ON SOME MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL MATRIX ARGUMENTS

Lalit Mohan Upadhyaya* & H.S.Dhami**
Department of Mathematics,
University of Kumaun,
Almora Campus,
Almora (Uttaranchal)
India -263601

AMS Subject Classification no. 33C65,33D99,44

ABSTRACT

In continuation of our previous studies [6], here we propose to define the Exton's K_3, K_6 and K_{11} quadruple hypergeometric functions, the Exton's ${}^{(k)}_{(1)}E_D^{(n)}$ and the Chandel's ${}^{(k)}_{(1)}E_C^{(n)}$ functions, the generalized Horn's ${}^{(k)}H_4^{(n)}$ function and the generalized Srivastava $H_B^{(n)}$ and $H_C^{(n)}$ functions of matrix arguments.

INTRODUCTION

The systematic treatment of hypergeometric functions of four variables has been given in the work of Exton [1], where different functions have been defined as generalization of certain classes of quadruple hypergeoemtric functions. We have defined matrix arguments for some of them including the generalized Srivastava functions $H_B^{(n)}$ and $H_C^{(n)}$ and the function ${}^{(k)}H_4^{(n)}$. Possible limiting forms, transformation relations and cases of reducibility have been dealt with. All the matrices appearing in this paper are $(p\times p)$ real symmetric positive definite matrices. The meanings of all other symbols used are the same as in the works of Mathai [2-4].

---

* Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun (Uttaranchal), India, 248179.
** To whom all the correspondence may be addressed.

1. Preliminary Definitions and Results

In this section we first quote some of the definitions and a result of Mathai [3] which will be required by us in proving our results in this paper. References to the other results of Mathai, which will be used by us, will be made at the appropriate places in the paper.

DEFINITION 1.1 :- The Ψ_2-function of matrix arguments

$$\Psi_2 = \Psi_2(a; c, c'; -X, -Y)$$

is defined as that function for which the matrix transform (M-transform) is the following :

$$M(\Psi_2) = [\int_{X>0} \int_{Y>0} |X|^{\rho_1 - (p+1)/2} |Y|^{\rho_2 - (p+1)/2} \times \Psi_2(a\,; c, c'; -X, -Y) dX dY]$$

$$= \frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(a)} \frac{\Gamma_p(a-\rho_1-\rho_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)}{\Gamma_p(c-\rho_1)\Gamma_p(c'-\rho_2)} \quad ....(1.1)$$

for Re $(a-\rho_1-\rho_2, c-\rho_1, c'-\rho_2, \rho_1, \rho_2) > (p-1)/2$.

where Re(.) denotes the real part of (.).

DEFINITION 1.2 :- The Lauricella function

$$F_C^{(n)} = F_C^{(n)}(a,b;c_1,....,c_n;-X_1,....,-X_n)$$

of matrix arguments is defined as that function for which the M-transform is the following :

$$M(F_C^{(n)}) = [\int_{X_1>0} ... \int_{X_n>0} \left|X_1\right|^{\rho_1-(p+1)/2} ... \left|X_n\right|^{\rho_n-(p+1)/2} \times F_C^{(n)}(a,b;c_1,...,c_n;-X_1,...,-X_n) dX_1 ... dX_n]$$

$$= [\frac{\{\prod_{j=1}^{n} \Gamma_p(c_j)\Gamma_p(\rho_j)\}}{\Gamma_p(a)\Gamma_p(b)} \frac{\Gamma_p(a-\rho_1-\cdots-\rho_n)\Gamma_p(b-\rho_1-\cdots-\rho_n)}{\{\prod_{j=1}^{n} \Gamma_p(c_j-\rho_j)\}}] \quad ...(1.2)$$

for $\mathrm{Re}(a-\rho_1-\cdots-\rho_n, b-\rho_1-\cdots-\rho_n, c_j-\rho_j, \rho_j) > (p-1)/2, j=1,...,n.$

DEFINITION 1.3 : The $\Psi_2^{(n)}$-function of matrix arguments

$$\Psi_2^{(n)} = \Psi_2^{(n)}(a; c_1,\dots,c_n; -X_1,\dots,-X_n)$$

is defined as a function for which the M-transform is the following :

$$M(\Psi_2^{(n)}) = [\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \dots |X_n|^{\rho_n-(p+1)/2} \times \Psi_2^{(n)}(a; c_1,\dots,c_n; -X_1,\dots,-X_n)\, dX_1 \dots dX_n]$$

$$= \frac{\{\prod_{j=1}^{n} \Gamma_p(c_j)\}}{\Gamma_p(a)} \; \frac{\Gamma_p(a-\rho_1-\cdots-\rho_n)\{\prod_{j=1}^{n} \Gamma_p(\rho_j)\}}{\{\prod_{j=1}^{n} \Gamma_p(c_j-\rho_j)\}} \qquad \dots\dots\dots\dots(1.3)$$

for Re $(a-\rho_1-\cdots-\rho_n, c_j-\rho_j, \rho_j) > (p-1)/2$, $j=1,\dots,n$.

DEFINITION 1.4 : The $\Phi_2^{(n)}$-function of matrix arguments

$$\Phi_2^{(n)} = \Phi_2^{(n)}(b_1,\dots,b_n; c; -X_1,\dots,-X_n)$$

is defined as a function for which the M-transform is the following :

$$M(\Phi_2^{(n)}) = [\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \dots |X_n|^{\rho_n-(p+1)/2} \times \Phi_2^{(n)}(b_1,\dots,b_n; c; -X_1,\dots,-X_n)\, dX_1 \dots dX_n]$$

$$= \frac{\Gamma_p(c)\{\prod_{j=1}^{n} \Gamma_p(b_j-\rho_j)\Gamma_p(\rho_j)\}}{\{\prod_{j=1}^{n} \Gamma_p(b_j)\}\Gamma_p(c-\rho_1-\cdots-\rho_n)} \qquad \dots(1.4)$$

for Re $(c-\rho_1-\cdots-\rho_n, b_j-\rho_j, \rho_j) > (p-1)/2$, $j=1,\dots,n$.

THEOREM 1.1 :

$$ {}_1F_1(a\,;c;-X)=\Big[\frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}\int_0^I |Y|^{a-(p+1)/2}\times $$

$$ |I-Y|^{c-a-(p+1)/2}e^{-tr(XY)}\,dY\Big] \qquad ....(1.5) $$

for $\mathrm{Re}(a,c-a)>(p-1)/2$ and for $0<Y<I$.

2. The Exton's Quadruple Hypergeometric Functions

DEFINITION 2.1 :

The function $K_3 = K_3(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_2,c_1;-X,-Y,-Z,-T)$ of matrix arguments is defined as that function for which the M-transform is the following:

$$ M(K_3)=\Big[\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times $$

$$ |Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\times $$

$$ K_3(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_2,c_1;-X,-Y,-Z,-T)dXdYdZdT\Big] $$

$$ =\Big[\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b_1-\rho_1-\rho_2)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_3-\rho_4)}{\Gamma_p(b_2)}\times $$

$$ \frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1-\rho_4)}\frac{\Gamma_p(c_2)}{\Gamma_p(c_2-\rho_2-\rho_3)}\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)\Big] \qquad ...(2.1) $$

for $\mathrm{Re}(a-\rho_1-\rho_2-\rho_3-\rho_4, b_1-\rho_1-\rho_2, b_2-\rho_3-\rho_4, c_1-\rho_1-\rho_4,$
$c_2-\rho_2-\rho_3, \rho_i) > (p-1)/2, \quad i=1,2,3,4.$

DEFINITION 2.2 :

$$K_6 = K_6(a,a,a,a;b,b,c_1,c_2;e,d,d,d;-X,-Y,-Z,-T)$$

$$M(K_6)=[\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times |Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_6(a,a,a,a;b,b,c_1,c_2;e,d,d,d;-X,-Y,-Z,-T)dXdYdZdT\ ]$$

$$=[\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b-\rho_1-\rho_2)}{\Gamma_p(b)}\frac{\Gamma_p(c_1-\rho_3)}{\Gamma_p(c_1)}\frac{\Gamma_p(c_2-\rho_4)}{\Gamma_p(c_2)}\times$$

$$\frac{\Gamma_p(e)}{\Gamma_p(e-\rho_1)}\frac{\Gamma_p(d)}{\Gamma_p(d-\rho_2-\rho_3-\rho_4)}\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)] \qquad ...(2.2)$$

for Re $(a-\rho_1-\rho_2-\rho_3-\rho_4, b-\rho_1-\rho_2, c_1-\rho_3, c_2-\rho_4, e-\rho_1, d-\rho_2-\rho_3-\rho_4, \rho_i)>(p-1)/2$, $i=1,2,3,4$.

