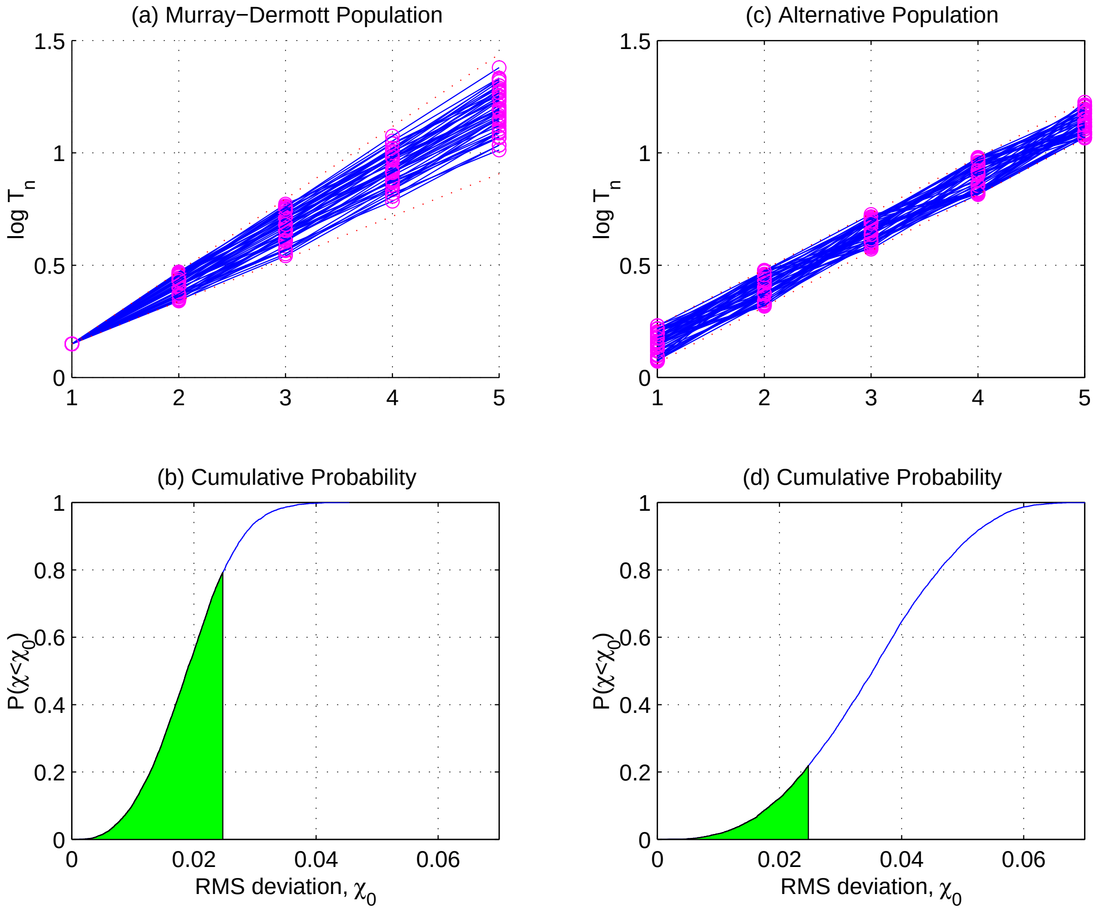

Figure 1: Distribution of the five principal Uranian satellites. (a) Logarithm of periods given by the Murray-Dermott formula (4) (only the first 50 random sets are shown). (b) Cumulative probability distribution calculated for 10^5 cases of (4). The shaded area is for $\chi < \chi_0$. (c) Logarithm of periods given by the alternative distribution (6) for $k = \frac{2}{3}$ (only the first 50 random sets are shown). (d) Cumulative probability distribution calculated for 10^5 cases of (6). The shaded area is for $\chi < \chi_0$.

(lower left panel) the cumulative probability distribution of χ is shown. The shaded area represents the cases where $\chi < \chi_0$. It confirms that most cases have rms deviation less than that of the actual system.

