An hp-Analysis of the Local Discontinuous Galerkin Method for Diffusion Problems*

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Abstract

We present an hp-analysis of the local discontinuous Galerkin method for diffusion problems, considering unstructured meshes with hanging nodes and two- and three-dimensional domains. Our estimates are optimal in the meshsize h and slightly suboptimal in the polynomial approximation order p.

Key words: hp-FEM, local discontinuous Galerkin methods

1 Introduction

The aim of this paper is to present an hp-error analysis of the local discontinuous Galerkin (LDG) method, introduced by Cockburn and Shu (1998), for the diffusion problem

$$-\nabla \cdot (\nu \nabla u) = f \text{ in } \Omega \qquad u = g_{\mathcal{D}} \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded polygonal or polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$, and $\nu \in L^\infty(\Omega)^{d\times d}$ a symmetric, uniformly positive definite diffusion tensor. The right-hand side f belongs to $L^2(\Omega)$, and the Dirichlet datum $g_{\mathcal{D}}$ to $H^{\frac{1}{2}}(\partial\Omega)$. Deriving error bounds that take into account both the elemental meshsize and approximation order completes previous work by Castillo, Cockburn, Perugia and Schötzau (2000) on the LDG method applied to

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purely elliptic problems. This is relevant since the LDG method, being based on discontinuous spaces, is ideally suited for hp-adaptivity.

The analysis in this paper follows the same lines as the one developed in Perugia and Schötzau (2001) in the more complicated situation of the low-frequency time-harmonic Maxwell equations, and uses the general framework introduced in Arnold, Brezzi, Cockburn and Marini (2001). For unstructured meshes with hanging nodes, we prove error estimates that are optimal in the mesh-size h and slightly suboptimal in the polynomial degree p (half a power of p is lost). We point out that, for two- or three-dimensional elliptic problems on unstructured grids, no better p-bounds can be found in the literature (see, e.g., Rivière, Wheeler and Girault (1999), Prudhomme, Pascal, Oden and Romkes (2000), Houston, Schwab and Süli (2000), where the same bounds have been obtained for different discontinuous Galerkin methods and with different analysis techniques). Optimal hp-bounds have been proved in Castillo, Cockburn, Schötzau and Schwab (2001) for one-dimensional convection-diffusion problems and, recently, in Georgoulis and Süli (2001) for two-dimensional reaction-diffusion problems on structured quadrilateral grids.

The outline of the paper is as follows. In section 2, we define the LDG method for the diffusion problem (1). We carry out the error analysis in section 3, obtaining hp-error estimates in a problem-related energy norm, as well as in the L^2-norm. We end our presentation in section 4 with concluding remarks.

2 LDG Discretization

2.1 Meshes and Finite Element Spaces

We consider shape-regular meshes $\mathcal{T}_h$ that partition the domain Ω into triangles and/or parallelograms, if $d = 2$, and tetrahedra and/or parallelepipeds, if $d = 3$, with possible hanging nodes and aligned with the possible discontinuities of the diffusion tensor ν, so that ν is smooth within each element of $\mathcal{T}_h$. We denote by h_K the diameter of the element $K \in \mathcal{T}_h$. We define the $(d-1)$-dimensional faces of $\mathcal{T}_h$ as follows. An *interior face* of $\mathcal{T}_h$ is the (non-empty) interior of $\partial K^+ \cap \partial K^-$, where K^+ and K^- are two adjacent elements of $\mathcal{T}_h$, not necessarily matching. Similarly, a *boundary face* of $\mathcal{T}_h$ is the (non-empty) interior of $\partial K \cap \partial\Omega$, where K is a boundary element of $\mathcal{T}_h$. We denote by $\mathcal{E}_\mathcal{I}$ the union of all interior faces of $\mathcal{T}_h$, by $\mathcal{E}_\mathcal{D}$ the union of all boundary faces, and set $\mathcal{E} = \mathcal{E}_\mathcal{I} \cup \mathcal{E}_\mathcal{D}$.

