Capillarity driven spreading of power-law fluids

S. I. Betelu
University of Minnesota
Institute for Mathematics and its Applications
Minneapolis, Minnesota 55455
M. A. Fontelos
Departamento de Matemática Aplicada
Universidad Rey Juan Carlos
Móstoles, Madrid 28933

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Abstract

We investigate the spreading of thin liquid films of power-law rheology. We construct an explicit travelling wave solution and source-type similarity solutions. We show that when the nonlinearity exponent λ for the rheology is larger than one, the governing dimensionless equation $h_t + (h^{\lambda+2}|h_{xxx}|^{\lambda-1}h_{xxx})_x = 0$ admits solutions with compact support and moving fronts. We also show that the solutions have bounded energy dissipation rate.

1 Introduction

This work deals with capillary spreadings of thin films of liquids of power-law rheology, also known as Ostwald-de Waele fluids [1]. The power-law rheology is one of the simplest generalizations of the Newtonian one, in which the effective viscosity on a point is assumed to be a function of the local rate of deformation $\dot{\gamma}$ as $\eta = m|\dot{\gamma}|^{1/\lambda-1}$. The values of m and λ depend on the physical properties of the liquid. When $\lambda > 1$ the viscosity tends to zero at high strain rates [1] and is larger at low strain rates (these fluids are called shear-thinning). This fact has been used, for instance, in the design of paints which have to present low viscosity under high stresses so they can be extended over a surface, and high viscosity under low stresses so that they do not drip under the action of gravity once extended.

Spreadings with this rheology were studied for gravity-driven currents ([2, 3]) where the height profile of the spreading $h(x,t)$ satisfies an equation of the form $h_t - (h^{\lambda+2}|h_x|^{\lambda-1}h_x)_x = 0$.

For a one-dimensional capillarity driven spreading, the dimensionless equation of motion is given by

$$h_t + (h^{\lambda+2}|h_{xxx}|^{\lambda-1}h_{xxx})_x = 0. \tag{1}$$

We derive this equation in the Appendix using the lubrication approximation. This approximation is valid when the film is much thinner than its horizontal length scale of the spreading.

The problem of drop spreading has been studied intensively in the last decades (see [4, 5, 6, 7, 8] and references therein). The motivation is that this class of flows plays a very important role in coating processes, painting and biology. They also represent a very interesting mathematical problem due to the high differential order and the degeneracy involved on the governing equations.

Remarkably, there are no compactly supported source-type solutions with moving interfaces for Eq. 1 and $\lambda = 1$ [6], corresponding to a newtonian fluid. This difficulty has been overcome by either assuming that the drop advances over a pre-existing thin film, by modifying the equation of motion to include slipping effects, or by adding terms corresponding to the influence of molecular forces.

Our goal is to show that for power-law fluids with $\lambda > 1$, there are solutions with compact support and moving interfaces, and that their energy dissipation rate is bounded. The qualitative reason lies in the fact that the flow creates a layer adjacent to the substrate where the strain rate diverges, and thus, for $\lambda > 1$ the effective viscosity η tends to zero. In other words, the drop *self-lubricates.*

2 Travelling-wave solutions

The main property that concerns us now is the existence and propagation of interfaces that separate regions where $h = 0$ from regions where $h > 0$. Next we introduce the simplest solution that exhibits this behaviour. We start with the *ansatz*

$$h(x,t) = S(x+ct)_+^{\alpha}, \tag{2}$$

that for $c > 0$ represents a wave moving to the left with velocity c. By substituting in Eq. 1 we obtain

$$c = S^{2\lambda+1}\left(\frac{3(\lambda-1)\lambda(\lambda+2)}{(2\lambda+1)^3}\right)^{\lambda}$$

and

$$\alpha = \frac{3\lambda}{2\lambda+1}.$$

For $\lambda > 1$ there is an interface that moves with finite velocity. The constant $S > 0$ governs the height scale. For a given value of S, the velocity c tends to zero as $\lambda \to 1$, which is consistent with the fact that there are no moving interfaces in the newtonian case (unless molecular forces or slipping effects are taken into account). The contact angle is zero, which is consistent with a fluid that wets the surface. The flux of mass F at the interface $x = -ct$ is zero because $F = h^{\lambda+2}|h_{xxx}|^{\lambda-1}h_{xxx} = 0$ there.

