On boundary regularity of the Navier-Stokes equations

Kyungkeun Kang

Abstract

We study boundary regularity of weak solutions of the Navier-Stokes equations in the half-space in dimension $n \geq 3$. We prove that a weak solution u which is locally in the class $L^{p,q}$ with $\frac{2}{p} + \frac{n}{q} = 1$, $q > n$ near boundary is Hölder continuous up to the boundary. Our main tool is a point-wise estimate for the fundamental solution of the Stokes system, which is of independent interest.

Key words. Navier-Stokes equations, Green tensor

AMS subject classifications. 76D03, 76D05

1 Introduction

In this paper we study the boundary regularity of weak solutions of the Navier-Stokes equations

$$\left.\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f \\ \text{div } u = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^n_+ \times (0,T) \tag{1}$$

where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, $n \geq 3$, and f is sufficiently smooth. In addition to (1) initial and boundary conditions are given as follows

$$u(x,t) = 0 \text{ on } \partial\mathbb{R}^n_+ \times (0,T), \qquad u(x,0) = u_0(x) \text{ at } \mathbb{R}^n_+ \times \{t = 0\} \tag{2}$$

where u_0 is also sufficiently smooth and satisfies the compatibility conditions, i.e. $u_0 = 0$ at $\partial\mathbb{R}^n_+$ and div $u_0 = 0$.

We will prove that if $u \in L^{p,q}_{\text{loc}}(\mathbb{R}^n_+ \times (0,T))$ in a neighborhood of $z \in \partial\mathbb{R}^n_+ \times (0,T)$, with $\frac{2}{p} + \frac{n}{q} = 1$, $q > n$, then u is regular in a neighborhood of z. The corresponding interior result was studied in [14] and [19]. In [7] Y. Giga investigated regularity under global assumptions. Partial results for the local problem near boundary were previously obtained in [4] and [20]. Recently, the interior partial regularity results in [2] were extended up to the boundary in [13]. These partial regularity results implies some (although not all) of our results, under an additional assumption that u is a *suitable weak solution*.

Our main tool is an estimate for the Green tensor of the Stokes system in the half-space, which seems to be of independent interest. We prove the following point-wise estimate of the Green tensor $\{T_{ij}\}_{i,j=1,2,\dots,n}$ of the Stokes system:

$$|D_t^k D_{x'}^l D_{x_n}^m T_{ij}(x,t)| \le \frac{C}{t^{k+\frac{l+m+\alpha}{2}}|x|^{n-1}x_n^{1-\alpha}}, \tag{3}$$

where $(x,t) \in \mathbb{R}^n_+ \times (0,T)$ and $C = C(k,l,m,\alpha,n)$ and α is any number with $0 < \alpha < 1$. Although such an estimate may probably not be surprising to experts, we were not able to find it in the literature. The estimate seems to be useful in other situations. Various other estimates on the Green tensors and solution formula can be found for example in [15], [16], [17], [21], and [3].

The plan of this paper is as follows. In section 2, we review results in [5] and [15], and obtain the estimate (3) of the Green tensor for the Stokes system in the half-space. In section 3, we study boundary regularity for the Navier-Stokes equations and prove our main result.

2 Green tensor for the Stokes system

2.1 Preliminaries

Let us begin with some definitions and notations used throughout this paper.

• We denote the whole space and the half-space by $\mathbb{R}^n$ and $\mathbb{R}^n_+, n \ge 3$, respectively. Let $T \in (0,\infty]$ and we denote $I = (0,T)$ for simplicity.

• For a given $z = (x,t) \in \partial\mathbb{R}^n_+ \times I$, we denote a half ball with radius r at x by $B^+_{x,r}$, namely $B^+_{x,r} = \{y \in \mathbb{R}^n : |x-y| < r, y_n > 0\}$ and we also write a parabolic cylinder at z by $Q^+_{z,\gamma} = B^+_{x,\gamma} \times (t-\gamma^2, t)$ with $0 < \gamma < \sqrt{t}$.

• Let $\Omega \subset \mathbb{R}^n$. For $1 \le q \le \infty$, $W^{k,q}(\Omega)$ denote the usual Sobolev space, i.e. $W^{k,q}(\Omega) = \{\, u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega), 0 \le |\alpha| \le k\,\}$. As usual, $W^{k,q}_0(\Omega)$ is the completion of $\mathcal{C}^\infty_0(\Omega)$ in $W^{k,q}(\Omega)$. For $q = \infty$, $L^\infty(\Omega)$ denote the Banach space of bounded functions on Ω with the norm $||u||_{L^\infty(\Omega)} = \sup_\Omega |u|$.

• Let $1 \le q, r \le \infty$ and $I' \subset I$. $L^r(I'; W^q(\Omega))$ denote the Banach space consisting of all measurable functions with a finite norm

$$||u||_{L^r(I';W^q(\Omega))} = (\int_{I'} ||u(\cdot,t)||^r_{W^q(\Omega)}\,\mathrm{d}t)^{\frac{1}{r}}.$$

We denote $L^{p,q}(Q^+_{z,\gamma}) = L^p((t-\gamma^2,t); L^q(B^+_{x,\gamma}))$ if there is no confusion.

• For a given parabolic domain $Q \subset \mathbb{R}^n \times I$, $\mathcal{C}^{\alpha,\frac{\alpha}{2}}_{x,t}(Q)$ indicates the Banach space of functions that are Hölder continuous with the exponent $\alpha \in (0,1)$. In the above definition,

$\mathbb{R}^n \times I$ is endowed with a parabolic metric $d(z,z') = |x-x'| + |t-t'|^{\frac{1}{2}}$ where $z=(x,t)$ and $z'=(x',t')$.

• Let $Q \subset \mathbb{R}^n_+ \times I$. We denote by χ_Q the characteristic function of Q defined by

$$\chi_Q(z) = \begin{cases} 1 & \text{if } z \in Q \\ 0 & \text{otherwise .} \end{cases}$$

• The convolution of two functions is denoted by $*$ with subscript. To be precise, suppose $f,g : \Omega \times (0,T) \to \mathbb{R}^n$ and $h : \Omega \to \mathbb{R}^n$ where $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$. Then $f *_1 g$ and $f *_2 h$ are defined as follows

$$f *_1 g(x,t) = \int_0^t \int_\Omega f(x-y,t-s) g(y,s)\, \mathrm{d}y\, \mathrm{d}s$$

$$f *_2 h(x,t) = \int_\Omega f(x-y,t) h(y)\, \mathrm{d}y.$$

In particular, in case of $\Omega = \mathbb{R}^n_+$, functions are understood as extended ones to the whole space by assigning zero in the lower half-space.

• We denote by Γ the fundamental solution of the heat equation in $\mathbb{R}^n$, namely $\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp^{\frac{-|x|^2}{4t}}$.

• Let k,l and m be nonnegative integers and $\mathbf{l} = (l_1,...,l_{n-1})$ a multi-index with $l = |\mathbf{l}| = \sum l_i$ where $l_i \geq 0$ is integer for each $i = 1,...,n-1$. we define $D^k_t D^l_{x'} D^m_{x_n} = \frac{\partial^k}{\partial t} \frac{\partial^l}{\partial x_1^{l_1} \cdots \partial x_{n-1}^{l_{n-1}}} \frac{\partial^m}{\partial x_n}$.

• If there is no confusion, summation convention is understood over repeated indices running from 1 to n, for instance $f_i g_i = \sum_{i=1}^n f_i g_i$.

• The capital letter C is used to denote the generic constant, the value of which may change from line to line.

In this section, we will find the Green tensor and its estimate for the Stokes system in the half-space $\mathbb{R}^n_+, n \geq 3$. Let u be in the class $u \in L^\infty(I; L^2(\mathbb{R}^n_+)) \cap L^2(I; W^{1,2}_0(\mathbb{R}^n_+))$ and solve the following Stokes system in a distribution sense:

$$\left.\begin{aligned} u_t - \Delta u + \nabla p = f \\ \mathrm{div}\ \mathrm{u} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^n_+ \times I \tag{4}$$

and initial and boundary conditions are

$$u(x',0,t) = 0 \text{ on } \partial\mathbb{R}^n_+ \times I, \qquad u(x,0) = g \text{ at } t = 0. \tag{5}$$

To avoid technical difficulties, we assume that f and g are smooth and compactly supported in $\mathbb{R}^n_+ \times [0,T)$ and $\mathbb{R}^n_+$, respectively (see Remark 2.1 for $f \in \mathcal{C}^k_0(\overline{\mathbb{R}^n_+} \times [0,T))$ and $g \in \mathcal{C}^k_0(\overline{\mathbb{R}^n_+})$).

Once we find the Green tensor T_{ij} and the pressure tensor P_j associated with T_{ij} solving the following Stokes system for each $i, j = 1, 2, ..., n$

$$\begin{cases} (\partial_t - \Delta) T_{ij}(x,t) + \partial_{x_i} P_j(x,t) = \delta_{ij}\delta(x,t) & \text{in } \mathbb{R}^n_+ \times I \\ \nabla \cdot T_{ij}(x,t) = \partial_{x_i} T_{ij}(x,t) = 0 & \text{in } \mathbb{R}^n_+ \times I \\ T_{ij}(x', 0, t) = 0 & \text{on } \partial\mathbb{R}^n_+ \times I \\ \int_{\mathbb{R}^n_+} T_{ij}(x-y,t) g_j(y)\,\mathrm{d}y \longrightarrow g_i(x) & \text{as } t \longrightarrow 0, \end{cases} \tag{6}$$

then the solution u of (4) and (5) can be represented as follows: For $i = 1, 2, ..., n$,

$$u_i(x,t) = T_{ij} *_1 f_j + T_{ij} *_2 g_j$$

$$= \int_0^t \int_{\mathbb{R}^n_+} T_{ij}(x,y,t,s) f_j(y,s)\,\mathrm{d}y\,\mathrm{d}s + \int_{\mathbb{R}^n_+} T_{ij}(x,y,t) g_j(y)\,\mathrm{d}y, \tag{7}$$

because of the uniqueness of weak solutions for the Stokes system (see e.g. Theorem 3.1 in [8, page 84] and Lemma 4.2 in [20, page 271]). To obtain the Green tensor T_{ij}, we shall combine the Green tensor for the whole space $\mathbb{R}^n$ proved in [5] and the Green tensor for the boundary value problem for the Stokes system in the half-space proved in [15]. Let us first recall the Green tensor of the Stokes system for the whole space $\mathbb{R}^n$ (see [5]). Let f and g be vector fields given in (4) and (5) (note that f and g are easily extended to a whole space because they are compactly supported in a upper half-space). We consider the following Stokes system in $\mathbb{R}^n$:

$$\begin{cases} v_t - \Delta v + \nabla q = f, \quad \text{div v} = 0 & \text{in } \mathbb{R}^n \times I \\ v(x,0) = g & \text{when } t = 0. \end{cases} \tag{8}$$

