On the Number of Positive Solutions to a Class of Integral Equations *

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Abstract: By using the complete discrimination system for polynomials, we study the number of positive solutions in $C[0,1]$ to the integral equation $\varphi(x) = \int_0^1 k(x,y)\varphi^n(y)dy$, where $k(x,y) = \varphi_1(x)\phi_1(y) + \varphi_2(x)\phi_2(y), \varphi_i(x) > 0, \phi_i(y) > 0, 0 < x, y < 1, i = 1, 2$, are continuous functions on $[0,1]$, n is a positive integer. We prove the following results: when $n = 1$, either there does not exist, or there exist infinitely many positive solutions in $C[0,1]$; when $n \geq 2$, there exist at least 1, at most $n+1$ positive solutions in $C[0,1]$. Necessary and sufficient conditions are derived for the cases: 1) $n = 1$, there exist positive solutions; 2) $n \geq 2$, there exist exactly $m(m \in \{1, 2, \cdots, n+1\})$ positive solutions. Our results generalize the existing results in the literature, and their usefulness is shown by examples presented in this paper.

Keywords: Integral Equations, Positive Solutions, the Complete Discrimination System for Polynomials, the Number of Solutions

1 Introduction

The existence of positive solutions to integral equations is an active research field and has important applications in the stability of feedback systems [1,2]. In 1991, the number of positive solutions to the following integral equation

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$$\varphi(x) = \int_0^1 k(x,y)\varphi^2(y)dy \tag{1}$$

was discussed in [3]. In this paper, we will study the number of positive solutions in $C[0,1]$ to following more general integral equation

$$\varphi(x) = \int_0^1 k(x,y)\varphi^n(y)dy \tag{2}$$

where

$$k(x,y) = \varphi_1(x)\phi_1(y) + \varphi_2(x)\phi_2(y), \varphi_i(x) > 0, \phi_i(y) > 0, 0 < x, y < 1, i = 1,2$$

are continuous functions on $[0,1]$, n is a positive integer. We prove the following results: when $n = 1$, either there does not exist, or there exist infinitely many positive solutions in $C[0,1]$; when $n \geq 2$, there exist at least 1, at most $n+1$ positive solutions in $C[0,1]$. Especially, when n is an odd number greater than 2, there exist at least 1, at most n positive solutions in $C[0,1]$. Necessary and sufficient conditions are derived for the cases: 1) $n = 1$, there exist positive solutions in $C[0,1]$; 2) $n \geq 2$, there exist exactly $m(m \in \{1,2,\cdots,n+1\})$ positive solutions in $C[0,1]$. Our results generalize the existing results in the literature, and their usefulness is shown by examples presented in this paper.

In essence, the number of positive solutions to (2) can be transformed into determination of real roots of a certain polynomial, which is a century-long, albeit still active research area in mathematics. The classical Sturm method or Newton formula can be employed to determine the real root distribution of polynomials [4-7], but the Sturm method is inefficient in establishing discriminant systems for high-order polynomials with symbolic coefficients [4,6,7], and Newton formula involves in recursive procedure to determine the real roots, thus it is difficult to establish explicit criteria [4,5,7].

More recently, Yang et al. established the complete discrimination system for polynomials, which can give a set of explicit expressions based on the coefficients of polynomials to determine the root distribution of polynomials [6,7].

Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \in P^n,$$

the Sylvester matrix of $f(x)$ and its derivative $f^{'}(x)$ [6,7]

$$\begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & & & & \\
0 & na_0 & (n-1)a_1 & \cdots & 2a_{n-2} & a_{n-1} & & & & \\
 & a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} & a_n & & & \\
 & 0 & na_0 & \cdots & 3a_{n-3} & 2a_{n-2} & a_{n-1} & & & \\
 & & & \cdots & \cdots & \cdots & & & & \\
 & & & \cdots & \cdots & \cdots & & & & \\
 & & & & a_0 & a_1 & a_2 & \cdots & a_n \\
 & & & & 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1}
\end{bmatrix}$$

is called the discrimination matrix of $f(x)$, denoted as $Discr(f)$.

$$[D_1(f), D_2(f), \cdots, D_n(f)]$$

the even-order principal minor sequence of $Discr(f)$, is called the discriminant sequence of $f(x)$.