3 Source-type similarity solutions

Now we solve the problem in which a finite volume of fluid is initially concentrated on a point on a surface. As there are no external length-scales in this problem, we shall look for similarity solutions [9] of the form

$$h(x,t) = \frac{A}{t^\beta} H\left(\frac{x}{t^\beta}\right) \tag{3}$$

where A is a constant and β is the *similarity exponent.* If the function $H(\eta)$ has a zero at η_f then the interface of the spreading will be given by

$$x_f = \eta_f t^\beta.$$

Eq. 3 guarantees that the mass is conserved, i.e. that $\int_0^{x_f} h(x,t)dx = const.$ By substituting Eq. 3 in Eq. 1 we find that the similarity exponent is given by

$$\beta = \frac{1}{5\lambda + 2}.$$

Moreover, by choosing $A^{2\lambda+1} = 1/(5\lambda+2)$ and integrating once, we obtain the following simple ODE for $H(\eta)$

$$H^{\lambda+2} H'''^{\lambda} = \eta H, \tag{4}$$

where we have also assumed that $H''' > 0$.

We solve Eq. 4 numerically with a fourth order Runge-Kutta scheme, with a variable stepsize equal to $H/10000$. We start the integration at the center of the drop $\eta = 0$, where the initial conditions are

$$H(0) = 1, \qquad H'(0) = 0, \qquad H''(0) = \kappa. \tag{5}$$

The first condition represents the scaled height of the drop at the center, the second is the bilateral symmetry condition, and the third is the initial value for the second derivative, which is a variable parameter κ.

In Fig. 1 we show several height profiles for different values of κ and $\lambda = 5/2$. For $\kappa > -1.67998$ the solution is always positive: first decreases and finally increases. Thus, for this choice of κ there are no interfaces.

For $\kappa < -1.67998$, the drop profile tends to zero linearly for $\eta = \eta_f$. This case represents a spreading of extension $x_f(t) = \eta_f t^\beta$ that has a variable contact slope that scales as $t^{-2\beta}$.

For a limiting value of $\kappa \sim -1.67998$ the profile tends to zero with zero contact angle. This case represents a drop of a liquid that wets the surface.

It can be shown that the limiting solution that has the zero contact angle has two interesting properties: (a) maximizes the dissipation of energy for a given mass of spreading fluid and (b) it maximizes the rate at which the fluid covers the surface.

Point (a) can be seen as follows: using the fact that all the functions $CH(\eta/C^\delta)$ are solutions of Eq. 4 for $\delta = (1+2\lambda)/(1+3\lambda)$ and any positive C, we can compute their rate of dissipation of energy

$$\int_0^{x_f} h^{\lambda+2}|h'''|^{\lambda+1}dx = A^{2\lambda+3}t^{-5\beta(\lambda+1)}C^{(1+4\lambda)/(1+3\lambda)}\int_0^{\eta_f} \eta^{1+1/\lambda}H^{-1/\lambda}(\eta)d\eta.$$

On the other hand, the volume of the spreading is $AC^{1+\delta}\int_0^{\eta_f} H(\eta)d\eta$. Thus, the dissipation for a fixed volume can be obtained by eliminating C, and it is proportional to $t^{-5\beta(\lambda+1)}Y$ with

$$Y = \frac{W}{Z^{(1+4\lambda)/(2+5\lambda)}}, \tag{6}$$

where

$$W = \int_0^{\eta_f} \eta^{1+1/\lambda}H^{-1/\lambda}(\eta)d\eta$$

and

$$Z = \int_0^{\eta_f} H(\eta)d\eta.$$

We observe numerically that the quotient of integrals is maximum for the zero contact angle solution. A similar argument yields point (b). For example, in Table 1 we show the numerical values of η_f, W, Z and Y for $\lambda = 5/2$ and several values of κ.