Let $\underline{p} = \{p_K\}_{K\in\mathcal{T}_h}$ be a degree vector that assigns to each element $K \in \mathcal{T}_h$ a polynomial approximation order $p_K \geq 1$. The generic hp-finite element space of piecewise polynomials is given by $S^{\underline{p}}(\mathcal{T}_h) := \{u \in L^2(\Omega) : u|_K \in$

$\mathcal{S}^{p_K}(K),\ \forall K \in \mathcal{T}_h\}$, where $\mathcal{S}^{p_K}(K)$ is the space $\mathcal{P}^{p_K}(K)$ of polynomials of degree at most p_K in K, if K is a triangle or a tetrahedron, and the space $\mathcal{Q}^{p_K}(K)$ of polynomials of degree at most p_K in each variable in K, if K is a quadrilateral or a parallelepiped.

2.2 The Flux Formulation of the LDG Method

By introducing the auxiliary variables $\boldsymbol{q} = \nu \boldsymbol{s}$ and $\boldsymbol{s} = \nabla u$, the diffusion problem (1) can be rewritten as

$$\begin{array}{rclcrcll} \boldsymbol{q} & = & \nu \boldsymbol{s} & \text{in } \Omega \qquad & -\nabla \cdot \boldsymbol{q} & = & f & \text{in } \Omega \\ \boldsymbol{s} & = & \nabla u & \text{in } \Omega \qquad & u & = & g_{\mathcal{D}} & \text{on } \partial\Omega. \end{array}$$

We approximate the variables ($\boldsymbol{q}$,$\boldsymbol{s}$,u) by discrete functions $(\boldsymbol{q}_h, \boldsymbol{s}_h, u_h)$ in the hp-finite element space $\boldsymbol{Q}_h \times \boldsymbol{Q}_h \times V_h$, where $\boldsymbol{Q}_h = S^{\underline{p}}(\mathcal{T}_h)^d$ and $V_h = S^{\underline{p}}(\mathcal{T}_h)$, for a given degree distribution $\underline{p}$.

The LDG method then consists in finding $(\boldsymbol{q}_h, \boldsymbol{s}_h, u_h) \in \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times V_h$ such that for any $(\boldsymbol{r}, \boldsymbol{t}, v) \in \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times V_h$ and for any element $K \in \mathcal{T}_h$

$$\begin{aligned} &\int_K \boldsymbol{q}_h \cdot \boldsymbol{r}\, d\boldsymbol{x} = \int_K \nu \boldsymbol{s}_h \cdot \boldsymbol{r}\, d\boldsymbol{x} \\ &\int_K \boldsymbol{s}_h \cdot \boldsymbol{t}\, d\boldsymbol{x} + \int_K u_h \nabla \cdot \boldsymbol{t}\, d\boldsymbol{x} - \int_{\partial K} \widehat{u}_h\, \boldsymbol{t} \cdot \boldsymbol{n}_K\, ds = 0 \\ &\int_K \boldsymbol{q}_h \cdot \nabla v\, d\boldsymbol{x} - \int_{\partial K} \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}_K\, v\, ds = \int_K f\, v\, d\boldsymbol{x}. \end{aligned} \tag{2}$$

Here, $\boldsymbol{n}_K$ denotes the outward unit normal vector to ∂K. The quantities $\widehat{u}_h$ and $\widehat{\boldsymbol{q}}_h$ are the so-called *numerical fluxes*, which are approximations to the traces of u and $\boldsymbol{q}$ on ∂K, and are defined as follows.

Consider an interior face e shared by two elements K^+ and K^-. Denoting by $v^\pm$ and $\boldsymbol{r}^\pm$ the traces on $\partial K^\pm$ of functions v and $\boldsymbol{r}$ that are smooth in $K^\pm$, we define the averages and jumps of v and $\boldsymbol{r}$ across e by

$$\begin{array}{ll} \{\!\{v\}\!\} = (v^+ + v^-)/2 \qquad & \{\!\{\boldsymbol{r}\}\!\} = (\boldsymbol{r}^+ + \boldsymbol{r}^-)/2 \\ [\![v]\!] = v^+ \boldsymbol{n}_{K^+} + v^- \boldsymbol{n}_{K^-} \qquad & [\![\boldsymbol{r}]\!] = \boldsymbol{r}^+ \cdot \boldsymbol{n}_{K^+} + \boldsymbol{r}^- \cdot \boldsymbol{n}_{K^-}. \end{array}$$

If $v \in H^1(\Omega)$, then $[\![v]\!] = \boldsymbol{0}$ on $\mathcal{E}_{\mathcal{I}}$. Similarly, if $\boldsymbol{r} \in H(\mathrm{div};\Omega)$, then $[\![\boldsymbol{r}]\!] = 0$ on $\mathcal{E}_{\mathcal{I}}$. In particular, for the exact solution u, we have $[\![\nu \nabla u]\!] = 0$ on $\mathcal{E}_{\mathcal{I}}$.