It is well known that unknowns v and q are represented as follows:

$$v_i(x,t) = (E_{ij} *_2 f_j + E_{ij} *_1 g_j)$$

$$= \int_0^t \int_{\mathbb{R}^n} E_{ij}(x,y,t,s) f_j(y,s)\,\mathrm{d}y\,\mathrm{d}s + \int_{\mathbb{R}^n} E_{ij}(x,y,t) g_j(y)\,\mathrm{d}y$$

and

$$q(x,t) = (Q_j *_2 f_j) = \int_0^t \int_{\mathbb{R}^n} Q_j(x,y,t,s) f_j(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

where E_{ij} and Q_j solve the following system:

$$\begin{cases} \partial_t E_{ij}(x,t) - \Delta E_{ij}(x,t) + \partial_{x_i} Q_j(x,t) = \delta_{ij}\delta(x,t) & \text{in } \mathbb{R}^n \times I \\ \nabla \cdot E_i(x,t) = D_{x_i} E_{ij}(x,t) = 0 & \text{in } \mathbb{R}^n \times I \\ \int_{\mathbb{R}^n} E_{ij}(x-y,t) g_j(y)\,\mathrm{d}y \longrightarrow g_i(x) & \text{as } t \longrightarrow 0. \end{cases}$$

Moreover, E_{ij} and Q_j are explicitly given as follows:

$$E_{ij}(x,t) = \delta_{ij}\Gamma(x,t) - \mathcal{R}_i\mathcal{R}_j\Gamma(x,t)$$
$$= \delta_{ij}\Gamma(x,t) - \int_0^{\frac{1}{t}} D^2_{x_ix_j}\frac{1}{(4\pi)^{\frac{n}{2}}}\exp(-\frac{|x|^2 s}{4})s^{\frac{n}{2}-1}\,\mathrm{d}s \tag{9}$$

and

$$Q_j(x,t) = -\delta(t)\frac{\gamma(\frac{n-2}{2})}{4\pi^{\frac{n}{2}}}\frac{\partial}{\partial x_j}\frac{1}{|x|^{n-2}} = \delta(t)(n-2)\frac{\gamma(\frac{n-2}{2})}{4\pi^{\frac{n}{2}}}\frac{x_j}{|x|^n},$$

where Γ is the fundamental solution of the heat equation and $\mathcal{R}_i, \mathcal{R}_j$ are Riesz transformations, namely

$$\mathcal{R}_i(h)(x) = \lim_{\epsilon\to 0} C\int_{|x-y|>\epsilon}\frac{x_i - y_i}{|x-y|^{n+1}}h(y)\,\mathrm{d}y,$$

where h is in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ and $C = \gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$ with $\gamma(z) = \int_0^\infty e^{-s}s^{z-1}\,\mathrm{d}s$. To sum up, for a given f and g in (4) and (5), $v = E *_1 f + E *_2 g$ and $q = Q *_2 f$, solves (8). Besides when it is restricted to the half-space, it solves

$$\begin{cases} v_t - \Delta v + \nabla q = f, \quad \text{div v} = 0 & \text{in } \mathbb{R}^n_+ \times I \\ v(x,0) = g & \text{when } t = 0 \\ v(x',0,t) = (E *_2 f + E *_1 g)(x',0) & \text{on } \partial\mathbb{R}^n_+ \times I \end{cases} \tag{10}$$

Next we set $T_{ij}(x,t) = E_{ij}(x,t) - K_{ij}(x,t)$ and $P_j = Q_j - R_j$. To find T_{ij} and P_j, it suffices to find K_{ij} and R_j satisfying the following system for $i,j = 1,2,...,n$

$$\begin{cases} (\partial_t - \Delta)K_{ij}(x,t) + \partial_{x_i}R_j(x,t) = 0 & \text{in } \mathbb{R}^n_+ \times I \\ \nabla\cdot K_{ij}(x,t) = D_{x_j}K_{ij}(x,t) = 0 & \text{in } \mathbb{R}^n_+ \times I \\ K_{ij}(x,0) = 0 & \text{when } t = 0 \\ K_{ij}(x',0,t) = E_{ij}(x',0,t) & \text{on } \partial\mathbb{R}^n_+ \times I. \end{cases} \tag{11}$$

With the aid of the Green tensor and their estimates for the half-space in [15], we can have explicitly the integral representation of K_{ij}, which enable us to calculate point-wise estimates of K_{ij}, and thus we also obtain those of T_{ij}.

Before we analyze K_{ij}, we first review results in [15] for the boundary value problem of the Stokes system in the half-space. Although it was studied for $n = 3$, it can be easily generalized to a general dimension $n \ge 3$. Let us recall some functionals and their estimates used later (see [15, page 37, 41]).

$$|D^l_x D^m_t A(x,t)| \le \frac{C}{t^{m+\frac{1}{2}}(x^2+t)^{\frac{l+(n-2)}{2}}} \tag{12}$$

$$|D^l_{x'} D^k_{x_n} D^m_t B(x,t)| \le \frac{C}{(x_n^2+t)^{m+\frac{k+1}{2}}(x^2+t)^{\frac{l+(n-2)}{2}}} \tag{13}$$

$$|D^l_{x'} D^k_{x_n} D^m_t \mathcal{C}_i(x,t)| \le \frac{C}{t^{m+\frac{1}{2}}(x^2+t)^{\frac{l+(n-1)}{2}}(x_n^2+t)^{\frac{k}{2}}}, \tag{14}$$

where A, B and $\mathcal{C}_i$ are defined as follows (see [15, page 37, 40]).

$$A(x,t) = (\frac{1}{|x|^{n-2}} * \Gamma(x',0,t)) = \int_{\mathbb{R}^{n-1}} \frac{\Gamma(y',0,t)}{|x-y'|^{n-2}} \, \mathrm{d}y',$$

$$B(x,t) = (\frac{1}{|x'|^{n-2}} * \Gamma(x',t)) = \int_{\mathbb{R}^{n-1}} \frac{\Gamma(x-y',t)}{|y'|^{n-2}} \, \mathrm{d}y',$$

$$\mathcal{C}_i(x,t) = \int_{\mathbb{R}_{n-1}} \mathrm{d}y' \int_0^{x_n} \frac{\partial \Gamma(y,t)}{\partial y_n} \frac{y_i - x_i}{|y-x|^n} \, \mathrm{d}y_n.$$

In addition, they satisfy the following relations (see [15, page 40]):

$$\frac{\partial \mathcal{C}_\alpha}{x_n} = \frac{\partial \mathcal{C}_n}{x_\alpha} + \frac{\partial^2 B}{x_\alpha x_n} \text{ if } \alpha \neq n, \qquad \sum_{i=1}^n \frac{\partial \mathcal{C}_i}{\partial x_i} = -\frac{2}{d_n}\frac{\partial \Gamma}{\partial x_n}, \tag{15}$$

where $d_n = \gamma(\frac{n-2}{2})/2\pi^{\frac{n}{2}}$. Now we consider the boundary value problem of the Stokes system in the half-space:

$$\begin{cases} w_t - \Delta w + \nabla q = 0, \quad \text{div w} = 0 & \text{in } \mathbb{R}^n_+ \times I \\ w(x,0) = 0 & \text{when } t = 0 \\ w(x',0,t) = b(x',t) & \text{on } \partial \mathbb{R}^n_+ \times I \end{cases}$$

The solution of the problem above is given by (see [15, page 53])

$$w_i(x,t) = (G_{ij} * b_j) = \int_0^t \int_{\mathbb{R}^{n-1}} G_{ij}(x-y',t-\tau) b_j(y',\tau) \, \mathrm{d}y \, \mathrm{d}\tau,$$

where G_{ij} solves the system below for each $i,j = 1,2,...,n$ (see [15, page 48]):

$$\begin{cases} \frac{\partial G_{ij}}{\partial t} - \Delta G_{ij} + \frac{\partial P_j}{\partial x_i} = 0 & \text{in } \mathbb{R}^n_+ \times I \\ \nabla \cdot G_{ij} = \sum_{i=1}^n \partial_{x_i} G_{ij} = 0 & \text{in } \mathbb{R}^n_+ \times I \\ G_{ij}|_{x_n=0} = \delta_{ij}\delta(t)\delta(x_1)\delta(x_2)...\delta(x_{n-1}) & \text{on } \partial\mathbb{R}^n_+ \times I. \end{cases} \tag{16}$$

Moreover, G_{ij} is explicitly given as follows (see [15, page 48]):

$$G_{ij}(x,t) = -2\frac{\partial \Gamma}{\partial x_n}\delta_{ij} - d_n \frac{\partial \mathcal{C}_i}{\partial x_j} - d_n \frac{\partial}{\partial x_i}\frac{1}{|x|^{n-2}}\delta_{jn}\delta(t), \quad i,j = 1,...,n. \tag{17}$$

Furthermore, the following estimate holds (see [15, page 42]):

$$|D_{x'}^l D_{x_n}^k D_t^m G_{ij}| \le \frac{C}{t^{m+\frac{1}{2}}(x^2+t)^{\frac{l+n}{2}}(x_n^2+t)^{\frac{k}{2}}}, \tag{18}$$

where $i = 1,2,...,n$ and $j = 1,2,...,n-1$. Therefore, with the aid of the representation formula, we have, for $i, j = 1,2,...,n$,

$$K_{ij}(x,t) = G_{i\alpha} * (E_{\alpha j}|_{x_n=0}) = \int_0^t \int_{\mathbb{R}^{n-1}} G_{i\alpha}(x-y', t-\tau) E_{\alpha j}(y', 0, \tau)\, \mathrm{d}y'\, \mathrm{d}\tau. \tag{19}$$

Although the representation of a pressure R_j is also available (see [15, page 48, 52]), it will not be needed, and therefore omitted. To sum up, $w_i = (K_{ij} *_1 f_j + K_{ij} *_2 g_j)$ and $r = R_j *_1 f_j + R_j *_2 g_j$ solves the following boundary value problem of the Stokes system:

$$\begin{cases} w_t - \Delta w + \nabla r = 0, \quad \mathrm{div\ w} = 0 & \text{in } \mathbb{R}_+^n \times I \\ w(x,0) = 0 & \text{when } t = 0 \\ w(x', 0, t) = (E *_2 f + E *_1 g) & \text{on } \partial\mathbb{R}_+^n \times I. \end{cases}$$

Indeed, it can be checked by using change of variables. For example, we consider $T_{ij} *_2 f_j$.

$$T_{ij} *_2 f_j(x,t) = [G_{i\alpha} * (E_{\alpha j}|_{x_n=0})] * f(x,t)$$

$$= \int_0^t \int_{\mathbb{R}_+^n} [\int_0^{t-s} \int_{R^{n-1}} G_{i\alpha}(x-y-z', t-s-\tau) E_{\alpha j}(z', \tau)\, \mathrm{d}z'\, \mathrm{d}\tau] f_j(y,s)\, \mathrm{d}y\, \mathrm{d}s$$

$$= \int_0^t \int_{\mathbb{R}_+^n} [\int_0^{t-s} \int_{R^{n-1}} G_{i\alpha}(x-z', t-\tau) E_{\alpha j}(z'-y, \tau-s)\, \mathrm{d}z'\, \mathrm{d}\tau] f_j(y,s)\, \mathrm{d}y\, \mathrm{d}s$$

$$= \int_0^t \int_{R^{n-1}} G_{i\alpha}(x-z', t-\tau) [\int_0^\tau \int_{\mathbb{R}^n} E_{\alpha j}(z'-y, \tau-s) f_j(y,s)\, \mathrm{d}y\, \mathrm{d}s]\, \mathrm{d}z'\, \mathrm{d}\tau$$

$$= G_{i\alpha} * [(E_{\alpha j} * f_j)|_{x_n=0}](x,t).$$

Since $T_{ij} * g_j$ can be verified in a similar manner, we omit the details. To sum up arguments above, we obtain the Green tensor for the Stokes system (4) and (5):

$$T_{ij} = E_{ij} - G_{i\alpha} * (E_{\alpha j}|_{x_n=0}) \ \text{ for } \ i, j = 1,2,...,n. \tag{20}$$

Namely, $T_{ij} *_2 f_j + T_{ij} *_1 g_j$ solves the Stokes system (4) and (5).

