

The Initial Functions in a Subdivision Scheme

Bin Han

Abstract. In this paper we shall study the initial functions in a subdivision scheme in a Sobolev space. By investigating the mutual relations among the initial functions in a subdivision scheme, we are able to study in a relatively unified approach several questions related to a subdivision scheme in a Sobolev space such as convergence, error estimate and convergence rate of a subdivision scheme in a Sobolev space with a general dilation matrix. A generalized definition of convergence of subdivision schemes in Banach spaces is also introduced.

§1. Introduction

Refinable functions play an important role in wavelet analysis. As widely used in computer aided geometric design, a subdivision scheme is a powerful tool in investigating various properties of refinable functions.

An $s \times s$ integer matrix M is called a dilation matrix if $\lim_{k \rightarrow \infty} M^{-k} = 0$. We say that a is a mask on \mathbb{Z}^s if a is a finitely supported sequence on \mathbb{Z}^s such that $\sum_{\beta \in \mathbb{Z}^s} a(\beta) = 1$. Wavelets are derived from refinable functions via a standard multiresolution technique. A refinable function ϕ is a solution to the following refinement equation

$$\phi = |\det M| \sum_{\beta \in \mathbb{Z}^s} a(\beta) \phi(M \cdot -\beta), \quad (1)$$

where a is a mask and M is a dilation matrix. For a mask a on \mathbb{Z}^s and an $s \times s$ dilation matrix M , it is known that there exists a unique compactly supported distributional solution, denoted by ϕ_a^M throughout the paper, to the refinement equation (1) such that $\hat{\phi}_a^M(0) = 1$, where the Fourier transform of $f \in L_1(\mathbb{R}^s)$ is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix \cdot \xi} dx$, $\xi \in \mathbb{R}^s$ and can be naturally extended to tempered distributions.

Let \mathbb{N}_0 denote all the nonnegative integers. For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, the length of μ is $|\mu| := \mu_1 + \dots + \mu_s$, and $\xi^\mu := \xi_1^{\mu_1} \cdots \xi_s^{\mu_s}$ for $\xi = (\xi_1, \dots, \xi_s) \in \mathbb{R}^s$. The partial derivative of a differentiable function f with respect to the j th

coordinate is denoted by $D_j f$, $j = 1, \dots, s$, and for $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, D^μ is the differential operator $D_1^{\mu_1} \dots D_s^{\mu_s}$. By $W_p^k(\mathbb{R}^s)$ we denote the Sobolev space that consists of all functions f such that $D^\mu f \in L_p(\mathbb{R}^s)$ for all $\mu \in \mathbb{N}_0^s$ with $|\mu| \leq k$, equipped with the norm defined by

$$\|f\|_{W_p^k(\mathbb{R}^s)} := \sum_{|\mu| \leq k} \|D^\mu f\|_p.$$

An iteration scheme can be employed to solve the refinement equation (1). Start with some initial function $\phi_0 \in W_p^k(\mathbb{R}^s)$ such that $\hat{\phi}_0(0) = 1$. We employ the iteration scheme $Q_{a,M}^n \phi_0$, $n = 0, 1, 2, \dots$, where $Q_{a,M}$ is the linear operator on $L_p(\mathbb{R}^s)$ ($1 \leq p \leq \infty$) given by

$$Q_{a,M} f := |\det M| \sum_{\beta \in \mathbb{Z}^s} a(\beta) f(M \cdot -\beta), \quad f \in L_p(\mathbb{R}^s).$$

This iteration scheme is called a **subdivision scheme** associated with mask a and dilation M . When the sequence $Q_{a,M}^n \phi_0$ converges in the Sobolev space $W_p^k(\mathbb{R}^s)$, then the limit function must be ϕ_a^M —the unique compactly supported distributional solution to the refinement equation (1) with $\hat{\phi}_a^M(0) = 1$.

It is the purpose of this paper to investigate the mutual relations among different initial functions in a subdivision scheme and then use such results to improve some results and to give alternative proofs to some known results on subdivision schemes in a Sobolev space.

The structure of the paper is as follows. In Section 2, we shall investigate the mutual relations of initial functions in a subdivision scheme in a Sobolev space. Then we shall give a proof of results in [6, 7, 8] on the characterization of convergence of a subdivision scheme. In Section 3, following the line developed in [3, 4], we shall investigate error estimates of a subdivision scheme in a Sobolev space with a perturbed mask. Therefore, we improve the results in [1, 2, 3, 4]. Finally, in Section 4, we shall discuss rate of convergence of a subdivision scheme in a Sobolev space and introduce a general definition of convergence of a subdivision scheme in a general Banach space.

