Projectable Multivariate Wavelets

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Abstract:

We demonstrate that many multivariate wavelets are projectable wavelets; that is, they essentially carry the tensor product (separable) structure though themselves may be non-tensor product (nonseparable) wavelets. We show that a projectable wavelet can be replaced by a tensor product wavelet without loss of desirable properties such as spatial localization, smoothness and vanishing moments.

Key words: Projectable wavelets, tensor product wavelets, dilation matrix, sum rules.

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1 Introduction

A simple way to obtain wavelets in multiple dimensions is to use the tensor product of univariate wavelets. The resulting wavelets are called tensor product (separable) multivariate wavelets. Tensor product multivariate wavelets consist only a small subset of all the multidimensional wavelets. There are many non-tensor product (nonseparable) multivariate wavelets which can not be derived by the tensor product method from univariate wavelets. In this paper, we shall introduce the concept of projectable wavelets. Roughly speaking, a projectable wavelet has the property that it can be easily replaced by a tensor product wavelet such that the tensor product wavelet function has no larger support (measured in size of squares), no worse L_p smoothness for all $1 \le p \le \infty$, and no less order of vanishing moments (or approximation order) than those of the given projectable wavelet function. In particular, all the tensor product wavelets are projectable wavelets. Other examples of projectable wavelets are wavelets derived from refinable functions which are convolutions of tensor product refinable functions with distributions.

Comparison between multivariate wavelets and tensor product ones has been discussed in [5]. In this paper, we shall generalize the results in [5] to demonstrate that many multivariate wavelets are projectable wavelets though themselves may be non-tensor product wavelets.

Tensor product method is a way to obtain high dimensional wavelets from lower dimensional ones. Conversely, we can also obtain lower dimensional wavelets with a general dilation matrix from higher dimensional ones via a simple linear transform, as discussed in [6]. Another purpose of this paper is to provide analysis of the wavelets which are constructed via a special class of linear transforms in [6, Theorem 3.2]. Such analysis is important in our study of projectable wavelets in this paper.

The structure of this paper is as follows. In Section 2, we shall study various properties such as the L_p convergence of subdivision schemes and L_p smoothness exponents associated with wavelets obtained via a special class of linear transforms in [6]. For the seek of completeness, we shall investigate wavelets with a general dilation matrix. In Section 3, we shall discuss projectable wavelets with the commonly used dilation matrix $2I_s$. We demonstrate that many nonseparable wavelets are projectable wavelets. Finally, we shall discuss how the concept of projectable wavelets helps us in designing multivariate wavelets. Examples will be given to illustrate the general theory.

2 Obtain Wavelets via a Linear Transform

In this section, we shall discuss how to obtain wavelets via a linear transform between two dilation matrices. Though construction of masks with certain desirable properties via a linear transform was discussed in [6, Theorem 3.2], the properties of their corresponding refinable functions are not discussed there. Our investigation in this section closely follows the line developed in [5, 6]. We shall generalize our analysis of optimal multivariate wavelets in [5] to the wavelets obtained via a special class of linear transforms in [6].

Before proceed further, let us introduce some notation. An $s \times s$ integer matrix M is called a **dilation matrix** if $\lim_{k\to\infty} M^{-k} = 0$. We say that a is a **mask** on $\mathbb{Z}^s$ if a is a finitely supported sequence on $\mathbb{Z}^s$ such that $\sum_{k\in\mathbb{Z}^s} a_k = 1$. Wavelets are derived from refinable functions via a standard multiresolution technique. A **refinable function** ϕ is a solution to the following refinement equation

$$\phi = |\det M| \sum_{k\in\mathbb{Z}^s} a_k \phi(M\cdot -k), \tag{2.1}$$

where a is a mask and M is a dilation matrix. For a mask a on $\mathbb{Z}^s$ and an $s\times s$ dilation matrix M, it is known that there exists a unique compactly supported distributional solution, denoted by ϕ_a^M throughout the paper, to the refinement equation (2.1) such that $\widehat{\phi_a^M}(0) = 1$, where the Fourier transform is defined to be

$$\widehat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^s,\ f \in L_1(\mathbb{R}^s)$$

and can be extended to tempered distributions naturally.