The pattern (4) chosen by Murray and Dermott is only one of limitless possibilities. We consider now an alternative choice. The values of the parameters, $T_0 = 0.7919$ and $A = 1.777$ are those which yield the best fit to the observed system. The best-fit periods are those in column 3 of Table 2, given by

$$\log T_n^{\rm fit} = \log T_0 + n \log A \,, \qquad n = 1, 2, 3, 4, 5 \,. \tag{5}$$

To generate the alternative population of sets of periods, we allow the logarithm of the period of each satellite to take a value at random within a band centered on the best-fit value:

$$\log T_n = \log T_0 + (n + k y_n) \log A \,. \tag{6}$$

Here y_n is a random number in the range $[-\frac{1}{2}, +\frac{1}{2}]$ and k is a fixed positive parameter which determines the width of the band. For $k = 1$, the bands abut each other.

In Fig. 1(c), we show a sample of the population of sets of orbital periods in the alternative population (upper right panel; only 50 cases are shown). The orbital band-width in (6) is $k = \frac{2}{3}$. The limiting cases permitted under the imposed restrictions are again indicated by the dashed lines.

The permitted satellite periods fall within a strip with parallel sides. In Fig. 1(d) (lower right panel), the corresponding cumulative probability distribution of χ is shown. The shaded area represents the cases where $\chi < \chi_0$. It confirms that, in marked contrast to the Murray-Dermott population, most cases have rms deviation greater than that of the actual system. From the sample of 10^5 cases, we estimate that $P(\chi < \chi_0) = 0.20$. When the band-width parameter is increased to $k = 1$, the estimated probability of the observed pattern is reduced to $P(\chi < \chi_0) = 0.05$, indicating that the actual disposition of the satellites is very unlikely to have arisen by chance.

4 The Solar System

We now apply the above analysis to the planets of the solar system. There are arguments about whether the asteroid belt, which may be the residue of a former planet, or may have been prevented by the tidal stresses of Jupiter from ever forming a planet, should be included or omitted. We recall that the gap in the pattern of the Titius-Bode Law was noted long before the observation of the first asteroid, Ceres, and indeed contributed to the detection of this celestial body. It appears reasonable to include Ceres as a representative of the putative former planet. In a similar vein Pluto, which is nowhere near the position expected from the Titius-Bode Law, may be a recently captured interloper and this may explain its large deviation from the prediction. However, its omission would seem artificial, and might justify criticism that inconvenient data was being disregarded. In summary, we have decided that the most objective choice is to include in the analysis all ten 'planets' listed in Table 1 above.

We postulate a geometric progression of planetary orbital radii

$$R_n = R_0 A^n\,, \quad n = 1, 2, \ldots, 10 \tag{7}$$

and choose the parameters R_0 and A by mimimising the root mean square deviation from the observed periods of the planets. Since the planetary radii R_n and periods T_n are related by Kepler's Third Law, (7) is equivalent in form to (2). The resulting values of the parameters are $R_0 = 0.2139$ and $A = 1.706$; for these parameters the discrepancy is $\chi_0 = 0.0544$ and the periods given by (7) are in *broad agreement* with the observed values (Table 3). (We note that the rms discrepancy for the original Titus-Bode formula (1) is $\chi_{\rm TB} = 0.0993$, so the geometric progression (7) is a better fit!)

Table 3: Planetary radii given by the best-fit geometric progression (7) compared to the Observed values.

n	Planetary Body	Radius from the Best Fit (7)	Observed mean Radius (AU)
$-\infty$	Mercury	0.37	0.39
0	Venus	0.63	0.72
1	Earth	1.07	1.00
2	Mars	1.83	1.52
3	(Ceres)	3.13	2.77
4	Jupiter	5.36	5.20
5	Saturn	9.17	9.54
6	Uranus	15.68	19.18
7	Neptune	26.82	30.06
8	Pluto	45.88	39.44