§2. Initial Functions in a Subdivision Scheme

In this section we shall investigate the mutual relations of initial functions in a subdivision scheme. By $\ell_0(\mathbb{Z}^s)$ we denote the linear space of all finitely supported sequences on \mathbb{Z}^s . For $\alpha \in \mathbb{Z}^s$ and $y \in \mathbb{R}^s$, we define

$$\nabla_\alpha \lambda := \lambda - \lambda(\cdot - \alpha), \quad \nabla_y f := f - f(\cdot - y), \quad \lambda \in \ell_0(\mathbb{Z}^s), f \in L_p(\mathbb{R}^s). \quad (2)$$

Define $[\lambda_1 * \lambda_2](\alpha) := \sum_{\beta \in \mathbb{Z}^s} \lambda_1(\beta) \lambda_2(\alpha - \beta)$ for $\lambda_1, \lambda_2 \in \ell_0(\mathbb{Z}^s)$. Define a semi-convolution as follows:

$$\lambda * f := \sum_{\beta \in \mathbb{Z}^s} \lambda(\beta) f(\cdot - \beta), \quad \lambda \in \ell_0(\mathbb{Z}^s), f \in L_p(\mathbb{R}^s). \quad (3)$$

For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, $\nabla^\mu := \nabla_{e_1}^{\mu_1} \cdots \nabla_{e_s}^{\mu_s}$, where e_j is the j th coordinate unit vector in \mathbb{R}^s . By δ we denote the Dirac sequence such that $\delta(0) = 1$ and $\delta(\beta) = 0$ for all $\beta \in \mathbb{Z}^s \setminus \{0\}$. It is easy to verify that $\nabla^\mu f = \nabla^\mu \delta * f$, $\nabla^\mu \lambda = \nabla^\mu \delta * \lambda$, $\nabla_\alpha \lambda = \nabla_\alpha \delta * \lambda$ and $\lambda_1 * (\lambda_2 * f) = (\lambda_1 * \lambda_2) * f$. Our discussion in this paper is based on the following fact.

Theorem 1. *Let f be a compactly supported function in $L_p(\mathbb{R}^s)$, where $1 \leq p \leq \infty$. Then for $k \in \mathbb{N}$, the following statements are equivalent*

- 1) $D^\mu \hat{f}(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s$ and $\mu \in \mathbb{N}_0^s$ with $|\mu| < k$;
- 2) $\sum_{\beta \in \mathbb{Z}^s} q(\beta) f(\cdot - \beta) = 0$ for all $q \in \Pi_{k-1}$, where Π_{k-1} denotes the set of all polynomials of total degree less than k ;
- 3) $f = \sum_{j=1}^r \lambda_j * g_j$ for some compactly supported functions $g_j \in L_p(\mathbb{R}^s)$ and some finitely supported sequences λ_j in the linear space V_k which consists of all $\lambda \in \ell_0(\mathbb{Z}^s)$ such that $\sum_{\beta \in \mathbb{Z}^s} q(\beta) \lambda(\beta) = 0$ for all $q \in \Pi_{k-1}$;
- 4) $f = \sum_{|\mu|=k} \nabla^\mu g_\mu$ for some compactly supported functions $g_\mu \in L_p(\mathbb{R}^s)$.

Proof: By the Poisson summation formula, it is known that 1) \iff 2). Note that $\{\nabla^\mu \delta(\cdot - \alpha) : \alpha \in \mathbb{Z}^s, |\mu| = k\}$ spans the linear space V_k . So 3) \iff 4). Observe that $\lambda \in V_k$ if and only if $\lambda * q = 0$ for all $q \in \Pi_{k-1}$. When 3) holds, it is straightforward to see that

$$\sum_{\alpha \in \mathbb{Z}^s} q(\alpha) f(\cdot - \alpha) = \sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}^s} g_j(\cdot - \alpha) \sum_{\beta \in \mathbb{Z}^s} \lambda_j(\beta) q(\alpha - \beta) = 0 \quad \forall q \in \Pi_{k-1}$$

since $\sum_{\beta \in \mathbb{Z}^s} \lambda_j(\beta) q(\alpha - \beta) = 0$. So 3) \Rightarrow 2). To complete the proof, it suffices to show that 2) \Rightarrow 3). Since f is compactly supported, we can write

$$f = \sum_{\alpha \in \mathbb{Z}^s} f|_{\alpha+[0,1]^s} = \sum_{j=1}^n f_j(\cdot - \alpha_j),$$

where $\alpha_j \in \mathbb{Z}^s$ and all $f_j \in L_p(\mathbb{R}^s)$ are supported on $[0, 1]^s$. Without loss of generality, we can assume that f_1, \dots, f_r are linearly independent and for some numbers c_j^m ,

$$f_m = \sum_{j=1}^r c_j^m f_j, \quad m = r+1, \dots, n.$$

Let $g_j := f_j, j = 1, \dots, r$ and choose sequences $\lambda_j, j = 1, \dots, r$ as follows:

$$\lambda_j(\beta) := \delta(\beta - \alpha_j) + \sum_{m=r+1}^n c_j^m \delta(\beta - \alpha_m), \quad \beta \in \mathbb{Z}^s, j = 1, \dots, r.$$

