APPELL'S AND HUMBERT'S FUNCTIONS OF MATRIX ARGUMENTS – II

Lalit Mohan Upadhyaya* & H. S. Dhami**
Department of Mathematics,
University of Kumaun,
Almora Campus,
Almora (Uttaranchal),
India – 263601.

2000 AMS Mathematics Subject Classification :
Primary: 33C65, 33C99.
Secondary: 60E, 62H, 44A05.
Key Words : Appell's functions, Humbert's functions, matrix arguments, matrix-transform.

ABSTRACT

In this paper we have established a result for the Appell's function F_1 and three results for Appell's function F_3 and a result for the Appell's function F_4 of matrix arguments. Some reducibility cases of F_1 and transformation relations for F_1 and Appell's function F_2 of matrix arguments have also been discussed.

INTRODUCTION

The present work is in continuation of our previous study [12] of Appell's functions of matrix arguments. First we shall give some definitions of Mathai [5] and one of his results which shall be used by us in deducing our results. All the matrices appearing in this paper are (pxp) real symmetric positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [4,5].

* Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun (Uttaranchal), India, 248179.
** To whom all the correspondence may be addressed.

1. Preliminary Results and Definitions

DEFINITION 1.1: The Humbert's Φ_2 - function of matrix arguments

$$\Phi_2 = \Phi_2(b, b^{'}; c; -X, -Y)$$

is defined as that function for which the matrix - transform (M-transform) is the following:

$$M(\Phi_2) = \int_{X>0} \int_{Y>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \Phi_2(b, b^{'}; c; -X, -Y) dX dY$$

$$= \frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(b-\rho_1)\Gamma_p(b^{'}-\rho_2)}{\Gamma_p(b)\Gamma_p(b^{'})\Gamma_p(c-\rho_1-\rho_2)} \qquad \cdots(1.1)$$

for Re $(b-\rho_1, b^{'}-\rho_2, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2.$

DEFINITION 1.2: Humbert's function $\Xi_1 = \Xi_1(a, a^{'}, b; c; -X, -Y)$

$$M(\Xi_1) = \int_{X>0} \int_{Y>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \Xi_1(a, a^{'}, b; c; -X, -Y) dX dY$$

$$= \frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1)\Gamma_p(b-\rho_1)\Gamma_p(a^{'}-\rho_2)}{\Gamma_p(a)\Gamma_p(a^{'})\Gamma_p(b)\Gamma_p(c-\rho_1-\rho_2)} \qquad \cdots(1.2)$$

for Re $(a-\rho_1, a^{'}-\rho_2, b-\rho_1, c-\rho_1-\rho_2, \rho_1, \rho_2) > (p-1)/2.$

THEOREM 1.1:

$${}_2F_1(a, b; c; -X) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \int_0^I |Y|^{a-(p+1)/2} |I-Y|^{c-a-(p+1)/2} \times$$

$$|I+XY|^{-b} dY \qquad \cdots\cdots(1.3)$$

where $0 < X < I$ and for Re$(a, c-a) > (p-1)/2.$

2.Appell's Functions of Matrix Arguments

THEOREM 2.1:

$$|P|^{-\alpha} F_1(\alpha,\beta,\beta';\gamma;-P^{-1/2}XP^{-1/2},-P^{-1/2}YP^{-1/2})$$

$$=\frac{1}{\Gamma_p(\alpha)}\int_{T>0} e^{-tr(PT)}|T|^{\alpha-(p+1)/2}\Phi_2(\beta,\beta';\gamma;$$

$$-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dT \qquad \cdots\cdots(2.1)$$

for $\mathrm{Re}(\alpha)>(p-1)/2.$

PROOF: Taking the M-transform of the right side of eq.(2.1) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\Phi_2(\beta,\beta';\gamma;-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dXdY \qquad \cdots\cdots(2.2)$$

