Synthesizing the Static Feedback Gain Matrix to Minimize the System H Infinity Norm Using Linear Matrix Inequality Approach

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Abstract: In this paper, I propose a method to synthesize a static state feedback gain matrix to minimize the system H infinity norm using linear matrix inequality (LMI) approach. To illustrate the result, an example using the dynamics of a double integrator is used. The final result shows that the solution achieved by LMI approach is the same as solution done manually. Furthermore, the LMI approach opens a direction to promote the sparsity structure of the feedback gain matrix.

1 Introduction

H infinity norm is one type of system norms which is defined as the worst case amplification from input to the output. In single input single output (SISO) system, H infinity norm can be computed by finding the supremum of the transfer function on all the frequencies. In case of multi input, multi output system, H infinity norm can be computed as the supremum of the singular values of the transfer map on all the frequencies.

To compute the H infinity norm of a system, bounded real lemma is used. The precise value of H infinity norm is hardly be determined but the approximate value can be found by bisection search algorithm. Basically, the H infinity norm of a system is below a certain level if and only if the associated Riccati equation has a positive definite solution. Using this property, the H infinity norm of a system can be estimated with a certain level of precision.

In this paper, I propose another method of estimating the minimum H infinity norm that a system can reach and at the same time synthesizing static feedback gain matrix using linear matrix inequality approach. Firstly, the bounded real lemma, Schur complement and change of variable are used to form a LMI. Secondly, the optimization problem is formed and solved by CVX, a Matlab optimization tool. And finally, the static state feedback gain matrix is reconstructed.

2 Problem formation

2.1 Mathematics preliminaries

2.1.1 Bounded real lemma

Given the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Bounded real lemma states that A is Hurwitz and

$$||C(sI-A)^{-1}B||_\infty < 1$$

If and only if there exists a matrix $P \succ 0$ such that

$$\begin{bmatrix} C^T \\ 0 \end{bmatrix} \begin{bmatrix} C & 0 \end{bmatrix} + \begin{bmatrix} A^TP + PA & PB \\ B^TP & -I \end{bmatrix} \prec 0$$

2.1.2 Schur complement

$$\begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix} \succ 0 \Longleftrightarrow \begin{cases} Y \succ 0 \\ X - ZY^{-1}Z^T \succ 0 \end{cases}$$

and similarly, for the negative definite case

$$\begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix} \prec 0 \Longleftrightarrow \begin{cases} Y \prec 0 \\ X - ZY^{-1}Z^T \prec 0 \end{cases}$$

2.1.3 Multiplication property of positive definite matrices

Given $M = M^T \succ 0$ and $N = N^T \succ 0$, then

$$MNM \succ 0$$

2.2 Problem formulation

A linear dynamical system is represented by the following equation

$$\dot{x} = Ax + B_1 d + B_2 u$$

where $A \in R^n$, B_1 and $B_2 \in R^{n\times m}$, x is the state vector, d represents disturbances, and u is the input vector. In this project, I want to design a static state feedback controller that minimize H infinity norm from disturbances to the outputs. The complete state space representation of such a system is as follow

$$\begin{aligned} \dot{x} &= Ax + B_1 d + B_2 u \\ z &= \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\ u &= -Kx \end{aligned}$$

where z is the control objective function and K is the feedback gain matrix relating the states to the inputs. The closed loop system is

$$\dot{x} = (A - B_2K)x + B_1 d \tag{1}$$

$$z = \begin{bmatrix} C \\ -K \end{bmatrix} x = C_z x \tag{2}$$

Using the above closed loop system, I design a feedback gain matrix K that minimizes the H infinity norm of the system with z the outputs and d the inputs.

2.3 Solution

My solution follows these 4 steps:

- Use bounded real lemma, change of variable and Schur complement to form a linear matrix inequality.
- Form the optimization objective function
- Use CVX to solve the optimization problem
- Reconstruct feedback gain matrix

Apply bounded real lemma to the system given by (1) and (2), the necessary and sufficient condition for H infinity norm of that system to be smaller than κ is there exist a $P = P^T \succ 0$ such that

$$\begin{bmatrix} \frac{1}{\kappa} C_z^T \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\kappa} C_z & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} (A - B_2K)^T P + P(A - B_2K) & PB_1 \\ B_1^T P & -I \end{bmatrix} \prec 0 \tag{3}$$

To eliminate the nonlinear term KP in (3), pre-multiply and post-multiply the left hand side of (3) by a positive definite matrix $\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}$. Carry out the multiplication, apply Schur complement, substitute C_z by $\begin{bmatrix} C \\ K \end{bmatrix}$, and make the change of variable $X = P^{-1}$, $Y = -KP^{-1}$ the equivalent matrix inequality to (3) is

$$\begin{bmatrix} XA^T + AX + Y^T B_2 T + B_2 Y & B_1 & XC^T & Y^T \\ B_1^T & -I & 0 & 0 \\ CX & 0 & -\kappa^2 I & 0 \\ Y & 0 & 0 & -\kappa^2 I \end{bmatrix} \prec 0 \tag{4}$$

To find the feedback gain matrix that minimize the system H infinity norm, the following optimization problem is formed

$$\begin{array}{ll} \underset{\kappa^2, Y_{ij}}{\text{minimize}} & \kappa^2 \\ \text{subject to} & X > \sigma I \\ \text{and} & (4) \end{array} \tag{5}$$

3 Illustration example

For illustration purpose, consider a double integrator with the following matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using CVX, a Matlab optimization toolbox to solve the optimization problem (5) with the above A, B, C matrices, σ is chosen to be 10^{-3}. Then the P, K, and the minimum H_∞ norm of the system are

$$\begin{aligned} P &= \begin{bmatrix} 12.5982 & 78.8658 \\ 78.8658 & 993.6908 \end{bmatrix} \\ K &= \begin{bmatrix} 78.8785993.8505 \end{bmatrix} \\ H_\infty &= 1.0001 \end{aligned}$$

If σ is chosen to be 0, the H_∞ norm of the system will be 1 but elements of P and K will be very large. Additionally, the manual computation shows that the minimum H_∞ norm of the system is 1, which is consistent with the result getting from the optimization problem. The consistency verifies the accuracy of this method.

4 Conclusion

Another approach to synthesize a feedback gain matrix to minimize the H infinity norm of a system using linear matrix inequality (LMI) is proposed in this paper. The manual computation of smallest H infinitive norm is the illustrative example verifies the accuracy of the proposed approach. In comparison solving Riccati equation approach, the LMI approach can be extended to find the sparse feedback gain matrix by modifying the optimization objective.

References

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