$$[sign(D_1), sign(D_2), \cdots, sign(D_n)]$$

is called the sign list of the discriminant sequence $[D_1, D_2, \cdots, D_n]$, where $sign(\cdot)$ is the sign function, i.e.,

$$sign(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Given a sign list $[s_1, s_2, \cdots, s_n]$, we can construct a revised sign list

$$[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n]$$

as follows:

1)If $[s_i, s_{i+1}, \cdots, s_{i+j}]$ is a section of the given sign list and $s_i \neq 0; s_{i+1} = s_{i+2} = \cdots = s_{i+j-1} = 0; s_{i+j} \neq 0$, then replace the subsection consisting of all 0 elements

$$[s_{i+1}, s_{i+2}, \cdots, s_{i+j-1}]$$

by the following subsection with equal number of terms

$$[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, -s_i, \cdots]$$

i.e.,$\varepsilon_{i+r} = (-1)^{[\frac{r+1}{2}]} \cdot s_i, r = 1, 2, \cdots, j-1$.

2)Let $\varepsilon_k = s_k$ for all other terms, i.e., all other terms remain the same.

Lemma 1 [6,7] Given the polynomial with real coefficients $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \in P^n$. If the number of sign changes in the revised sign list of its discriminant sequence is ν, and the number of non-zero elements in the revised sign list is μ, then the number of distinct real roots of $f(x)$ is $\mu - 2\nu$.

Remark 1 The discriminant sequence of $f(x)$ can also be constructed by the principal minors of the Bezout matrix of $f(x)$ and $f^{'}(x)$ [6,7]; the number of distinct real roots of $f(x)$ can also be determined by the sign difference of Bezout matrix of $f(x)$ and $f^{'}(x)$ [6,7].

Remark 2 The complete discrimination system for polynomials can also be used to determine the number and the multiplicity of complex roots [6,7].

Yang and Xia also proposed a method to determine the number of positive (negative) roots of a polynomial [8], which is similar to Lemma 1 in principle, but is more efficient.

Lemma 2 [8] Given the polynomial with real coefficients $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \in P^n$, $a_0 \neq 0, a_n \neq 0$. Let $h(x) = f(-x)$ and $\{d_1, d_2, \cdots, d_{2n+1}\}$ be the sequence of the principal minors of the discrimination matrix $Discr(h)$ of $h(x)$. If the number of sign changes in the revised sign list of the sequence $\{d_1d_2, d_2d_3, \cdots, d_{2n}d_{2n+1}\}$ is ν, and the number of non-zero elements in the revised sign list is μ, then the number of distinct positive roots of $f(x)$ is $\mu - 2\nu$.

2 Main Results

Consider the problem of determining the number of positive solutions in $C[0,1]$ to the integral equation of the following form

$$\varphi(x) = \int_0^1 k(x, y)\varphi^n(y)dy \tag{3}$$

where

$$k(x,y)=\varphi_1(x)\phi_1(y)+\varphi_2(x)\phi_2(y),\varphi_i(x)>0,\phi_i(y)>0,0<x,y<1,i=1,2$$

are continuous functions on $[0,1]$,n is a positive integer.

Denote

$$a_{n-i,i}=C_n^i\int_0^1\phi_1(y)\varphi_1^{n-i}(y)\varphi_2^i(y)dy,\quad i=0,1,\cdots,n,$$

$$b_{n-i,i}=C_n^i\int_0^1\phi_2(y)\varphi_1^{n-i}(y)\varphi_2^i(y)dy,\quad i=0,1,\cdots,n,$$

$$\alpha_i=b_{n-i,i}-a_{n-i+1,i-1},\ i=1,2,\cdots,n,\quad \alpha_0=b_{n,0},\quad \alpha_{n+1}=-a_{0,n},$$

where $C_n^i, i=0,1,\cdots,n$, stand for the combinatorial number. Our main result is as follows:

Theorem 1 When $n=1$, either there does not exist, or there exist infinitely many positive solutions in $C[0,1]$ to the integral equation (3). The necessary and sufficient conditions for the existence of positive solutions in $C[0,1]$ are $a_{1,0}-1<0$ and $(a_{1,0}-1)(b_{0,1}-1)-a_{0,1}b_{1,0}=0$.

Theorem 2 When $n\geq 2$, there exist at least 1, at most $n+1$ positive solutions in $C[0,1]$ to the integral equation (3). Especially, when n is an odd number greater than 2, there exist at least 1, at most n positive solutions in $C[0,1]$.