Applying the transformations,

$X_1=T^{1/2}XT^{1/2}, Y_1=T^{1/2}YT^{1/2}$ with $dX_1=|T|^{(p+1)/2}dX, dY_1=|T|^{(p+1)/2}dY$ and $|X_1|=|T||X|,|Y_1|=|T||Y|$

to the above expression and then using the definition (1.1) yields,

$$|T|^{-\rho_1-\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\beta')\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.3)$$

Substituting this expression on the right side of eq.(2.1) and then integrating out T in the resulting expression by using a Gamma integral gives,

$$|P|^{-(\alpha-\rho_1-\rho_2)}\frac{\Gamma_p(\gamma)\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\beta')\Gamma_p(\gamma-\rho_1-\rho_2)}\cdots(2.4)$$

Now taking the M-transform of the left side of eq.(2.1) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we have,

$$\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|P|^{-\alpha} F_1(\alpha,\beta,\beta';\gamma;-P^{-1/2}XP^{-1/2},-P^{-1/2}YP^{-1/2})dXdY \qquad \cdots\cdots(2.5)$$

which, under the transformations

$X_2 = P^{-1/2}XP^{-1/2}, Y_2 = P^{-1/2}YP^{-1/2}$ with $dX_2 = |P|^{-(p+1)/2}dX,$

$dY_2 = |P|^{-(p+1)/2}dY$ and $|X_2| = |P|^{-1}|X|, |Y_2| = |P|^{-1}|Y|$

and then using the eq.(1.1) of the authors' paper [12] yields the same result as in eq.(2.4) above.

THEOREM 2.2 :

$$F_3(\alpha,\alpha',\beta,\beta';\alpha+\alpha';-X,-Y)$$

$$= \frac{\Gamma_p(\alpha+\alpha')}{\Gamma_p(\alpha)\Gamma_p(\alpha')}\int_0^I |U|^{\alpha-(p+1)/2}|I-U|^{\alpha'-(p+1)/2}\left|I+U^{1/2}XU^{1/2}\right|^{-\beta}\times$$

$$\left|I+(I-U)^{1/2}Y(I-U)^{1/2}\right|^{-\beta'} dU \qquad \cdots\cdots(2.6)$$

for $\mathrm{Re}(\alpha,\alpha') > (p-1)/2$ and $0 < U < I$.

PROOF: Taking the M-transform of the right side of eq.(2.6) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\left|I+U^{1/2}XU^{1/2}\right|^{-\beta}\times$$

$$\left|I+(I-U)^{1/2}Y(I-U)^{1/2}\right|^{-\beta'} dXdY \qquad \cdots\cdots(2.7)$$

This expression, on the application of the transformations

$X_1 = U^{1/2} X U^{1/2}, Y_1 = (I-U)^{1/2} Y (I-U)^{1/2}$ with $dX_1 = |U|^{(p+1)/2} dX$,

$dY_1 = |I-U|^{(p+1)/2} dY$ and $|X_1| = |U||X|, |Y_1| = |I-U||Y|$

and then integrating out X_1 and Y_1 by using a type-2 Beta integral gives,

$$|U|^{-\rho_1} |I-U|^{-\rho_2} \frac{\Gamma_p(\beta-\rho_1)\Gamma_p(\beta'-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\beta)\Gamma_p(\beta')} \qquad \cdots\cdots(2.8)$$

Substituting this expression on the right side of eq.(2.6) and then integrating out U by using a type -1 Beta integral gives $M(F_3)$ as given by eq.(1.3) of the authors' paper [12].

Similarly we may also establish the following theorem:

THEOREM 2.3 :

$$F_3(\alpha,\alpha',\beta,\beta';\beta+\beta';-X,-Y)$$

$$= \frac{\Gamma_p(\beta+\beta')}{\Gamma_p(\beta)\Gamma_p(\beta')} \int_0^I |U|^{\beta-(p+1)/2} |I-U|^{\beta'-(p+1)/2} \left| I + U^{1/2} X U^{1/2} \right|^{-\alpha} \times$$

$$\left| I + (I-U)^{1/2} Y (I-U)^{1/2} \right|^{-\alpha'} dU \qquad \cdots\cdots(2.9)$$

for $\mathrm{Re}(\beta,\beta') > (p-1)/2$ and $0 < U < I$.