A biorthogonal wavelet is derived from two refinable functions ϕ_a^M and $\phi_{a^d}^M$ in $L_2(\mathbb{R}^s)$ such that

$$\int_{\mathbb{R}^s} \overline{\phi_a^M(x)} \phi_{a^d}^M(x+j)\, dx = \delta_j \qquad \forall\, j \in \mathbb{Z}^s, \tag{2.2}$$

where δ is the **Dirac sequence** such that $\delta_0 = 1$ and $\delta_j = 0$ for all $j \in \mathbb{Z}^s\backslash\{0\}$. When (2.2) holds, $\phi_{a^d}^M$ is called a **dual refinable function** of ϕ_a^M. A necessary condition for the two refinable functions ϕ_a^M and $\phi_{a^d}^M$ to satisfy the biorthogonal relation in (2.2) is that their masks a and a^d satisfy the following discrete biorthogonal relation:

$$\sum_{k\in\mathbb{Z}^s} \overline{a_{j+k}} a_k^d = \delta_j / |\det M| \qquad \forall\, j \in M\mathbb{Z}^s. \tag{2.3}$$

Let a be a finitely supported sequence on $\mathbb{Z}^s$ such that $\sum_{k\in\mathbb{Z}^s} a_k = 1$. If there exists a finitely supported sequence a^d on $\mathbb{Z}^s$ such that (2.3) holds, then a is called a **primal mask** (with respect to the lattice $M\mathbb{Z}^s$) and a^d is called a **dual mask** of a (with respect to the lattice $M\mathbb{Z}^s$). When a is a dual mask of itself, a is called an **orthogonal mask**.

The concept of sum rules is closely related to the order of vanishing moments of a biorthogonal wavelet (see [2, 3, 8]). We say that a satisfies the **sum rules** of order ℓ (with respect to the lattice $M\mathbb{Z}^s$) if

$$\sum_{k\in M\mathbb{Z}^s} a_{j+k} q(j+k) = \sum_{k\in M\mathbb{Z}^s} a_k q(k) \qquad \forall\, j \in \mathbb{Z}^s,\ q \in \Pi_{\ell-1}, \tag{2.4}$$

where $\Pi_{\ell-1}$ denotes the set of all polynomials of (total) degree less than ℓ.

Let P be an $r \times s$ integer matrix. For any mask a on $\mathbb{Z}^s$, we define another mask Pa on $\mathbb{Z}^r$ as follows:

$$[Pa]_j := \sum_{\{k \in \mathbb{Z}^s \,:\, Pk = j\}} a_k, \qquad j \in \mathbb{Z}^r, \tag{2.5}$$

where by convention $[Pa]_j := 0$ when $\{k \in \mathbb{Z}^s \;:\; Pk = j\}$ is the empty set. The **symbol** of a mask a is defined to be

$$\widetilde{a}(\xi) := \sum_{k \in \mathbb{Z}^s} a_k e^{-ik\cdot\xi}, \qquad \xi \in \mathbb{R}^s. \tag{2.6}$$

Using the symbol of a mask, (2.5) can be rewritten as $\widetilde{Pa}(\xi) = \widetilde{a}(P^*\xi)$, where P^* denotes the complex conjugate of the transpose of the matrix P.

Lemma 2.1 *Let M be an $r \times r$ integer matrix and N be an $s \times s$ integer matrix such that both M and N have nonzero determinants. Let P be an $r \times s$ integer matrix such that*

$$PN\mathbb{Z}^s \subseteq M\mathbb{Z}^r \quad \text{and} \quad (k + M\mathbb{Z}^r) \cap P\mathbb{Z}^s \neq \emptyset \qquad \forall\, k \in \mathbb{Z}^r. \tag{2.7}$$

Let Pa be defined in (2.5)*. If a satisfies the sum rules of order ℓ with respect to the lattice $N\mathbb{Z}^s$, then Pa must satisfy the sum rules of order at least ℓ with respect to the lattice $M\mathbb{Z}^r$.*

Moreover, for any primal mask a with respect to the lattice $N\mathbb{Z}^s$, the following statements are equivalent:

(1) *For any dual mask a^d of a with respect to the lattice $N\mathbb{Z}^s$, Pa^d is a dual mask of Pa with respect to the lattice $M\mathbb{Z}^r$;*