Remark 2.1 *As mentioned earlier, to avoid technical difficulty, we assume that f and g are smooth and compactly supported in $\mathbb{R}_+^n \times [0,T), \mathbb{R}_+^n$, respectively. However, the assumptions on f and g can be relaxed. Suppose that $f \in \mathcal{C}_0^k(\overline{\mathbb{R}_+^n} \times [0,T))$ where k is a positive integer (since it can be treated similarly, we omit the details for g). Then f can* *be extended to $\mathbb{R}^n$ such that the extension of f, say $\tilde{f}$, is $\mathcal{C}^k$ differentiable, for example $\tilde{f}(x', x_n, t) = \sum_{i=1}^k c_i f(x', -x_n/i, t)$ for $x_n < 0$ where constant c_i is determined by the system of equation (see e.g. Theorem 7.25 in [6] or Lemma 4 in [15]). Green tensor for such case can be expressed finite sum of tensors, which basically have the same form as (20) with a slight modification because $\tilde{f}$ is more or less the finite sum of f in a lower space. Since our purpose is to obtain point-wise estimates of Green tensor, it suffices, for simplicity, to estimate (20) by the reason mentioned above.*

2.2 Point-wise estimates of Green tensor

In this subsection, we prove the point-wise estimate of Green tensor T_{ij} in (6). To obtain point-wise estimates of T_{ij}, we need to analyze E_{ij} in (9) and K_{ij} in (16) separately. We consider first E_{ij}. In [5, page 230], the point-wise estimate of ∇E_{ij} was mentioned and the estimate of higher derivatives of E_{ij} can be also easily calculated. Since such estimates of E_{ij} are elementary and straightforward compared to K_{ij}, we only state the result and omit the details. For convenience, let k and l be nonnegative integers and $\mathbf{l} = (l_1, ..., l_n)$ a multi-index with $l = |\mathbf{l}| = \sum l_i$ where $l_i \geq 0$ is integer for each i and we denote $D_t^k D_x^l = \frac{\partial^k}{\partial t} \frac{\partial^l}{\partial x_1^{l_1} \cdots \partial x_n^{l_n}}$.

Lemma 2.2 *Let E_{ij} be the Green tensor in (9) where $i, j = 1, 2, ..., n$ and let $(x, t) \in \mathbb{R}_+^n \times I$. Then for every nonnegative integer $k, l, m \geq 0$, E_{ij} satisfies the following estimate:*

$$|D_t^k D_x^l E_{ij}(x, t)| \leq \frac{C}{t^k (|x|^2 + t)^{\frac{n+l}{2}}}, \tag{21}$$

where $C = C(k, l, n)$.

Next it remains to analyze the estimate of K_{ij}, which automatically implies the estimate of T_{ij}. Let us start with a simple observation on T_{ij}.

Lemma 2.3 *Let T_{ij} be the Green tensor for (6) where $i, j = 1, 2, ..., n$ and let $(x, t) \in \mathbb{R}_+^n \times I$. Then for every nonnegative integers $k, l \geq 0$, T_{ij} satisfies*

$$D_t^k D_x^l T_{ij}(x, t) = t^{-\frac{n+l}{2} - k} D_t^k D_x^l T_{ij}\left(\frac{x}{\sqrt{t}}, 1\right). \tag{22}$$

Proof. We note that, first of all, it is easy to prove that

$$D_t^k D_x^l G_{ij}(\mu x, \mu^2 t) = \mu^{-n-1-2k-l} D_t^k D_x^l G_{ij}(x, t), \tag{23}$$

where G_{ij} is the Green tensor in (16). In addition, we easily see that

$$D_t^k D_x^l E_{ij}(\mu x, \mu^2 t) = \mu^{-n-2k-l} D_t^k D_x^l E_{ij}(x, t), \tag{24}$$

where E_{ij} is the Green tensor in (9). Since $T_{ij} = E_{ij} + K_{ij}$, it remains to check that $D_t^k D_x^l K_{ij}$ satisfies such scaling property. Using the identities (23) and (24), and substituting $y' = \sqrt{t}z', \mu = t\tau$, we have

$$D_t^k D_x^l K_{ij}(\frac{x}{\sqrt{t}}, 1) = D_t^k D_x^l [G_{i\beta} * (E_{\beta j}|_{x_n=0})](\frac{x}{\sqrt{t}}, 1)$$

$$= \int_0^1 \int_{\mathbb{R}^{n-1}} D_t^k D_x^l G_{i\beta}(\frac{x}{\sqrt{t}} - z', 1-\tau) E_{\beta j}(z', \tau)\, \mathrm{d}z'\, \mathrm{d}\tau$$

$$= \int_0^t \int_{\mathbb{R}^{n-1}} D_t^k D_x^l G_{i\beta}(\frac{x}{\sqrt{t}} - \frac{y'}{\sqrt{t}}, \frac{t-\mu}{t}) E_{\beta j}(\frac{y'}{\sqrt{t}}, \frac{\mu}{t}) t^{-\frac{n+1}{2}}\, \mathrm{d}y'\, \mathrm{d}\mu$$

$$= t^{\frac{n+l}{2}+k} \int_0^t \int_{\mathbb{R}^{n-1}} D_t^k D_x^l G_{i\beta}(x-y', t-\mu) E_{\beta j}(y', \mu)\, \mathrm{d}y'\, \mathrm{d}\mu$$

$$= t^{\frac{n+l}{2}+k} D_t^k D_x^l \int_0^t \int_{\mathbb{R}^{n-1}} G_{i\beta}(x-y', t-\mu) E_{\beta j}(y', \mu)\, \mathrm{d}y'\, \mathrm{d}\mu$$

$$= t^{\frac{n+l}{2}+k} D_t^k D_x^l [G_{i\beta} * (E_{\beta j}|_{x_n=0})](x,t) = t^{\frac{n+l}{2}+k} D_t^k D_x^l K_{ij}(x,t),$$

where we used the fact that $D_t^p D_x^q G_{i\beta}(x-y', 0) = 0$ for all integers $p, q \geq 0$, because $|x-y'| \neq 0$. This completes the proof. $\square$

Thanks to the observations above, it suffices to estimate $T_{ij}(x,t)$ when $t=1$. Now we are ready to prove the estimate of green tensor T_{ij}.

Theorem 2.4 *Let T_{ij} be the Green tensor for (6) where $i, j = 1, 2, ..., n$ and let $(x,t) \in \mathbb{R}^n_+ \times I$. Then for every nonnegative integer $k, l, m \geq 0, T_{ij}$ satisfies the following estimate:*

$$|D_t^k D_{x'}^l D_{x_n}^m T_{ij}(x,t)| \leq \frac{C}{t^{k+\frac{l+m+\alpha}{2}} |x|^{n-1} x_n^{1-\alpha}}, \tag{25}$$

where $C = C(k, l, m, \alpha, n)$ and α is any number with $0 < \alpha < 1$.

Proof. We note first that it suffices to prove (25) for cases of $l, m \geq 0, k = 0$, and $t = 1$ because of Lemma 2.3. Moreover, in this proof, we only consider the case $0 < x_n < 1$ because Green tensor T_{ij} is more singular when x_n is close to zero. For the case that $x_n \geq 1$, the details are omitted because similar procedure leads to the same estimate as the case $0 < x_n < 1$. In addition, we will prove (25) only for the case $l = 0$ and $m = 1$, that is

$$|\frac{\partial}{\partial x_n} T_{ij}(x, 1)| \leq \frac{C}{|x|^{n-1} x_n^{1-\alpha}}. \tag{26}$$

Indeed, the tangential derivatives (for example $l > 0, m = 0$) are easily estimated and the case $m \geq 2$, can be reduced to the case $l > 0$ and $m = 1$ by using the integration by parts successively. To be precise, we recall first the representation of G_{ij} already mentioned in (17).

$$G_{ij}(x,t) = -2\frac{\partial\Gamma}{\partial x_n}\delta_{ij} - 2d_n\frac{\partial\mathcal{C}_i}{\partial x_j} - d_n\frac{\partial}{\partial x_i}\frac{1}{|x|^{n-2}}\delta_{jn}\delta(t).$$

Note that the first term of G_{ij} satisfies heat equation and the last one holds the Laplace equation. In addition, for the second term, we can use the relation (15). To sum up, in this proof, we consider only the case that $k = l = 0, m = 1$, and $t = 1$, which is sufficient for proving (25) by arguments above.