DEFINITION 2.3 :

$$K_{11} = K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

$$M(K_{11})=[\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times |Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)dXdYdZdT\ ]$$

$$=[\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(b_1-\rho_1)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)}\frac{\Gamma_p(b_3-\rho_3)}{\Gamma_p(b_3)}\times$$

$$\frac{\Gamma_p(b_4-\rho_4)}{\Gamma_p(b_4)}\frac{\Gamma_p(c)}{\Gamma_p(c-\rho_1-\rho_2-\rho_3)}\frac{\Gamma_p(d)}{\Gamma_p(d-\rho_4)}\times$$

$$\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)] \qquad .....(2.3)$$

for Re$(a-\rho_1-\rho_2-\rho_3-\rho_4, b_i-\rho_i, c-\rho_1-\rho_2-\rho_3, d-\rho_4, \rho_i) > (p-1)/2$, $i=1,2,3,4$.

THEOREM 2.1 :

$$K_3(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_2,c_1;-X,-Y,-Z,-T)$$

$$=\left[\frac{1}{\Gamma_p(b_1)\Gamma_p(b_2)}\int_{S_1>0}\int_{S_2>0} e^{-\mathrm{tr}(S_1+S_2)}|S_1|^{b_1-(p+1)/2}\times\right.$$

$$|S_2|^{b_2-(p+1)/2}\Psi_2(a;c_1,c_2;-S_1^{1/2}XS_1^{1/2}-S_2^{1/2}TS_2^{1/2},$$

$$\left.-S_1^{1/2}YS_1^{1/2}-S_2^{1/2}ZS_2^{1/2})dS_1dS_2\right] \qquad ....(2.4)$$

for Re $(b_1,b_2)>(p-1)/2$.

PROOF : We take the M-transform of the right side of eq.(2.4) with respect to the variables X,Y,Z,T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively to get ,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\Psi_2(a;c_1,c_2;-S_1^{1/2}XS_1^{1/2}-S_2^{1/2}TS_2^{1/2},$$

$$-S_1^{1/2}YS_1^{1/2}-S_2^{1/2}ZS_2^{1/2})dXdYdZdT \qquad ....(2.5)$$

Applying the transformations,

$$X_1=S_1^{1/2}XS_1^{1/2}; \qquad Y_1=S_1^{1/2}YS_1^{1/2}; \qquad Z_1=S_2^{1/2}ZS_2^{1/2};$$

$$T_1=S_2^{1/2}TS_2^{1/2}$$

then, $dX_1=|S_1|^{(p+1)/2}dX$; $dY_1=|S_1|^{(p+1)/2}dY$; $dZ_1=|S_2|^{(p+1)/2}dZ$;

$dT_1=|S_2|^{(p+1)/2}dT$; and, $|X_1|=|S_1||X|$; $|Y_1|=|S_1||Y|$; $|Z_1|=|S_2||Z|$;

$|T_1|=|S_2||T|$

which renders the expression (2.5) as below,

$$|S_1|^{-\rho_1-\rho_2}|S_2|^{-\rho_3-\rho_4}\int_{X_1>0}\int_{Y_1>0}\int_{Z_1>0}\int_{T_1>0}|X_1|^{\rho_1-(p+1)/2}$$
$$|Y_1|^{\rho_2-(p+1)/2}|Z_1|^{\rho_3-(p+1)/2}|T_1|^{\rho_4-(p+1)/2}\times$$
$$\Psi_2(a;c_1,c_2;-X_1-T_1,-Y_1-Z_1)dX_1dY_1dZ_1dT_1 \qquad \text{.........(2.6)}$$

Now making use of another set of transformations,

$X_2=X_1,\ X_3=X_1+T_1;\quad Y_2=Y_1,\ Y_3=Y_1+Z_1$

so that, from eq.(6.2) page 91 of Mathai [2], we have,

$dX_1dT_1=dX_2dX_3$ and $dY_1dZ_1=dY_2dY_3$

and, $|X_1|=|X_2|$, $|T_1|=|X_3-X_2|$; $|Y_1|=|Y_2|$, $|Z_1|=|Y_3-Y_2|$

where, $0<X_2<X_3$ and $0<Y_2<Y_3$,

which render the expression (2.6) as below,

$$|S_1|^{-\rho_1-\rho_2}|S_2|^{-\rho_3-\rho_4}\int_{0<X_2<X_3}\int_{X_3>0}\int_{0<Y_2<Y_3}\int_{Y_3>0}\times$$
$$|X_2|^{\rho_1-(p+1)/2}|X_3-X_2|^{\rho_4-(p+1)/2}|Y_2|^{\rho_2-(p+1)/2}\times$$
$$|Y_3-Y_2|^{\rho_3-(p+1)/2}\Psi_2(a;c_1,c_2;-X_3,-Y_3)dX_2dX_3dY_2dY_3 \qquad \text{....(2.7)}$$

Integrating out the variables X_2 and Y_2 by using a type-1 Beta integral in the expression (2.7) and then using the definition (1.1) in the resulting expression we obtain,

$$|S_1|^{-\rho_1-\rho_2}|S_2|^{-\rho_3-\rho_4}\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)}\frac{\Gamma_p(\rho_1)}{\Gamma_p(c_1-\rho_4-\rho_1)}\times$$
$$\frac{\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(c_2-\rho_3-\rho_2)} \qquad \text{.........(2.8)}$$

Substituting this expression on the right side of eq.(2.4) and integrating out the variables S_1 and S_2 by using a Gamma integral in the resulting expression ,we finally have $M(K_3)$ as given by eq.(2.1). Therefore, the theorem is established.

THEOREM 2.2 :

$$K_6(a,a,a,a;b,b,c_1,c_2;e,d,d,d;-X,-Y,-Z,-T)$$

$$=\Big[\frac{1}{\Gamma_p(b)\Gamma_p(c_1)\Gamma_p(c_2)}\int_{S_1>0}\int_{S_2>0}\int_{S_3>0} e^{-tr(S_1+S_2+S_3)}\times$$

$$|S_1|^{b-(p+1)/2}|S_2|^{c_1-(p+1)/2}|S_3|^{c_2-(p+1)/2}\Psi_2(a;e,d;-S_1^{1/2}XS_1^{1/2},$$

$$-S_1^{1/2}YS_1^{1/2}-S_2^{1/2}ZS_2^{1/2}-S_3^{1/2}TS_3^{1/2})dS_1dS_2dS_3\Big] \qquad ....(2.9)$$

for Re $(b,c_1,c_2)>(p-1)/2$.

PROOF : Taking M-transform of the right side of eq.(2.9) with respect to the variables X,Y,Z,T and the parameters $\rho_1,\rho_2,\rho_3,\rho_4$ respectively we get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\Psi_2(a;e,d;-S_1^{1/2}XS_1^{1/2},$$

$$-S_1^{1/2}YS_1^{1/2}-S_2^{1/2}ZS_2^{1/2}-S_3^{1/2}TS_3^{1/2})dXdYdZdT \qquad ....(2.10)$$

Applying the transformations,

$$X_1=S_1^{1/2}XS_1^{1/2};\ Y_1=S_1^{1/2}YS_1^{1/2};\ Z_1=S_2^{1/2}ZS_2^{1/2};\ T_1=S_3^{1/2}TS_3^{1/2}$$

whence, $dX_1=|S_1|^{(p+1)/2}dX;\ dY_1=|S_1|^{(p+1)/2}dY;\ dZ_1=|S_2|^{(p+1)/2}dZ;$

$dT_1=|S_3|^{(p+1)/2}dT$; and $|X_1|=|S_1||X|;\ |Y_1|=|S_1||Y|;\ |Z_1|=|S_2||Z|;\ |T_1|=|S_3||T|$

which render the expression (2.10) as below,

$$\left|S_1\right|^{-\rho_1-\rho_2}\left|S_2\right|^{-\rho_3}\left|S_3\right|^{-\rho_4}\times$$

$$\int_{X_1>0}\int_{Y_1>0}\int_{Z_1>0}\int_{T_1>0}\left|X_1\right|^{\rho_1-(p+1)/2}\left|Y_1\right|^{\rho_2-(p+1)/2}\times$$

$$\left|Z_1\right|^{\rho_3-(p+1)/2}\left|T_1\right|^{\rho_4-(p+1)/2}\times$$

$$\Psi_2(a;e,d;-X_1,-Y_1-Z_1-T_1)dX_1dY_1dZ_1dT_1 \quad ....(2.11)$$

Now making use of another set of transformations,

$$Y_2=Y_1, Y_3=Y_1+Z_1, Y_4=Y_1+Z_1+T_1$$

so that, we have, from eq. (6.7) page 95 of Mathai [2],

$$dY_1dZ_1dT_1=dY_2dY_3dY_4 \text{ and } \left|Y_1\right|=\left|Y_2\right|,\left|Z_1\right|=\left|Y_3-Y_2\right|,\left|T_1\right|=\left|Y_4-Y_3\right|,$$

where, $0<Y_2<Y_3<Y_4$.