(2) *The following condition holds*

$$\widetilde{a}(P^*\xi + 2\pi\varepsilon) = 0 \qquad \forall\, \xi \in \mathbb{R}^r,\ \varepsilon \in (N^*)^{-1}\mathbb{Z}^s \backslash [P^*(M^*)^{-1}\mathbb{Z}^r + \mathbb{Z}^s]. \tag{2.8}$$

When a is an orthogonal mask with respect to the lattice $N\mathbb{Z}^s$, Pa is an orthogonal mask with respect to the lattice $M\mathbb{Z}^r$ if and only if (2.8) *holds.*

Proof: It follows from [6, Theorem 3.2] that if a satisfies the sum rules of order ℓ with respect to the lattice $N\mathbb{Z}^s$, then Pa must satisfy the sum rules of order at least ℓ with respect to the lattice $M\mathbb{Z}^r$.

For an integer $j \in \mathbb{Z}^r$, if $\langle P^*(M^*)^{-1}j, k\rangle = \langle (M^*)^{-1}j, Pk\rangle$ is an integer for all $k \in \mathbb{Z}^s$, then the condition in (2.7) will force $(M^*)^{-1}j$ to be an integer. Regarding $P^* : \mathbb{Z}^r \mapsto \mathbb{Z}^s$ as a linear mapping, we observe that the condition in (2.7) is equivalent to that the induced linear mapping P^* from the quotient group $(M^*)^{-1}\mathbb{Z}^r/\mathbb{Z}^r$ to the quotient group $(N^*)^{-1}\mathbb{Z}^s/\mathbb{Z}^s$ is well defined and is one-to-one.

Let $n := |\det N|$ and $m := |\det M|$. Note that a^d is a dual mask of a with respect to the lattice $N\mathbb{Z}^s$ if and only if

$$\sum_{j=1}^{n} \overline{\widetilde{a}(\xi + 2\pi\varepsilon_j)}\widetilde{a^d}(\xi + 2\pi\varepsilon_j) = 1 \qquad \forall\, \xi \in \mathbb{R}^s, \tag{2.9}$$

where $\{\varepsilon_j \;:\; j = 1, \dots, n\}$ is a complete set of representatives of the distinct cosets of the quotient group $(N^*)^{-1}\mathbb{Z}^s/\mathbb{Z}^s$. Since $P^* : (M^*)^{-1}\mathbb{Z}^r/\mathbb{Z}^r \mapsto (N^*)^{-1}\mathbb{Z}^s/\mathbb{Z}^s$ is one-to-one, we may assume that $\varepsilon_j = P^*\eta_j, j = 1, \dots, m$ where $\{\eta_j \;:\; j = 1, \dots, m\}$ is a complete set of representatives of the distinct cosets of the quotient group $(M^*)^{-1}\mathbb{Z}^r/\mathbb{Z}^r$. In particular, from (2.9), we deduce that

$$\sum_{j=1}^{m} \overline{\widetilde{a}(P^*(\xi + 2\pi\eta_j))}\widetilde{a^d}(P^*(\xi + 2\pi\eta_j)) + \sum_{j=m+1}^{n} \overline{\widetilde{a}(P^*\xi + 2\pi\varepsilon_j)}\widetilde{a^d}(P^*\xi + 2\pi\varepsilon_j) = 1. \tag{2.10}$$

Note that $\widetilde{Pa}(\xi) = \widetilde{a}(P^*\xi)$ and $\widetilde{Pa^d}(\xi) = \widetilde{a^d}(P^*\xi)$. That Pa^d is a dual mask of Pa with respect to the lattice $M\mathbb{Z}^r$ if and only if

$$\sum_{j=1}^{m} \overline{\widetilde{a}(P^*(\xi + 2\pi\eta_j))}\widetilde{a^d}(P^*(\xi + 2\pi\eta_j)) = 1 \qquad \forall\, \xi \in \mathbb{R}^r.$$

Hence, it follows from (2.10) that Pa^d is a dual mask of Pa with respect to the lattice $M\mathbb{Z}^r$ if and only if

$$\sum_{j=m+1}^{n} \overline{\widetilde{a}(P^*\xi + 2\pi\varepsilon_j)}\widetilde{a^d}(P^*\xi + 2\pi\varepsilon_j) = 0 \qquad \forall\, \xi \in \mathbb{R}^r. \tag{2.11}$$

Note that (2.8) is equivalent to that $\widetilde{a}(P^*\xi + 2\pi\varepsilon_j) = 0$ for all $\xi \in \mathbb{R}^r$ and $j = m+1, \dots, n$. If (2.8) holds, it is straightforward to see that (2.11) holds and Pa^d is a dual mask of Pa.