Obviously, all λ_j are finitely supported sequences on \mathbb{Z}^s . All g_j are compactly supported and are supported on $[0, 1]^s$. By computation, we have $f = \sum_{j=1}^r \lambda_j * g_j$. To complete the proof, we prove $\lambda_j \in V_k$. By 2), we have

$$0 = \sum_{\alpha \in \mathbb{Z}^s} q(\alpha) f(\cdot - \alpha) = \sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}^s} g_j(\cdot - \alpha) \sum_{\beta \in \mathbb{Z}^s} \lambda_j(\beta) q(\alpha - \beta) \quad \forall q \in \Pi_{k-1}.$$

Since all g_j are supported on $[0, 1]^s$ and are linearly independent, it follows that $\sum_{\beta \in \mathbb{Z}^s} \lambda_j(\beta) q(\alpha - \beta) = 0$ and therefore, all λ_j are in V_k . \square

When $s = 1$, Theorem 1 is evident since $f = \nabla^k g$ with a compactly supported function $g \in L_p(\mathbb{R})$ given by $g = \sum_{j=1}^{\infty} j^{k-1} f(\cdot - j + 1)/(k-1)!$.

It was proved in Jia, Jiang and Lee [8] that if a subdivision scheme with an initial function f converges in $W_p^k(\mathbb{R}^s)$, then f must satisfy the moment conditions of order $k+1$; that is, $\hat{f}(0) = 1$ and

$$D^\mu \hat{f}(2\pi\beta) = 0 \quad \forall |\mu| < k+1, \beta \in \mathbb{Z}^s \setminus \{0\}.$$

As a consequence of Theorem 1, we have the following result.

Corollary 2. *Let $f \in W_p^k(\mathbb{R}^s)$ be compactly supported such that f satisfies the moment conditions of order $k+1$. For any compactly supported function $g \in W_p^k(\mathbb{R}^s)$, g satisfies the moment conditions of order $k+1$ if and only if*

$$D^\mu g = D^\mu f + \sum_{|\nu|=|\mu|+1} \nabla^\nu h_\nu^\mu, \quad \forall |\mu| \leq k, \mu \in \mathbb{N}_0^s \quad (4)$$

for some compactly supported functions $h_\nu^\mu \in L_p(\mathbb{R}^s)$.

Proof: For $|\mu| \leq k$, let $h := D^\mu g - D^\mu f$. Since $\hat{h}(\xi) = (-i\xi)^\mu [\hat{g}(\xi) - \hat{f}(\xi)]$, if both f and g satisfy the moment conditions of order $k+1$, then $D^\nu \hat{h}(2\pi\beta) = 0$ for all $|\nu| < |\mu| + 1$ and for all $\beta \in \mathbb{Z}^s$. Therefore, by Theorem 1, (4) holds.

Let us prove the converse statement. Note that $\widehat{\nabla^\nu h_\nu^\mu}(\xi_1, \dots, \xi_s) = \widehat{h_\nu^\mu}(\xi_1, \dots, \xi_s) \prod_{j=1}^s (1 - e^{-i\xi_j})^{\nu_j}$ for $\nu = (\nu_1, \dots, \nu_s) \in \mathbb{N}_0^s$. Consequently, $D^\eta [\widehat{\nabla^\nu h_\nu^\mu}](2\pi\beta) = 0$ for all $|\eta| < |\nu|$ and $\beta \in \mathbb{Z}^s$. When (4) holds, then in particular (4) holds for $|\mu| = k$. Let $h := g - f$. Hence, $D^\eta (\widehat{D^\mu h})(2\pi\beta) = D^\eta [(-i\cdot)^\mu \hat{h}(\cdot)](2\pi\beta) = 0$ for all $|\eta| \leq |\mu| = k$ and $\beta \in \mathbb{Z}^s$. Therefore, $\hat{h}(0) = 0$ and $D^\eta \hat{h}(2\pi\beta) = 0$ for all $|\eta| < k+1$ and $\beta \in \mathbb{Z}^s \setminus \{0\}$. Since f satisfies the moment conditions of order $k+1$, so does g by $g = f + h$. \square

Let f be a compactly supported function in $L_p(\mathbb{R}^s)$. We say that the shifts of f are **stable** if there exist two positive constants A and B such that $A\|\lambda\|_p \leq \|\lambda * f\|_p \leq B\|\lambda\|_p$ for all finitely supported sequences λ on \mathbb{Z}^s .

The B-spline function B_k of order k is defined by the recursive formula: $B_0 := \chi_{[0,1]}$ and $B_k := \int_0^1 B_{k-1}(\cdot - t) dt$. Define a function on \mathbb{R}^s as follows

$$\varphi(x_1, \dots, x_s) := B_{k+1}(x_1) \cdots B_{k+1}(x_s), \quad x_1, \dots, x_s \in \mathbb{R}. \quad (5)$$

Then $\varphi \in W_p^k(\mathbb{R}^s)$, the shifts of φ are stable, and φ satisfies the moment conditions of order $k+1$.