These transformations on substitution in the expression (2.11) convert it into the following form,

$$\left|S_1\right|^{-\rho_1-\rho_2}\left|S_2\right|^{-\rho_3}\left|S_3\right|^{-\rho_4}\times$$

$$\int_{X_1>0}\int_{0<Y_2<Y_3}\int_{0<Y_3<Y_4}\int_{Y_4>0}\left|X_1\right|^{\rho_1-(p+1)/2}\left|Y_2\right|^{\rho_2-(p+1)/2}\times$$

$$\left|Y_3-Y_2\right|^{\rho_3-(p+1)/2}\left|Y_4-Y_3\right|^{\rho_4-(p+1)/2}\times$$

$$\Psi_2(a;e,d;-X_1,-Y_4)dX_1dY_2dY_3dY_4 \quad ....(2\ .12)$$

Now, integrating out the variables Y_2 and Y_3 in the expression (2.12), one- by- one and in order, by using a type-1 Beta integral and then using definition (1.1) in the resulting expression, we have,

$$\left|S_1\right|^{-\rho_1-\rho_2}\left|S_2\right|^{-\rho_3}\left|S_3\right|^{-\rho_4}\frac{\Gamma_p(\rho_4)\Gamma_p(\rho_3)\Gamma_p(\rho_2)\Gamma_p(e)\Gamma_p(d)\Gamma_p(\rho_1)}{\Gamma_p(a)\Gamma_p(e-\rho_1)\Gamma_p(d-\rho_4-\rho_3-\rho_2)}\times$$

$$\Gamma_p(a-\rho_1-\rho_4-\rho_3-\rho_2) \quad ....(2.13)$$

Substituting this expression on the right side of eq.(2.9) and then integrating out the variables S_1, S_2, S_3 in the resulting expression by using a Gamma integral,we finally have $M(K_6)$ as given by eq.(2.2), which proves the theorem.

THEOREM 2.3:

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

$$=\left[\frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(d-b_4)\Gamma_p(c-b_1-b_2-b_3)}\iiiint \times\right.$$

$$|U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}|S|^{b_4-(p+1)/2}\times$$

$$|I-U-V-W|^{c-b_1-b_2-b_3-(p+1)/2}|I-S|^{d-b_4-(p+1)/2}\times$$

$$|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}ZW^{1/2}+S^{1/2}TS^{1/2}|^{-a}\times$$

$$\left.dUdVdWdS\right] \qquad ....(2.14)$$

where $U>0, V>0, W>0, 0<S<I$ and $0<U+V+W<I$ and, for $\mathrm{Re}(b_i, c-b_1-b_2-b_3, d-b_4)>(p-1)/2,\ i=1,2,3,4.$

PROOF: We take the M-transform of the right side of eq.(2.14) with respect to the variables X,Y,Z,T and the parameters $\rho_1, \rho_2, \rho_3, \rho_4$ to get

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{T>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$

$$|Z|^{\rho_3-(p+1)/2}|T|^{\rho_4-(p+1)/2}\times$$

$$|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}ZW^{1/2}+S^{1/2}TS^{1/2}|^{-a}dXdYdZdT \quad ....(2.15)$$

Appealing to the transformations,

$$X_1=U^{1/2}XU^{1/2};Y_1=V^{1/2}YV^{1/2};Z_1=W^{1/2}ZW^{1/2};T_1=S^{1/2}TS^{1/2},$$

so that, $dX_1=|U|^{(p+1)/2}dX; dY_1=|V|^{(p+1)/2}dY; dZ_1=|W|^{(p+1)/2}dZ;$

$dT_1=|S|^{(p+1)/2}dT$ and, $|X_1|=|U||X|; |Y_1|=|V||Y|; |Z_1|=|W||Z|; |T_1|=|S||T|,$

which convert the expression (2.15) as below ,

$$|U|^{-\rho_1}|V|^{-\rho_2}|W|^{-\rho_3}|S|^{-\rho_4}\int_{X_1>0}\int_{Y_1>0}\int_{Z_1>0}\int_{T_1>0}|X_1|^{\rho_1-(p+1)/2}\times$$

$$|Y_1|^{\rho_2-(p+1)/2}|Z_1|^{\rho_3-(p+1)/2}|T_1|^{\rho_4-(p+1)/2}|I+X_1+Y_1+Z_1+T_1|^{-a}\times$$

$$dX_1dY_1dZ_1dT_1 \qquad \text{........}(2.16)$$

This expression, on integrating out the variables X_1, Y_1, Z_1, T_1 by using a type –2 Dirichlet integral, reduces to,

$$|U|^{-\rho_1}|V|^{-\rho_2}|W|^{-\rho_3}|S|^{-\rho_4}\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)\Gamma_p(\rho_4)$$

$$\frac{\Gamma_p(a-\rho_1-\rho_2-\rho_3-\rho_4)}{\Gamma_p(a)} \qquad \text{.......}(2.17)$$

Substituting this expression on the right side of eq.(2.14) and then integrating out the variables U,V,W and S in the resulting expression by using a type –1 Dirichlet integral and a type –1 Beta integral respectively we finally have $M(K_{11})$ as given by eq. (2.3),which proves the theorem.

A limiting case of eq.(2.14) has the following form :

$$\lim_{a\to\infty} K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;\frac{-X}{a},\frac{-Y}{a},\frac{-Z}{a},\frac{-T}{a})$$

$$=[\frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(d-b_4)\Gamma_p(c-b_1-b_2-b_3)}\iiiint\times$$

$$|U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}|S|^{b_4-(p+1)/2}\times$$

$$|I-U-V-W|^{c-b_1-b_2-b_3-(p+1)/2}|I-S|^{d-b_4-(p+1)/2}\times$$

$$e^{-tr(UX+VY+WZ+ST)}dUdVdWdS\,] \qquad \text{.........}(2.18)$$

$$=[\frac{\Gamma_p(c)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c-b_1-b_2-b_3)}\times{}_1F_1(b_4;d;-T)\iiint\times$$

$$|U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}\times$$

$$|I-U-V-W|^{c-b_1-b_2-b_3-(p+1)/2}e^{-tr(UX+VY+WZ)}dUdVdW\,] \quad \text{...}(2.19)$$

which follows from eq.(2.18) by the use of theorem (1.1).

THEOREM 2.4: A case of reducibility-

$$\lim_{a\to\infty} K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;\frac{-X}{a},\frac{-X}{a},\frac{-X}{a},\frac{-T}{a})$$

$$={}_1F_1(b_4;d;-T)\cdot{}_1F_1(b_1+b_2+b_3;c;-X) \qquad ....(2.20)$$

PROOF: This theorem follows by putting Z=Y=X in eq.(2.19) and applying the transformations

$W_1 = U, W_2 = U+V, W_3 = U+V+W$, then $dUdVdW = dW_1dW_2dW_3$, from eq.(6.7) page 95 of Mathai [2], and,

$$|U| = |W_1|; |V| = |W_2 - W_1|; |W| = |W_3 - W_2|$$

where, $0 < W_1 < W_2 < W_3 < I$,

which renders the right side of eq.(2.19) as ,

$$[\frac{\Gamma_p(c)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c-b_1-b_2-b_3)}\times{}_1F_1(b_4;d;-T)\iiint\times$$

$$|W_1|^{b_1-(p+1)/2}|W_2-W_1|^{b_2-(p+1)/2}|W_3-W_2|^{b_3-(p+1)/2}\times$$

$$|I-W_3|^{c-b_1-b_2-b_3-(p+1)/2}e^{-tr(W_3X)}dW_1dW_2dW_3] \quad ...(2.21)$$

Now, integrating out W_1 and W_2 in the expression (2.21), one-by-one and in order, by using a type-1 Beta integral and then applying theorem (1.1) to the resulting expression, the desired result is obtained.