Theorem 3 When $n\geq 2$, the necessary and sufficient conditions for the existence of exactly $m(m\in\{1,2,\cdots,n+1\})$ positive solutions in $C[0,1]$ to the integral equation (3) are: the number of sign changes ν in the revised sign list of the discriminant sequence of the polynomial $f(s):=\sum\limits_{i=0}^{n+1}\alpha_i s^{2(n+1-i)}$ and the number of its non-zero elements μ satisfy $m=\frac{\mu-2\nu}{2}$; or, equivalently, the number of sign changes ν in the revised sign list of the sequence $\{d_1d_2,d_2d_3,\cdots,d_{2n+2}d_{2n+3}\}$ and the number of its non-zero elements μ satisfy $m=\frac{\mu-2\nu}{2}$, where $\{d_1,d_2,\cdots,d_{2n+3}\}$ is the sequence of the principal minors of the discriminant matrix $Discr(h)$ of $h(s):=\sum\limits_{i=0}^{n+1}\alpha_i(-s)^{n+1-i}$.

Specifically, when $n=2$, denote

$$p=\frac{\alpha_1}{\alpha_0},r=\frac{\alpha_2}{\alpha_0},t=\frac{\alpha_3}{\alpha_0}<0,$$

$$\Delta_1=p^2-3r,\Delta_2=rp^2+3tp-4r^2,$$

$$\Delta_3=-4r^3+18rtp+p^2r^2-4p^3t-27t^2,$$

$$[D_1,D_2,D_3,D_4,D_5,D_6]=[1,-p,-p\Delta_1,\Delta_1\Delta_2,\Delta_2\Delta_3,-t\Delta_3^2]. \tag{4}$$

then we have

Corollary 1 There exist at least 1, at most 3 positive solutions in $C[0,1]$ to the integral equation (1).

Corollary 2 The necessary and sufficient conditions for the integral equation (1) to have exactly 3 positive solutions in $C[0,1]$ are $p<0,\Delta_1>0,\Delta_2>0,\Delta_3>0$.

Corollary 3 The necessary and sufficient conditions for the integral equation (1) to have exactly 2 positive solutions in $C[0,1]$ are $p<0,\Delta_1>0,\Delta_2>0,\Delta_3=0$.

Corollary 4 The necessary and sufficient conditions for the integral equation (1) to have exactly 1 positive solutions in $C[0,1]$ are $p\geq 0$, or $\Delta_1\leq 0$, or $\Delta_2\leq 0$, or $\Delta_3<0$.

Remark 3 If n is even, the integral equation (3) does not have any negative solutions in $C[0,1]$.

Remark 4 If n is odd, since $\varphi(x)$ is a positive solution in $C[0,1]$ to the integral equation (3) if and only if $-\varphi(x)$ is a negative solution in $C[0,1]$ to the integral equation (3), thus, when $n=1$, the integral equation (3) either does not have, or has infinitely many negative solutions in $C[0,1]$; when n is odd and greater than 2, the integral equation (3) has at least 1, at most n negative solutions in $C[0,1]$.

Remark 5 When $n=1$, the necessary and sufficient conditions for existence of negative solutions in $C[0,1]$ to the integral equation (3) are the same as in Theorem 1. When n is odd and greater than 2, the necessary and sufficient conditions for existence of exactly $m(m \in \{1,2,\cdots,n\})$ negative solutions in $C[0,1]$ to the integral equation (3) are the same as in Theorem 3.

Remark 6 Our method can be extended to the case when the integral kernel $k(x,y)$ is taken as $\sum\limits_{i=1}^{l} \varphi_i(x)\phi_i(y)$, where $\varphi_i(x)>0, \phi_i(y)>0$, $0<x,y<1, i=1,2,\cdots,l$, are continuous functions on $[0,1]$.

Remark 7 The conclusions in [3] are equivalent to Corollaries 1,2,3 above.