Corollary 3. *Let $f \in W_p^k(\mathbb{R}^s)$ be a compactly supported function such that $\hat{f}(0) = 1$. If f satisfies the moment conditions of order $k+1$, then*

$$D^\mu f = \sum_{|\nu|=|\mu|} \nabla^\nu h_\nu^\mu \quad \forall |\mu| \leq k, \mu \in \mathbb{N}_0^s \quad (6)$$

for some compactly supported functions h_ν^μ in $L_p(\mathbb{R}^s)$.

Proof: Using a similar argument as in Corollary 2, (6) can be proved directly. On the other hand, take $g = \varphi$ in (5). Then it is obvious that $g \in W_p^k(\mathbb{R}^s)$ and satisfies the moment conditions of order $k+1$. Observe that $D^\mu \varphi = \nabla^\mu \psi$, where $\psi(x_1, \dots, x_s) := B_{k+1-\mu_1}(x_1) \cdots B_{k+1-\mu_s}(x_s)$ for all $\mu = (\mu_1, \dots, \mu_s)$ with $|\mu| \leq k$. The claim in (6) now follows directly from (4). \square

A characterization of convergence of a subdivision scheme in the L_p norm and in the Sobolev norm for $p = 2$ was given in Han and Jia [6] and Jia, Jiang and Lee [8], respectively. Also see Jia [7] for the univariate L_p case. Here, we present another proof for such characterization using Theorem 1. To do so, we need the following facts.

Proposition 4. *Let A be an $s \times s$ real-valued (or complex-valued) matrix. Then A is isotropic (i.e., A is similar to a diagonal matrix $\text{diag}(\sigma_1, \dots, \sigma_s)$ with $|\sigma_1| = \dots = |\sigma_s|$) if and only if there exists a norm $\|\cdot\|$ on \mathbb{R}^s (or on \mathbb{C}^s) such that $\|Ax\| = |\det A|^{1/s}\|x\|$ for all $x \in \mathbb{R}^s$ (or for all $x \in \mathbb{C}^s$).*

Proof: When A is isotropic, we can write $A = \Lambda^{-1} \text{diag}(\sigma_1, \dots, \sigma_s) \Lambda$ for some nonsingular matrix Λ . Let $\|\cdot\|_E$ be the Euclidean norm on \mathbb{C}^s . Define an equivalent norm $\|x\| := \|\Lambda x\|_E, x \in \mathbb{C}^s$. Obviously, $\|Ax\| = |\det A|^{1/s}\|x\|$ for all $x \in \mathbb{C}^s$.

Conversely, when A is real-valued and there is a norm on \mathbb{R}^s such that $\|Ax\| = |\det A|^{1/s}\|x\|$ for all $x \in \mathbb{R}^s$. Then we can define a norm on \mathbb{C}^s by $\|x + iy\| := \sqrt{\|x\|^2 + \|y\|^2}$ for all $x, y \in \mathbb{R}^s$, where i is the imaginary unit such that $i^2 = -1$. Obviously, we have $\|A^n x\| = |\det A|^{n/s}\|x\|$ for all $x \in \mathbb{C}^s$ and $n \in \mathbb{N}$. Using the Jordan canonical form of A , it is easy to see that A is isotropic. \square

When A is an $r \times r$ isotropic matrix, throughout, we always denote $\|\cdot\|$ to be the norm on \mathbb{C}^r such that $\|Ax\| = |\det A|^{1/r}\|x\|$ for all $x \in \mathbb{C}^r$.

The following claim can be easily verified by a simple computation.

Proposition 5. *Let $D := [D_1, \dots, D_s]^T$. Define a column vector of differential operators by $\otimes^k D := D \otimes \dots \otimes D$ with k copies of D , where \otimes denotes the (right) Kronecker product. Let A be an $s \times s$ real-valued matrix. Then*

$$\otimes^k D[f(A \cdot)] = (\otimes^k A^T)(\otimes^k Df)(A \cdot), \quad k \in \mathbb{N}, f \in C^\infty(\mathbb{R}^s).$$

Now, (4) can be rewritten as $\otimes^j D(g - f) = \sum_{|\nu|=j+1} \nabla^\nu h_\nu^j$ for all $j = 0, \dots, k$ for some vectors h_ν^j of compactly supported functions in $L_p(\mathbb{R}^s)$. We can also rewrite (6) into a similar form. Define a norm on $W_p^k(\mathbb{R}^s)$ as follows: $\|f\|_* := \sum_{j=0}^k \|\otimes^j Df\|_p$, where $\|[f_1, \dots, f_r]^T\|_p := \|[\|f_1\|_p, \dots, \|f_r\|_p]^T\|$ and the norm $\|\cdot\|$ on \mathbb{C}^r is determined in Proposition 4. It is evident that $\|\cdot\|_*$ is an equivalent norm to $\|\cdot\|_{W_p^k(\mathbb{R}^s)}$ in $W_p^k(\mathbb{R}^s)$.

