# HUMBERT'S FUNCTIONS OF MATRIX ARGUMENTS –I

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## ABSTRACT

Five results concerning the Humbert's functions $\Phi_2, \Phi_3, \Psi_1, \Psi_2,$ and $\Xi_1$ of matrix arguments have been established in this paper.

## INTRODUCTION

Mathai has earlier studied the Humbert's functions of matrix arguments in conjunction with the Appell's functions of matrix arguments [4,5,6].We have also studied them together with the Appell's functions in our earlier studies [9,10].But in the present and its consequent study we have ventured in the direction of studying some properties of Humbert's functions independently without any reference at all to the Appell's functions of matrix arguments. All the matrices appearing in this paper are (pxp) real symmetric positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [3,4].

## 1. Preliminary Definition

We have already given the Mathai's definitions of Humbert's functions of matrix arguments in our previous two papers [9, 10]. Only one definition which has not been given previously in our works shall be given here. For other definitions which shall be required by us for proving the results in this paper, references to our earlier papers shall be made at the appropriate places.

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**DEFINITION 1.1:** The Humbert's $\Phi_3$- function of matrix arguments,

$$\Phi_3 = \Phi_3(b;c;-X,-Y)$$

is defined as that function for which the matrix- transform (M-transform) is the following:

$$M(\Phi_3) = \int_{X>0} \int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \Phi_3(b;c;-X,-Y)dXdY$$

$$= \frac{\Gamma_p(c)\Gamma_p(b-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(b)\Gamma_p(c-\rho_1-\rho_2)} \qquad \cdots\cdots(1.1)$$

for $\mathrm{Re}\,(b-\rho_1, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2.$

## 2. Results

**THEOREM 2.1:**

$$|P|^{-\beta}\Xi_1(\alpha,\alpha',\beta;\gamma;-P^{-1/2}XP^{-1/2},-Y)$$

$$= \frac{1}{\Gamma_p(\beta)}\int_{T>0} e^{-\mathrm{tr}(PT)}|T|^{\beta-(p+1)/2}\Phi_2(\alpha,\alpha';\gamma;-T^{1/2}XT^{1/2},-Y)dT \cdots\cdots(2.1)$$

for $\mathrm{Re}\,(\beta) > (p-1)/2.$

**PROOF:** Taking the M-transform of the right side of eq. (2.1) with respect to the variables X and Y and the parameters $\rho_1$ and $\rho_2$ respectively, we get,

$$\int_{X>0} \int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$

$$\Phi_2(\alpha,\alpha';\gamma;-T^{1/2}XT^{1/2},-Y)dXdY \qquad \cdots\cdots(2.2)$$

Applying the transformation

$X_1 = T^{1/2}XT^{1/2}$ (with $dX_1 = |T|^{(p+1)/2}dX$ and $|X_1| = |T||X|$) and then using eq.(1.1) of the authors' paper [10], the above expression yields,

$$|T|^{-\rho_1}\frac{\Gamma_p(\gamma)\Gamma_p(\alpha-\rho_1)\Gamma_p(\alpha'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\alpha')\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.3)$$

Substituting this expression on the right side of eq.(2.1) and then integrating out T in the resulting expression by using a Gamma integral generates,

$$|P|^{-(\beta-\rho_1)}\frac{\Gamma_p(\gamma)\Gamma_p(\alpha-\rho_1)\Gamma_p(\alpha'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\beta-\rho_1)}{\Gamma_p(\beta)\Gamma_p(\alpha)\Gamma_p(\alpha')\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.4)$$

Now, taking the M-transform of the left side of eq.(2.1) with respect to the variables X and Y and the parameters $\rho_1$ and $\rho_2$ respectively, we have,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|P|^{-\beta}\Xi_1(\alpha,\alpha',\beta;\gamma;-P^{-1/2}XP^{-1/2},-Y)dXdY \qquad \cdots\cdots(2.5)$$

which under the transformation

$X_2=P^{-1/2}XP^{-1/2}$ (with $dX_2=|P|^{-(p+1)/2}dX$ and $|X_2|=|P|^{-1}|X|$) and then using eq.(1.2) of the authors' paper [10] yields the same result as in eq.(2.4) above.