The numerical fluxes are defined by

$$\widehat{u}|_e = \begin{cases} \{\!\{u\}\!\} + \boldsymbol{b} \cdot [\![u]\!] & \text{if } e \subset \mathcal{E}_{\mathcal{I}} \\ g_{\mathcal{D}} & \text{if } e \subset \mathcal{E}_{\mathcal{D}} \end{cases} \quad \widehat{\boldsymbol{q}}|_e = \begin{cases} \{\!\{\boldsymbol{q}\}\!\} - \mathsf{a}[\![u]\!] - \boldsymbol{b}[\![\boldsymbol{q}]\!] & \text{if } e \subset \mathcal{E}_{\mathcal{I}} \\ \boldsymbol{q} - \mathsf{a}(u - g_{\mathcal{D}})\boldsymbol{n} & \text{if } e \subset \mathcal{E}_{\mathcal{D}}, \end{cases}$$

with parameters a and $\boldsymbol{b}$ to be properly chosen. This completes the definition of the LDG method. Notice that the flux in u is independent of $\boldsymbol{q}$.

This allows for an element-by-element elimination of the auxiliary variables $\boldsymbol{q}$ and $\boldsymbol{s}$, giving rise to the so-called *primal formulation* of the method in the variable u only. This local solvability gives the name to the LDG method. We refer to Castillo (2001) for a discussion of this elimination process from a computational point of view. Let us also point out that the LDG method defined above is consistent and, if $\mathtt{a}$ is strictly positive, defines a unique discrete solution $(\boldsymbol{q}_h, \boldsymbol{s}_h, v) \in \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times V_h$; see Castillo et al. (2000).

2.3 The Primal Formulation

We develop the error analysis of the LDG method in the framework introduced in Arnold et al. (2001), by considering its primal formulation.

For v belonging to $V(h) := V_h + H^1(\Omega)$, we define $\boldsymbol{\mathcal{L}}(v) \in \boldsymbol{Q}_h$ by

$$\int_\Omega \boldsymbol{\mathcal{L}}(v) \cdot \boldsymbol{r}\, d\boldsymbol{x} = \int_{\mathcal{E}_\mathcal{I}} (\{\!\{\boldsymbol{r}\}\!\} - \boldsymbol{b}\, [\![\boldsymbol{r}]\!]) \cdot [\![v]\!]\, ds + \int_{\mathcal{E}_\mathcal{D}} v\, \boldsymbol{r} \cdot \boldsymbol{n}\, ds \qquad \forall \boldsymbol{r} \in \boldsymbol{Q}_h.$$

Similarly, we define the lifting $\boldsymbol{\mathcal{G}}_\mathcal{D} \in \boldsymbol{Q}_h$ of the boundary datum $g_\mathcal{D}$ by

$$\int_\Omega \boldsymbol{\mathcal{G}}_\mathcal{D} \cdot \boldsymbol{r}\, d\boldsymbol{x} = \int_{\mathcal{E}_\mathcal{D}} g_\mathcal{D}\, \boldsymbol{r} \cdot \boldsymbol{n}\, ds \qquad \forall \boldsymbol{r} \in \boldsymbol{Q}_h.$$

Notice that, for the exact solution u, we have $\boldsymbol{\mathcal{L}}(u) = \boldsymbol{\mathcal{G}}_\mathcal{D}$. Adding the first and second equations in (2) over all elements and simple calculations yield

$$\boldsymbol{q}_h = \boldsymbol{\Pi}\big[\nu\big(\nabla_h u_h - \boldsymbol{\mathcal{L}}(u_h) + \boldsymbol{\mathcal{G}}_\mathcal{D}\big)\big], \tag{3}$$

with $\boldsymbol{\Pi}$ denoting the L^2-projection onto $\boldsymbol{Q}_h$ and ∇_h the elementwise gradient. Inserting this expression in the third equation of (2) yields the primal form of the LDG method: find $u_h \in V_h$ such that