Let M be an $s \times s$ dilation matrix and a be a finitely supported sequence on \mathbb{Z}^s . For $1 \leq p \leq \infty$ and $k \in \mathbb{N}_0$, we define

$$\rho_k(a, M; p) := \max\left\{\lim_{n \rightarrow \infty} \|\nabla^\mu S_{a, M}^n \delta\|_p^{1/n} : \mu \in \mathbb{N}_0^s, |\mu| = k\right\}, \quad (7)$$

$$\rho(a, M; p) := \inf\{\rho_k(a, M; p) : k \in \mathbb{N}_0\}, \quad (8)$$

where $\|\lambda\|_p := (\sum_{\beta \in \mathbb{Z}^s} |\lambda(\beta)|^p)^{1/p}$ and $S_{a,M}$ is the subdivision operator

$$[S_{a,M}\lambda](\alpha) := |\det M| \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta)\lambda(\beta), \quad \alpha \in \mathbb{Z}^s, \lambda \in \ell_0(\mathbb{Z}^s).$$

A mask a satisfies the sum rules of order k with respect to the lattice $M\mathbb{Z}^s$ if

$$\sum_{\beta \in M\mathbb{Z}^s} a(\alpha + \beta)q(\alpha + \beta) = \sum_{\beta \in M\mathbb{Z}^s} a(\beta)q(\beta) \quad \forall \alpha \in \mathbb{Z}^s, q \in \Pi_{k-1}.$$

For simplicity, we omit the proof of the following result.

Proposition 6. *Let M be an $s \times s$ isotropic dilation matrix. For any finitely supported sequence a on \mathbb{Z}^s such that $\sum_{\beta \in \mathbb{Z}^s} a(\beta) = 1$, then*

$$\rho_k(a, M; p) = \max \left\{ |\det M|^{1/p-k/s}, \rho_{k+j}(a, M; p) \right\} \quad \forall j, k \in \mathbb{N}_0.$$

Moreover, if $\rho(a, M; p) < |\det M|^{1/p-k/s}$ for some integer k , then a must satisfy the sum rules of order $k+1$ with respect to the lattice $M\mathbb{Z}^s$.

The following characterizes convergence of subdivision schemes in $W_p^k(\mathbb{R}^s)$.

Theorem 7. *Let M be an $s \times s$ isotropic dilation matrix. Let a be a finitely supported mask on \mathbb{Z}^s . Then the following statements are equivalent*

- 1) *For every compactly supported function $f \in W_p^k(\mathbb{R}^s)$ such that f satisfies the moment conditions of order $k+1$, the subdivision scheme converges in $W_p^k(\mathbb{R}^s)$; that is, $\lim_{n \rightarrow \infty} \|Q_{a,M}^n f - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} = 0$;*
- 2) *For some compactly supported function $f \in W_p^k(\mathbb{R}^s)$ such that f satisfies the moment conditions of order $k+1$ and the shifts of f are stable (one example of such f is φ in (5)), $\lim_{n \rightarrow \infty} \|Q_{a,M}^n f - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} = 0$;*
- 3) $\rho_{k+1}(a, M; p) < |\det M|^{1/p-k/s}$;
- 4) $\rho(a, M; p) < |\det M|^{1/p-k/s}$.

Proof: Obviously, 1) \Rightarrow 2). Suppose 2) holds. Let $f_n := Q_{a,M}^n f$. By induction, for $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, we have

$$\nabla_{M^{-n}e_1}^{\mu_1} \cdots \nabla_{M^{-n}e_s}^{\mu_s} f_n = \sum_{\beta \in \mathbb{Z}^s} [\nabla^\mu S_{a,M}^n \delta](\beta) f(M^n \cdot -\beta) \quad \forall n \in \mathbb{N}_0. \quad (9)$$

Note that 2) implies that $\phi_a^M \in W_p^k(\mathbb{R}^s)$. Let $\nabla^{\mu,n} := \nabla_{M^{-n}e_1}^{\mu_1} \cdots \nabla_{M^{-n}e_s}^{\mu_s}$ and $m := |\det M|$ for short. Since the shifts of f are stable, from (9), there exists a positive constant C depending only of f such that

$$m^{-n/p} \|\nabla^\mu S_{a,M}^n \delta\|_p \leq C \|\nabla^{\mu,n} f_n\|_p \leq C \|\nabla^{\mu,n} \phi_a^M\|_p + C \|\nabla^{\mu,n} (f_n - \phi_a^M)\|_p.$$

Note that all the functions f_n and ϕ_a^M are supported on $[-N, N]^s$ for some integer N independent of n . Since M is isotropic, there is a constant C_1 independent of n such that $\|\nabla^{\mu,n} (f_n - \phi_a^M)\|_p \leq C_1 m^{-nk/s} \|f_n - \phi_a^M\|_{W_p^k(\mathbb{R}^s)}$ for

all $|\mu| = k + 1$. Since $\phi_a^M \in W_p^k(\mathbb{R}^s)$, we have $\lim_{n \rightarrow \infty} m^{-nk/s} \|\nabla^{\mu, n} \phi_a^M\|_p = 0$ for all $|\mu| = k + 1$. Therefore, by $\lim_{n \rightarrow \infty} \|f_n - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} = 0$, we deduce that $\lim_{n \rightarrow \infty} m^{n(k/s-1/p)} \|\nabla^\mu S_{a,M}^n \delta\|_p = 0$ for all $|\mu| = k + 1$. By joint spectral radius as in [6, Theorem 3.2], we have $\lim_{n \rightarrow \infty} \|\nabla^\mu S_{a,M}^n \delta\|_p^{1/n} < m^{1/p-k/s}$ and therefore, 3) holds. By Proposition 6, 3) \iff 4).