**THEOREM 2.2:**

$$|P|^{-\gamma}\left|I+P^{-1/2}XP^{-1/2}\right|^{-\beta}\left|I+P^{-1/2}YP^{-1/2}\right|^{-\beta'}$$

$$=\frac{1}{\Gamma_p(\gamma)}\int_{T>0}e^{-tr(PT)}|T|^{\gamma-(p+1)/2}\times$$

$$\Phi_2(\beta,\beta';\gamma;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dT\cdots\cdots(2.6)$$

for $Re(\gamma)>(p-1)/2$.

**PROOF:** Taking the M-transform of the right side of eq.(2.6) with respect to the variables X and Y and the parameters $\rho_1$ and $\rho_2$ respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\Phi_2(\beta,\beta';\gamma;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dXdY \qquad \cdots\cdots(2.7)$$

Applying the same transformation as for eq.(2.2) above along with a similar transformation for the Y-variable and then using eq.(1.1) of the authors' paper [10], the last expression produces,

$$|T|^{-\rho_1-\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\beta')\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.8)$$

Substituting this expression on the right side of eq.(2.6) and then integrating out T in the resulting expression by using a Gamma integral gives,

$$|P|^{-(\gamma-\rho_1-\rho_2)}\frac{\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\beta')} \qquad \cdots\cdots(2.9)$$

Now, taking the M-transform of the left side of eq.(2.6) with respect to the variables X and Y and the parameters $\rho_1$ and $\rho_2$ respectively, we obtain,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|P|^{-\gamma}\left|I+P^{-1/2}XP^{-1/2}\right|^{-\beta}\left|I+P^{-1/2}YP^{-1/2}\right|^{-\beta'}dXdY\cdots\cdots(2.10)$$

On making use of the same transformation as for eq.(2.5) along with a similar transformation for the Y-variable in eq.(2.10) and integrating out the new variables of integration by using a type-2 Beta integral, we get the same result as in eq.(2.9).

**THEOREM 2.3:**

$$|P|^{-\beta'}\Phi_2(\beta,\beta';\gamma;-X,-P^{-1/2}YP^{-1/2})$$

$$=\frac{1}{\Gamma_p(\beta')}\int_{T>0}e^{-\mathrm{tr}(PT)}|T|^{\beta'-(p+1)/2}\Phi_3(\beta;\gamma;-X,-T^{1/2}YT^{1/2})dT\cdots\cdots(2.11)$$

for $\mathrm{Re}(\beta')>(p-1)/2$.

**PROOF:** We take the M-transform of the function on the right side of eq.(2.11) with respect to the variables X and Y and the parameters $\rho_1$ and $\rho_2$ respectively, to obtain,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\Phi_3(\beta;\gamma;-X,-T^{1/2}YT^{1/2})dXdY \qquad \cdots\cdots(2.12)$$

Applying the transformation

$Y_1 = T^{1/2}YT^{1/2}$ (with $dY_1 = |T|^{(p+1)/2}dY$ and $|Y_1| = |T||Y|$) and then using the definition (1.1) we obtain from the above expression,

$$|T|^{-\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.13)$$

which, on substitution on the right side of eq.(2.11) and then integrating out T in the resulting expression by using a Gamma integral gives,

$$|P|^{-(\beta'-\rho_2)}\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\beta')\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.14)$$

Now, taking the M-transform of the left side of eq.(2.11) with respect to the variables X and Y and the parameters $\rho_1$ and $\rho_2$ respectively, we obtain,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|P|^{-\beta'}\Phi_2(\beta,\beta';\gamma;-X,-P^{-1/2}YP^{-1/2})dXdY \qquad \cdots\cdots(2.15)$$

Applying the transformation

$Y_2 = P^{-1/2}YP^{-1/2}$ (with $dY_2 = |P|^{-(p+1)/2}dY$ and $|Y_2| = |P|^{-1}|Y|$) and then using eq.(1.1) of the authors' paper [10], the above expression generates the same result as in eq.(2.14) above.

**THEOREM 2.4:** For p=2,

$$|P|^{-\alpha}\Xi_1[(\alpha+1)/2,\beta,(2\alpha+1)/4;\gamma;-4P^{-1}YP^{-1},-X)$$

$$=\frac{1}{\Gamma_p(\alpha)}\int_{T>0}e^{-tr(PT)}|T|^{\alpha-(p+1)/2}\Phi_3(\beta;\gamma;-X,-TYT')dT \cdots\cdots(2.16)$$

where, $Re(\alpha) > (p-1)/2$.