We first denote disjoint sets A and B, for simplicity, in $\mathbb{R}^{n-1}$ defined as follows: For given $x \in \mathbb{R}^n_+$,

$$\begin{aligned} A &\equiv \{z' \in \mathbb{R}^{n-1} : |z'| \le \frac{|x|}{2} \text{ or } |z'| \ge 2|x|\}, \\ B &\equiv \{z' \in \mathbb{R}^{n-1} : \frac{|x|}{2} < |z'| < 2|x|\}. \end{aligned}$$

It is easy to see that, on each separated set A or B, the following relations hold:

$$\frac{C}{|x-z'|^2+1} \le \frac{C}{x^2+1} \ \text{ if } z' \in A, \qquad \frac{1}{|z'|} \sim \frac{1}{|x|} \ \text{ if } z' \in B, \tag{27}$$

where $f \sim g$ means that they are comparable, i.e. $C_1|f| \le |g| \le C_2|f|$ where C_1 and C_2 are absolute constants. According to (20) and Lemma 2.2, it suffices to estimate $D_{x_n}K_{ij}(x,t)$. Since the estimate (18) holds for $j \neq n$, it is required to analyze G_{ij} separately depending on $j = n$ or $j \neq n$. Reminding the expression (19), we have

$$\begin{aligned} D_{x_n}K_{ij}(x,1) &= D_{x_n}[G_{i\beta} * (E_{\beta j}|_{x_n=0})](x,1) \\ &= \sum_{\beta=1}^{n-1}\int_0^1\int_{\mathbb{R}^{n-1}} D_{x_n}G_{i\beta}(x-z',1-\tau)E_{\beta j}(z',\tau)\,\mathrm{d}z'\,\mathrm{d}\tau \\ &+ \int_0^1\int_{\mathbb{R}^{n-1}} D_{x_n}G_{in}(x-z',1-\tau)E_{nj}(z',\tau)\,\mathrm{d}z'\,\mathrm{d}\tau \equiv \mathcal{F}_1(x) + \mathcal{F}_2(x). \end{aligned} \tag{28}$$

Let us first consider the first term $\mathcal{F}_1(x)$, which is also split into two parts depending on the range of time interval.

$$|\mathcal{F}_1(x)| \le \sum_{\beta=1}^{n-1}\Big|\int_0^{\frac12}\int_{\mathbb{R}^{n-1}}\Big| + \sum_{\beta=1}^{n-1}\Big|\int_{\frac12}^1\int_{\mathbb{R}^{n-1}}\Big| \equiv \mathcal{F}_{11}(x) + \mathcal{F}_{12}(x). \tag{29}$$

For $\mathcal{F}_{11}(x)$, using the estimate (18) and relations (27), we obtain

$$\mathcal{F}_{11}(x) \le \int_0^{\frac12}\int_{\mathbb{R}^{n-1}} \frac{C\,\mathrm{d}z'\,\mathrm{d}\tau}{((x-z')^2+1)^{\frac{n}{2}}(x_n^2+1)^{\frac12}(z'^2+\tau)^{\frac{n}{2}}} = \int_0^{\frac12}\Big(\int_A + \int_B\Big)$$

$$\leq \frac{C}{(x^2+1)^{\frac{n}{2}}}\int_0^{\frac{1}{2}}\int_A \frac{\mathrm{d}z'\,\mathrm{d}\tau}{(z'^2+\tau)^{\frac{n}{2}}} + \frac{C}{|x|^n}\int_0^{\frac{1}{2}}\int_B \frac{\mathrm{d}z'\,\mathrm{d}\tau}{((x-z')^2+1)^{\frac{n}{2}}}$$

$$\leq \frac{C}{(x^2+1)^{\frac{n}{2}}} + \frac{C}{|x|^n} \leq \frac{C}{|x|^n}.$$

Before considering $\mathcal{F}_{12}(x)$ in (29), we recall the representation of $G_{i\beta}$ where $\beta \neq n$ (see (17)):

$$G_{i\beta}(x,t) = -2\frac{\partial \Gamma}{\partial x_n}\delta_{i\beta} - d_n\frac{\partial \mathcal{C}_i}{\partial x_\beta} \qquad i = 1, 2, ..., n, \quad \beta = 1, ..., n-1.$$

Changing the variable $1-\tau$ by τ, we have

$$\mathcal{F}_{12}(x) \leq C|\int_0^{\frac{1}{2}}\int_{\mathbb{R}^{n-1}} D_{x_n}\frac{\partial \Gamma}{\partial x_n}\delta_{i\beta}(x-z',\tau)E_{\beta j}(z',1-\tau)\,\mathrm{d}z'\,\mathrm{d}\tau|$$

$$+C|\int_0^{\frac{1}{2}}\int_{\mathbb{R}^{n-1}} D_{x_n}\frac{\partial \mathcal{C}_i}{\partial x_\beta}(x-z',\tau)E_{\beta j}(z',1-\tau)\,\mathrm{d}z'\,\mathrm{d}\tau| \equiv \mathcal{F}_{121}(x) + \mathcal{F}_{122}(x). \quad (30)$$

Let us first consider $\mathcal{F}_{121}(x)$. Note first that $D_{x_n}\frac{\partial \Gamma}{\partial x_n} = (\partial_\tau - \sum_{j=1}^{n-1}\partial_{x_j x_j})\Gamma$ because Γ is the fundamental solution of the heat equation. In addition, since $\Gamma(x-z',\tau)$ is zero at $\tau = 0$ and for the spatial variable the decay rate of integrand is $C|z'|^{-(2n+2)}$ at infinity, using the integration by parts, we get

$$\mathcal{F}_{121}(x) \leq C|\int_0^{\frac{1}{2}}\int_{\mathbb{R}^{n-1}} \Gamma(x-z',\tau)(\partial_\tau - \sum_{j=1}^{n-1}\partial_{z_j z_j})E_{\beta j}(z',1-\tau)\,\mathrm{d}z'\,\mathrm{d}\tau|$$

$$+C|\int_{\mathbb{R}^{n-1}} \Gamma(x-z',\frac{1}{2})E_{\beta i}(z',\frac{1}{2})\,\mathrm{d}z'|$$

Using $|(\partial_\tau - \sum_{j=1}^{n-1}\partial_{z_j z_j})E_{\beta j}(z',1-\tau)| \leq C(z'^2+1)^{-\frac{n+2}{2}}$ for $0 < \tau < \frac{1}{2}$, we obtain

$$\mathcal{F}_{121}(x) \leq \int_0^{\frac{1}{2}}\int_{\mathbb{R}^{n-1}} \frac{C}{((x-z')^2+\tau)^{\frac{n}{2}}(z'^2+1)^{\frac{n+2}{2}}}\,\mathrm{d}z'\,\mathrm{d}\tau$$

$$+\int_{\mathbb{R}^{n-1}} \frac{C}{((x-z')^2+1)^{\frac{n}{2}}(z'^2+1)^{\frac{n}{2}}}\,\mathrm{d}z' \equiv \mathcal{F}_{1211}(x) + \mathcal{F}_{1212}(x). \quad (31)$$

The second term $\mathcal{F}_{1212}(x)$ is easily estimated by using (18) and (21). Indeed,

$$\mathcal{F}_{1212}(x) \leq (\int_A + \int_B) \leq \frac{C}{(x^2+1)^{\frac{n}{2}}}(\int_A \frac{\mathrm{d}z'}{(z'^2+1)^{\frac{n}{2}}}$$

$$+\int_B \frac{\mathrm{d}z'}{((x-z')^2+1)^{\frac{n}{2}}}) \leq \frac{C}{(x^2+1)^{\frac{n}{2}}}.$$

Similarly, using the relations (27), $\mathcal{F}_{1211}$ in (31) is estimated as follows.

$$\mathcal{F}_{1211}(x) \le \int_0^{\frac{1}{2}} \int_A \mathrm{d}z'\,\mathrm{d}\tau + \int_0^{\frac{1}{2}} \int_B \mathrm{d}z'\,\mathrm{d}\tau$$

$$\le \frac{C}{|x|^n} \int_0^{\frac{1}{2}} \int_A \frac{\mathrm{d}z'\,\mathrm{d}\tau}{\tau^{\frac{\alpha}{2}}(z'^2+1)^{\frac{n+2}{2}}} + \frac{C}{(x^2+1)^{\frac{n+2}{2}}} \int_0^{\frac{1}{2}} \int_B \frac{\mathrm{d}z'\,\mathrm{d}\tau}{((x-z')^2+\tau)^{\frac{n}{2}}}$$

$$\le \frac{C}{|x|^n} + \frac{C}{(x^2+1)^{\frac{n+2}{2}}} \le \frac{C}{|x|^n}.$$

Before estimating $\mathcal{F}_{122}(x)$ in (30), we note that, on B, the following relation holds

$$((x-z')^2+\tau)^{\mu} \le C_{\mu}(z'^2+1)^{\mu} \quad \text{for } \mu > 0 \text{ on } B, \tag{32}$$

for any $\mu > 0$ where $0 < \tau < 1/2$ and $x_n < 1$ are used. Let $\alpha \in (0,1)$. According to the estimate (14) of $\mathcal{C}_i$ and the relation (32) for a sufficiently small $\mu > 0$ (we can take any μ satisfying $0 < \mu < 1-\alpha$), we have

$$\mathcal{F}_{122}(x) \le \int_0^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} \frac{C\,\mathrm{d}z'\,\mathrm{d}\tau}{\tau^{\frac{1}{2}}((x-z')^2+\tau)^{\frac{n-1}{2}}(x_n^2+\tau)^{\frac{1}{2}}(z'^2+1)^{\frac{n+1}{2}}}$$

$$= \int_0^{\frac{1}{2}} (\int_A + \int_B) \le \frac{C}{|x|^{n-1}x_n^{1-\alpha}} \int_0^{\frac{1}{2}} \int_A \frac{\mathrm{d}z'\,\mathrm{d}\tau}{\tau^{\frac{1+\alpha}{2}}(z'^2+1)^{\frac{n+1}{2}}}$$

$$+ \frac{C}{x_n^{1-\alpha}(|x|^2+1)^{\frac{n}{2}}} \int_0^{\frac{1}{2}} \int_B \frac{\mathrm{d}z'\,\mathrm{d}\tau}{\tau^{\frac{1+\alpha}{2}}((x-z')^2+\tau)^{\frac{n-1+\mu}{2}}}$$

$$\le \frac{C}{|x|^{n-1}x_n^{1-\alpha}} + \frac{C}{x_n^{1-\alpha}(|x|^2+1)^{\frac{n}{2}}} \le \frac{C}{|x|^{n-1}x_n^{1-\alpha}},$$

where we used $C/(|x|^2+1)^{\frac{n}{2}} \le C/|x|^{n-1}$. To sum up, we obtain

$$|\mathcal{F}_{12}(x)| \le |\mathcal{F}_{121}(x)| + |\mathcal{F}_{122}(x)| \le \frac{C}{|x|^{n-1}x_n^{1-\alpha}} \quad \text{for } 0 < \alpha < 1,$$

where we used $|x|^{n-1}x_n^{1-\alpha} \le |x|^{n-\alpha}$. Adding all the above estimates, we get

$$|\mathcal{F}_1(x)| \le |\mathcal{F}_{11}(x)| + |\mathcal{F}_{12}(x)| \le \frac{C}{|x|^{n-1}x_n^{1-\alpha}} \quad \text{for } 0 < \alpha < 1.$$

It remains to estimate $\mathcal{F}_2(x)$ in (28). Note that, due to the second equation in (16), the case $i = n$ is reduced to the case $k = 0$ and $m = 1$. To be precise, since $\partial_{x_n} G_{nn} = -\sum_{k=1}^{n-1} \partial_{x_k} G_{kn}$, we have

$$\int_0^1 \int_{\mathbb{R}^{n-1}} D_{x_n} G_{nn}(x-z', 1-\tau) E_{nj}(z',\tau)\,\mathrm{d}z'\,\mathrm{d}\tau$$

$$= \sum_{k=1}^{n-1} \int_0^1 \int_{\mathbb{R}^{n-1}} D_{z'_k} G_{kn}(x - z', 1 - \tau) E_{nj}(z', \tau)\, \mathrm{d}z'\, \mathrm{d}\tau.$$