THEOREM 2.5 : A transformation theorem –

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

$$=|I+X|^{-a}K_{11}[a,a,a,a;c-b_1-b_2-b_3,b_2,b_3,b_4;c,c,c,d;$$

$$(I+X)^{-1/2}X(I+X)^{-1/2},-(I+X)^{-1/2}(Y-X)(I+X)^{-1/2},$$

$$-(I+X)^{-1/2}(Z-X)(I+X)^{-1/2},-(I+X)^{-1/2}T(I+X)^{-1/2}] \qquad ....(2.22)$$

where $Y-X>0$ and $Z-X>0$.

$$=\left|I+Y\right|^{-a}K_{11}[a,a,a,a;b_1,c-b_1-b_2-b_3,b_3,b_4;c,c,c,d;$$
$$-(I+Y)^{-1/2}(X-Y)(I+Y)^{-1/2},(I+Y)^{-1/2}Y(I+Y)^{-1/2},$$
$$-(I+Y)^{-1/2}(Z-Y)(I+Y)^{-1/2},-(I+Y)^{-1/2}T(I+Y)^{-1/2}] \qquad ....(2.23)$$

where $Z-Y>0$ and $X-Y>0$.

$$=\left|I+Z\right|^{-a}K_{11}[a,a,a,a;b_1,b_2,c-b_1-b_2-b_3,b_4;c,c,c,d;$$
$$-(I+Z)^{-1/2}(X-Z)(I+Z)^{-1/2},-(I+Z)^{-1/2}(Y-Z)(I+Z)^{-1/2},$$
$$(I+Z)^{-1/2}Z(I+Z)^{-1/2},-(I+Z)^{-1/2}T(I+Z)^{-1/2}] \qquad ....(2.24)$$

where $X-Z>0$ and $Y-Z>0$.

$$=\left|I+T\right|^{-a}K_{11}[a,a,a,a;b_1,b_2,b_3,d-b_4;c,c,c,d;$$
$$-(I+T)^{-1/2}X(I+T)^{-1/2},-(I+T)^{-1/2}Y(I+T)^{-1/2},$$
$$-(I+T)^{-1/2}Z(I+T)^{-1/2},(I+T)^{-1/2}T(I+T)^{-1/2}] \qquad ....(2.25)$$

$$=\left|I+X+T\right|^{-a}K_{11}[a,a,a,a;c-b_1-b_2-b_3,b_2,b_3,d-b_4;c,c,c,d;$$
$$(I+X+T)^{-1/2}X(I+X+T)^{-1/2},-(I+X+T)^{-1/2}(Y-X)(I+X+T)^{-1/2},$$
$$-(I+X+T)^{-1/2}(Z-X)(I+X+T)^{-1/2},(I+X+T)^{-1/2}T(I+X+T)^{-1/2}] .....(2.26)$$

where $Y-X>0$ and $Z-X>0$.

PROOF : To prove this theorem we define the function K_{11} through an integral representation,

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Z,-T)$$

$$=\Big[\frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(d-b_4)\Gamma_p(c-b_1-b_2-b_3)}\iiiint \times$$

$$|U|^{b_1-(p+1)/2}|V|^{b_2-(p+1)/2}|W|^{b_3-(p+1)/2}|S|^{b_4-(p+1)/2}\times$$

$$|I-U-V-W|^{c-b_1-b_2-b_3-(p+1)/2}|I-S|^{d-b_4-(p+1)/2}\times$$

$$|I+X^{1/2}UX^{1/2}+Y^{1/2}VY^{1/2}+Z^{1/2}WZ^{1/2}+T^{1/2}ST^{1/2}|^{-a}\times$$

$$dUdVdWdS\Big] \qquad \ldots\ldots\ldots\ldots(2.27)$$

where $U>0, V>0, W>0, 0<S<I$ and $0<U+V+W<I$ and for $\mathrm{Re}(b_i, c-b_1-b_2-b_3, d-b_4)>(p-1)/2$, $i=1,2,3,4$.

To prove eq.(2.22), we apply the transformations $U_1 = I-U-V-W, V_1 = V, W_1 = W$, so that, $dU_1dV_1dW_1 = dUdVdW$, to eq.(2.27) and observing that,

$$|I+X^{1/2}(I-U_1-V_1-W_1)X^{1/2}+Y^{1/2}V_1Y^{1/2}+Z^{1/2}W_1Z^{1/2}$$

$$+T^{1/2}ST^{1/2}| = |I+X|.|I-(I+X)^{-1/2}X^{1/2}U_1X^{1/2}(I+X)^{-1/2}+$$

$$(I+X)^{-1/2}(Y-X)^{1/2}V_1(Y-X)^{1/2}(I+X)^{-1/2}+$$

$$(I+X)^{-1/2}(Z-X)^{1/2}W_1(Z-X)^{1/2}(I+X)^{-1/2}+$$

$$(I+X)^{-1/2}T^{1/2}ST^{1/2}(I+X)^{-1/2}|$$

the desired result follows after a suitable reinterpretation of eq.(2.27). The results in eqs.(2.23) and (2.24) follow similarly.

To prove the result of eq.(2.25) we observe that,

$$\left| I + X^{1/2} U X^{1/2} + Y^{1/2} V Y^{1/2} + Z^{1/2} W Z^{1/2} + T^{1/2} S T^{1/2} \right|$$

$$= |I+T| . |I + (I+T)^{-1/2} X^{1/2} U X^{1/2} (I+T)^{-1/2} + (I+T)^{-1/2} Y^{1/2} V Y^{1/2} (I+T)^{-1/2} +$$
$$(I+T)^{-1/2} Z^{1/2} W Z^{1/2} (I+T)^{-1/2} - (I+T)^{-1/2} T^{1/2} (I-S) T^{1/2} (I+T)^{-1/2} |$$

Applying this result along with the transformation $S_1 = I - S$, so that, $dS_1 = dS$, to eq.(2.27) and suitably interpreting the resulting expression in the light of eq.(2.27), the result of eq.(2.25) follows immediately.

The result of eq.(2.26) is a combination of the results of eqs.(2.22) and (2.25). It is established by applying the transformations
$U_1 = I - U - V - W, V_1 = V, W_1 = W, S_1 = I - S,$ so that, $dUdVdW = dU_1 dV_1 dW_1$ and $dS = dS_1$, to eq.(2.27) and observing that

$$\left| I + X^{1/2} (I - U_1 - V_1 - W_1) X^{1/2} + Y^{1/2} V_1 Y^{1/2} + Z^{1/2} W_1 Z^{1/2} + T^{1/2} (I - S_1) T^{1/2} \right|$$

$$= |I+X+T| . |I - (I+X+T)^{-1/2} X^{1/2} U_1 X^{1/2} (I+X+T)^{-1/2} +$$
$$(I+X+T)^{-1/2} (Y-X)^{1/2} V_1 (Y-X)^{1/2} (I+X+T)^{-1/2} +$$
$$(I+X+T)^{-1/2} (Z-X)^{1/2} W_1 (Z-X)^{1/2} (I+X+T)^{-1/2}$$
$$- (I+X+T)^{-1/2} T^{1/2} S_1 T^{1/2} (I+X+T)^{-1/2} |$$

and then suitably interpreting the resulting expression as per eq.(2.27) gives the result of eq.(2.26). Two similar results of the type of eq.(2.26) in the variables Y and T and Z and T also exist.

THEOREM 2.6: Another case of reducibility-

(i) $K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Y,-T)$

$= F_G(a,a,a,b_4,b_1,b_2+b_3;d,c,c;-T,-X,-Y)$ ....(2.28)

(ii) $K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-Y,-Y,-Y,-T)$

$= F_2(a,b_4,b_1+b_2+b_3;d,c;-T,-Y)$ ....(2.29)

PROOF : (i) In eq.(2.14) we put Z=Y and observe that

$$|I+U^{1/2}XU^{1/2}+V^{1/2}YV^{1/2}+W^{1/2}ZW^{1/2}+S^{1/2}TS^{1/2}|$$

$$=|I+U^{1/2}XU^{1/2}+(V+W)^{1/2}Y(V+W)^{1/2}+S^{1/2}TS^{1/2}|$$

Now, applying the transformations

$V_1 = V,\ W_1 = V+W$, so that, $dVdW = dV_1dW_1$ and $0 < V_1 < W_1$, the eq.(2.14) transforms into,

$$K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;-X,-Y,-Y,-T)$$

$$=\Big[\frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(b_4)\Gamma_p(d-b_4)\Gamma_p(c-b_1-b_2-b_3)}\iiiint \times$$

$$|U|^{b_1-(p+1)/2}|V_1|^{b_2-(p+1)/2}|W_1-V_1|^{b_3-(p+1)/2}|S|^{b_4-(p+1)/2}\times$$

$$|I-U-W_1|^{c-b_1-b_2-b_3-(p+1)/2}|I-S|^{d-b_4-(p+1)/2}\times$$

$$|I+U^{1/2}XU^{1/2}+W_1^{1/2}YW_1^{1/2}+S^{1/2}TS^{1/2}|^{-a}dUdV_1dW_1dS\Big] \text{ ....(2.30)}$$

Integrating out V_1 in the above equation by using a type-1 Beta integral and comparing the resulting expression with eq.(1.9) of the authors' paper [6], the required result follows.