κ	η_f	W	Z	Y
-2.0	1.08145	0.952197	0.695057	1.25480
-1.9	1.12093	1.052875	0.716376	1.35603
-1.8	1.16698	1.186311	0.740071	1.49063
-1.7	1.22363	1.396952	0.766721	1.70883
-1.679984	1.23740	1.483208	0.772482	1.80407

Table 1

For the cases where the height profile tends to zero at η_f, we can compute the asymptotic behaviour of H. In the linear case, we write $H(\eta) = a(\eta_f - \eta) + F(\eta_f - \eta)$, where F is a small correction. By substituting in Eq. 4 we obtain that for $\lambda > 3/2$,

$$H(\eta) = a(\eta_f - \eta) + \frac{1}{a^{1+1/\lambda}(2-1/\lambda)(1-1/\lambda)/\lambda}\eta_f^{1/\lambda}(\eta_f-\eta)^{2-1/\lambda} + o((\eta_f-\eta)^{\mu-\epsilon}),$$

where $\mu = \min(3-2/\lambda, 2)$ and ϵ is any small positive number. The slope is finite and the curvature diverges at the contact line.

For the zero contact angle case, the front will locally behave as a travelling wave (see Eq. 2), thus we write $H(\eta) = B(\eta_f - \eta)^\alpha + G(\eta)$, where $\alpha = 3\lambda/(2\lambda+1)$ and G is a small correction. By substituting on Eq. 4 we obtain

$$H(\eta) = \left(\frac{(2\lambda+1)^3}{3\lambda(\lambda-1)(\lambda+2)}\right)^{\lambda/(2\lambda+1)} \eta_f^{1/(2\lambda+1)} (\eta_f - \eta)^{3\lambda/(2\lambda+1)} + o((\eta_f - \eta)^2).$$

In both cases the local rate of dissipation of energy Eq. 6 is bounded near the interface.

4 Conclusions

In this work we studied two properties of the spreading of thin films of fluids with power-law rheology: a) the existence of interfaces separating regions where $h > 0$ from regions where $h = 0$ and b) existence of compact-supported source-type similarity solutions.

The first property has a straightforward physical interpretation: if a drop of fluid is deposited on a plane surface, then there will be a sharp interface separating the drop from the dry surface. This interface is naturally given by the equation of motion. In contrast, for a newtonian fluid, a moving interface cannot be explained without the description of the molecular interactions between the liquid and the substrate.

Eq. 1 poses several interesting open mathematical problems: existence and uniqueness of the solutions, finite speed of propagation and convergence to the self-similar solutions of the type described in this paper. In particular, the multiplicity of selfsimilar solutions poses the question of whether a particular one is selected by the dynamics or not. The zero angle solution that maximizes the dissipation rate is a good candidate, but this remains an open problem.

5 Appendix: Derivation of Eq. 1

In the power-law fluids, the deviatoric stress tensor is related to the strain tensor according to the following constitutive relation [1, 3]:

$$\tau_{ij} = m|\dot{\gamma}|^{1/\lambda - 1}\dot{\gamma}_{ij}, \tag{7}$$

where $|\dot{\gamma}| = \sqrt{\frac{1}{2}\sum_{i,j}\dot{\gamma}_{ij}\dot{\gamma}_{ij}}$ and $\dot{\gamma}_{ij}$ is the strain tensor, given by

$$\dot{\gamma}_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}.$$

Here v_i is the fluid velocity field. When $\lambda = 1$, one has a Newtonian fluid. If $\lambda > 1$, the fluid is said to be *shear-thinning*, which is the most common case that includes many polymer solutions. The values of λ are typically between 1.7 and 6.7. Just to give a concrete example, a solution of 0.5 percent Hydroxyethylcellulose at $293^o K$ has $m = 0.84 N s^{1/\lambda}/m^2$ and $\lambda = 1.96$ [1].

In order to derive the equation of motion of a thin film, we shall assume that the film is much thinner than its horizontal dimension L, that the motion is nearly horizontal and that the inertial effects are negligible so that the flow is governed by a balance between capillary and viscous forces. We shall neglect systematically all y components of the velocities when they are compared with the x components. We also suppose that the stresses are mainly due to high gradients of the horizontal velocity u in the y direction. Consistently, we shall assume that the components $\dot{\gamma}_{xy}$ and τ_{xy} are much larger than all its other respective components. This approximation is called the *lubrication approximation*, and has been widely used in the case of Newtonian flows (for which $\lambda = 1$) (see for instance [8, 4] and its references). Then, the x-component of the momentum equation can be written as

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \tag{8}$$

where p is the pressure. Let $y = h(x,t)$ be the fluid free surface. The conservation of mass can be written within the lubrication approximation as

$$h_t + (Uh)_x = 0, \tag{9}$$

where U is the horizontal velocity averaged in the y coordinate and $h(x,t)$ is the fluid thickness. The pressure under the free surface will be given by

$$p(x,t) = -\gamma h_{xx} \tag{10}$$

where γ is the surface tension and we have approximated the curvature of the free surface by the second partial derivative of h. If the thickness of the film is small and the volume forces are neglected, the pressure does not depend substantially on y.