After simple calculations, we have

$$\int_0^1 \int_{\mathbb{R}^{n-1}} |D_{x_n} G_{nn}(x - z', 1 - \tau) E_{nj}(z', \tau)|\, \mathrm{d}z'\, \mathrm{d}\tau \leq \frac{C}{|x|^{n-1} x_n^{1-\alpha}}$$

where α is any number with $0 < \alpha < 1$. Since computation is similar to the previous case, we omit the details. Next, it remains to estimate $\mathcal{F}_2(x)$ for the case of $i \neq n$. Recalling the representation (17) of G_{in}, we define, for convenience, U_{in} and V_{in} as follows:

$$U_{in}(x, t) \equiv -d_n \frac{\partial \mathcal{C}_i}{\partial x_n}, \qquad V_{in}(x, t) \equiv -d_n \frac{\partial}{\partial x_i} \frac{1}{|x|^{n-2}} \delta(t).$$

Using notations above, we can write $\mathcal{F}_2(x)$ in (28) as follows:

$$\mathcal{F}_2(x) = \int_0^1 \int_{\mathbb{R}^{n-1}} D_{x_n} U_{in}(x - z', 1 - \tau) E_{nj}(z', \tau)\, \mathrm{d}z'\, \mathrm{d}\tau$$

$$+ \int_0^1 \int_{\mathbb{R}^{n-1}} D_{x_n} V_{in}(x - z', 1 - \tau) E_{nj}(z', \tau)\, \mathrm{d}z'\, \mathrm{d}\tau \equiv \tilde{U}(x) + \tilde{V}(x). \tag{33}$$

We first estimate $\tilde{U}(x)$ by separating time interval into two parts as before.

$$|\tilde{U}(x)| \leq |\int_0^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} \mathrm{d}z'\, \mathrm{d}\tau| + |\int_{\frac{1}{2}}^1 \int_{\mathbb{R}^{n-1}} \mathrm{d}z'\, \mathrm{d}\tau| \equiv \tilde{U_1}(x) + \tilde{U_2}(x). \tag{34}$$

With the aid of (14) and (27), $\tilde{U_1}(x)$ can be estimated as follows.

$$\tilde{U_1}(x) \leq \int_0^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} \frac{C\, \mathrm{d}z'\, \mathrm{d}\tau}{((x - z')^2 + 1)^{\frac{n-1}{2}} (x_n^2 + 1)(z'^2 + \tau)^{\frac{n}{2}}}$$

$$= \int_0^{\frac{1}{2}} (\int_A + \int_B) \leq \frac{C}{(x^2 + 1)^{\frac{n-1}{2}} (x_n^2 + 1)} \int_0^{\frac{1}{2}} \int_A \frac{\mathrm{d}z'\, \mathrm{d}\tau}{(z'^2 + \tau)^{\frac{n}{2}}}$$

$$+ \frac{C}{(x_n^2 + 1)|x|^n} \int_0^{\frac{1}{2}} \int_B \frac{\mathrm{d}z'\, \mathrm{d}\tau}{((x - z')^2 + 1)^{\frac{n-2}{2}}}$$

$$\leq \frac{C}{(x^2 + 1)^{\frac{n-1}{2}} (x_n^2 + 1)} + \frac{C}{(x_n^2 + 1)|x|^{n-1}} \leq \frac{C}{(x_n^2 + 1)|x|^{n-1}}.$$

Next we estimate $\tilde{U_2}(x)$ in (34). Note that, using (15), we have the following relation:

$$\frac{\partial^2 \mathcal{C}_i}{\partial^2 x_n} = -\sum_{k=1}^{n-1} \frac{\partial}{\partial x_i} (\frac{\partial \mathcal{C}_k}{\partial x_k}) - \frac{\partial^2 \Gamma}{\partial x_i \partial x_n} - \sum_{k=1}^{n-1} \frac{\partial^3 B}{\partial x_i \partial^2 x_k} + \frac{\partial^2 B}{\partial x_i \partial \tau}, \tag{35}$$

where we used $i \neq n$. Changing the variable τ by $1-\tau$ and using (35), we get

$$\tilde{U}_2(x) \le C|\int_0^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} \sum_{k=1}^{n-1} \frac{\partial}{\partial x_i}((\frac{\partial \mathcal{C}_k}{\partial x_k}) + \frac{\partial^3 B}{\partial x_i \partial^2 x_k}))(x-z',\tau)E_{nj}(z',1-\tau)|$$

$$+C|\int_0^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} \frac{\partial^2 \Gamma}{\partial x_i \partial x_n}(x-z',\tau)E_{nj}(z',1-\tau)\,\mathrm{d}z'\,\mathrm{d}\tau|$$

$$+C|\int_0^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} \frac{\partial^2 B}{\partial x_i \partial \tau}(x-z',\tau)E_{nj}(z',1-\tau)\,\mathrm{d}z'\,\mathrm{d}\tau|.$$

We note that the first and second integrals except for the last one can be estimated as before, because the former has only tangential derivatives and the latter is the second derivative of the fundamental solution of the heat equation, and thus we can use the same estimate as that in Lemma 2.2. The details are omitted for those cases because the most difficult case is the last one. Disregarding the three terms and only considering the last term $\frac{\partial^2 B}{\partial x_i \partial \tau}$ in (35), we have

$$\tilde{U}_2(x) \le C|\int_0^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} \frac{\partial B}{\partial x_i}(x-z',\tau)\frac{\partial E_{nj}}{\partial \tau}(z',1-\tau)\,\mathrm{d}z'\,\mathrm{d}\tau|$$

$$+C|\int_{\mathbb{R}^{n-1}} \frac{\partial B}{\partial x_i}(x-z',\frac{1}{2})E_{nj}(z',\frac{1}{2})\,\mathrm{d}z'| \equiv \tilde{U}_{21} + \tilde{U}_{22}, \tag{36}$$

where we used the integration by parts and the facts that $\frac{\partial B}{\partial x_i}(x-z',\tau)$ is zero at $\tau = 0$ and for the spatial variable the decay rate of integrand is $C|z'|^{-(2n-1)}$ at infinity. Let us first consider the second term $\tilde{U}_{22}$ in (36). Using the estimate (13) and (21), and properties of (27), we have

$$\tilde{U}_{22} \le \int_{\mathbb{R}^{n-1}} \frac{C\,\mathrm{d}z'}{((x-z')^2+1)^{\frac{n-1}{2}}(x_n^2+1)^{\frac{1}{2}}(z'^2+1)^{\frac{n}{2}}}$$

$$= (\int_A + \int_B) \le \frac{C}{(x^2+1)^{\frac{n-1}{2}}(x_n^2+1)^{\frac{1}{2}}} \int_A \frac{\mathrm{d}z'}{(z'^2+1)^{\frac{n}{2}}}$$

$$+\frac{C}{(x^2+1)^{\frac{n}{2}}(x_n^2+1)^{\frac{1}{2}}} \int_B \frac{\mathrm{d}z'}{((x-z')^2+1)^{\frac{n-1}{2}}}$$

$$\le \frac{C}{(|x|^2+1)^{\frac{n-1}{2}}(x_n^2+1)^{\frac{1}{2}}} + \frac{C|x|}{(|x|^2+1)^{\frac{n}{2}}(x_n^2+1)^{\frac{1}{2}}} \le \frac{C}{(|x|^2+1)^{\frac{n-1}{2}}(x_n^2+1)^{\frac{1}{2}}}.$$

Thus, it remains to estimate the first term $\tilde{U}_{21}$ in (36). According to estimates (13), (21), and (27), we have

$$\tilde{U}_{21} \le \int_0^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} \frac{C\,\mathrm{d}z'\,\mathrm{d}\tau}{(x_n^2+\tau)^{\frac{1}{2}}((x-z')^2+\tau)^{\frac{n-1}{2}}(z'^2+1)^{\frac{n+2}{2}}}$$

$$= \int_0^{\frac{1}{2}} (\int_A + \int_B) \le \frac{C}{|x|^{n-1} x_n^{1-\alpha}} \int_0^{\frac{1}{2}} \int_A \frac{\mathrm{d}z'\,\mathrm{d}\tau}{\tau^{\frac{\alpha}{2}} (z'^2+1)^{\frac{n+2}{2}}}$$

$$+ \frac{C}{(x^2+1)^{\frac{n+2}{2}}} \int_0^{\frac{1}{2}} \int_B \frac{\mathrm{d}z'\,\mathrm{d}\tau}{\tau^{\frac{1+\alpha}{2}} ((x-z')^2+\tau)^{\frac{n-1-\alpha}{2}}}$$

$$\le \frac{C}{|x|^{n-1} x_n^{1-\alpha}} + \frac{C|x|^\alpha}{(x^2+1)^{\frac{n+2}{2}}} \le \frac{C}{|x|^{n-1} x_n^{1-\alpha}},$$

where $0 < \alpha < 1$ and we used $|x|^\alpha \le C(x^2+1)$. To summarize above estimates, we have

$$\tilde{U}(x) \le \tilde{U}_1(x) + \tilde{U}_2(x) \le \frac{C}{|x|^{n-1} x_n^{1-\alpha}},$$

where α is any number satisfying $0 < \alpha < 1$. Next we will estimate $\tilde{V}(x)$ in (33). Since $i \ne n$, we get

$$|\tilde{V}(x)| = |\int_{\mathbb{R}^{n-1}} D_{x_n} \frac{\partial}{\partial x_i} \frac{1}{|x-z'|^{n-2}} E_{nj}(z',1)\,\mathrm{d}z'|$$

$$\le \int_A \frac{C\,\mathrm{d}z'}{|x-z'|^{n-1}} |\frac{\partial}{\partial x_i} E_{nj}(z',1)| + \int_B \frac{C\,\mathrm{d}z'}{|x-z'|^{n-1}} |\frac{\partial}{\partial x_i} E_{nj}(z',1)|$$

$$\equiv \tilde{V}_1(x) + \tilde{V}_2(x). \tag{37}$$

Considering first $\tilde{V}_1$, we have

$$\tilde{V}_1(x) \le \frac{C}{|x|^{n-1}} \int_A \frac{1}{(|z'^2+1)^{\frac{n+1}{2}}} \le \frac{C}{x_n^{1-\alpha} |x|^{n-1}},$$

where α is any number with $0 < \alpha < 1$ and $x_n \le 1$ is used in the last inequality. Finally, we estimate $\tilde{V}_2$ in (37).