(ii) This result follows by putting X=Y in eq.(2.28) and using the theorem (1.2) of the authors' paper [6] .

3. The Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$ Function and The Chandel's ${}^{(k)}_{(1)}E^{(n)}_{C}$ Function.

DEFINITION (3.1) :

$${}^{(k)}_{(1)}E^{(n)}_{D}={}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)$$

$$M[{}^{(k)}_{(1)}E^{(n)}_{D}]=[\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$

$${}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)dX_1\cdots dX_n]$$

$$=[\frac{\Gamma_p(a-\rho_1-\cdots-\rho_n)\Gamma_p(b_1-\rho_1)\cdots\Gamma_p(b_n-\rho_n)\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(a)\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-\rho_1-\cdots-\rho_k)\Gamma_p(c'-\rho_{k+1}-\cdots-\rho_n)}\times$$

$$\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)] \qquad ....(3.1)$$

for $\mathrm{Re}(a-\rho_1-\cdots-\rho_n,b_i-\rho_i,c-\rho_1-\cdots-\rho_k,c'-\rho_{k+1}-\cdots-\rho_n,\rho_i)$ $>(p-1)/2, i=1,\cdots,n.$

DEFINITION (3.2) :

$${}^{(k)}_{(1)}E^{(n)}_{C}={}^{(k)}_{(1)}E^{(n)}_{C}(a,a',b;c_1,\cdots,c_n;-X_1,\cdots,-X_n)$$

$$M[{}^{(k)}_{(1)}E^{(n)}_{C}]=[\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$

$${}^{(k)}_{(1)}E^{(n)}_{C}(a,a',b;c_1,\cdots,c_n;-X_1,\cdots,-X_n)dX_1\cdots dX_n]$$

$$=[\frac{\Gamma_p(a-\rho_1-\cdots-\rho_k)\Gamma_p(a'-\rho_{k+1}-\cdots-\rho_n)\Gamma_p(b-\rho_1-\cdots-\rho_n)\Gamma_p(c_1)\cdots\Gamma_p(c_n)}{\Gamma_p(a)\Gamma_p(a')\Gamma_p(b)\Gamma_p(c_1-\rho_1)\cdots\Gamma_p(c_n-\rho_n)}\times$$

$$\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)] \qquad ....(3.2)$$

for $\mathrm{Re}(a-\rho_1-\cdots-\rho_k,a'-\rho_{k+1}-\cdots-\rho_n,b-\rho_1-\cdots-\rho_n,c_i-\rho_i,\rho_i)$ $>(p-1)/2,\ i=1,\cdots,n.$

THEOREM 3.1 :

$$
\begin{aligned}
&{}_{(1)}^{(k)}E_D^{(n)}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)\\
&=\Big[\frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-b_1-\cdots-b_k)\Gamma_p(c'-b_{k+1}-\cdots-b_n)}\int\cdots(n)\cdots\int\ \times\\
&\left|U_1\right|^{b_1-(p+1)/2}\cdots\left|U_n\right|^{b_n-(p+1)/2}\left|I-U_1-\cdots-U_k\right|^{c-b_1-\cdots-b_k-(p+1)/2}\times\\
&\left|I-U_{k+1}-\cdots-U_n\right|^{c'-b_{k+1}-\cdots-b_n-(p+1)/2}\left|\ I+U_1^{1/2}X_1U_1^{1/2}+\cdots+\right.\\
&\left.U_n^{1/2}X_nU_n^{1/2}\ \right|^{-a}dU_1\cdots dU_k dU_{k+1}\cdots dU_n\Big] \qquad ....(3.3)
\end{aligned}
$$

for $U_i>0,\ 0<U_1+\cdots+U_k<I,\ 0<U_{k+1}+\cdots+U_n<I$ and

for $\mathrm{Re}\,(b_i, c-b_1-\cdots-b_k, c'-b_{k+1}-\cdots-b_n)>(p-1)/2,\ i=1,\cdots,n.$

PROOF : Taking the M-transform of the right side of eq.(3.3) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$, we have,

$$
\begin{aligned}
&\int_{X_1>0}\cdots\int_{X_n>0}\left|X_1\right|^{\rho_1-(p+1)/2}\cdots\left|X_n\right|^{\rho_n-(p+1)/2}\left|\ I+U_1^{1/2}X_1U_1^{1/2}+\cdots+\right.\\
&\left.U_n^{1/2}X_nU_n^{1/2}\ \right|^{-a}dX_1\cdots dX_n \qquad ....(3.4)
\end{aligned}
$$

Making use of the transformations,

$Y_i=U_i^{1/2}X_iU_i^{1/2}$, then , $dY_i=\left|U_i\right|^{(p+1)/2}dX_i$, $\left|Y_i\right|=\left|U_i\right|\left|X_i\right|$ for $i=1,\cdots,n$,in eq.(3.4) and integrating out the variables $Y_i\ (i=1,\cdots,n)$, in the resulting expression by using a type-2 Dirichlet integral, we obtain,

$$
\left|U_1\right|^{-\rho_1}\cdots\left|U_n\right|^{-\rho_n}\frac{\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\Gamma_p(a-\rho_1-\cdots-\rho_n)}{\Gamma_p(a)} \qquad ....(3.5)
$$

which, on substitution on the right side of eq.(3.3) and integrating out the variables $U_1,\cdots,U_k$ and $U_{k+1},\cdots,U_n$ in the resulting expression by using a type-1 Dirichlet integral finally gives $M[{}_{(1)}^{(k)}E_D^{(n)}]$ as given by eq.(3.1), thereby proving the theorem.

A limiting form of this function can be seen to have the following form :

$$\lim_{\alpha\to\infty} {}_{(1)}^{(k)}E_D^{(n)}(\alpha, b_1, \cdots, b_n; c, c'; \frac{-X_1}{\alpha}, \cdots, \frac{-X_n}{\alpha})$$

$$= [\frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-b_1-\cdots-b_k)\Gamma_p(c'-b_{k+1}-\cdots-b_n)} \int\cdots(n)\cdots\int \times$$

$$|U_1|^{b_1-(p+1)/2}\cdots|U_n|^{b_n-(p+1)/2}|I-U_1-\cdots-U_k|^{c-b_1-\cdots-b_k-(p+1)/2} \times$$

$$|I-U_{k+1}-\cdots-U_n|^{c'-b_{k+1}-\cdots-b_n-(p+1)/2} e^{-tr(U_1X_1+\cdots+U_kX_k)} \times$$

$$e^{-tr(U_{k+1}X_{k+1}+\cdots+U_nX_n)} dU_1\cdots dU_n] \qquad ....(3.6)$$

THEOREM 3.2 : A case of reducibility-

$$\lim_{\alpha\to\infty} {}_{(1)}^{(k)}E_D^{(n)}[\alpha, b_1, \cdots, b_n; c, c'; \frac{-X}{\alpha}, \cdots (k) \cdots \frac{-X}{\alpha}, \frac{-Y}{\alpha}, \cdots (n-k) \cdots, \frac{-Y}{\alpha}]$$

$$= {}_1F_1(b_1+\cdots+b_k; c; -X).{}_1F_1(b_{k+1}+\cdots+b_n; c'; -Y) \qquad ......(3.7)$$