THEOREM 2.4:

$$|P|^{-\beta'} F_3(\alpha,\alpha',\beta,\beta';\gamma;-X,-P^{-1/2} Y P^{-1/2})$$

$$= \frac{1}{\Gamma_p(\beta')} \int_{T>0} e^{-\mathrm{tr}(PT)} |T|^{\beta'-(p+1)/2} \Xi_1(\alpha,\alpha',\beta;\gamma;$$

$$-X,-T^{1/2} Y T^{1/2}) dT \qquad \cdots\cdots(2.10)$$

for $\mathrm{Re}(\beta') > (p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(2.10) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0} \int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \Xi_1(\alpha,\alpha',\beta;\gamma; -X,-T^{1/2}YT^{1/2})dXdY \qquad \cdots\cdots(2.11)$$

Making use of the transformation

$Y_1 = T^{1/2}YT^{1/2}$, with $dY_1 = |T|^{(p+1)/2}dY$ and $|Y_1| = |T||Y|$ and then using the definition (1.2), the above expression produces,

$$|T|^{-\rho_2} \frac{\Gamma_p(\gamma)\Gamma_p(\alpha-\rho_1)\Gamma_p(\alpha'-\rho_2)\Gamma_p(\beta-\rho_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\alpha')\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots\cdots(2.12)$$

Substituting this expression on the right side of eq.(2.10) and integrating out T in the resulting expression by using a Gamma integral gives,

$$|P|^{-(\beta'-\rho_2)} \frac{\Gamma_p(\gamma)\Gamma_p(\alpha-\rho_1)\Gamma_p(\alpha'-\rho_2)\Gamma_p(\beta-\rho_1)}{\Gamma_p(\beta')\Gamma_p(\alpha)\Gamma_p(\alpha')\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \times \Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\beta'-\rho_2) \qquad \cdots\cdots(2.13)$$

Now taking the M- transform of the left side of eq.(2.10) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively yields,

$$\int_{X>0} \int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |P|^{-\beta'} \times F_3(\alpha,\alpha',\beta,\beta';\gamma;-X,-P^{-1/2}YP^{-1/2})dXdY \qquad \cdots\cdots(2.14)$$

which, under the transformation

$Y_2 = P^{-1/2}YP^{-1/2}$ with $dY_2 = |P|^{-(p+1)/2}dY$ and $|Y_2| = |P|^{-1}|Y|$

and then using eq.(1.3) of the authors' paper [12] generates the same result as in eq.(2.13).

THEOREM 2.5:

(i) $F_1(a,b,b';c;-X,-X) = {}_2F_1(a,b+b';c;-X)$ ……(2.15)

$$\text{(ii) } F_1(a,b,b';c;-X,I)=\frac{\Gamma_p(c)\Gamma_p(c-a-b')}{\Gamma_p(c-a)\Gamma_p(c-b')}\times {}_2F_1(a,b;c-b';-X) \qquad \cdots\cdots(2.16)$$

$$\text{(iii) } F_1(a,b,b';c;I,I)=\frac{\Gamma_p(c)\Gamma_p(c-a-b-b')}{\Gamma_p(c-a)\Gamma_p(c-b-b')} \qquad \cdots\cdots(2.17)$$

PROOF :(i) This result follows by putting Y=X in theorem (3.5) page 58 of Mathai [5] and then using the theorem (1.1).

(ii) This result can be obtained by letting $Y \to -I$ in theorem (3.5) page 58 of Mathai [5], and then using the theorem (1.1).

(iii) On letting $X \to -I$ in eq. (2.16) above and then using the theorem (2.3.2) page 39 of Mathai [5], this result is obtained. Alternatively, we can let both X and Y approach to $-I$ in theorem (3.5) page 58 of Mathai [5] and then use a type -1 Beta integral to see the result.