$$A_h(u_h, v) + I_h(u_h, v) = F_h(v) \qquad \forall v \in V_h, \tag{4}$$

where

$$\begin{aligned} A_h(u,v) &= \int_\Omega \nu \left(\nabla_h u - \boldsymbol{\mathcal{L}}(u)\right) \cdot \left(\nabla_h v - \boldsymbol{\mathcal{L}}(v)\right) d\boldsymbol{x} \\ I_h(u,v) &= \int_{\mathcal{E}_\mathcal{I}} \mathtt{a}\, [\![u]\!] \cdot [\![v]\!]\, ds + \int_{\mathcal{E}_\mathcal{D}} \mathtt{a}\, u\, v\, ds \\ F_h(v) &= \int_\Omega f\, v\, d\boldsymbol{x} + \int_{\partial\Omega} \mathtt{a}\, g_\mathcal{D}\, v\, ds - \int_\Omega \nu\, \boldsymbol{\mathcal{G}}_\mathcal{D} \cdot \left(\nabla_h v - \boldsymbol{\mathcal{L}}(v)\right) d\boldsymbol{x}. \end{aligned}$$

For *discrete* trial and test functions, the primal form (4), together with identity (3), is equivalent to the original flux form (2) of the LDG method. However, unlike (2), the formulation (4) is no longer consistent. Nevertheless, the form $A_h + I_h$ has continuity and coercivity properties that allow us to carry out an error analysis in a straightforward way by using Strang's lemma.

3 Error Analysis

In this section, we develop the hp-error analysis of the LDG method. We start by specifying the parameters $\mathtt{a}$ and $\boldsymbol{b}$ in the definition of the method. Then we prove abstract error estimates in a broken norm and derive the actual hp-error bounds. Finally, we address the issue of the stability of the LDG formulation.

3.1 The Discontinuity Stabilization Parameter

We introduce the functions $\mathtt{h}$ and $\mathtt{p}$ in $L^\infty(\mathcal{E})$ related to the local meshsize and approximation degree as

$$\mathtt{h} = \mathtt{h}(\boldsymbol{x}) := \begin{cases} \min\{h_K, h_{K'}\} & \text{if } \boldsymbol{x} \text{ in the interior of } \partial K \cap \partial K' \\ h_K & \text{if } \boldsymbol{x} \text{ in the interior of } \partial K \cap \partial\Omega \end{cases}$$

$$\mathtt{p} = \mathtt{p}(\boldsymbol{x}) := \begin{cases} \max\{p_K, p_{K'}\} & \text{if } \boldsymbol{x} \text{ in the interior of } \partial K \cap \partial K' \\ p_K & \text{if } \boldsymbol{x} \text{ in the interior of } \partial K \cap \partial\Omega. \end{cases}$$

Regarding the diffusivity, we assume ν to be Lipschitz continuous in K, for any $K \in \mathcal{T}_h$. This implies that $\nu|_K$ can be extended up to ∂K, and we denote this extension by ν_K. Therefore, for any $K \in \mathcal{T}_h$, there are positive constants n_K and N_K such that $n_K \leq \lambda_i(\nu_K(\boldsymbol{x})) \leq N_K$ for all $\boldsymbol{x} \in \overline{K}$, where $\lambda_i(\nu_K(\boldsymbol{x}))$, $i = 1, 2, 3$, are the eigenvalues of $\nu_K(\boldsymbol{x})$. For any $K \in \mathcal{T}_h$, the constants n_K and N_K are assumed to satisfy

$$N_K \leq \kappa\, n_K, \qquad \forall K \in \mathcal{T}_h,$$

with a constant $\kappa > 0$. Whenever ν is a piecewise constant scalar function, this assumption holds true with $\kappa = 1$. We set

$$\mathtt{n} = \mathtt{n}(\boldsymbol{x}) := \begin{cases} \max\{|\nu_K(\boldsymbol{x})|, |\nu_{K'}(\boldsymbol{x})|\} & \text{if } \boldsymbol{x} \text{ is in the interior of } \partial K \cap \partial K' \\ |\nu_K(\boldsymbol{x})| & \text{if } \boldsymbol{x} \text{ is in the interior of } \partial K \cap \partial\Omega, \end{cases}$$

where $|\nu(\boldsymbol{x})|$ is the spectral norm of the tensor $\nu(\boldsymbol{x})$.