3 Proof of the Theorems

Proof of Theorem 1 When $n=1$, the integral equation (3) becomes

$$\varphi(x)=\int_0^1 k(x,y)\varphi(y)dy \tag{5}$$

Thus, we have

$$\varphi(x)=\varphi_1(x)\int_0^1 \phi_1(y)\varphi(y)dy+\varphi_2(x)\int_0^1 \phi_2(y)\varphi(y)dy$$

If $\varphi(x)$ is a positive solution in $C[0,1]$ to equation (5), then $\varphi(x)$ can be expressed as $\varphi(x)=\lambda_1\varphi_1(x)+\lambda_2\varphi_2(x)$, where $\lambda_1>0, \lambda_2>0$ are coefficients to be determined. Taking it into equation (5), we get the following system of algebraic equations

$$\begin{cases} a_{1,0}\lambda_1+a_{0,1}\lambda_2=\lambda_1 \\ b_{1,0}\lambda_1+b_{0,1}\lambda_2=\lambda_2 \end{cases} \tag{6}$$

where $a_{1,0}=\int_0^1 \phi_1(y)\varphi_1(y)dy$, $a_{0,1}=\int_0^1 \phi_1(y)\varphi_2(y)dy$, $b_{1,0}=\int_0^1 \phi_2(y)\varphi_1(y)dy$, $b_{0,1}=\int_0^1 \phi_2(y)\varphi_2(y)dy$. Apparently, the necessary and sufficient conditions for the system of algebraic equations (6) to have positive solutions λ_1, λ_2 are $a_{1,0}-1<0$, and $(a_{1,0}-1)(b_{0,1}-1)-a_{0,1}b_{1,0}=0$. Moreover, if $\varphi(x)$ is a positive solution to equation (5), then, obviously, for any positive constant number $c, c\varphi(x)$ is also a positive solution to equation (5). Thus, there are infinitely many positive solutions in $C[0,1]$ to equation (5). This completes the proof.

Lemma 3 The system of equations

$$\begin{cases} a_{n,0}x^n+a_{n-1,1}x^{n-1}y+a_{n-2,2}x^{n-2}y^2+\cdots+a_{1,n-1}xy^{n-1}+a_{0,n}y^n=x \\ b_{n,0}x^n+b_{n-1,1}x^{n-1}y+b_{n-2,2}x^{n-2}y^2+\cdots+b_{1,n-1}xy^{n-1}+b_{0,n}y^n=y \end{cases} \tag{7}$$

$$a_{n-i,i}>0, \quad b_{n-i,i}>0, \quad i=0,1,2,\cdots,n.$$

has at least 1, at most $n+1$ (at most n, when n is odd) positive solutions, where $n\geq 2$.

Proof Let

$$p(x,y)=\frac{x}{a_{n,0}x^n+a_{n-1,1}x^{n-1}y+\cdots+a_{0,n}y^n}, \quad q(x,y)=\frac{y}{b_{n,0}x^n+b_{n-1,1}x^{n-1}y+\cdots+b_{0,n}y^n},$$

$$x > 0,\ y > 0,$$

then

$$p(\kappa x, \kappa y) = \frac{1}{\kappa^{n-1}} p(x,y), \quad q(\kappa x, \kappa y) = \frac{1}{\kappa^{n-1}} q(x,y),\ \kappa > 0.$$

Let

$$E = \{x | p(x,1) = q(x,1)\},$$

then the number of positive solutions to the system of equations (7) is equal to the number of elements in E. In fact, if (x, y) is a positive solution to (7), then

$$p(x,y) = q(x,y) = 1,$$

thus $\frac{x}{y} \in E$. Conversely, if $x \in E$, since $n \geq 2$, it is easy to verify that $(\sqrt[n-1]{p(x,1)}x,\ \sqrt[n-1]{p(x,1)})$ is a positive solution to (7).

Suppose $x \in E$, by $p(x,1) = q(x,1)$, we have

$$b_{n,0}x^{n+1} + (b_{n-1,1} - a_{n,0})x^n + (b_{n-2,2} - a_{n-1,1})x^{n-1} + \cdots + (b_{0,n} - a_{1,n-1})x - a_{0,n} = 0$$

Namely

$$\alpha_0 x^{n+1} + \alpha_1 x^n + \alpha_2 x^{n-1} + \cdots + \alpha_n x + \alpha_{n+1} = 0 \tag{8}$$

where

$$\alpha_i = b_{n-i,i} - a_{n-i+1,i-1},\ i = 1, 2, \cdots, n, \quad \alpha_0 = b_{n,0}, \quad \alpha_{n+1} = -a_{0,n},$$

Since $\alpha_0 = b_{n,0} > 0, \alpha_{n+1} = -a_{0,n} < 0$, equation (8) has at least 1, at most $n+1$ positive roots. Especially, when $n > 2$ and is odd, since equation (8) has at least 1 negative root, it has at most n positive roots. This completes the proof.

