APPELL'S AND HUMBERT'S FUNCTIONS OF MATRIX ARGUMENTS – I

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ABSTRACT

We have proved six results for the Appell's functions of matrix arguments-two each for the functions F_1 and F_3 and one each for the functions F_2 and F_4.

INTRODUCTION

Appell's functions of matrix arguments have earlier been studied by Mathai [4, 5, 6] and also by Saxena, Sethi and Gupta [7]. In the present paper we have utilized Mathai's definitions for all the functions studied. All the matrices appearing in this paper are (pxp) real symmetric positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [3,4].

1. Preliminary Definitions

DEFINITION 1.1: The Appell's function $F_1 = F_1(a, b, b^{'}; c; -X, -Y)$ of matrix arguments is defined as that function for which the M-transform (matrix-transform) is the following:

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$$M(F_1)=\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}F_1(a,b,b';c;-X,-Y)dXdY$$

$$=\frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1-\rho_2)\Gamma_p(b-\rho_1)\Gamma_p(b'-\rho_2)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-\rho_1-\rho_2)} \quad \cdots(1.1)$$

for $\operatorname{Re}(a-\rho_1-\rho_2, b-\rho_1, b'-\rho_2, c-\rho_1-\rho_2, \rho_1, \rho_2)>(p-1)/2.$

DEFINITION 1.2: $F_2=F_2(a,b,b';c,c';-X,-Y)$

$$M(F_2)=\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}F_2(a,b,b';c,c';-X,-Y)dXdY$$

$$=\frac{\Gamma_p(c)\Gamma_p(c')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1-\rho_2)\Gamma_p(b-\rho_1)\Gamma_p(b'-\rho_2)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-\rho_1)\Gamma_p(c'-\rho_2)} \quad \cdots(1.2)$$

for $\operatorname{Re}(a-\rho_1-\rho_2, b-\rho_1, b'-\rho_2, c-\rho_1, c'-\rho_2, \rho_1, \rho_2)>(p-1)/2.$

DEFINITION 1.3: $F_3=F_3(a,a',b,b';c;-X,-Y)$

$$M(F_3)=\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}F_3(a,a',b,b';c;-X,-Y)dXdY$$

$$=\frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1)\Gamma_p(a'-\rho_2)\Gamma_p(b-\rho_1)\Gamma_p(b'-\rho_2)}{\Gamma_p(a)\Gamma_p(a')\Gamma_p(b)\Gamma_p(b')\Gamma_p(c-\rho_1-\rho_2)} \quad \cdots(1.3)$$

for $\operatorname{Re}(a-\rho_1, a'-\rho_2, b-\rho_1, b'-\rho_2, c-\rho_1-\rho_2, \rho_1, \rho_2)>(p-1)/2.$

DEFINITION 1.4: $F_4=F_4(a,b;c,c';-X,-Y)$

$$M(F_4)=\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}F_4(a,b;c,c';-X,-Y)dXdY$$

$$=\frac{\Gamma_p(c)\Gamma_p(c')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a-\rho_1-\rho_2)\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c-\rho_1)\Gamma_p(c'-\rho_2)} \quad \cdots(1.4)$$

for $\operatorname{Re}(a-\rho_1-\rho_2, b-\rho_1-\rho_2, c-\rho_1, c'-\rho_2, \rho_1, \rho_2)>(p-1)/2.$

DEFINITION 1.5: The Humbert's function $\Phi_1 = \Phi_1(a, b; c; -X, -Y)$ of matrix arguments is defined as that function which has the following M-transform:

$$M(\Phi_1) = \int_{X>0} \int_{Y>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \Phi_1(a, b; c; -X, -Y) dX dY$$

$$= \frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a - \rho_1 - \rho_2)\Gamma_p(b - \rho_1)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c - \rho_1 - \rho_2)} \qquad \cdots(1.5)$$

for Re $(a - \rho_1 - \rho_2, b - \rho_1, c - \rho_1 - \rho_2, \rho_1, \rho_2) > (p-1)/2$.