Finally, we prove 3) \Rightarrow 1). Let f be a compactly supported function in $W_p^k(\mathbb{R}^s)$ such that f satisfies the moment conditions of order $k + 1$. Then 3) implies that $Q_{a,M}f$ also has such properties. By 3), there exist two constants $0 < \rho < 1$ and $C > 0$ such that

$$\|\nabla^\mu S_{a,M}^n \delta\|_p \leq C m^{n(1/p-k/s)} \rho^n \quad \forall n \in \mathbb{N}, |\mu| = k + 1. \quad (10)$$

Let $h = Q_{a,M}f - f$. By Corollary 2, $\otimes^k D h = \sum_{|\nu|=k+1} \nabla^\nu h_\nu^k$ for some vectors h_ν^k of compactly supported functions in $L_p(\mathbb{R}^s)$. By the equation in (9) and Proposition 5, we have

$$\otimes^k D[f_{n+1} - f_n] = (\otimes^k M^T) \left(\sum_{|\nu|=k+1} \sum_{\beta \in \mathbb{Z}^s} [\nabla^\nu S_{a,M}^n \delta](\beta) h_\nu^k(M^n \cdot -\beta) \right).$$

Note that the isotropic matrix $\otimes^k (M^T)^n$ has the operator norm $\|\otimes^k (M^T)^n\| = m^{nk/s}$. Since all the components in h_ν^k are compactly supported functions in $L_p(\mathbb{R}^s)$, there exists a positive constant C_1 depending only on $h_\nu^k, |\nu| = k + 1$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned} \|\otimes^k D[f_{n+1} - f_n]\|_p &\leq C_1 m^{\frac{-n}{p}} \|\otimes^k (M^T)^n\| \max\{\|\nabla^\nu S_{a,M}^n \delta\|_p : |\nu| = k + 1\} \\ &\leq C C_1 m^{-n/p} m^{nk/s} m^{n(1/p-k/s)} \rho^n = C C_1 \rho^n. \end{aligned}$$

Thus, $D^\mu f_n$ converges in the L_p norm for all $|\mu| = k$. Therefore, we must have $\lim_{n \rightarrow \infty} \|Q_{a,M}^n f - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} = \lim_{n \rightarrow \infty} \|f_n - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} = 0$.

An alternative proof of 3) \Rightarrow 1) is that 3) implies $\phi_a^M \in W_p^k(\mathbb{R}^s)$ and $\lim_{n \rightarrow \infty} \|Q_{a,M}^n(f - \phi_a^M)\|_{W_p^k(\mathbb{R}^s)} = 0$ by a similar argument. \square

By a similar argument as in the proof of Theorem 7, we see that when a compactly supported refinable function $\phi_a^M \in W_p^k(\mathbb{R}^s)$ (for $p = \infty$, the k th derivatives of ϕ_a^M must be continuous) has stable shifts, then the subdivision scheme converges in $W_p^k(\mathbb{R}^s)$ and all the claims in Theorem 7 hold. Moreover, the quantities $\rho_k(a, M; 2)$ in Theorem 7 can be computed by [6, Theorem 4.1].

§3. Error Estimate of a Subdivision Scheme in a Sobolev Space

In applications, when the coefficients in a mask (such as the Daubechies' orthogonal masks) are irrational numbers, one often needs to truncate such a mask. Daubechies and Huang [2] first studied how such truncation affects the associated refinable function in the univariate L_∞ case. In [3, 4], Han first provided a sharp error estimate for multivariate refinable functions and

their subdivision schemes with a perturbed mask in any L_p norm. The main idea in Han [3, 4] was used in Han and Hogan [5] to obtain error estimate for vector subdivision schemes in the univariate L_p case, and recently was generalized by Chen and Plonka [1] to establish error estimates for subdivision schemes in a Sobolev space with the particular initial function φ defined in (5). The estimate in [1] essentially relies on the relation $D^\mu \varphi = \nabla^\mu \psi$, where $\psi(x_1, \dots, x_s) = B_{k+1-\mu_1}(\xi_1) \cdots B_{k+1-\mu_s}(x_s)$ for all $\mu = (\mu_1, \dots, \mu_s)$ with $|\mu| \leq k$. In the following result, we shall remove such restriction on the initial functions with the help of Theorem 1 and Corollary 3.