THEOREM 2.6:

$$F_1(a,b,b';c;-X,-Y)=|I+X|^{-a}F_1[a,c-b-b',b';c;(I+X)^{-1/2}X(I+X)^{-1/2},$$
$$-(I+X)^{-1/2}(Y-X)(I+X)^{-1/2}] \quad \cdots\cdots(2.18)$$

where $Y-X>0$.

$$=|I+Y|^{-a}F_1[a,b,c-b-b';c;-(I+Y)^{-1/2}(X-Y)(I+Y)^{-1/2},$$
$$(I+Y)^{-1/2}Y(I+Y)^{-1/2}] \quad \cdots\cdots(2.19)$$

where $X-Y>0$.

$$=|I+X|^{-b}|I+Y|^{-b'}F_1[c-a,b,b';c;(I+X)^{-1/2}X(I+X)^{-1/2},$$
$$(I+Y)^{-1/2}Y(I+Y)^{-1/2}] \quad \cdots\cdots(2.20)$$

$$= |I+X|^{c-a-b}|I+Y|^{-b'} F_1[c-a, c-b-b', b'; c; -X,$$

$$(I+X)^{1/2}(I+Y)^{-1/2} Y(I+Y)^{-1/2}(I+X)^{1/2} - X] \quad \cdots\cdots (2.21)$$

$$= |I+X|^{-b}|I+Y|^{c-a-b'} F_1[c-a, b, c-b-b'; c;$$

$$(I+Y)^{1/2}(I+X)^{-1/2} X(I+X)^{-1/2}(I+Y)^{1/2} - Y, -Y] \quad \cdots\cdots (2.22)$$

PROOF: To prove this theorem we first give two definitions of the F_1 function through integral representations:

$$F_1(a,b,b';c;-X,-Y) = \frac{\Gamma_p(c)}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-b-b')} \iint |U_1|^{b-(p+1)/2} \times$$

$$|U_2|^{b'-(p+1)/2} |I-U_1-U_2|^{c-b-b'-(p+1)/2} \times$$

$$\left| I + X^{1/2} U_1 X^{1/2} + Y^{1/2} U_2 Y^{1/2} \right|^{-a} dU_1 dU_2 \cdots\cdots (2.23)$$

for $\operatorname{Re}(b, b', c-b-b') > (p-1)/2$.

Also, $$F_1(a,b,b';c;-X,-Y) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \int_0^I |V|^{a-(p+1)/2} |I-V|^{c-a-(p+1)/2} \times$$

$$\left| I + X^{1/2} V X^{1/2} \right|^{-b} \left| I + Y^{1/2} V Y^{1/2} \right|^{-b'} dV \qquad \cdots\cdots (2.24)$$

for $\operatorname{Re}(a, c-a) > (p-1)/2$.

To prove eq.(2.18) we apply the transformations

$V_1 = I - U_1 - U_2, V_2 = U_2$ (so that, $dV_1 dV_2 = dU_1 dU_2$) to eq.(2.23)

and observe that,

$$\left|I + X^{1/2}(I - V_1 - V_2)X^{1/2} + Y^{1/2}V_2Y^{1/2}\right|$$

$$= |I + X|\Big|I - (I + X)^{-1/2}X^{1/2}V_1X^{1/2}(I + X)^{-1/2} +$$

$$(I + X)^{-1/2}(Y - X)^{1/2}V_2(Y - X)^{1/2}(I + X)^{-1/2}\Big| \text{ where, } Y - X > 0.$$

The desired result then follows immediately after a suitable interpretation of the resulting expression in the light of eq.(2.23).

The result of eq. (2.19) also follows in a similar manner from eq. (2.23).

The result in eq.(2.20) can be proved by using eq.(2.24) by observing that

$$\left|I + X^{1/2}VX^{1/2}\right| = |I + X|\left|I - (I + X)^{-1/2}X^{1/2}(I - V)X^{1/2}(I + X)^{-1/2}\right|$$

and a similar expression for $\left|I + Y^{1/2}VY^{1/2}\right|$ and then applying the transformation $V_1 = I - V$ and suitably interpreting the resulting expression as per eq.(2.24).