PROOF : To prove this theorem, we put $X_1 = \cdots = X_k = X$ and $X_{k+1} = \cdots = X_n = Y$ in eq.(3.6) and apply the following sets of transformations on its right side,

$$V_1 = U_1, V_2 = U_1 + U_2, \cdots, V_k = U_1 + \cdots + U_k, \text{ and } W_1 = U_{k+1},$$

$$W_2 = U_{k+1} + U_{k+2}, \cdots, W_{n-k} = U_{k+1} + \cdots + U_n$$

so that from eq.(6.7) page 95 of Mathai [2],

$$dU_1\cdots dU_k = dV_1\cdots dV_k, dU_{k+1}\cdots dU_n = dW_1\cdots dW_{n-k} \text{ and}, |U_1| = |V_1|, |U_2| = |V_2 - V_1|,$$

$$\cdots, |U_k| = |V_k - V_{k-1}|; |U_{k+1}| = |W_1|, |U_{k+2}| = |W_2 - W_1|, \cdots, |U_n| = |W_{n-k} - W_{n-k-1}|$$

where $0 < V_1 < V_2 < \cdots < V_k < I$ and $0 < W_1 < W_2 < \cdots < W_{n-k} < I$,

which render the right side of eq.(3.6) as below,

$$=\Big[\frac{\Gamma_p(c)\Gamma_p(c')}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-b_1-\cdots-b_k)\Gamma_p(c'-b_{k+1}-\cdots-b_n)}\int\cdots(n)\cdots\int \times$$

$$\left|V_1\right|^{b_1-(p+1)/2}\left|V_2-V_1\right|^{b_2-(p+1)/2}\cdots\left|V_k-V_{k-1}\right|^{b_k-(p+1)/2}\times$$

$$\left|W_1\right|^{b_{k+1}-(p+1)/2}\left|W_2-W_1\right|^{b_{k+2}-(p+1)/2}\cdots\left|W_{n-k}-W_{n-k-1}\right|^{b_n-(p+1)/2}\times$$

$$\left|I-V_k\right|^{c\text{-}b_1-\cdots-b_k-(p+1)/2}\left|I-W_{n-k}\right|^{c'\text{-}b_{k+1}-\cdots-b_n-(p+1)/2}e^{-\mathrm{tr}(V_kX)}\times$$

$$e^{-\mathrm{tr}(W_{n-k}Y)}dV_1\cdots dV_k dW_1\cdots dW_{n-k}\Big] \qquad \text{..........(3.8)}$$

Now, integrating out the variables $V_1,\cdots,V_{k-1}$ and $W_1,\cdots,W_{n-k-1}$, one-by-one and in order, in the above expression by using a type-1 Beta integral and using theorem (1.1) in the resulting expression, the required result follows.

A transformation theorem for the function ${}^{(k)}_{(1)}E^{(n)}_{D}$ of matrix arguments will appear in one of our future communications.

THEOREM 3.3 :

$${}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,\cdots,b_n;c,c';-X_1,\cdots,-X_n)$$

$$=\Big[\frac{1}{\Gamma_p(a)}\int_{U>0} e^{-\mathrm{tr}(U)}|U|^{a-(p+1)/2}\Phi_2^{(k)}(b_1,\cdots,b_k;c;-U^{1/2}X_1U^{1/2},\cdots,$$

$$-U^{1/2}X_kU^{1/2})\Phi_2^{(n-k)}(b_{k+1},\cdots,b_n;c';-U^{1/2}X_{k+1}U^{1/2},\cdots,$$

$$-U^{1/2}X_nU^{1/2})dU\Big] \qquad \text{........(3.9)}$$

for $\mathrm{Re}\,(a)>(p-1)/2.$

PROOF : Taking the M-transform of the right side of eq.(3.9) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$ respectively, we get,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_k|^{\rho_k-(p+1)/2}|X_{k+1}|^{\rho_{k+1}-(p+1)/2}\times$$
$$\cdots|X_n|^{\rho_n-(p+1)/2}\Phi_2^{(k)}(b_1,\cdots,b_k;c;-U^{1/2}X_1U^{1/2},\cdots,-U^{1/2}X_kU^{1/2})\times$$
$$\Phi_2^{(n-k)}(b_{k+1},\cdots,b_n;c';-U^{1/2}X_{k+1}U^{1/2},\cdots,-U^{1/2}X_nU^{1/2})dX_1\cdots dX_n \quad .....(3.10)$$

Applying the transformations, $Y_i = U^{1/2}X_iU^{1/2}$, so that , $dY_i = |U|^{(p+1)/2}dX_i$ and $|Y_i| = |U||X_i|$ for $i = 1,\cdots,n$ in the expression (3.10) and using the definition(1.4) in the resulting expression gives,

$$|U|^{-\rho_1-\cdots-\rho_n}\frac{\Gamma_p(b_1-\rho_1)\cdots\Gamma_p(b_n-\rho_n)\Gamma_p(c)\Gamma_p(c')\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)}{\Gamma_p(b_1)\cdots\Gamma_p(b_n)\Gamma_p(c-\rho_1-\cdots-\rho_k)\Gamma_p(c'-\rho_{k+1}-\cdots-\rho_n)} \quad ....(3.11)$$

Using this expression on the right side of eq.(3.9) and integrating out the variable U in the resulting expression by using a Gamma integral we are led to $M[{}^{(k)}_{(1)}E^{(n)}_D]$ as given by eq.(3.1), which concludes the proof.

THEOREM 3.4 :

$${}^{(k)}_{(1)}E^{(n)}_C(a,a',b;c_1,\cdots,c_n;-X_1,\cdots,-X_n)$$
$$=[\frac{1}{\Gamma_p(a)\Gamma_p(a')}\int_{U>0}\int_{V>0} e^{-\mathrm{tr}(U+V)}|U|^{a-(p+1)/2}|V|^{a'-(p+1)/2}\times$$
$$\Psi_2^{(n)}(b;c_1,\cdots,c_n;-U^{1/2}X_1U^{1/2},\cdots,-U^{1/2}X_kU^{1/2},-V^{1/2}X_{k+1}V^{1/2},\cdots,$$
$$-V^{1/2}X_nV^{1/2})dUdV] \quad ....(3.12)$$

for $\mathrm{Re}(a,a') > (p-1)/2$.

PROOF : Taking the M-transform of the $\Psi_2^{(n)}$-function on the right side of eq.(3.12) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$, we have,

$$\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_k|^{\rho_k-(p+1)/2} |X_{k+1}|^{\rho_{k+1}-(p+1)/2} \times$$
$$\cdots |X_n|^{\rho_n-(p+1)/2} \Psi_2^{(n)}(b;c_1,\cdots,c_n;-U^{1/2}X_1U^{1/2},\cdots,-U^{1/2}X_kU^{1/2},$$
$$-V^{1/2}X_{k+1}V^{1/2},\cdots,-V^{1/2}X_nV^{1/2})dX_1\cdots dX_k dX_{k+1}\cdots dX_n \quad .....(3.13)$$

Making use of the transformations $Y_i = U^{1/2}X_iU^{1/2}$ and $Y_j = V^{1/2}X_jV^{1/2}$, so that,

$dY_i = |U|^{(p+1)/2}dX_i, dY_j = |V|^{(p+1)/2}dX_j$, and $|Y_i| = |U||X_i|$, $|Y_j| = |V||X_j|$

for $i = 1, \cdots, k; \quad j = k+1, \cdots, n$

in the expression (3.13) and using the definition (1.3) in the resulting expression, we are led to

$$|U|^{-\rho_1-\cdots-\rho_k}|V|^{-\rho_{k+1}-\cdots-\rho_n} \frac{\Gamma_p(b-\rho_1-\cdots-\rho_n)\Gamma_p(c_1)\cdots\Gamma_p(c_n)}{\Gamma_p(b)\Gamma_p(c_1-\rho_1)\cdots\Gamma_p(c_n-\rho_n)} \times$$
$$\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n) \quad ....(3.14)$$

Using this expression on the right side of eq.(3.12) and then integrating out the variables U and V in the resulting expression by using a Gamma integral, the outcome is $M[{}_{(1)}^{(k)}E_C^{(n)}]$ as given by eq.(3.2), thus finishing the proof.