Proof of Theorems 2 and 3 When $n \geq 2$, since

$$\begin{aligned} \varphi(x) &= \textstyle\int_0^1 k(x,y)\varphi^n(y)dy \\ &= \varphi_1(x) \textstyle\int_0^1 \phi_1(y)\varphi^n(y)dy + \varphi_2(x) \int_0^1 \phi_2(y)\varphi^n(y)dy \end{aligned}$$

similar to the proof of Theorem 1, the positive solution $\varphi(x)$ in $C[0,1]$ to the integral equation (3) can be expressed as $\varphi(x) = \lambda_1\varphi_1(x) + \lambda_2\varphi_2(x)$, where $\lambda_1 > 0, \lambda_2 > 0$ are coefficients to be determined. Taking it into equation (3), by a simple but lengthy calculation, we see that λ_1, λ_2 should be positive solutions to the following system of algebraic equations

$$\begin{cases} a_{n,0}\lambda_1^n + a_{n-1,1}\lambda_1^{n-1}\lambda_2 + a_{n-2,2}\lambda_1^{n-2}\lambda_2^2 + \cdots + a_{1,n-1}\lambda_1\lambda_2^{n-1} + a_{0,n}\lambda_2^n = \lambda_1 \\ b_{n,0}\lambda_1^n + b_{n-1,1}\lambda_1^{n-1}\lambda_2 + b_{n-2,2}\lambda_1^{n-2}\lambda_2^2 + \cdots + b_{1,n-1}\lambda_1\lambda_2^{n-1} + b_{0,n}\lambda_2^n = \lambda_2 \end{cases} \tag{9}$$

where

$$a_{n-i,i} = C_n^i \int_0^1 \phi_1(y)\varphi_1^{n-i}(y)\varphi_2^i(y)dy, \quad i = 0, 1, \cdots, n,$$

$$b_{n-i,i} = C_n^i \int_0^1 \phi_2(y)\varphi_1^{n-i}(y)\varphi_2^i(y)dy, \quad i = 0, 1, \cdots, n.$$

By Lemma 3, we complete the proof of Theorem 2.

Moreover, from the proof of Lemma 3, we know that finding the positive solutions to the system of algebraic equations (9) or (7) can be transformed into finding the positive solutions to equation (8). Applying Lemmas 1 and 2 to equation (8), we complete the proof of Theorem 3.

Proof of Corollaries 1,2,3,4 Some notations in this proof are defined in Section 2.

Corollary 1 is a direct consequence of Theorem 2.

When $n = 2$, equation (8) becomes

$$\alpha_0 x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3 = 0 \tag{10}$$

By a direct computation, we know that the discriminant sequence $[D_1, D_2, D_3, D_4, D_5, D_6]$ of the polynomial $f(s) := \alpha_0 s^6 + \alpha_1 s^4 + \alpha_2 s^2 + \alpha_3$ is determined by (4) (up to a positive factor). Since $t < 0$, it is easy to see that the number of sign changes ν in the revised sign list of $[D_1, D_2, D_3, D_4, D_5, D_6]$ and the number of its non-zero elements μ satisfy $6 = \mu - 2\nu$ if and only if the revised sign list of $[D_1, D_2, D_3, D_4, D_5, D_6]$ is $[1, 1, 1, 1, 1, 1]$, which is equivalent to $p < 0, \Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0$. This completes the proof of Corollary 2.

Similarly, the number of sign changes ν in the revised sign list of $[D_1, D_2, D_3, D_4, D_5, D_6]$ and the number of its non-zero elements μ satisfy $4 = \mu - 2\nu$ if and only if the revised sign list of $[D_1, D_2, D_3, D_4, D_5, D_6]$ is $[1, 1, 1, 1, 0, 0]$, which is equivalent to $p < 0, \Delta_1 > 0, \Delta_2 > 0, \Delta_3 = 0$. This completes the proof of Corollary 3.

Combining Corollaries 1,2,3, we get Corollary 4.

4 Some Illustrative Examples

Example 1 Consider the integral equation

$$\varphi(x) = \int_0^1 (\frac{6}{5}xy + \frac{3}{5}y)\varphi(y)dy, \quad 0 \le x \le 1 \tag{11}$$

Let

$$\varphi_1(x) = \frac{6}{5}x,\ \phi_1(y) = y,\ \varphi_2(x) = \frac{3}{5},\ \phi_2(y) = y.$$

Then, it is easy to get $a_{1,0} = b_{1,0} = \frac{2}{5},\ a_{0,1} = b_{0,1} = \frac{3}{5}$. The conditions in Theorem 1 are met. Hence, there are infinitely many positive solutions in $C[0, 1]$. In fact, $\varphi(x) = c(\frac{6}{5}x + \frac{3}{5}), \forall c > 0$ are such solutions.