We define the discontinuity stabilization parameter $\mathtt{a} \in L^\infty(\mathcal{E})$ in terms of $\mathtt{h}$, $\mathtt{p}$ and $\mathtt{n}$ by

$$\mathtt{a} = \alpha\, \frac{\mathtt{p}^2 \mathtt{n}}{\mathtt{h}},$$

with $\alpha > 0$ independent of meshsize, approximation order and diffusion. Moreover, the parameter $\boldsymbol{b}$ is taken to be of order one, i.e.,

$$\|\boldsymbol{b}\|_{\infty, \mathcal{E}_\mathcal{I}} \leq \delta,$$

with $\delta > 0$ independent of meshsize, approximation order and diffusion.

3.2 Continuity and Stability

We introduce the energy norm

$$|\!|\!| u |\!|\!|_h^2 = \|\nu^{\frac{1}{2}} \nabla_h u\|_{0,\Omega}^2 + \alpha \,\|\mathrm{h}^{-\frac{1}{2}} \mathrm{p}\, \mathrm{n}^{\frac{1}{2}} [\![u]\!]\|_{0,\mathcal{E}_\mathcal{I}}^2 + \alpha \,\|\mathrm{h}^{-\frac{1}{2}} \mathrm{p}\, \mathrm{n}^{\frac{1}{2}} u\|_{0,\mathcal{E}_\mathcal{D}}^2.$$

We have the following continuity and coercivity properties.

Proposition 3.1 Assume the above hypotheses on ν and on the coefficients in the definition of the numerical fluxes. Then

$$\begin{aligned} |A_h(w,v) + I_h(w,v)| &\leq C_{\mathrm{cont}} |\!|\!| w |\!|\!|_h |\!|\!| v |\!|\!|_h \qquad && \forall w, v \in V(h) \\ A_h(v,v) + I_h(v,v) &\geq C_{\mathrm{coer}} |\!|\!| v |\!|\!|_h^2 && \forall v \in V_h, \end{aligned}$$

with C_{cont} and C_{coer} only depending on α, δ, κ and the shape-regularity of the mesh.

Proof. With arguments similar to the ones in Perugia and Schötzau (2001, Proposition 4.2), we have

$$\|\nu^{\frac{1}{2}} \boldsymbol{\mathcal{L}}(v)\|_{0,\Omega} \leq C_{\mathrm{lift}}\, \kappa\, (\delta + 1) \Big[\|\mathrm{h}^{-\frac{1}{2}} \mathrm{p}\, \mathrm{n}^{\frac{1}{2}} [\![v]\!]\|_{0,\mathcal{E}_\mathcal{I}} + \|\mathrm{h}^{-\frac{1}{2}} \mathrm{p}\, \mathrm{n}^{\frac{1}{2}} v\|_{0,\mathcal{E}_\mathcal{D}} \Big] \quad (5)$$

for all $v \in V(h)$. The constant $C_{\mathrm{lift}} > 0$ only depends on the shape-regularity of the mesh. The continuity and coercivity of $A_h + I_h$ are now easy consequences of estimate (5). □

From Proposition 3.1 and Strang's lemma, we immediately have the following abstract error bound.

Theorem 3.1 Assume the above hypotheses on ν and on the coefficients in the definition of the numerical fluxes. Then we have

$$|\!|\!| u - u_h |\!|\!|_h \leq \left(1 + \frac{C_{\mathrm{cont}}}{C_{\mathrm{coer}}}\right) \inf_{v \in V_h} |\!|\!| u - v |\!|\!|_h + \frac{1}{C_{\mathrm{coer}}} \sup_{w \in V_h} \frac{|R_h(u,w)|}{|\!|\!| w |\!|\!|_h},$$

with the residual $R_h(u,w) = A_h(u,w) + I_h(u,w) - F_h(w)$.

3.3 hp-Error Estimates

We make the assumption that the local meshsizes and approximation degrees have bounded variation, i.e., there exists a constant $\ell > 0$ such that $\ell^{-1} h_K \leq h_{K'} \leq \ell h_K$ and $\ell^{-1} p_K \leq p_{K'} \leq \ell p_K$ for all K and K' sharing a $(d-1)$-dimensional face. This assumption forbids the situation where the mesh is indefinitely refined in only one of two adjacent subdomains, but allows for geometric refinement and linearly increasing approximation orders. For any element K, we define $N_{\delta K} := \max\{N_{K'} :$ K and K' share at least one face$\}$. We have the following error bound.