DEFINITION 1.6: $\Psi_1 = \Psi_1(a, b; c, c'; -X, -Y)$

$$M(\Psi_1) = \int_{X>0} \int_{Y>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \Psi_1(a, b; c, c'; -X, -Y) dX dY$$

$$= \frac{\Gamma_p(c)\Gamma_p(c')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a - \rho_1 - \rho_2)\Gamma_p(b - \rho_1)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c - \rho_1)\Gamma_p(c' - \rho_2)} \qquad \cdots(1.6)$$

for Re $(a - \rho_1 - \rho_2, b - \rho_1, c - \rho_1, c' - \rho_2, \rho_1, \rho_2) > (p-1)/2$.

DEFINITION 1.7: $\Psi_2 = \Psi_2(a; c, c'; -X, -Y)$

$$M(\Psi_2) = \int_{X>0} \int_{Y>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \Psi_2(a; c, c'; -X, -Y) dX dY$$

$$= \frac{\Gamma_p(c)\Gamma_p(c')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a - \rho_1 - \rho_2)}{\Gamma_p(a)\Gamma_p(c - \rho_1)\Gamma_p(c' - \rho_2)} \qquad \cdots(1.7)$$

for Re $(a - \rho_1 - \rho_2, c - \rho_1, c' - \rho_2, \rho_1, \rho_2) > (p-1)/2$.

DEFINITION 1.8: $\Xi_2 = \Xi_2(a, b; c; -X, -Y)$

$$M(\Xi_2) = \int_{X>0} \int_{Y>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \Xi_2(a, b; c; -X, -Y) dX dY$$

$$= \frac{\Gamma_p(c)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(a - \rho_1)\Gamma_p(b - \rho_1)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c - \rho_1 - \rho_2)} \qquad \cdots(1.8)$$

for Re $(a - \rho_1, b - \rho_1, c - \rho_1 - \rho_2, \rho_1, \rho_2) > (p-1)/2$.

2. Appell's Functions of Matrix Arguments.

THEOREM 2.1:

$$F_1(\alpha,\beta,\beta^{'};\gamma;-X,-Y)$$

$$=\frac{\Gamma_p(\gamma)}{\Gamma_p(\beta)\Gamma_p(\beta^{'})\Gamma_p(\gamma-\beta-\beta^{'})}\int_0^I\int_0^I |U|^{\beta+\beta^{'}-(p+1)/2}|V|^{\beta^{'}-(p+1)/2}\times$$

$$|I-U|^{\gamma-\beta-\beta^{'}-(p+1)/2}|I-V|^{\beta-(p+1)/2}\Big|I+(I-V)^{1/2}U^{1/2}XU^{1/2}(I-V)^{1/2}$$

$$+V^{1/2}U^{1/2}YU^{1/2}V^{1/2}\Big|^{-\alpha}dUdV \qquad \cdots\cdots(2.1)$$

for $0<U<I,\ 0<V<I$ and for $\mathrm{Re}\,(\beta,\beta^{'},\gamma-\beta-\beta^{'})>(p-1)/2$.

PROOF: We take the M-transform of the right side of eq. (2.1) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively to obtain,

$$\int_{X>0}\int_{Y>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\Big|I+(I-V)^{1/2}U^{1/2}XU^{1/2}(I-V)^{1/2}$$

$$+V^{1/2}U^{1/2}YU^{1/2}V^{1/2}\Big|^{-\alpha}dXdY \qquad \cdots\cdots(2.2)$$

Making use of the transformations,

$X_1=(I-V)^{1/2}U^{1/2}XU^{1/2}(I-V)^{1/2},\ Y_1=V^{1/2}U^{1/2}YU^{1/2}V^{1/2}$ with,

$dX_1=|I-V|^{(p+1)/2}|U|^{(p+1)/2}dX,\ dY_1=|V|^{(p+1)/2}|U|^{(p+1)/2}dY$ and

$|X_1|=|I-V||U||X|,\ |Y_1|=|V||U||Y|$

in the above expression and then integrating out the variables X_1 and Y_1 by using a type-2 Dirichlet integral we get,

$$|U|^{-\rho_1-\rho_2}|V|^{-\rho_2}|I-V|^{-\rho_1}\frac{\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)}{\Gamma_p(\alpha)} \qquad \cdots(2.3)$$

Substituting this expression on the right side of eq. (2.1) and then integrating out the variables U and V in the resulting expression by using a type-1 Beta integral generates $M(F_1)$ as given by eq. (1.1).