Let a be a finitely supported mask on \mathbb{Z}^s such that a vanishes outside a finite subset Ω of \mathbb{Z}^s . For any $\eta > 0$, we define the neighborhood $N_\eta(a; k, M, \Omega)$ to be the set of all sequences b on \mathbb{Z}^s such that $\sum_{\beta \in \mathbb{Z}^s} b(\beta) = 1$, b vanishes outside Ω , b satisfies the sum rules of order $k+1$ with respect to the lattice $M\mathbb{Z}^s$, and $\|a - b\|_1 < \eta$.

Following the line developed in [3, 4], we have the following result.

Theorem 8. *Let M be an $s \times s$ isotropic dilation matrix with $m := |\det M|$. Let a be a mask on \mathbb{Z}^s such that a vanishes outside a finite subset Ω of \mathbb{Z}^s . Suppose that the subdivision scheme associated with mask a and dilation M converges in the Sobolev space $W_p^k(\mathbb{R}^s)$ (i.e., $\rho_{k+1}(a, M; p) < m^{1/p-k/s}$). Then there exists $\eta > 0$ such that for every $b \in N_\eta(a; k, M, \Omega)$, the subdivision scheme associated with mask b and dilation M converges in $W_p^k(\mathbb{R}^s)$. Moreover, for any compactly supported function f in $W_p^k(\mathbb{R}^s)$ such that f satisfies the moment conditions of order $k+1$, there is a constant $C > 0$ such that*

$$\|Q_{a,M}^n f - Q_{b,M}^n f\|_{W_p^k(\mathbb{R}^s)} \leq C \|a - b\|_1 \quad \forall n \in \mathbb{N}, b \in N_\eta(a; k, M, \Omega),$$

where the neighborhood $N_\eta(a; k, M, \Omega)$ of a is defined above. Consequently, $\|\phi_a^M - \phi_b^M\|_{W_p^k(\mathbb{R}^s)} \leq C \|a - b\|_1$ for all $b \in N_\eta(a; k, M, \Omega)$.

Proof: By assumption and Theorem 7, we have $\rho_{k+1}(a, M; p) < m^{1/p-k/s}$. Using an argument of joint spectral radius, there exists a small enough $\eta > 0$ such that for any mask $b \in N_\eta(a; k, M, \Omega)$, $\rho_{k+1}(b, M; p) < m^{1/p-k/s}$ and therefore, the subdivision scheme associated with mask b and dilation M converges in $W_p^k(\mathbb{R}^s)$. Similarly, using joint spectral radius again (for detail, see [1, 3, 4]), there exists a constant $C > 0$ such that

$$\|\nabla^\mu S_{a,M}^n \delta - \nabla^\mu S_{b,M}^n \delta\|_p \leq C m^{n(1/p-|\mu|/s)} \|a - b\|_1 \quad \forall |\mu| \leq k. \quad (11)$$

By induction, we have

$$Q_{a,M}^n f - Q_{b,M}^n f = \sum_{\beta \in \mathbb{Z}^s} [S_{a,M}^n \delta(\beta) - S_{b,M}^n \delta(\beta)] f(M^n \cdot -\beta) \quad \forall n \in \mathbb{N}.$$

By Proposition 5 and Corollary 3, for $j = 0, \dots, k$, we have

$$\begin{aligned} & \otimes^j D[Q_{a,M}^n f - Q_{b,M}^n f] \\ &= \sum_{\beta \in \mathbb{Z}^s} [S_{a,M}^n \delta(\beta) - S_{b,M}^n \delta(\beta)] (\otimes^j (M^T)^n) (\otimes^j Df)(M^n \cdot -\beta) \\ &= \sum_{|\nu|=j} \sum_{\beta \in \mathbb{Z}^s} [\nabla^\nu S_{a,M}^n \delta(\beta) - \nabla^\nu S_{b,M}^n \delta(\beta)] (\otimes^j (M^T)^n) h_\nu^j(M^n \cdot -\beta). \end{aligned}$$

Since all the components of h_ν^j are compactly supported functions in $L_p(\mathbb{R}^s)$, there exists a constant $C_1 > 0$ depending only on h_ν^j such that

$$\begin{aligned} & \|\otimes^j D[Q_{a,M}^n f - Q_{b,M}^n f]\|_p \\ & \leq C_1 m^{n(j/s-1/p)} \max\{\|\nabla^\nu S_{a,M}^n \delta - \nabla^\nu S_{b,M}^n \delta\|_p : |\nu| = j\} \\ & \leq C_1 C \|a - b\|_1, \end{aligned}$$

where we used (11) in the last inequality. This completes the proof. \square

§4. Convergence Rate and Generalized Definition of Convergence

In this section, we study the rate of convergence of a subdivision scheme. We also generalize the definition of convergence of a subdivision scheme.