Remark 8 From the proof of theorems and the example above, we can see that, for a given integral equation, we can not only determine the number of its positive solutions, but also find the positive solutions explicitly by solving the algebraic equation (8).

Example 2 Consider the integral equation

$$\begin{aligned}\varphi(x) = &\textstyle\int_0^1 [18\max\{\varepsilon, -2x + 1 + \varepsilon\} + \max\{\frac{\varepsilon}{3}, \frac{1}{3}(2x - 1 + \varepsilon)\} \\ &\times \max\{6, 272y - 130\}]\varphi^n(y)dy, \quad 0 \le x \le 1,\end{aligned} \tag{12}$$

where $\varepsilon \ge 0, n = 1$ or 2.

Let

$$\varphi_1(x) = \max\{\varepsilon, -2x + 1 + \varepsilon\},\ \phi_1(y) = 18,$$

$$\varphi_2(x) = \max\{\frac{\varepsilon}{3}, \frac{1}{3}(2x - 1 + \varepsilon)\},\ \phi_2(y) = \max\{6, 272y - 130\}.$$

When $n = 1$, using the notations in Section 2 and by a simple computation, we can get

$$a_{1,0} = \int_0^1 \phi_1(y)\varphi_1(y)dy = 18\varepsilon + \frac{9}{2},\ a_{0,1} = \int_0^1 \phi_1(y)\varphi_2(y)dy = 6\varepsilon + \frac{3}{2};$$

$$b_{1,0} = \int_0^1 \phi_2(y)\varphi_1(y)dy = 40\varepsilon + \frac{3}{2},\ b_{0,1} = \int_0^1 \phi_2(y)\varphi_2(y)dy = \frac{40}{3}\varepsilon + \frac{145}{18}.$$

Since $a_{1,0}-1>0$, the conditions in Theorem 1 are not met. Thus, equation (12) does not have any positive solutions in $C[0,1]$.
Similarly, when $n=2$, by a simple computation, we can get

$$a_{2,0}=3+9\varepsilon+18\varepsilon^2,\ a_{1,1}=6\varepsilon+12\varepsilon^2,\ a_{0,2}=\frac{1}{3}+\varepsilon+2\varepsilon^2;$$

$$b_{2,0}=1+3\varepsilon+40\varepsilon^2,\ b_{1,1}=\frac{154}{9}\varepsilon+\frac{80}{3}\varepsilon^2,\ b_{0,2}=2+\frac{145}{27}\varepsilon+\frac{40}{9}\varepsilon^2;$$

$$\alpha_0=b_{2,0}=1+3\varepsilon+40\varepsilon^2,\ \alpha_1=b_{1,1}-a_{2,0}=\frac{26}{3}\varepsilon^2+\frac{73}{9}\varepsilon-3,$$

$$\alpha_2=b_{0,2}-a_{1,1}=-\frac{68}{9}\varepsilon^2-\frac{17}{27}\varepsilon+2,\ \alpha_3=-a_{0,2}=-2\varepsilon^2-\varepsilon-\frac{1}{3}.$$

Let

$$p=\frac{\alpha_1}{\alpha_0},\ r=\frac{\alpha_2}{\alpha_0},\ t=\frac{\alpha_3}{\alpha_0};$$

$$\Delta_1=p^2-3r,\ \Delta_2=rp^2+3tp-4r^2,$$

$$\Delta_3=-4r^3+18rtp+p^2r^2-4p^3t-27t^2,$$

we have (up to a positive factor)

$$p=-1+2.7037\varepsilon+2.8889\varepsilon^2,$$

$$\Delta_1=-65.959\varepsilon^2+94.716\varepsilon^3-21.593\varepsilon+327.26\varepsilon^4+1,$$

$$\Delta_2=702.22\varepsilon^4-1291.9\varepsilon^5+255.78\varepsilon^3-2356.3\varepsilon^6-78.517\varepsilon^2-26.207\varepsilon+1,$$

$$\Delta_3=1-27.778\varepsilon-1.4371\times10^5\varepsilon^6-23275.0\varepsilon^4-1.0374\times10^5\varepsilon^5-63.724\varepsilon^2-1222.6\varepsilon^3.$$

By numerical computations, it is easy to get

The real roots of $p=0$ are -1.2197 and 0.2838;
The real roots of $\Delta_1=0$ are 0.041426 and 0.45024;
The real roots of $\Delta_2=0$ are -0.70495 and 0.034952;
The real roots of $\Delta_3=0$ are -0.21287 and 0.03143.