$$\tilde{V}_2(x) \le \int_B \frac{C\,\mathrm{d}z'}{|x-z'|^{n-1} (z'^2+1)^{\frac{1+n}{2}}} \le \int_B \frac{C\,\mathrm{d}z'}{x_n^{1-\alpha} |x-z'|^{n-2+\alpha} (z'^2+1)^{\frac{n+1}{2}}}$$

$$\le \frac{C|x|^{1-\alpha}}{x_n^{1-\alpha} (x^2+1)^{\frac{n+1}{2}}} \le \frac{C}{x_n^{1-\alpha} |x|^{n-1}},$$

where α is any number with $0 < \alpha < 1$. Therefore, to sum up, we have

$$\mathcal{F}_2(x) \le |\tilde{U}(x)| + |\tilde{V}(x)| \le \frac{C}{x_n^{1-\alpha} |x|^{n-1}}.$$

Adding up above all estimates, we obtain

$$|D_{x_n} K_{ij}(x,1)| \le \frac{C}{x_n^{1-\alpha} |x|^{n-1}} \quad \text{for } \ 0 < \alpha < 1.$$

With the aid of the Lemma 2.3, we have

$$|D_{x_n} K_{ij}(x,t)| \le \frac{C}{t^{\frac{1+\alpha}{2}} |x|^{n-1} x_n^{1-\alpha}} \quad \text{for } \ 0 < \alpha < 1.$$

Combining the estimate of E_{ij} in Lemma 2.3 and K_{ij} above, we obtain

$$|D_{x_n} T_{ij}| \le |D_{x_n} E_{ij}| + |D_{x_n} K_{ij}| \le \frac{C}{t^{\frac{1+\alpha}{2}} |x|^{n-1} x_n^{1-\alpha}},$$

where we used $t^{\frac{1+\alpha}{2}} |x|^{n-1} x_n^{1-\alpha} \le C(t+|x|^2)^{\frac{n+1}{2}}$. This completes the proof. $\square$

Remark 2.5 *It is interesting to compare the estimates of T_{ij} and E_{ij}. With the aid of Lemma 2.2 and the Young's inequality, E_{ij} can be estimated as follows: For any α with $0 < \alpha < 1$,*

$$|D_t^k D_x^l E_{ij}(x,t)| \le \frac{C}{t^k (|x|^2 + t)^{\frac{n+l}{2}}} \le \frac{C}{t^{k+\frac{l+\alpha}{2}} |x|^{n-\alpha}}. \tag{38}$$

Although (25) is weaker than (38), it will be sufficient for our estimates of the Riesz potentials, which will be used next section for investigating the boundary regularity of the Navier-Stokes equations.

Remark 2.6 *In [21] Ukai proved the following $L^p - L^q$ estimates of a solution u in (4) for $f = 0$: For any p, q with $1 < q < p < \infty$,*

$$\begin{cases} ||u(t)||_{L^p(\mathbb{R}^n_+)} \le C t^{-\alpha} ||g||_{L^q(\mathbb{R}^n_+)} \\ ||\nabla u(t)||_{L^p(\mathbb{R}^n_+)} \le C t^{-\alpha - \frac{1}{2}} ||g||_{L^q(\mathbb{R}^n_+)} \end{cases} \tag{39}$$

for any $t > 0$ with $\alpha = (n/2)(1/q - 1/p)$ (see theorem 3.1 in [21]). Our point-wise estimate of the Green tensor also gives the simple proof of the estimate (39).

3 Navier-Stokes equations

In this section, using the estimates of the Green tensor T_{ij} proved in Theorem 2.4 for the Stokes system (6), we study the boundary regularity of the Navier-Stokes equations (1) with initial and boundary conditions (2) near boundary in the half-space $\mathbb{R}^n_+, n \ge 3$. As usual, a weak solution u means u is in the class

$$u \in L^\infty(I; L^2(\mathbb{R}^n_+)) \cap L^2(I; W_0^{1,2}(\mathbb{R}^n_+))$$

and u solves the Navier-Stokes equations (1) and (2) in a distribution sense, i.e.

$$\int_0^T \int_{\mathbb{R}^n_+} (u\xi_t - \nabla u \nabla \xi - (u\nabla)u\xi + f\xi)\, \mathrm{d}x\, \mathrm{d}t = 0 \tag{40}$$

for all $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^n_+ \times I : \mathbb{R}^n)$ with div $\xi = 0$ and

$$\int_0^T \int_{\mathbb{R}^n_+} u \nabla \phi \, \mathrm{d}x = 0 \text{ for all } \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n_+).$$

In addition, u satisfies the global energy inequality

$$\sup_{t \in I} \int_{\mathbb{R}^n_+} |u(\cdot,t)|^2 \, \mathrm{d}x + \int_{\mathbb{R}^n_+ \times I} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \le C\Big(\int_{\mathbb{R}^n_+} |u_0|^2 \, \mathrm{d}x + \int_{\mathbb{R}^n_+ \times I} |f|^2 \, \mathrm{d}x \, \mathrm{d}t\Big). \quad (41)$$

Remark 3.1 *The pressure p does not appear explicitly in the above weak formulation, but p can be chosen to belong to the class (see Theorem 3.1 in [8, page 84])*

$$p \in L^r(I; L^s(\mathbb{R}^n_+)) \qquad \textit{if } \frac{2}{r} + \frac{n}{s} = n \textit{ and } 1 < r < 2.$$

An equivalent approach would be to bring p into the definition of weak solutions and replace (40) by

$$\int_0^T \int_{\mathbb{R}^n_+} (u\xi_t - \nabla u \nabla \xi - (u\nabla)u\xi + p\nabla \cdot \xi + f\xi) \, \mathrm{d}x \, \mathrm{d}t = 0$$

for all $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^n_+ \times I : \mathbb{R}^n)$. □

Here we state our Main Theorem.

Main Theorem Let u be a weak solution of the Navier-Stokes equations (1) and (2). Let $z_0 = (x_0, t_0) \in \partial\mathbb{R}^n_+ \times (0,T)$. Suppose that there exists $\gamma > 0$ such that u satisfies

$$u \in L^{p,q}(Q^+_{z_0,\gamma}) \text{ for some } p, q \text{ with } \frac{2}{p} + \frac{n}{q} = 1 \text{ and } q > n.$$

Then u is Hölder continuous in $\overline{Q^+_{z_0,\frac{\gamma}{2}}}$ with the exponent $\alpha \in (0,1)$, i.e.

$$|u(x,t) - u(x',t')| \le C(|x - x'| + |t - t'|^{\frac{1}{2}})^\alpha$$

for all $(x,t), (x',t') \in \overline{Q^+_{z_0,\frac{\gamma}{2}}}$ where C depends on n, α and given data f, u_0, but not on u.

Remark 3.2 *Unlike in the interior case, it is not clear that Hölder continuity implies higher regularity in space variables at a boundary point.* □

To complete the proof of Main Theorem, we first prove next Lemma.

Lemma 3.3 *Let u be a weak solution of the Navier-Stokes equations (1) and (2). Let $z_0 = (x_0, t_0) \in \partial\mathbb{R}^n_+ \times (0, T)$. Suppose that there exists $\gamma > 0$ such that u satisfies*

$$u \in L^{p,q}(Q^+_{z_0,\gamma}) \ \text{ for some } \ p, q \ \text{ with } \ \frac{2}{p} + \frac{n}{q} = 1 \text{ and } q > n.$$

Then $u \in L^{r,s}(Q^+_{z_0,\frac{\gamma}{2}})$ for any r, s with $p < r < \infty$ and $q < s < \infty$.

Proof. Using the Green tensor T_{ij} in (6), we can have representation of u as follows:

$$u_i(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \partial_k T_{ij}(x,y,t,s) u_k(y,s) u_j(y,s) \,\mathrm{d}y \,\mathrm{d}s$$
$$+ \int_{\mathbb{R}^n_+} T_{ij}(x,y,t) u_{0j}(y) \,\mathrm{d}y \equiv \mathcal{H}_1(x,t) + \mathcal{H}_2(x,t), \tag{42}$$

where $i = 1, 2, ..., n$. We first consider the second term $\mathcal{H}_2$ and we will show that $\mathcal{H}_2$ is bounded in $Q^+_{z_0,\gamma/2}$. Note first that since u_0 is smooth and compactly supported in $\mathbb{R}^n_+$, there exists a sufficiently large $\rho > 0$ such that support of u_0 is contained in $B^+_\rho \subset \mathbb{R}^n_+$. For convenience we recall the estimate of T_{ij} (see Theorem 2.4).

$$|D^k_t D^l_{x'} D^m_{x_n} T_{ij}(x,t)| \le \frac{C}{t^{k+\frac{l+m+\alpha}{2}} |x|^{n-1} x_n^{1-\alpha}}$$

For any $z = (x,t) \in Q^+_{z_0,\gamma/2}$, using the estimate above, we obtain

$$|\mathcal{H}_2(x,t)| \le \int_{\mathbb{R}^n_+} |T_{ij}(x,y,t)||u_0(y)| \,\mathrm{d}y \le ||u_0||_{L^\infty(B^+_\rho)} \int_{B^+_\rho} |T_{ij}(x,y,t)| \,\mathrm{d}y$$
$$\le ||u_0||_{L^\infty(B^+_\rho)} \int_{B^+_\rho} \frac{C}{t^{\frac{\alpha}{2}} |x-y|^{n-1} |x_n - y_n|^{1-\alpha}} \,\mathrm{d}y$$
$$\le ||u_0||_{L^\infty(B^+_\rho)} \frac{C}{t^{\frac{\alpha}{2}}} \le ||u_0||_{L^\infty(B^+_\rho)} \frac{C}{(t_0 - \frac{\gamma^2}{4})^{\frac{\alpha}{2}}}$$

where we used $t_0 - (\gamma^2/4) < t$ because $(x,t) \in Q^+_{z_0,\gamma/2}$. It is obvious that $\mathcal{H}_2 \in L^{r,s}(Q^+_{z_0,\gamma/2})$ for every r, s with $p < r < \infty$ and $q < s < \infty$ because $\mathcal{H}_2$ is bounded in $Q^+_{z_0,\gamma/2}$. Next, we consider the first term $\mathcal{H}_1(x,t)$ in (42). To utilize the assumption $u \in L^{p,q}(Q^+_{z_0,\gamma})$, we split $\mathbb{R}^n_+$ into two parts $Q^+_{z_0,\gamma}$ and $(\mathbb{R}^n_+ \times I) \setminus Q^+_{z_0,\gamma}$, which are, for simplicity, denoted by Q and Q^c, respectively. Namely, we express $\mathcal{H}_1$ as follows:

$$\mathcal{H}_1(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \partial_k T_{ij}(x,y,t,s) u_k(y,s) u_j(y,s) \,\mathrm{d}y \,\mathrm{d}s$$
$$= \int_0^t \int_{\mathbb{R}^n_+} \partial_k T_{ij} u_k u_j \chi_Q + \int_0^t \int_{\mathbb{R}^n_+} \partial_k T_{ij} u_k u_j \chi_{Q^c} \equiv \mathcal{H}_{11}(x,t) + \mathcal{H}_{12}(x,t), \tag{43}$$

where χ is the characteristic function defined at the beginning of section 2. It is easy to see that the second term $\mathcal{H}_{12}$ in (43) is bounded in $Q^+_{z_0,\gamma/2}$. Indeed it is due to the fact that T_{ij} has no singularity in Q^c because $(x,t) \in Q^+_{z_0,\gamma/2}$ and integration is performed in Q^c. More precisely, $|x-y| \geq \gamma/2$ and $|t-s| \geq \gamma^2/4$ for any $(x,t) \in Q^+_{z_0,\gamma/2}$ and $(y,s) \in Q^c$. Thus, for any $z = (x,t) \in Q_{z_0,\gamma/2}$, we obtain

$$
\begin{aligned}
|\mathcal{H}_{12}(x,t)| &\leq \int\int_{Q^c} |\partial_k T_{ij}(x,y,t,s)||u_k(y,s)u_j(y,s)|\, \mathrm{d}y\, \mathrm{d}s \\
&\leq \int\int_{Q^c} \frac{C|u_k(y,s)u_j(y,s)|}{(t-s)^{\frac{1+\alpha}{2}}|x-y|^{n-1}|x_n-y_n|^{1-\alpha}}\, \mathrm{d}y\, \mathrm{d}s \\
&\leq \frac{C}{\gamma^n}\int\int_{Q^c} |u(y,s)|^2\, \mathrm{d}y\, \mathrm{d}s \leq \frac{C(t_0+\gamma^2)}{\gamma^n}\sup_{t\in I}\int_{\mathbb{R}^n_+} |u(\cdot,s)|^2\, \mathrm{d}y.
\end{aligned}
$$