The result in eq.(2.21) is obtained by applying eq.(2.18) to the F_1- function on the right side of eq.(2.20) and similarly, the result in eq.(2.22) follows from eqs.(2.19) and (2.20).

THEOREM 2.7:

$$F_2(a, b, b';c, c';-X,-Y) = |I + X|^{-a} F_2[a, c - b, b';c, c';(I + X)^{-1/2}X(I + X)^{-1/2},$$

$$-(I + X)^{-1/2}Y(I + X)^{-1/2}] \qquad \cdots\cdots(2.25)$$

$$= |I + Y|^{-a} F_2[a, b, c - b';c, c';-(I + Y)^{-1/2}X(I + Y)^{-1/2},$$

$$(I + Y)^{-1/2}Y(I + Y)^{-1/2}] \qquad \cdots\cdots(2.26)$$

$$= \left|I+X+Y\right|^{-a} F_2[a, c-b, c'-b'; c, c'; (I+X+Y)^{-1/2} X (I+X+Y)^{-1/2},$$

$$(I+X+Y)^{-1/2} Y (I+X+Y)^{-1/2}] \qquad \cdots\cdots (2.27)$$

PROOF: To prove this theorem we define the function F_2 through an integral representation:

$$F_2(a, b, b'; c, c'; -X, -Y) = \frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-b)\Gamma_p(c'-b')} \int_0^I \int_0^I \left|U_1\right|^{b-(p+1)/2} \times$$

$$\left|U_2\right|^{b'-(p+1)/2} \left|I-U_1\right|^{c-b-(p+1)/2} \left|I-U_2\right|^{c'-b'-(p+1)/2} \times$$

$$\left|I + X^{1/2} U_1 X^{1/2} + Y^{1/2} U_2 Y^{1/2}\right|^{-a} dU_1 dU_2 \cdots\cdots (2.28)$$

for $\operatorname{Re}(b, b', c-b, c'-b') > (p-1)/2.$

The result in eq.(2.25) is obtained by applying the transformation, $V_1 = I - U_1$ to eq.(2.28) and observing that,

$$\left|I + X^{1/2}(I - V_1)X^{1/2} + Y^{1/2} U_2 Y^{1/2}\right|$$

$$= \left|I+X\right| \Big| I - (I+X)^{-1/2} X^{1/2} V_1 X^{1/2} (I+X)^{-1/2} +$$

$$(I+X)^{-1/2} Y^{1/2} U_2 Y^{1/2} (I+X)^{-1/2} \Big|$$

and then interpreting the resulting expression in view of eq.(2.28). The result in eq.(2.26) also follows similarly from eq.(2.28) while, the result in eq.(2.27) is a combination of the results in eqs.(2.25) and (2.26).

THEOREM 2.8:

$$F_4(\alpha,\beta;\gamma,\gamma';-X,-Y)=\frac{\Gamma_p(\gamma)\Gamma_p(\gamma')}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma'-\beta)}\iint |U|^{\alpha-(p+1)/2}\times$$

$$|V|^{\beta-(p+1)/2}|I-U|^{\gamma-\alpha-(p+1)/2}|I-V|^{\gamma'-\beta-(p+1)/2}\times$$

$$\left|I-(I-V)^{-1/2}V^{1/2}U^{1/2}XU^{1/2}V^{1/2}(I-V)^{-1/2}\right|^{\gamma'-\beta-(p+1)/2}\times$$

$$\left|I-(I-U)^{-1/2}V^{1/2}U^{1/2}YU^{1/2}V^{1/2}(I-U)^{-1/2}\right|^{\gamma-\alpha-(p+1)/2}dUdV\cdots\cdots(2.29)$$

for $0<U<I$ and $0<V<I$ and for $\mathrm{Re}(\alpha,\beta,\gamma-\alpha,\gamma'-\beta)>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(2.29) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\left|I-(I-V)^{-1/2}V^{1/2}U^{1/2}XU^{1/2}V^{1/2}(I-V)^{-1/2}\right|^{\gamma'-\beta-(p+1)/2}\times$$

$$\left|I-(I-U)^{-1/2}V^{1/2}U^{1/2}YU^{1/2}V^{1/2}(I-U)^{-1/2}\right|^{\gamma-\alpha-(p+1)/2}dXdY\cdots\cdots(2.30)$$