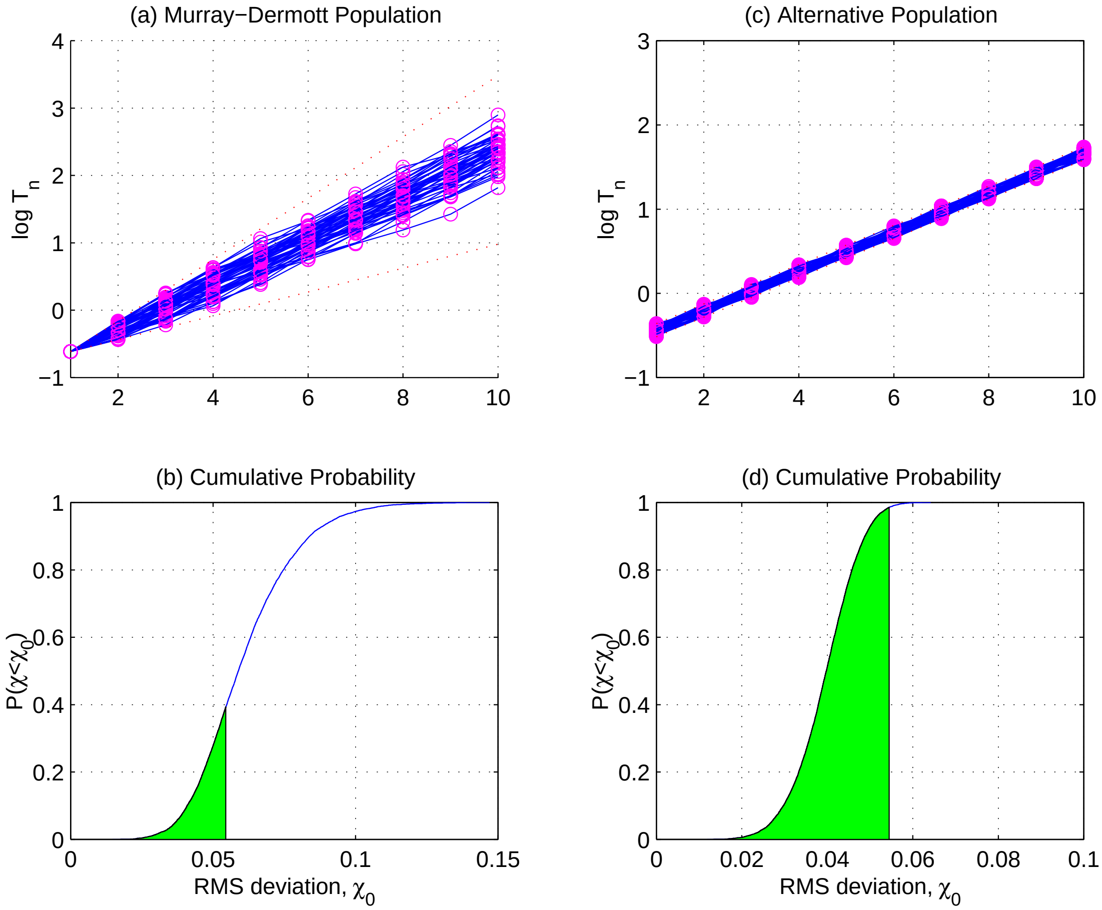

Figure 2: Distribution of the solar system planetary radii. (a) Logarithm of periods given by the Murray-Dermott formula (4) (only the first 50 random sets are shown). (b) Cumulative probability distribution calculated for 10^5 cases of (4). The shaded area is for $\chi < \chi_0$. (c) Logarithm of periods given by the alternative distribution (6) for $k = \frac{2}{3}$ (only the first 50 random sets are shown). (d) Cumulative probability distribution calculated for 10^5 cases of (6). The shaded area is for $\chi < \chi_0$.