THEOREM 2.2:

$$|P|^{-\beta'}F_1(\alpha,\beta,\beta';\gamma;-X,-P^{-1/2}YP^{-1/2})$$

$$=\frac{1}{\Gamma_p(\beta')}\int_{T>0}e^{-tr(PT)}|T|^{\beta'-(p+1)/2}\Phi_1(\alpha,\beta;\gamma;-X,-T^{1/2}YT^{1/2})dT \qquad \cdots(2.4)$$

for $\mathrm{Re}(\beta')>(p-1)/2.$

PROOF: Taking the M-transform of the right side of eq. (2.4) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we have,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\Phi_1(\alpha,\beta;\gamma;-X,-T^{1/2}YT^{1/2})dXdY \cdots(2.5)$$

which, under the transformation

$$Y_1=T^{1/2}YT^{1/2}(\text{with } dY_1=|T|^{(p+1)/2}dY \text{ and } |Y_1|=|T||Y|)$$

and then using the definition (1.5) yields,

$$|T|^{-\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)\Gamma_p(\beta-\rho_1)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots(2.6)$$

Substituting this expression on the right side of eq. (2.4) and then integrating out T in the resulting expression by using a Gamma integral gives,

$$|P|^{-(\beta'-\rho_2)}\frac{\Gamma_p(\beta'-\rho_2)\Gamma_p(\gamma)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)\Gamma_p(\beta-\rho_1)}{\Gamma_p(\beta')\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \quad \cdots(2.7)$$

Now, taking the M-transform of the left side of eq. (2.4) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|P|^{-\beta'}\times$$

$$F_1(\alpha,\beta,\beta';\gamma;-X,-P^{-1/2}YP^{-1/2})dXdY \qquad \cdots\cdots(2.8)$$

which, under the transformation

$$Y_2=P^{-1/2}YP^{-1/2}\,(\text{with } dY_2=|P|^{-(p+1)/2}dY \text{ and } |Y_2|=|P|^{-1}|Y|\,)$$

and then using the definition (1.1) leads us to the same result as in eq. (2.7).

THEOREM 2.3:

$$|P|^{-\beta'}F_2(\alpha,\beta,\beta';\gamma,\gamma';-X,-P^{-1/2}YP^{-1/2})$$

$$=\frac{1}{\Gamma_p(\beta')}\int_{T>0}e^{-\mathrm{tr}(PT)}|T|^{\beta'-(p+1)/2}\Psi_1(\alpha,\beta;\gamma,\gamma';-X,-T^{1/2}YT^{1/2})dT \qquad \cdots(2.9)$$

for $\mathrm{Re}\,(\beta')>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq. (2.9) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\Psi_1(\alpha,\beta;\gamma,\gamma';-X,-T^{1/2}YT^{1/2})dXdY \qquad \cdots\cdots(2.10)$$

The application of the same transformation to this expression as we have applied to the expression (2.5) above and then the use of definition (1.6) leads us to,

$$|T|^{-\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\gamma')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)\Gamma_p(\beta-\rho_1)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1)\Gamma_p(\gamma'-\rho_2)} \qquad \cdots(2.11)$$

Substituting this expression on the right side of eq. (2.9) and then integrating out T in the resulting expression by using a Gamma integral gives,

$$|P|^{-(\beta'-\rho_2)}\frac{\Gamma_p(\beta'-\rho_2)\Gamma_p(\gamma')\Gamma_p(\gamma)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)\Gamma_p(\beta-\rho_1)}{\Gamma_p(\beta')\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1)\Gamma_p(\gamma'-\rho_2)} \quad \cdots(2.12)$$

Now, taking the M-transform of the left side of eq. (2.9) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we have,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|P|^{-\beta'}\times$$
$$F_2(\alpha,\beta,\beta';\gamma,\gamma';-X,-P^{-1/2}YP^{-1/2})dXdY \quad \cdots\cdots(2.13)$$

which, on the application of the same transformation as in eq.(2.8) above and then using the definition (1.2) yields the same result as in eq.(2.12) above.