One can integrate Eq. 8 with respect to y to obtain the shear stress

$$\tau_{xy} = p_x(y-h), \tag{11}$$

that satisfies the zero-tangential stress condition at the free surface $y = h$. By using the constitutive relation given by Eq. 7,

$$\tau_{xy} = m|u_y|^{1/\lambda - 1} u_y,$$

and by substituting τ_{xy} from Eq. 11 we can compute the y derivative of the horizontal velocity,

$$u_y = -m^{-\lambda}|p_x|^{\lambda-1} p_x (h-y)^\lambda.$$

After integration and using the no-slip condition $u(y=0) = 0$ one gets

$$u = m^{-\lambda}|p_x|^{\lambda-1} p_x \left(\frac{(h-y)^{\lambda+1}}{\lambda+1} - \frac{h^{\lambda+1}}{\lambda+1} \right).$$

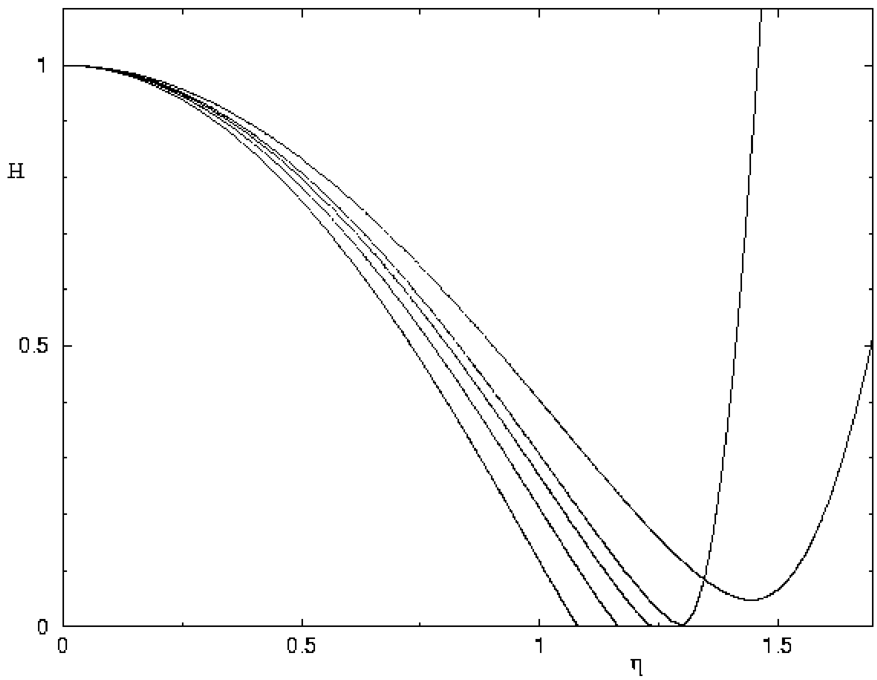

Figure 1: Selfsimilar drop profiles, for different values of the second derivative at the center of the drop, for $\lambda = 5/2$. From left profile to right: $\kappa = -2$, $\kappa = -1.8$, $\kappa = -1.679984$, $\kappa = -1.6$ and $\kappa = -1.4$. The darker line is the solution with zero contact angle.

Now, in order to apply the equation for the conservation of the mass Eq. 9, we need the averaged velocity U,

$$U = \frac{1}{h}\int_0^h u dy = -m^{-\lambda}|p_x|^{\lambda-1}p_x\frac{h^{\lambda+1}}{\lambda+2}. \tag{12}$$

Finally, using Eq. 10 and Eq. 9, one obtains

$$h_t + \frac{1}{\lambda+2}\left(\frac{\gamma}{m}\right)^{\lambda}(h^{\lambda+2}|h_{xxx}|^{\lambda-1}h_{xxx})_x = 0. \tag{13}$$

By rescaling we finally get Eq. 1.

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