Hence, by Corollaries 2,3,4, it is easy to know that there exists a positive number $r_0\approx0.03143$ (note here the difference between the exactness of the conditions in Corollaries 2,3,4 and the inexactness of the numerical computations above), such that: when $0\le\varepsilon<r_0$, equation (12) has 3 positive solutions in $C[0,1]$; when $\varepsilon=r_0$, equation (12) has 2 positive solutions in $C[0,1]$; when $\varepsilon>r_0$, equation (12) has 1 positive solutions in $C[0,1]$.

Remark 9 The case when $n=2$ in the example above has also been studied in [3]. Our result here is completely consistent with the result in [3].

Example 3 Consider the integral equation (12) in the example above. When $n=3,\varepsilon$ is 2 or 0.2, determine the number of its positive solutions in $C[0,1]$.
Similar to Example 2, when $n=3$, using the notations in Section 2 and by a simple computation, we can get

$$a_{3,0}=\frac{9}{4}+9\varepsilon+\frac{27}{2}\varepsilon^2+18\varepsilon^3,\ a_{2,1}=3\varepsilon+\frac{27}{2}\varepsilon^2+18\varepsilon^3,$$

$$a_{1,2}=\frac{9}{2}\varepsilon^2+6\varepsilon^3+\varepsilon,\ a_{0,3}=\frac{2}{3}\varepsilon^3+\frac{1}{12}+\frac{1}{3}\varepsilon+\frac{1}{2}\varepsilon^2;$$

$$b_{3,0}=\frac{3}{4}+3\varepsilon+\frac{9}{2}\varepsilon^2+40\varepsilon^3,\ b_{2,1}=\varepsilon+\frac{163}{6}\varepsilon^2+40\varepsilon^3,$$

$$b_{1,2} = \frac{299}{18}\varepsilon^2 + \frac{40}{3}\varepsilon^3 + 6\varepsilon,\ b_{0,3} = \frac{163}{54}\varepsilon^2 + \frac{287}{540} + 2\varepsilon + \frac{37}{27}\varepsilon^3.$$

$$\alpha_0 = \frac{3}{4} + 3\varepsilon + \frac{9}{2}\varepsilon^2 + 40\varepsilon^3,\ \alpha_1 = -8\varepsilon + \frac{41}{3}\varepsilon^2 + 22\varepsilon^3 - \frac{9}{4},$$
$$\alpha_2 = \frac{28}{9}\varepsilon^2 - \frac{14}{3}\varepsilon^3 + 3\varepsilon,\ \alpha_3 = -\frac{40}{27}\varepsilon^2 + \frac{287}{540} + \varepsilon - \frac{125}{27}\varepsilon^3,$$
$$\alpha_4 = -\frac{2}{3}\varepsilon^3 - \frac{1}{12} - \frac{1}{3}\varepsilon - \frac{1}{2}\varepsilon^2.$$

Hence, when $\varepsilon = 2$, we have $\alpha_0 s^8 + \alpha_1 s^6 + \alpha_2 s^4 + \alpha_3 s^2 + \alpha_4 = 344.75s^8 + 212.42s^6 - 18.889s^4 - 40.431s^2 - 8.0833$. By a simple computation, the revised sign list of its discriminant sequence is

$$[1, -1, -1, -1, 1, 1, 1, -1].$$

By Theorem 3, equation (12) has only 1 positive solution in $C[0,1]$.
When $\varepsilon = 0.2$, we have $\alpha_0 s^8 + \alpha_1 s^6 + \alpha_2 s^4 + \alpha_3 s^2 + \alpha_4 = 1.85s^8 - 3.1273s^6 + 0.68711s^4 + 0.63519s^2 - 0.17533$. By a simple computation, the revised sign list of its discriminant sequence is

$$[1, 1, 1, -1, -1, -1, -1, -1].$$

By Theorem 3, equation (12) has 3 positive solutions in $C[0,1]$.

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