THEOREM 2.4:

$$F_3(\alpha,\alpha',\beta,\beta';\gamma+\gamma';-X,-Y)$$
$$=\frac{\Gamma_p(\gamma+\gamma')}{\Gamma_p(\gamma)\Gamma_p(\gamma')}\int_0^I|U|^{\gamma-(p+1)/2}|I-U|^{\gamma'-(p+1)/2}{}_2F_1(\alpha,\beta;\gamma;-U^{1/2}XU^{1/2})\times$$
$${}_2F_1[\alpha',\beta';\gamma';-(I-U)^{1/2}Y(I-U)^{1/2}]dU \quad \cdots\cdots(2.14)$$

for $0<U<I$ and for $\mathrm{Re}\,(\gamma,\gamma')>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq. (2.14) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}{}_2F_1(\alpha,\beta;\gamma;-U^{1/2}XU^{1/2})\times$$
$${}_2F_1[\alpha',\beta';\gamma';-(I-U)^{1/2}Y(I-U)^{1/2}]dXdY \quad \cdots\cdots(2.15)$$

Applying the transformations,

$X_1=U^{1/2}XU^{1/2},\ Y_1=(I-U)^{1/2}Y(I-U)^{1/2}$ with $dX_1=|U|^{(p+1)/2}dX$,

$dY_1=|I-U|^{(p+1)/2}dY$ and $|X_1|=|U||X|$, $|Y_1|=|I-U||Y|$

to the above expression and then applying eq. (2.3.5) page 38 of Mathai [4] leads us to,

$$|U|^{-\rho_1}|I-U|^{-\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\alpha'-\rho_2)\Gamma_p(\beta'-\rho_2)\Gamma_p(\gamma')}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1)\Gamma_p(\alpha')}\times$$
$$\frac{\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1)\Gamma_p(\beta-\rho_1)}{\Gamma_p(\beta')\Gamma_p(\gamma'-\rho_2)} \qquad \cdots\cdots(2.16)$$

Substituting this expression on the right side of eq.(2.14) and then integrating out the variable U in the resulting expression by using a type -1 Beta integral results in $M(F_3)$ (definition (1.3)).

THEOREM 2.5: For p =2**,**

$$|P|^{-\alpha'}F_3[\alpha,(\alpha'+1)/2,\beta,(2\alpha'+1)/4;\gamma;-X,-4P^{-1}YP^{-1})$$
$$=\frac{1}{\Gamma_p(\alpha')}\int_{T>0}e^{-tr(PT)}|T|^{\alpha'-(p+1)/2}\Xi_2(\alpha,\beta;\gamma;-X,-TYT')dT \qquad \cdots(2.17)$$

where Re $(\alpha')>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq. (2.17) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we have,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$\Xi_2(\alpha,\beta;\gamma;-X,-TYT')dXdY \qquad \cdots\cdots(2.18)$$

Applying the transformation $Y_1=TYT'$ with $dY_1=|T|^{p+1}dY$ and $|Y_1|=|T|^2|Y|$ to the above expression and then using the definition (1.8) produces,

$$|T|^{-2\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1)\Gamma_p(\beta-\rho_1)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \qquad \cdots(2.19)$$

Substituting this expression on the right side of eq. (2.17) and then integrating out T in the resulting expression by using a Gamma integral leads us to,

$$|P|^{-(\alpha'-2\rho_2)} \frac{\Gamma_p(\gamma)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1)\Gamma_p(\beta-\rho_1)\Gamma_p(\alpha'-2\rho_2)}{\Gamma_p(\alpha')\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1-\rho_2)} \cdots\cdots (2.20)$$