Now, applying the transformations,

$$X_1=(I-V)^{-1/2}V^{1/2}U^{1/2}XU^{1/2}V^{1/2}(I-V)^{-1/2},$$

$$Y_1=(I-U)^{-1/2}V^{1/2}U^{1/2}YU^{1/2}V^{1/2}(I-U)^{-1/2},$$

with $dX_1=|I-V|^{-(p+1)/2}|V|^{(p+1)/2}|U|^{(p+1)/2}dX$,

$$dY_1 = |I-U|^{-(p+1)/2}|V|^{(p+1)/2}|U|^{(p+1)/2}dY \text{ and } |X_1| = |I-V|^{-1}|V||U||X|,$$
$$|Y_1| = |I-U|^{-1}|V||U||Y|$$

to the last expression and then integrating out X_1 and Y_1 by using a type -1 Beta integral yields,

$$|U|^{-\rho_1-\rho_2}|V|^{-\rho_1-\rho_2}|I-V|^{\rho_1}|I-U|^{\rho_2} \times$$
$$\frac{\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma'-\beta)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(\gamma'-\beta+\rho_1)\Gamma_p(\gamma-\alpha+\rho_2)} \qquad \cdots\cdots(2.31)$$

Substituting this expression on the right side of eq.(2.29) and then integrating out U and V in the resulting expression by using a type-1 Beta integral produces M(F_4) as given by eq.(1.4) of the authors' paper [12].

References

1. Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G. (1953). Higher Transcendental Functions, Vol. I, McGraw Hill, New York, Toronto and London.
2. Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G. (1954). Tables of Integral Transforms, Vol. I, McGraw Hill, New York, Toronto and London.
3. Exton H. (1976). Multiple Hypergeometric Functions and Applications, Ellis Horwood Limited, Publishers, Chichester.
4. Mathai A.M. (1992). Jacobians of Matrix Transformations I, Centre for Mathematical Sciences, Trivandrum, India.
5. Mathai A.M. (1993). Hypergeometric Functions of Several Matrix Arguments, Centre for Mathematical Sciences, Trivandrum, India.
6. Mathai A.M. (1993). Appell's and Humbert's Functions of Matrix Arguments, Linear Algebra and its Applications; 183, pp. 201-221.
7. Mathai A.M. (1995). Special Functions of Matrix Arguments-III; Proceedings of the National Academy of Sciences, India; LXV (IV) pp. 367-393.
8. Mathai A.M., Pederzoli G. (1996). Some Transformations for Functions of Matrix Arguments; Indian J. Pure Appl. Math. 27(3), pp. 277-284.

9. Saxena R.K., Sethi P.L. & Gupta O.P. (1997). Appell's Functions of Matrix Arguments. Indian J. Pure Appl. Math.; 28, no. 3, pp. 371-380.
10. Srivastava H.M., Karlsson P.W. (1985). Multiple Gaussian Hypergeometric Series. Ellis Horwood Limited, Publishers, Chichester.
11. Upadhyaya Lalit Mohan, Dhami H.S.(Nov.2001). Matrix Generalizations of Multiple Hypergeometric Functions, # 1818 IMA Preprints Series, University of Minnesota, Minneapolis, U.S.A.
12. Upadhyaya Lalit Mohan, Dhami H.S.(Mar. 2002). Appell's and Humbert's Functions of Matrix Arguments-I. # 1848 IMA Preprints Series, University of Minnesota, Minneapolis, U.S.A.