4. The Generalized Horn's Function ${}^{(k)}H_4^{(n)}$

DEFINITION 4.1 :

$${}^{(k)}H_4^{(n)} = {}^{(k)}H_4^{(n)}(a, b_{k+1}, \cdots, b_n; c_1, \cdots, c_n; -X_1, \cdots, -X_n)$$

$$M[{}^{(k)}H_4^{(n)}] = [\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$

$${}^{(k)}H_4^{(n)}(a, b_{k+1}, \cdots, b_n; c_1, \cdots, c_n; -X_1, \cdots, -X_n)dX_1 \cdots dX_n]$$

$$= [\frac{\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n)\Gamma_p(b_{k+1}-\rho_{k+1})\cdots\Gamma_p(b_n-\rho_n)}{\Gamma_p(a)\Gamma_p(b_{k+1})\cdots\Gamma_p(b_n)\Gamma_p(c_1-\rho_1)\cdots\Gamma_p(c_n-\rho_n)} \times$$

$$\Gamma_p(c_1)\cdots\Gamma_p(c_n)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)] \qquad ......(4.1)$$

for $\mathrm{Re}(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n, b_{k+1}-\rho_{k+1}, \cdots, b_n-\rho_n,$
$c_i-\rho_i, \rho_i) > (p-1/2,\ i=1, \cdots, n.$

THEOREM 4.1 :

$$
{}^{(k)}H_4^{(n)}(a, b_{k+1}, \cdots, b_n; c_1, \cdots, c_n; -X_1, \cdots, -X_n)
$$

$$
= [\frac{\Gamma_p(c_{k+1})\cdots\Gamma_p(c_n)}{\Gamma_p(b_{k+1})\cdots\Gamma_p(b_n)\Gamma_p(c_{k+1}-b_{k+1})\cdots\Gamma_p(c_n-b_n)}\int\cdots(n-k)\cdots
$$

$$
\int |U_{k+1}|^{b_{k+1}-(p+1)/2}\cdots|U_n|^{b_n-(p+1)/2}\ |I-U_{k+1}|^{c_{k+1}-b_{k+1}-(p+1)/2}\times
$$

$$
\cdots|I-U_n|^{c_n-b_n-(p+1)/2}\left|I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2}\right|^{-a}
$$

$$
F_C^{(k)}[(a+1)/2, (2a+1)/4; c_1, \cdots, c_k;
$$

$$
-4(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}\times
$$

$$
X_1(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}, \cdots,
$$

$$
-4(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}X_k(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+
$$

$$
U_n^{1/2}X_nU_n^{1/2})^{-1}]\,dU_{k+1}\cdots dU_n] \qquad ....(4.2)
$$

for $p = 2$, for $0 < U_j < I$, $j = k+1, \cdots, n$ and

for $\mathrm{Re}(b_{k+1}, \cdots, b_n, c_{k+1}-b_{k+1}, \cdots, c_n-b_n) > (p-1)/2.$

PROOF : We take the M-transform of the right side of eq.(4.2) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ to obtain,

$$
\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_k|^{\rho_k-(p+1)/2}|X_{k+1}|^{\rho_{k+1}-(p+1)/2}\cdots\times
$$

$$
|X_n|^{\rho_n-(p+1)/2}\left|I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2}\right|^{-a}\times
$$

Continued to the next page…………..

$$F_C^{(k)}[(a+1)/2,(2a+1)/4;c_1,\cdots,c_k;$$
$$-4(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}\times$$
$$X_1(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1},\cdots,$$
$$-4(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}X_k\times$$
$$(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}]dX_1\cdots dX_k\,dX_{k+1}\cdots dX_n \qquad ....(4.3)$$

Making use of the transformations,

$$Y_i=4(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1}X_i\times$$
$$(I+U_{k+1}^{1/2}X_{k+1}U_{k+1}^{1/2}+\cdots+U_n^{1/2}X_nU_n^{1/2})^{-1};i=1,\cdots,k$$

$$Y_j=U_j^{1/2}X_jU_j^{1/2};\quad j=k+1,\cdots,n$$

so that, $Y_i=4(I+Y_{k+1}+\cdots+Y_n)^{-1}X_i(I+Y_{k+1}+\cdots+Y_n)^{-1}$

$$dY_i=4^{p(p+1)/2}\left|I+Y_{k+1}+\cdots+Y_n\right|^{-(p+1)}dX_i;\ i=1,\cdots,k$$

$$dY_j=\left|U_j\right|^{(p+1)/2}dX_j;\ j=k+1,\cdots,n$$

and $\left|Y_i\right|=4^p\left|I+Y_{k+1}+\cdots+Y_n\right|^{-2}\left|X_i\right|;\ i=1,\cdots,k$

$$\left|Y_j\right|=\left|U_j\right|\left|X_j\right|;\ j=k+1,\cdots,n$$

in the expression (4.3) and then using the definition (1.2) in the resulting expression and integrating out the variables $Y_{k+1},\cdots,Y_n$ by using a type-2 Dirichlet integral,we have,

$$4^{-p(\rho_1+\cdots+\rho_k)}\left|U_{k+1}\right|^{-\rho_{k+1}}\cdots\left|U_n\right|^{-\rho_n}\frac{\Gamma_p(c_1)\cdots\Gamma_p(c_k)}{\Gamma_p(c_1-\rho_1)\cdots\Gamma_p(c_k-\rho_k)}\times$$

$$\frac{\Gamma_p[(a+1)/2-\rho_1-\cdots-\rho_k]}{\Gamma_p[(a+1)/2]}\frac{\Gamma_p[(2a+1)/4-\rho_1-\cdots-\rho_k]}{\Gamma_p[(2a+1)/4]}\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)\times$$

$$\frac{\Gamma_p(a-2\rho_1-\cdots-2\rho_k-\rho_{k+1}-\cdots-\rho_n)}{\Gamma_p(a-2\rho_1-\cdots-2\rho_k)} \qquad ....(4.4)$$

Using this expression on the right side of eq.(4.2) and integrating out the variables $U_{k+1},\cdots,U_n$ in the resulting expression by using a type-1 Beta integral and observing that,

$$4^{-p(\rho_1+\cdots+\rho_k)}\frac{\Gamma_p[(a+1)/2-\rho_1-\cdots-\rho_k]}{\Gamma_p[(a+1)/2]}\frac{\Gamma_p[(2a+1)/4-\rho_1-\cdots-\rho_k]}{\Gamma_p[(2a+1)/4]\Gamma_p(a-2\rho_1-\cdots-2\rho_k)}$$

$$=\frac{1}{\Gamma_p(a)} \qquad \text{for } p=2 \qquad ....(4.5)$$

from eq.(6.13) page 84 of Mathai [3], finally we have $M[{}^{(k)}H_4^{(n)}]$ as given by eq.(4.1), thereby completing the proof. This result is different from the corresponding result in the scalar case.

5. The Generalized Srivastava $H_B^{(n)}$ and $H_C^{(n)}$ Functions

DEFINITION 5.1 :

$$H_B^{(n)}=H_B^{(n)}(\alpha_1,\cdots,\alpha_n;\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_n)$$

$$M[H_B^{(n)}]=[\int_{X_1>0}\cdots\int_{X_n>0}\left|X_1\right|^{\rho_1-(p+1)/2}\cdots\left|X_n\right|^{\rho_n-(p+1)/2}\times$$

$$H_B^{(n)}(\alpha_1,\cdots,\alpha_n;\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_n)dX_1\cdots dX_n]$$

Continued in the next page………..

$$=[\frac{\Gamma_p(\alpha_1-\rho_1-\rho_n)\Gamma_p(\alpha_2-\rho_1-\rho_2)\cdots\Gamma_p(\alpha_n-\rho_{n-1}-\rho_n)\Gamma_p(\gamma_1)\cdots\Gamma_p(\gamma_n)}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)\cdots\Gamma_p(\alpha_n)\Gamma_p(\gamma_1-\rho_1)\cdots\Gamma_p(\gamma_n-\rho_n)}\times$$

$$\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)] \qquad ....(5.1)$$

for $\mathrm{Re}(\alpha_1-\rho_1-\rho_n,\alpha_2-\rho_1-\rho_2,\cdots,\alpha_n-\rho_{n-1}-\rho_n,\gamma_i-\rho_i,\rho_i)>(p-1)/2,$

$i=1,\cdots,n$

DEFINITION 5.2 :

$$H_C^{(n)}=H_C^{(n)}(\alpha_1,\cdots,\alpha_n;\gamma;-X_1,\cdots,-X_n)$$

$$M[H_C^{(n)}]=[\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$

$$H_C^{(n)}(\alpha_1,\cdots,\alpha_n;\gamma;-X_1,\cdots,-X_n)dX_1\cdots dX_n]$$

$$=[\frac{\Gamma_p(\alpha_1-\rho_1-\rho_n)\Gamma_p(\alpha_2-\rho_1-\rho_2)\cdots\Gamma_p(\alpha_n-\rho_{n-1}-\rho_n)\Gamma_p(\gamma)}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)\cdots\Gamma_p(\alpha_n)\Gamma_p(\gamma-\rho_1-\cdots-\rho_n)}\times$$

$$\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)] \qquad ....(5.2)$$

for $\mathrm{Re}(\alpha_1-\rho_1-\rho_n,\alpha_2-\rho_1-\rho_2,\cdots,\alpha_n-\rho_{n-1}-\rho_n,\gamma-\rho_1-\cdots-\rho_n,\rho_i)$