Theorem 9. *Let M be an $s \times s$ isotropic dilation matrix with $m := |\det M|$. Let a be a finitely supported mask on \mathbb{Z}^s and k be a nonnegative integer such that $\rho(a, M; p) < m^{1/p-k/s}$. Let $K := \inf\{j \in \mathbb{N}_0 : \rho_{j+1}(a, M; p) = \rho(a, M; p)\}$ and $r := \rho(a, M; p)m^{k/s-1/p} < 1$. Suppose that f is a compactly supported function in $W_p^k(\mathbb{R}^s)$ such that $D^\mu \hat{f}(0) = D^\mu \hat{\phi}_a^M(0)$ for all $|\mu| \leq K - k$, and $D^\mu \hat{f}(2\pi\beta) = 0$ for all $|\mu| \leq K$ and $\beta \in \mathbb{Z}^s \setminus \{0\}$. Then for any $0 < \epsilon < 1 - r$, there exists a constant $C > 0$ such that*

$$\|Q_{a,M}^n f - \phi_a^M\|_{W_p^k(\mathbb{R}^s)} \leq C(r + \epsilon)^n \quad \forall n \in \mathbb{N}.$$

Consequently, $\|Q_{a,M}^{n+1} f - Q_{a,M}^n f\|_{W_p^k(\mathbb{R}^s)} \leq 2C(r + \epsilon)^n$ for all $n \in \mathbb{N}$.

Proof: Let $h = f - \phi_a^M$. By assumption, $D^\mu \hat{h}(0) = 0$ for all $|\mu| \leq K - k$ and $D^\mu \hat{h}(2\pi\beta) = 0$ for all $|\mu| \leq K$ and $\beta \in \mathbb{Z}^s \setminus \{0\}$. According to Corollary 2 and Theorem 1, we have $D^\mu h = \sum_{|\nu|=|\mu|+K-k+1} \nabla^\nu h_\nu^\mu$ for all $|\mu| \leq k$ and for some compactly supported functions h_ν^μ in $L_p(\mathbb{R}^s)$. The rest of the proof is the same as that in Theorem 7. \square

By definition, $\rho(a, M; p)$ is isotropic, but each $\lim_{n \rightarrow \infty} \|\nabla^\mu S_{a,M}^n \delta\|_p$ depends on $\mu \in \mathbb{N}_0^s$. Therefore, better convergence rate than that in Theorem 9 can be easily achieved as long as the initial function matches the refinable functions well in certain directions. Also $D^\mu \hat{\phi}_a^M(0), |\mu| < k$ can be easily obtained by a recursive formula derived from the refinement equation (1).

Finally, the discussion in this paper motivates us to give a general definition about the convergence of a subdivision scheme. Let $(X, \|\cdot\|)$ be a Banach space with $X \subseteq L_1(\mathbb{R}^s)$. Let M be a dilation matrix and a be a finitely supported mask on \mathbb{Z}^s . We say that a linear space X_0 is an **admissible space** if for every $f \in X_0$, f is a compactly supported function in X and $Q_{a,M} f \in X_0$. We say that a subdivision scheme associated with mask a and dilation M converges in $(X, \|\cdot\|)$ with the admissible space X_0 if $\phi_a^M \in X$ and $\lim_{n \rightarrow \infty} \|Q_{a,M}^n f\| = 0$ for all $f \in X_0$. This definition implies that $\lim_{n \rightarrow \infty} \|Q_{a,M}^n (f + \phi_a^M) - \phi_a^M\| = 0$ for all $f \in X_0$.

Let X_0 denote the set of all compactly supported functions $f \in W_p^k(\mathbb{R}^s)$ such that $\hat{f}(0) = 0$ and $D^\mu \hat{f}(2\pi\beta) = 0$ for all $|\mu| \leq k$ and $\beta \in \mathbb{Z}^s \setminus \{0\}$. Then the definition of convergence of subdivision schemes in Theorem 7 can be restated as that the subdivision scheme converges in $(W_p^k(\mathbb{R}^s), \|\cdot\|_{W_p^k(\mathbb{R}^s)})$ with the admissible space X_0 .

Example. Let $a(0) = a(2) = 1/2$ and $a(\beta) = 0$ for all $\beta \in \mathbb{Z} \setminus \{0, 2\}$. Let X_0 denote the set of all compactly supported functions $f \in L_p(\mathbb{R})$ such that $\hat{f}(\pi\beta) = 0$ for all $\beta \in \mathbb{Z}$. Then it is not difficult to verify that for any $1 \leq p < \infty$, the subdivision scheme associated with mask a and dilation 2 converges in $(L_p(\mathbb{R}), \|\cdot\|_p)$ with the admissible space X_0 . However, the subdivision scheme with mask a and dilation 2 does not converge in the L_p norm in the sense of Theorem 7 with $k = 0$.

From the proofs, all results hold for general dilation matrices when $k = 0$.

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Bin Han

Department of Mathematical Sciences

University of Alberta

Edmonton, Alberta, Canada T6G 2G1

bhan@ualberta.ca, <http://www.ualberta.ca/~bhan>