**PROOF:** Taking the M-transform of the function on the right side of eq.(2.16) with respect to the variables X and Y and the parameters $\rho_1$ and $\rho_2$ respectively, we have,

$$\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\Phi_3(\beta;\gamma;-X,-TYT')dXdY \qquad \cdots\cdots(2.17)$$

Applying the transformation

$Y_1 = TYT'$ (with $dY_1 = |T|^{p+1}dY$ and $|Y_1| = |T|^2|Y|$) and then using the definition (1.1), the above expression yields,

$$|T|^{-2\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.18)$$

Substituting this expression on the right side of eq.(2.16) and then integrating out T in the resulting expression by using a Gamma integral produces,

$$|P|^{-(\alpha-2\rho_2)}\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\alpha-2\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\alpha)\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.19)$$

Now, taking the M-transform of the left side of eq.(2.16) with respect to the variables X and Y and the parameters $\rho_1$ and $\rho_2$ respectively, we obtain,

$$\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|P|^{-\alpha}\Xi_1[(\alpha+1)/2,\beta,(2\alpha+1)/4;\gamma;-4P^{-1}YP^{-1},-X)dXdY \qquad \cdots\cdots(2.20)$$

which, under the transformation

$Z_1 = 4P^{-1}YP^{-1}$ (with $dZ_1 = 4^{p(p+1)/2}|P|^{-(p+1)}dY$ and $|Z_1| = 4^p|P|^{-2}|Y|$)

and then using eq.(1.2) of the authors' paper [10] along with the observation that for p=2,

$$4^{-p\rho_2}\times\frac{\Gamma_p[(\alpha+1)/2-\rho_2]\Gamma_p[(2\alpha+1)/4-\rho_2]}{\Gamma_p[(\alpha+1)/2]\Gamma_p[(2\alpha+1)/4]}=\frac{\Gamma_p(\alpha-2\rho_2)}{\Gamma_p(\alpha)} \cdots\cdots(2.21)$$

( from eq.(6.13) page 84 of Mathai [4] ) yields the same result as in eq.(2.19) above.

This result is different from the corresponding result in the scalar case.

**THEOREM 2.5:**

$$|P|^{-\beta}\Psi_1(\alpha,\beta;\gamma,\gamma';-P^{-1/2}XP^{-1/2},-Y)$$

$$=\frac{1}{\Gamma_p(\beta)}\int_{T>0} e^{-tr(PT)}|T|^{\beta-(p+1)/2}\Psi_2(\alpha;\gamma,\gamma';-T^{1/2}XT^{1/2},-Y)dT\cdots\cdots(2.22)$$

for $\mathrm{Re}(\beta) > (p-1)/2$.

**PROOF:** Taking the M-transform of the right side of eq.(2.22) with respect to the variables X and Y and the parameters $\rho_1$ and $\rho_2$ respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\Psi_2(\alpha;\gamma,\gamma';-T^{1/2}XT^{1/2},-Y)dXdY \qquad\cdots\cdots(2.23)$$

Applying the same transformation as for eq.(2.2) and using eq.(1.7) of the authors' paper [9], the above expression yields,

$$|T|^{-\rho_1}\frac{\Gamma_p(\gamma)\Gamma_p(\gamma')\Gamma_p(\alpha-\rho_1-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\gamma-\rho_1)\Gamma_p(\gamma'-\rho_2)} \qquad\cdots\cdots(2.24)$$

which, on substitution on the right side of eq.(2.22) and then integrating out T in the resulting expression by using a Gamma integral produces,

$$|P|^{-(\beta-\rho_1)}\frac{\Gamma_p(\gamma)\Gamma_p(\gamma')\Gamma_p(\alpha-\rho_1-\rho_2)\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\alpha)\Gamma_p(\gamma'-\rho_2)\Gamma_p(\gamma-\rho_1)} \qquad\cdots\cdots(2.25)$$

The same result is also obtained by taking the M-transform of the left side of eq.(2.22) with respect to the variables X and Y and the parameters $\rho_1$ and $\rho_2$ respectively and then applying the same transformation as for eq.(2.5) along with the use of eq.(1.6) of the authors' paper [9].

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