We now estimate the probability that the agreement between the observed planetary distribution and that arising from the assumed geometric law might result from chance. The random population chosen by Murray and Dermott (1999) was generated using (4), with the observed lower and upper limits of the ratios of successive planetary radii, $L = 1.503$ and $U = 2.851$. The probability $P(\chi < \chi_0)$ was thus estimated to be 0.39. One might conclude that the chance that a randomly chosen planetary configuration would fit the Titius-Bode Law as closely as the observed system is only about 40%. However, when the alternative population given by (6) is used, another conclusion suggests itself. The values of the parameters, $R_0 = 0.2139$ and $A = 1.706$, are those which yield the best fit to the observed system. The resulting best-fit periods are given in column 3 of Table 3.

In Fig. 2, a sample of the population of 10^5 sets of orbital periods of the planets in the population chosen by Murray and Dermott is illustrated in the upper left panel. In Fig. 2(b) (lower left panel) the cumulative probability distribution of χ is shown. The shaded area represents the cases where $\chi < \chi_0$. It confirms that most cases have rms deviation greater than that of the actual system. In Fig. 2(c), we show a sample of the population of sets of orbital periods in the alternative population (upper right panel) for orbital band-width $k = \frac{2}{3}$. In Fig. 2(d) (lower right panel), the corresponding cumulative probability distribution of χ is shown. In contrast to the Murray-Dermott population, most cases have rms deviation less than that of the actual system. From the sample of 10^5 cases, we estimate that $P(\chi < \chi_0) = 0.99$. This prompts the conclusion that the observed pattern is almost certainly due to chance. When the band-width parameter is increased to $k = 1$, the estimated probability of the observed pattern is reduced to $P(\chi < \chi_0) = 0.34$, about the same value as for the Murray-Dermott population.

5 Summary

The estimated probability of a chance agreement with a geometric progression was derived for the (principal) Uranian satellites and for the Solar system, using a Monte Carlo approach, with two distinct populations generated with different constraints. For the Uranian system, the Murray-Dermott method gave a greater probability of chance occurrence than the alternative method (with $k = \frac{2}{3}$). Surprisingly, for the Solar System, the opposite situation obtained: the alternative population indicated a very high probability of chance agreement with a geometric progression. However, the value varied strongly with the band-width parameter k. Murray and Dermott also found that the choice of L and U strongly affected the outcome. We conclude that the estimated probability is very sensitive to the method of defining the 'random' set of planetary systems.

Hayes and Tremaine (1998) studied simulated solar systems using a wide variety of radius exclusion laws. They found that the results were quite sensitive to details of the exclusion method chosen. They concluded that the significance of Bode's Law is simply that stable planetary systems tend to be regularly spaced. They conjectured that this conclusion could be strengthened by making long-term orbit integrations to reject unstable planetary configurations. Their conclusion may be looked at in another way: the stability of the solar system may yet be shown to 'explain' the regularity encapsulated in Bode's Law.

We make no claim as to the relative merits of the alternative methods of choosing the random populations. Indeed, there is unlimited scope for yet other choices. We note that, for the solar system, the alternative population with $k = \frac{2}{3}$ implies a minimum ratio of successive radii $R_{n+1}/R_n \geq 1.13$ and successive periods $T_{n+1}/T_n \geq 1.20$. Murray and Dermott stated that there is no compelling evidence that the Uranian satellite system obeys any Titius-Bode type relation, beyond what would be expected by chance. They go on to suggest that the law as applied to the planets is also without significance. The main result of the current study is that this conclusion is unsafe, and that the possibility that the observed regularity in the patterns of the planetary and satellite systems has some physical explanation is still open.

We will not attempt to review the extensive literature devoted to explaining the Titius-Bode law. Nieto (*loc. cit.*) summarises the main work up to 1970. We mention only the study of White (1972), who argued that jet streams may develop in a rotating gaseous disk at discrete orbital distances given by a geometric progression. It is arguable that such a hydrodynamic process could have determined the gross features of the planetary distribution of the solar system. Variations from this might well be associated with the apparent tendency of the system to move, over its lifetime, towards resonant configurations. The possible relationship between Bode's law and the well-known near-resonances between periods in the planetary and satellite systems (Molchanov, 1968) remains to be clarified. Molchanov's total resonance theory has been reviewed, in the light of more recent understanding, by Beletsky (2001, §4.5).

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