Theorem 3.2 Assume the above hypotheses on ν, on the coefficients in the definition of the numerical fluxes and on the meshes and polynomial degree distributions. Let the exact solution u satisfy $u|_K \in H^{s_K+1}(K)$ and $\nu \nabla u|_K \in H^{s_K}(K)$, for all $K \in \mathcal{T}_h$, with local regularity exponents $s_K \geq 1$. Then

$$\| u - u_h \|_h^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min(p_K, s_K)}}{p_K^{2s_K - 1}} \Big[N_{\delta K} \|u\|_{s_K+1,K}^2 + \frac{1}{n_K} \|\nu \nabla u\|_{s_K,K}^2 \Big],$$

with C independent of h_K and p_K. Moreover,

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,\Omega}^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min(p_K, s_K)}}{p_K^{2s_K - 1}} \Big[N_{\delta K} \|u\|_{s_K+1,K}^2 + (1 + n_K^{-1}) \|\nu \nabla u\|_{s_K,K}^2 \Big].$$

Remark 3.1 The bound in Theorem 3.2 is optimal in the meshsize and slightly suboptimal in the approximation order. On structured quadrilateral meshes in two dimensions, the p-bound can be improved by using the hp-projector of Georgoulis and Süli (2001), provided that the solution belongs to certain augmented Sobolev spaces.

In order to prove Theorem 3.2, we need the following hp-approximation result (see Babuška and Suri (1987, Lemma 4.5)).

Lemma 3.1 Let $K \in \mathcal{T}_h$ and suppose that $v \in H^{t_K}(K)$, $t_K \geq 1$. Then there exists a sequence of polynomials $\pi_{p_K}^{h_K} v$ in $\mathcal{S}^{p_K}(K)$, $p_K = 1, 2, \ldots$, satisfying

$$\|v - \pi_{p_K}^{h_K} v\|_{q,K} \leq C \frac{h_K^{\min(p_K+1, t_K) - q}}{p_K^{t_K - q}} \|v\|_{t_K,K} \qquad \forall 0 \leq q \leq t_K$$

$$\|v - \pi_{p_K}^{h_K} v\|_{0,\partial K} \leq C \frac{h_K^{\min(p_K+1, t_K) - \frac{1}{2}}}{p_K^{t_K - \frac{1}{2}}} \|v\|_{t_K,K},$$

with a constant C independent of v, h_K and p_K, but depending on the shape-regularity of the mesh and on t_K.

Let now $\Pi_p^h v$ be given by $\Pi_p^h v|_K = \pi_{p_K}^{h_K}(v|_K)$, for any $K \in \mathcal{T}_h$, with $\pi_{p_K}^{h_K}$ from Lemma 3.1. For a vector-valued function $\boldsymbol{r} = (r_1, \ldots, r_d)$ we set $\boldsymbol{\Pi}_p^h \boldsymbol{r} = (\Pi_p^h r_1, \ldots, \Pi_h r_d)$. First, we give an estimate of the residual.

Lemma 3.2 Let u be the exact solution. Assume $(\nu \nabla u)|_K \in H^{s_K}(K)^d$, $K \in \mathcal{T}_h$, with local regularity exponents $s_K \geq 1$. Then, for $w \in V(h)$,

$$|R_h(u, w)| \leq C \Big[\sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min(p_K+1, s_K)}}{p_K^{2s_K}} \frac{1}{n_K} \|\nu \nabla u\|_{s_K,K}^2 \Big]^{\frac{1}{2}} \| w \|_h,$$