Due to the global energy inequality (41), $\mathcal{H}_{12}$ is bounded by given data. To sum up, so far we have proved $\mathcal{H}_{12}(x,t)$ and $\mathcal{H}_2(x,t)$ are bounded in $Q^+_{z_0,\gamma/2}$ where

$$
u_i(x,t) = \mathcal{H}_{11}(x,t) + \mathcal{H}_{12}(x,t) + \mathcal{H}_2(x,t). \tag{44}
$$

For simplicity, we denote $\tilde{\mathcal{H}}(x,t) \equiv \mathcal{H}_{12}(x,t) + \mathcal{H}_2(x,t)$. To complete our assertion, we introduce an operator $A(u) : L^{r,s}(Q^+_{z_0,\gamma/2}) \to L^{r,s}(\mathbb{R}^n_+ \times I)$ as follows:

$$
(A(u)\phi)_i(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \partial_k T_{ij}(x,y,t,s)u_k(y,s)\chi_Q \phi_j(y,s)\, \mathrm{d}y\, \mathrm{d}s,
$$

where $\phi \in L^{r,s}(Q^+_{z_0,\gamma/2};\mathbb{R}^n_+)$ and r,s are numbers with $p < r < \infty$ and $q < s < \infty$. In the above integral, ϕ is understood to an extended one in $\mathbb{R}^n_+$ by assigning zero outside $Q^+_{z_0,\gamma/2}$. Our aim is to show that $A(u)\phi \in L^{r,s}(\mathbb{R}^n_+ \times I)$ and $A(u)$ is a bounded operator, and the following estimate holds:

$$
||A(u)\phi||_{L^{r,s}(\mathbb{R}^n_+\times I)} \leq C||u||_{L^{p,q}(Q^+_{z_0,\gamma})}||\phi||_{L^{r,s}(Q^+_{z_0,\frac{\gamma}{2}})}, \tag{45}
$$

which automatically implies

$$
||A(u)\phi||_{L^{r,s}(Q^+_{z_0,\frac{\gamma}{2}})} \leq C||u||_{L^{p,q}(Q^+_{z_0,\gamma})}||\phi||_{L^{r,s}(Q^+_{z_0,\frac{\gamma}{2}})}. \tag{46}
$$

For simplicity, we write, from now on, $u_j(y,s)\chi_{Q^+_{z_0,\gamma}}$ as $u_j(y,s)$ without any confusion. Indeed, using the estimate of $|\nabla T_{ij}(x,t)|$, we have

$$
\begin{aligned}
|A(u)\phi(x,t)| &\leq \int_0^t \int_{\mathbb{R}^n_+} |\nabla T_{ij}(x,y,t,s)| \cdot |u(y,s)||\phi(y,s)|\, \mathrm{d}y\, \mathrm{d}s \\
&\leq C\int_0^t \int_{\mathbb{R}^n_+} \frac{|u(y,s)||\phi(y,s)|}{|t-s|^{\frac{1+\alpha}{2}}|x-y|^{n-1}|x_n-y_n|^{1-\alpha}}\, \mathrm{d}y\, \mathrm{d}s.
\end{aligned}
$$

For an arbitrarily α with $0 < \alpha < 1$, we take $\beta = (n-1)\alpha/n$. With the aid of $|x' - y'|^{n-1-\beta}|x_n - y_n|^{\beta} \le C|x-y|^{n-1}$, the right side of the above inequality is estimated as follows:

$$\int_0^t \int_{\mathbb{R}^n_+} \frac{C|u(y,s)||\phi(y,s)|}{|t-s|^{\frac{1+\alpha}{2}}|x-y|^{n-1}|x_n-y_n|^{1-\alpha}}\,\mathrm{d}y\,\mathrm{d}s$$

$$\le \int_0^t \int_{\mathbb{R}^n_+} \frac{C|u(y,s)||\phi(y,s)|}{|t-s|^{\frac{1+\alpha}{2}}|x'-y'|^{(n-1)-\frac{(n-1)\alpha}{n}}|x_n-y_n|^{1-\frac{\alpha}{n}}}\,\mathrm{d}y\,\mathrm{d}s,$$

Since α is an arbitrary number between 0 and 1, we may specifically choose $\alpha = n/q$. Using the inequality of the Riesz potential (see e.g. Theorem 1 in [18, page 119-120]), we obtain

$$||A(u)\phi(\cdot,t)||_{L^s(\mathbb{R}^n_+)} \le C\int_0^t \frac{1}{|t-s|^{\frac{1+\frac{n}{q}}{2}}}||u(,s)\phi(,s)||_{L^{\kappa}(\mathbb{R}^n_+)}\,\mathrm{d}s$$

$$\le C\int_0^t \frac{||u(,s)||_{L^q(\mathbb{R}^n_+)}||\phi(,s)||_{L^s(\mathbb{R}^n_+)}}{|t-s|^{\frac{1+\frac{n}{q}}{2}}}\,\mathrm{d}s,$$

where $1/\kappa = 1/s + 1/q$. Applying the inequality of the Riesz potential again, we have

$$||A(u)\phi||_{L^{r,s}(\mathbb{R}^n_+\times I)} \le C||u||_{L^{p,q}(\mathbb{R}^n_+\times I)}||\phi||_{L^{r,s}(\mathbb{R}^n_+\times I)}.$$

The last inequality comes from $2/p + n/q = 1$, which is equivalent to $1/r = 1/p + 1/r - (1/2 - n/2q)$. This completes our assertion (45), and thus (46). Now we rewrite representation of u in (44) as follows:

$$(I - A(u))(u)(x,t) = \tilde{\mathcal{H}}(x,t) \text{ where } \tilde{\mathcal{H}}(x,t) \equiv \mathcal{H}_{12}(x,t) + \mathcal{H}_2(x,t).$$

The main point in this argument is that $I - A(u)$ is an invertible operator on $L^{r,s}(Q^+_{z_0,\gamma/2})$ provided that $||u||_{L^{p,q}(Q^+_{z_0,\gamma})}$ is sufficiently small because the operator norm of $A(u)$ between $L^{p,q}(Q^+_{z_0,\gamma})$ can be less than $\frac{1}{2}$ if $||u||_{L^{p,q}(Q^+_{z_0,\gamma})} < \frac{C}{2}$ where C is the constant in (45). Since $\tilde{\mathcal{H}}$ is bounded in $Q^+_{z_0,\gamma/2}$, which also implies it is in $L^{r,s}(Q^+_{z_0,\gamma/2})$ for all $r, s < \infty$, u must be also in the same space $L^{r,s}(Q^+_{z_0,\gamma/2})$ due to invertibility of $I - A(u)$. However, we still need the smallness of $||u||_{L^{p,q}(Q^+_{z_0,\gamma})}$. To complete the arguments, at the beginning we decompose $I_{t_0,\gamma/\sqrt{2}} = (t_0 - \gamma^2/2, t_0)$ as a union of m sectors, denoted by $I_j = (t_j - 2\delta, t_j), j = 1,2,...,m$ where $t_m = t_0, t_1 - 2\delta = t_0 - \gamma^2/2$ and $t_{j+1} - t_j = \delta$ such that $J_j \equiv (t_j - \delta, t_j) \subset I_j$ for $j = 1,2,...,m$. We note that $(t_0 - \gamma^2/4, t_0) \subset \cup_{j=1}^m J_j$. For given small $\epsilon > 0$, we can have $||u||_{L^{p,q}(B^+_\gamma \times I_j)} < \epsilon$ for all $j = 1,2,...,m$ if δ is taken sufficiently small. By repeating this procedure at each $B^+_\gamma \times I_j$, we obtain $u \in L^{r,s}(B^+_{\gamma/2} \times J_j)$, which leads to $u \in L^{r,s}(B^+_{\gamma/2} \times (\cup_{j=1}^m J_j))$. This completes the proof. □

The easy consequence of the above Lemma is the following.

Corollary 3.4 *Let u be a weak solution of the Navier-Stokes equations (1) and (2). Let $z_0 = (x_0, t_0) \in \partial\mathbb{R}^n_+ \times (0, T)$. Suppose that there exists $\gamma > 0$ such that u satisfies*

$$u \in L^{p,q}(Q^+_{z_0,\gamma}) \ \textit{ for some } \ p, q \ \textit{ with } \ \frac{2}{p} + \frac{n}{q} = 1 \textit{ and } q > n.$$

Then u is bounded in $Q^+_{z_0,\frac{\gamma}{2}}$, i.e. $u \in L^{\infty,\infty}(Q^+_{z_0,\frac{\gamma}{2}})$.

Proof. In the expression (44), we already verified that $\mathcal{H}_{12}, \mathcal{H}_2$ are bounded in $Q^+_{z_0,\gamma/2}$ in the Lemma above. Thus, it remains to show that $\mathcal{H}_{11}$ in (44) is bounded. As mentioned earlier, taking $\alpha = n/q$, for $(x,t) \in Q^+_{z_0,\gamma/2}$, we easily obtain

$$\mathcal{H}_{11}(x,t) = |A(u)u(x,t)| \leq \int_0^t \int_{\mathbb{R}^n_+} |\nabla T(x,y,t,s)| \cdot |u(y,s)\chi_Q|^2 \, \mathrm{d}y \, \mathrm{d}s$$

$$\leq \int_0^t \int_{\mathbb{R}^n_+} \frac{C|u(y,s)\chi_Q|^2 \, \mathrm{d}y \, \mathrm{d}s}{|t-s|^{\frac{1+\alpha}{2}} |x'-y'|^{n-1-\frac{(n-1)\alpha}{n}} |x_n - y_n|^{1-\frac{\alpha}{n}}} \leq C||u||^2_{L^{2r,2s}(Q)},$$

where $Q = Q^+_{z_0,\gamma}$ and r, s are sufficiently large numbers (in fact, it is enough to choose r, s satisfying $r > p$ and $s > q$). This completes the proof. $\square$

Now we are ready to prove the main theorem.