Now taking the M-transform of the left side of eq. (2.17) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0} \int_{Y>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |P|^{-\alpha'} \times$$
$$F_3[\alpha, (\alpha'+1)/2, \beta, (2\alpha'+1)/4; \gamma; -X, -4P^{-1}YP^{-1}) dX dY \qquad \cdots\cdots (2.21)$$

Making use of the transformation

$Y_2 = 4P^{-1}YP^{-1}$ with $dY_2 = 4^{p(p+1)/2}|P|^{-(p+1)}dY$ and $|Y_2| = 4^p|P|^{-2}|Y|$ in the above expression and then using the definition (1.3) along with the observation that for p=2,

$$4^{-p\rho_2} \frac{\Gamma_p[(\alpha'+1)/2-\rho_2]\Gamma_p[(2\alpha'+1)/4-\rho_2]}{\Gamma_p[(\alpha'+1)/2]\Gamma_p[(2\alpha'+1)/4]} = \frac{\Gamma_p(\alpha'-2\rho_2)}{\Gamma_p(\alpha')} \quad \cdots (2.22)$$

from eq. (6.13) page 84 of Mathai [4], finally leads us to the same result as in eq. (2.20) above.

It is to be noted that this result is different from the corresponding result in the scalar case.

THEOREM 2.6:

$$|P|^{-\alpha} F_4(\alpha, \beta; \gamma, \gamma'; -P^{-1/2}XP^{-1/2}, -P^{-1/2}YP^{-1/2})$$
$$= \frac{1}{\Gamma_p(\alpha)} \int_{T>0} e^{-\mathrm{tr}(PT)} |T|^{\alpha-(p+1)/2} \Psi_2(\beta; \gamma, \gamma'; -T^{1/2}XT^{1/2}, -T^{1/2}YT^{1/2}) dT \cdots (2.23)$$

for Re $(\alpha) > (p-1)/2$.

PROOF: Taking the M-transform of the right side of eq. (2.23) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we have,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$\Psi_2(\beta;\gamma,\gamma';-T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2})dXdY \qquad \cdots\cdots(2.24)$$

which, under the transformations

$$X_1=T^{1/2}XT^{1/2},\ Y_1=T^{1/2}YT^{1/2} \text{ with } dX_1=|T|^{(p+1)/2}dX,\ dY_1=|T|^{(p+1)/2}dY$$

and $|X_1|=|T||X|,\ |Y_1|=|T||Y|$

and then using the definition (1.7) yields,

$$|T|^{-\rho_1-\rho_2}\frac{\Gamma_p(\gamma)\Gamma_p(\gamma')\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\beta-\rho_1-\rho_2)}{\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1)\Gamma_p(\gamma'-\rho_2)} \qquad \cdots(2.25)$$

Substituting this expression on the right side of eq.(2.23) and then integrating out T in the resulting expression by using a Gamma integral produces,

$$|P|^{-(\alpha-\rho_1-\rho_2)}\frac{\Gamma_p(\gamma')\Gamma_p(\gamma)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1-\rho_2)\Gamma_p(\beta-\rho_1-\rho_2)}{\Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma-\rho_1)\Gamma_p(\gamma'-\rho_2)} \quad \cdots(2.26)$$

Now, taking the M-transform of the left side of eq.(2.23) with respect to the variables X and Y and the parameters ρ_1 and ρ_2 respectively, we get,

$$\int_{X>0}\int_{Y>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|P|^{-\alpha}F_4(\alpha,\beta;\gamma,\gamma';-P^{-1/2}XP^{-1/2},-P^{-1/2}YP^{-1/2})dXdY \qquad \cdots\cdots(2.27)$$

which, under the transformations

$$X_2=P^{-1/2}XP^{-1/2},Y_2=P^{-1/2}YP^{-1/2} \text{ with } dX_2=|P|^{-(p+1)/2}dX,$$

$$dY_2=|P|^{-(p+1)/2}dY \text{ and } |X_2|=|P|^{-1}|X|,\ |Y_2|=|P|^{-1}|Y|$$

and then using the definition (1.4) leads us to the same result as in eq. (2.26).

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