Proof. Simple calculations lead to

$$R_h(u,w) = \int_{\mathcal{E}_\mathcal{I}} \big(\{\!\!\{ \nu \nabla u - \mathbf{\Pi}(\nu \nabla u) \}\!\!\} - \boldsymbol{b} [\![\nu \nabla u - \mathbf{\Pi}(\nu \nabla u)]\!] \big) \cdot [\![w]\!] \, ds + \int_{\mathcal{E}_\mathcal{D}} w \, \big(\nu \nabla u - \mathbf{\Pi}(\nu \nabla u) \big) \cdot \boldsymbol{n} \, ds,$$

with $\mathbf{\Pi}$ being the L^2-projection onto $\boldsymbol{Q}_h$. By writing $\nu\nabla u - \mathbf{\Pi}(\nu\nabla u) = [\nu\nabla u - \mathbf{\Pi}_p^h(\nu\nabla u)] - \mathbf{\Pi}[\nu\nabla u - \mathbf{\Pi}_p^h(\nu\nabla u)]$, the assertion follows as in Perugia and Schötzau (2001, Lemma 4.11) by the triangle inequality, the Cauchy-Schwarz inequality, inverse estimates for $\mathbf{\Pi}[\nu\nabla u - \mathbf{\Pi}_p^h(\nu\nabla u)]$ from ∂K to K, $K \in \mathcal{T}_h$, the L^2-stability of $\mathbf{\Pi}$ and the approximation properties in Lemma 3.1. □

We are now able to prove Theorem 3.2.

Proof of Theorem 3.2. We start by estimating $|\!|\!| u - \Pi_p^h u |\!|\!|_h$. Our assumptions on ν, meshes and polynomial distributions and the approximation properties in Lemma 3.1 yield

$$|\!|\!| u - \Pi_p^h u |\!|\!|_h^2 \leq C \sum_{K \in \mathcal{T}_h} \frac{h_K^{2\min(p_K, s_K)}}{p_K^{2s_K - 1}} N_{\delta K} \|u\|_{s_K+1,K}^2.$$

By inserting this and the result of Lemma 3.2 in the bound of Theorem 3.1, we obtain the estimate of $|\!|\!| u - u_h |\!|\!|_h$.

To estimate $\|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,\Omega}$, we use (3), the triangle inequality and $\boldsymbol{\mathcal{L}}(u) = \boldsymbol{\mathcal{G}}_\mathcal{D}$ to obtain $\|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,\Omega} \leq \|\nu \nabla u - \mathbf{\Pi}(\nu \nabla_h u_h)\|_{0,\Omega} + \|\mathbf{\Pi}(\nu \boldsymbol{\mathcal{L}}(u - u_h))\|_{0,\Omega}$. From the L^2-stability of $\mathbf{\Pi}$ and (5), $\|\mathbf{\Pi}(\nu \boldsymbol{\mathcal{L}}(u - u_h))\|_{0,\Omega} \leq C |\!|\!| u - u_h |\!|\!|_h$. By the triangle inequality, the identity $\mathbf{\Pi}_p^h(\nu \nabla u) = \mathbf{\Pi}\big(\mathbf{\Pi}_p^h(\nu \nabla u)\big)$, and the L^2-stability of $\mathbf{\Pi}$, we get

$$\|\nu \nabla u - \mathbf{\Pi}(\nu \nabla_h u_h)\|_{0,\Omega} \leq 2\|\nu \nabla u - \mathbf{\Pi}_p^h(\nu \nabla u)\|_{0,\Omega} + |\!|\!| u - u_h |\!|\!|_h.$$

Therefore, the desired result follows from the bound for $|\!|\!| u - u_h |\!|\!|_h$ and Lemma 3.1. □

An estimate for the L^2-error in u can be obtained by using a standard duality argument. We assume that Ω and ν are such that the following elliptic regularity result holds true: for any $\lambda \in L^2(\Omega)$, the solution z to the problem

$$-\nabla \cdot (\nu \nabla z) = \lambda \text{ in } \Omega \qquad z = 0 \text{ on } \partial\Omega, \tag{6}$$

satisfies $z \in H^2(\Omega)$, $\nu \nabla z \in H^1(\Omega)^d$ and $\|z\|_{2,\Omega} \leq C \|\lambda\|_{0,\Omega}$, $\|\nu \nabla z\|_{1,\Omega} \leq C \|\lambda\|_{0,\Omega}$, with a constant $C > 0$.

Theorem 3.3 With the same assumptions as in Theorem 3.2 and the above hypothesis on Ω and ν, we have

$$\|u-u_h\|_{0,\Omega} \le C\,\frac{h^{\min(p,s)+1}}{p^{s+\frac{1}{2}}}\left(\|u\|_{s+1,\Omega}+\|\nu\,\nabla u\|_{s,\Omega}\right),$$

with $h=\max_{K\in\mathcal{T}_h} h_K$, $p=\min_{K\in\mathcal{T}_h} p_K$ and $s=\min_{K\in\mathcal{T}_h} s_K \ge 1$.