The Proof of Main Theorem According to Lemma 3.3 and corollary 3.4, we can prove that $u \in L^{\infty,\infty}(Q^+_{z_0,3\gamma/4})$ (this can be done by taking a characteristic function $\chi_{Q^+_{z_0,3\gamma/4}}$ instead of $\chi_{Q^+_{z_0,\gamma}}$ in (43)). Recalling the expression (44), we will show that $\mathcal{H}_{12}$ and $\mathcal{H}_2$ are Hölder continuous in $Q^+_{z_0,\gamma/2}$. In fact, they are more than Hölder continuous. Indeed, $\mathcal{H}_{12}$ and $\mathcal{H}_2$ are smooth in $Q^+_{z_0,\gamma/2}$. We consider first $\mathcal{H}_2$ in (42). Since u_0 is supported in B^+_ρ for a sufficiently large $\rho > 0$, using the estimate of T_{ij}, we have

$$|D^k_t D^l_{x'} D^m_{x_n} \mathcal{H}_2(x,t)| \leq \int_{\mathbb{R}^n_+} |D^k_t D^l_{x'} D^m_{x_n} T_{ij}(x,y,t)||u_0(y)| \, \mathrm{d}y$$

$$\leq ||u_0||_{L^\infty} \int_{B^+_\rho} \frac{C}{t^{k+\frac{l+m+\alpha}{2}} |x-y|^{n-1} |x_n - y_n|^{1-\alpha}} \, \mathrm{d}y$$

$$\leq ||u_0||_{L^\infty} \frac{C}{t^{k+\frac{l+m+\alpha}{2}}} \leq \frac{C||u_0||_{L^\infty}}{(t_0 - \frac{\gamma^2}{4})^{k+\frac{l+m+\alpha}{2}}}.$$

Now we consider $\mathcal{H}_{12}$ in (43). Here we again emphasize that a characteristic function considered here is $\chi_{Q^+_{z_0,3\gamma/4}}$ instead of $\chi_{Q^+_{z_0,\gamma}}$ in (43). As mentioned earlier, T_{ij} has no singularity provided that $(x,t) \in Q^+_{z_0,\gamma/2}$. Hence any higher derivatives including time variable of $\mathcal{H}_{12}$ is bounded by L^2 norm of u, which is finite due to the global energy inequality (41). Therefore, it remains to prove that $\mathcal{H}_{11}(x,t)$ is Hölder continuous. For clarity, we recall $\mathcal{H}_{11}(x,t)$:

$$\mathcal{H}_{11}(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \partial_k T_{ij}(x,y,t,s) u_k(y,s) u_j(y,s) \chi_{Q^+_{z_0,\frac{3\gamma}{4}}} \,\mathrm{d}y\,\mathrm{d}s.$$

From now on, we denote $Q^+_{z_0,3\gamma/4}$ by Q without any confusion. We first show the Hölder continuity for space variables. Let (x,t) and $(\tilde{x},t)$ be in $Q^+_{z_0,\gamma/2}$ and $\epsilon = |x-\tilde{x}|$. Since u is bounded in Q, using the estimate of T_{ij}, we have

$$|\mathcal{H}_{11}(x,t) - \mathcal{H}_{11}(\tilde{x},t)| \leq ||u||^2_{L^\infty(Q)} \int_0^t \int_Q |\partial_k T_{ij}(x,y,t,s) - \partial_k T_{ij}(\tilde{x},y,t,s)|$$

$$\leq C||u||^2_{L^\infty(Q)} \int_{B^+_{\frac{3\gamma}{4}}} |\frac{1}{|x-y|^{n-1}|x_n-y_n|^{1-\alpha}} - \frac{1}{|\tilde{x}-y|^{n-1}|\tilde{x}_n-y_n|^{1-\alpha}}| \,\mathrm{d}y$$

where $\int_0^t (t-s)^{-(1+\alpha)/2}\,\mathrm{d}s$ is finite because $\alpha \in (0,1)$. Let $w = x/\epsilon$ and $\tilde{w} = \tilde{x}/\epsilon$. Changing of variable $\tilde{y} = y/\epsilon$ and using the scaling method, we obtain

$$\int_{B^+_{\frac{3\gamma}{4}}} |\frac{1}{|x-y|^{n-1}|x_n-y_n|^{1-\alpha}} - \frac{1}{|\tilde{x}-y|^{n-1}|\tilde{x}_n-y_n|^{1-\alpha}}| \,\mathrm{d}y$$

$$= \epsilon^\alpha \int_{B^+_{\frac{3\gamma}{4\epsilon}}} |\frac{1}{|w-\tilde{y}|^{n-1}|w_n-\tilde{y}_n|^{1-\alpha}} - \frac{1}{|\tilde{w}-\tilde{y}|^{n-1}|\tilde{w}_n-\tilde{y}_n|^{1-\alpha}}| \,\mathrm{d}\tilde{y}$$

$$\leq \epsilon^\alpha \int_{\mathbb{R}^n_+} |\frac{1}{|w-\tilde{y}|^{n-1}|w_n-\tilde{y}_n|^{1-\alpha}} - \frac{1}{|\tilde{w}-\tilde{y}|^{n-1}|\tilde{w}_n-\tilde{y}_n|^{1-\alpha}}| \,\mathrm{d}\tilde{y}.$$

It can be easily checked that the last integral is finite regardless of ϵ because $|w-\tilde{w}| = 1$ and it is locally integrable, and its decay rate is $O(|x|^{-2n+2\alpha})$ at infinity. To sum up, we have the following Hölder estimate:

$$|\mathcal{H}_{11}(x,t) - \mathcal{H}_{11}(\tilde{x},t)| \leq C||u||_{L^\infty(Q)}\epsilon^\alpha = C|x-\tilde{x}|^\alpha.$$

For the time variable, we can also obtain the following Hölder estimate:

$$|\mathcal{H}_{11}(x,t_1) - \mathcal{H}_{11}(x,t_2)| \leq C||u||_{L^\infty(Q)}|t_1-t_2|^{\frac{\alpha}{2}}.$$

Since it follows the same argument as the case of space variables, the details are omitted. Since α is arbitrary between 0 and 1, we conclude that u is also Hölder continuous with exponent α for every $\alpha \in (0,1)$ in $Q^+_{z_0,\gamma/2}$. This completes the proof. □

ACKNOWLEDGEMENT

This research was supported in part by NSF Grant No. DMS-9877055. The author is deeply grateful to his Professor Vladimír Šverák for guidance and encouragement. The author would like to thank to Professor G. Seregin for showing him the preprint [13]. The author also would like to thank Professor V. Solonnikov for drawing his attention to the papers [16] and [17].

References

[1] S. AGMON, A. DOUGLIS & L. NIRENBERG *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math. **17** (1964), 35–92.

[2] L. CAFFARELLI, R. KOHN & L. NIRENBERG *Partial regularity of suitable weak solutions of the Navier-Stokes equations,* Comm. Pure Appl. Math. **35** (1982), 771–831.

[3] M. CANNONE, F. PLANCHON, & M. SCHONBEK *Strong solutions to the incompressible Navier-Stokes equations in the half-space*, Comm. in P.D.E. **25** (2000) 903-924.

[4] H. CHOE *Boundary regularity of weak solutions of the Navier-Stokes equations*, Journal of differential equations **149**, (1998), 211–247.

[5] E. B. FABES, B. F. JONES & N. M. RIVIERE *The initial value problem for the Navier-Stokes equations with data in L^p*, Arch. Rational Mech. Anal. **45**, (1972), 222–240.

[6] D. GILBARG, N. S. TRUDINGER, *Elliptic partial differential equations of second order,* Reprint of the 1998 edition, Springer-Verlag, Berlin, 1997.

[7] Y. GIGA *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations **62**, no. 2, (1986), 186–212.

[8] Y. GIGA & H. SOHR *Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains* J. Funct. Anal. **102** (1991), 72–94.

[9] E. HOPF *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen (German)* , Math. Nachr. **4**, (1951), 213–231.

[10] O. A. LADYZHENSKAYA, V. A. SOLONNIKOV & N. N. URALCEVA *Linear and Quasilinear Equations of Parabolic type.* Translations of Mathematical Monographs, **23**, Amer. Math. Soc., Providence, R.I., 1968.

[11] J. LERAY *Sur le mouvement d'un liquide visqueux emplissant l'espace,* Acta Math. **63**, (1934), 193–248.

[12] V. SCHEFFER *Boundary regularity for the Navier-Stokes equations in a half-space*, Comm. Math. Phys. **85**, no. 2, (1982), 275-299.

[13] G. A. SEREGIN *Local regularity of suitable weak solutions to the Navier-Stokes equations near the boundary*, preprint.

[14] J. SERRIN *On the interior regularity of weak solutions of the Navier-Stokes equations,* Arch. Rational Mech. Anal. **9**, (1962), 187–195.

[15] V. A. SOLONNIKOV *Estimates of the solutions of nonstationary linearized system of Navier-Stokes equations*, Trudy Mat. Inst. Steklov **70**, (1964), 213–317. (in English:A.M.S. Translations, Series II **75**, 1–117).

[16] V. A. SOLONNIKOV *Estimates of the solutions of the nonstationary Navier-Stokes system. (Russian) Boundary value problems of mathematical physics and related questions in the theory of functions, 7,* Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **38**, (1973), 153–231, Translated in Journal of Soviet Mathematics **8**, (1977), 467–529.

[17] V. A. SOLONNIKOV *Estimates of solutions of an initial- and boundary-value problem for the linear nonstationary system of Navier-Stokes equations (Russian)*, Zap. Naučn. Sem. Leningrad. Otdel Mat. Inst. Steklov. (LOMI) **59**, (1976), 178–254, Translated in Journal of Soviet Mathematics **10**, (1978), 336–393.

[18] E. M. STEIN *Singular integrals and differentiability properties of functions,* Princeton University Press, 1970.

[19] M. STRUWE *On partial regularity results for the Navier-Stokes equations,* Comm. Pure Appl. Math. **41**, (1988), 437–458.

[20] S. TAKAHASHI *On a regularity criterion up to the boundary for weak solutions of the Navier-Stokes equations*, Comm. Partial Differential Equations **17**, (1992), 261–285.

[21] S. UKAI *A solution formula for the Stokes equation in* $\mathbb{R}^n_+$, Comm. pure. Appl. Math. **40**, (1987), 611–621.

Kyungkeun Kang
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
USA
Email:kkang@math.umn.edu