$>(p-1)/2,\quad i=1,\cdots,n.$

THEOREM 5.1 :

$$H_B^{(n)}(\alpha_1,\cdots,\alpha_n;\gamma_1,\cdots,\gamma_n;-X_1,\cdots,-X_n)$$

$$=[\frac{1}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)\cdots\Gamma_p(\alpha_n)}\int_{T_1>0}\cdots\int_{T_n>0}e^{-tr(T_1+\cdots+T_n)}|T_1|^{\alpha_1-(p+1)/2}\times$$

$$|T_2|^{\alpha_2-(p+1)/2}\cdots|T_n|^{\alpha_n-(p+1)/2}\,{}_0F_1(\,;\gamma_1;-T_2^{1/2}T_1^{1/2}X_1T_1^{1/2}T_2^{1/2})\times$$

$${}_0F_1(\,;\gamma_2;-T_3^{1/2}T_2^{1/2}X_2T_2^{1/2}T_3^{1/2})\cdots{}_0F_1(\,;\gamma_n;-T_1^{1/2}T_n^{1/2}X_nT_n^{1/2}T_1^{1/2})\times$$

$$dT_1\cdots dT_n] \qquad ....(5.3)$$

for $\mathrm{Re}(\alpha_i)>(p-1)/2,\ i=1,\cdots,n.$

PROOF : Taking the M-transform of the right side of eq.(5.3) with respect to the variables $X_1, \cdots, X_n$ and the parameters $\rho_1, \cdots, \rho_n$ we obtain,

$$\int_{X_1>0} \cdots \int_{X_n>0} |X_1|^{\rho_1-(p+1)/2} |X_2|^{\rho_2-(p+1)/2} \cdots |X_n|^{\rho_n-(p+1)/2} \times$$
$${}_0F_1(\ ;\gamma_1; -T_2^{1/2} T_1^{1/2} X_1 T_1^{1/2} T_2^{1/2})\, {}_0F_1(\ ;\gamma_2; -T_3^{1/2} T_2^{1/2} X_2 T_2^{1/2} T_3^{1/2}) \times$$
$$\cdots {}_0F_1(\ ;\gamma_n; -T_1^{1/2} T_n^{1/2} X_n T_n^{1/2} T_1^{1/2}) dX_1 \cdots dX_n] \qquad ......(5.4)$$

Making use of the transformations

$$Y_1 = T_2^{1/2} T_1^{1/2} X_1 T_1^{1/2} T_2^{1/2},\ Y_2 = T_3^{1/2} T_2^{1/2} X_2 T_2^{1/2} T_3^{1/2}, \cdots$$
$$Y_n = T_1^{1/2} T_n^{1/2} X_n T_n^{1/2} T_1^{1/2};\ \text{so that, } dY_1 = |T_2|^{(p+1)/2} |T_1|^{(p+1)/2} dX_1,$$
$$dY_2 = |T_3|^{(p+1)/2} |T_2|^{(p+1)/2} dX_2, \cdots, dY_n = |T_1|^{(p+1)/2} |T_n|^{(p+1)/2} dX_n$$
$$\text{and, } |Y_1| = |T_2||T_1||X_1|,\ |Y_2| = |T_3||T_2||X_2|,\ \cdots, |Y_n| = |T_1||T_n||X_n|$$

in the expression (5.4), and using the definition of M-transform of a ${}_0F_1$-function from eq.(2.3.5) page 38 of Mathai [3] , we have,

$$|T_1|^{-\rho_1-\rho_n} |T_2|^{-\rho_1-\rho_2} \cdots |T_n|^{-\rho_{n-1}-\rho_n} \frac{\Gamma_p(\gamma_1)\Gamma_p(\rho_1)}{\Gamma_p(\gamma_1-\rho_1)} \frac{\Gamma_p(\gamma_2)\Gamma_p(\rho_2)}{\Gamma_p(\gamma_2-\rho_2)} \times$$
$$\cdots \frac{\Gamma_p(\gamma_n)\Gamma_p(\rho_n)}{\Gamma_p(\gamma_n-\rho_n)} \qquad ....(5.5)$$

which, on substitution on the right side of eq.(5.3) and integrating out the variables $T_1, \cdots, T_n$ in the resulting expression by using a Gamma integral leads to $M[H_B^{(n)}]$, as given by eq.(5.1) , thus concluding the proof.

THEOREM 5.2 :

$$H_C^{(n)}(\alpha_1,\cdots,\alpha_n;\gamma;-X_1,\cdots,-X_n)$$

$$=[\frac{1}{\Gamma_p(\alpha_1)\cdots\Gamma_p(\alpha_n)}\int_{T_1>0}\cdots\int_{T_n>0} e^{-tr(T_1+\cdots+T_n)}|T_1|^{\alpha_1-(p+1)/2}\times$$

$$\cdots|T_n|^{\alpha_n-(p+1)/2}\,{}_0F_1(;\gamma;-T_2^{1/2}T_1^{1/2}X_1T_1^{1/2}T_2^{1/2}-T_3^{1/2}T_2^{1/2}X_2T_2^{1/2}T_3^{1/2}$$

$$-\cdots-T_1^{1/2}T_n^{1/2}X_nT_n^{1/2}T_1^{1/2})dT_1\cdots dT_n] \quad ....(5.6)$$

for $\mathrm{Re}(\alpha_i)>(p-1)/2, i=1,\cdots,n.$

PROOF: Taking the M- transform of the right side of eq.(5.6) with respect to the variables $X_1,\cdots,X_n$ and the parameters $\rho_1,\cdots,\rho_n$,we have,

$$\int_{X_1>0}\cdots\int_{X_n>0}|X_1|^{\rho_1-(p+1)/2}\cdots|X_n|^{\rho_n-(p+1)/2}\times$$

$${}_0F_1(;\gamma;-T_2^{1/2}T_1^{1/2}X_1T_1^{1/2}T_2^{1/2}-T_3^{1/2}T_2^{1/2}X_2T_2^{1/2}T_3^{1/2}-\cdots$$

$$-T_1^{1/2}T_n^{1/2}X_nT_n^{1/2}T_1^{1/2})dX_1\cdots dX_n \quad ....(5.7)$$

Making use of the transformations,

$$Y_1=T_2^{1/2}T_1^{1/2}X_1T_1^{1/2}T_2^{1/2},\ Y_2=T_3^{1/2}T_2^{1/2}X_2T_2^{1/2}T_3^{1/2},\cdots,$$

$$Y_n=T_1^{1/2}T_n^{1/2}X_nT_n^{1/2}T_1^{1/2},$$

so that, $dY_1=|T_2|^{(p+1)/2}|T_1|^{(p+1)/2}dX_1; dY_2=|T_3|^{(p+1)/2}|T_2|^{(p+1)/2}dX_2;$

$\cdots; dY_n=|T_1|^{(p+1)/2}|T_n|^{(p+1)/2}dX_n;$

and, $|Y_1|=|T_1||T_2||X_1|;\ |Y_2|=|T_3||T_2||X_2|;\cdots;\ |Y_n|=|T_1||T_n||X_n|$

in the expression (5.7) and then using the M-transform of a ${}_0F_1$ function by applying theorem 3.3 page 55 of Mathai [3] , we obtain,

$$\left|T_1\right|^{-\rho_1-\rho_n}\left|T_2\right|^{-\rho_1-\rho_2}\cdots\left|T_n\right|^{-\rho_{n-1}-\rho_n}\frac{\Gamma_p(\gamma)\Gamma_p(\rho_1)\cdots\Gamma_p(\rho_n)}{\Gamma_p(\gamma-\rho_1-\cdots-\rho_n)} \quad ....(5.8)$$

Using this expression on the right of eq.(5.6) and integrating out the variables T_1, ……, T_n in the resulting expression by using a Gamma integral, finally yields $M[H_C^{(n)}]$, as given by eq.(5.2), whence the theorem is proved.

References

1. Exton H.(1976).Multiple Hypergeometric Functions and Applications. Ellis Horwood Limited, Publishers, Chichester.
2. Mathai A.M. (1992). Jacobians of Matrix Transformations I. Centre for Mathematical Sciences,Trivandrum, India.
3. Mathai A.M. (1993). Hypergeometric Functions of Several Matrix Arguments. Centre for Mathematical Sciences, Trivandrum,India.
4. Mathai A.M., Pederzoli G. (1996). Some Transformations for Functions of Matrix Arguments. Indian Journal of Pure and Applied Mathematics; 27(3), pp. 277-284.
5. Srivastava H.M. and Karlsson P.W.(1985).Multiple Gaussian Hypergeometric Series. Ellis Horwood Limited, Publishers, Chichester.
6. Upadhyaya Lalit Mohan and Dhami H.S.(Nov.2001). Matrix Generalizations of Multiple Hypergeometric Functions. # 1818 IMA Preprints Series, University of Minnesota, Minneapolis, U.S.A.