Proof. Let z be the solution to problem (6) with $\lambda = u-u_h$. Simple calculations give $\|u-u_h\|^2_{0,\Omega} = A_h(z,u-u_h)+I_h(z,u-u_h)-R_h(z,u-u_h)$. Since $A_h(z_h,u-u_h)+I_h(z_h,u-u_h) = R_h(u,z_h)$, for any $z_h \in V_h$ and $R_h(u,z_h) = -R_h(u,z-z_h)$, we obtain

$$\begin{aligned}\|u-u_h\|^2_{0,\Omega} =& A_h(z-z_h,u-u_h)+I_h(z-z_h,u-u_h)\\ &- R_h(u,z-z_h)-R_h(z,u-u_h).\end{aligned}$$

Therefore, from Proposition 3.1, Lemma 3.2 and the regularity of z,

$$\begin{aligned}\|u-u_h\|^2_{0,\Omega} \le& \Big[C_{\text{cont}}|\!|\!|\, z-z_h\,|\!|\!|_h + C\,\frac{h}{p}\,\|\nu\,\nabla z\|_{1,\Omega}\Big]|\!|\!|\, u-u_h\,|\!|\!|_h\\ &+ C\,\frac{h^{\min(p+1,s)}}{p^s}\,\|\nu\,\nabla u\|_{s,\Omega}|\!|\!|\, z-z_h\,|\!|\!|_h.\end{aligned}$$

By choosing $z_h = \Pi^h_p z$, from the estimates in Lemma 3.1 and the elliptic regularity assumption, $|\!|\!|\, z-z_h\,|\!|\!|_h \le C\,\frac{h}{p}\,\|z\|_{2,\Omega} \le C\,\frac{h}{p}\,\|u-u_h\|_{0,\Omega}$, in addition to $\|\nu\,\nabla z\|_{1,\Omega} \le C\,\|u-u_h\|_{0,\Omega}$. The result then follows from the estimate of $|\!|\!|\, u-u_h\,|\!|\!|_h$ in Theorem 3.2. □

The stability of the LDG formulation with respect to the right-hand side, under mild smoothness assumptions, is implied by the following result.

Proposition 3.2 Assume that Ω and ν are such that the solution z of (6) with right-hand side $\lambda \in V(h)$ satisfies $\nu\,\nabla z \in H^s(\Omega)^d$ and $\|\nu\,\nabla z\|_{s,\Omega} \le C\,\|\lambda\|_{0,\Omega}$ for $s > \frac{1}{2}$. Then, $|F_h(v)| \le C\,[\|f\|^2_{0,\Omega} + \|\mathtt{h}^{-\frac{1}{2}}\mathtt{p}\,\mathtt{n}^{\frac{1}{2}} g_{\mathcal{D}}\|^2_{0,\mathcal{E}_{\mathcal{D}}}]^{\frac{1}{2}}|\!|\!|\, v\,|\!|\!|_h$, for all $v \in V(h)$.

Proof. The assertion follows from the broken Poincaré inequality $\|v\|_{0,\Omega} \le C|\!|\!|\, v\,|\!|\!|_h$, $v \in V(h)$, that can be proved following Arnold (1982), and the estimate $\|\nu^{\frac{1}{2}}\boldsymbol{\mathcal{G}}_{\mathcal{D}}\|_{0,\Omega} \le C_{\text{lift}}\kappa\|\mathtt{h}^{-\frac{1}{2}}\mathtt{p}\,\mathtt{n}^{\frac{1}{2}} g_{\mathcal{D}}\|_{0,\mathcal{E}_{\mathcal{D}}}$, obtained as in Perugia and Schötzau (2001, Proposition 4.2). □

4 Conclusions

In this paper, we presented the first *hp*-error analysis of the LDG method for diffusion problems in several space dimensions and extended the previous *h*-analysis in Castillo et al. (2000). Although we used the setting of

Arnold et al. (2001) to cast the LDG method in its primal form, we proposed a new technique to actually derive error estimates which is based on Strang's lemma and which might be of independent interest in the analysis of discontinuous Galerkin methods.

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