Let us prove the converse statement. Since a is a primal mask with respect to the lattice $N\mathbb{Z}^s$, by the Quinlin-Suslin theorem, there exist finitely supported sequences $c^1, \dots, c^n$ such that $\det A(\xi) \neq 0$ for all $\xi \in \mathbb{R}^s$ and

$$[\overline{\widetilde{a}(\xi + 2\pi\varepsilon_1)}, \dots, \overline{\widetilde{a}(\xi + 2\pi\varepsilon_n)}]A(\xi) = [1, 0, \dots, 0] \qquad \forall\, \xi \in \mathbb{R}^s, \tag{2.12}$$

where

$$A(\xi) := \begin{bmatrix} \widetilde{a^d}(\xi + 2\pi\varepsilon_1) & \widetilde{c^1}(\xi + 2\pi\varepsilon_1) & \cdots & \widetilde{c^n}(\xi + 2\pi\varepsilon_1) \\ \widetilde{a^d}(\xi + 2\pi\varepsilon_2) & \widetilde{c^1}(\xi + 2\pi\varepsilon_2) & \cdots & \widetilde{c^n}(\xi + 2\pi\varepsilon_2) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a^d}(\xi + 2\pi\varepsilon_n) & \widetilde{c^1}(\xi + 2\pi\varepsilon_n) & \cdots & \widetilde{c^n}(\xi + 2\pi\varepsilon_n) \end{bmatrix}.$$

Since a^d is a dual mask of a, it follows from (2.12) that $a^d + c^j$ is also a dual mask of a. By assumption, $P(a^d + c^j)$ is a dual mask of Pa. It follows from (2.11) that

$$[\overline{\widetilde{a}(P^*\xi + 2\pi\varepsilon_{m+1})}, \dots, \overline{\widetilde{a}(P^*\xi + 2\pi\varepsilon_n)}]B(P^*\xi) = 0 \qquad \forall\, \xi \in \mathbb{R}^r, \tag{2.13}$$

where

$$B(\xi) := \begin{bmatrix} \widetilde{a^d}(\xi + 2\pi\varepsilon_{m+1}) & \widetilde{c^1}(\xi + 2\pi\varepsilon_{m+1}) & \cdots & \widetilde{c^n}(\xi + 2\pi\varepsilon_{m+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{a^d}(\xi + 2\pi\varepsilon_n) & \widetilde{c^1}(\xi + 2\pi\varepsilon_n) & \cdots & \widetilde{c^n}(\xi + 2\pi\varepsilon_n) \end{bmatrix}.$$

Since $\det A(\xi) \neq 0$, as a submatrix of $A(\xi)$, $B(\xi)$ is of full rank $n - m$ for all $\xi \in \mathbb{R}^s$. It follows from (2.13) that $\widetilde{a}(P^*\xi + 2\pi\varepsilon_j) = 0$ for all $j = m+1, \dots, n$ which is equivalent to (2.8). ∎

An iteration scheme can be employed to solve the refinement equation (2.1). Start with an initial function ϕ_0 given by $\phi_0(x_1, \dots, x_s) = \prod_{j=1}^s \chi(x_j)$, $(x_1, \dots, x_s) \in \mathbb{R}^s$, where $\chi(x) := \max\{1 - |x|, 0\}$, $x \in \mathbb{R}$. Then we employ the iteration scheme $Q_{a,M}^n \phi_0$, $n = 0, 1, 2, \dots$, where $Q_{a,M}$ is the linear operator on $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$ given by

$$Q_{a,M} f := \sum_{k \in \mathbb{Z}^s} a_k f(M \cdot - k), \qquad f \in L_p(\mathbb{R}^s).$$

This iteration scheme is called a **subdivision scheme** or **cascade algorithm** associated with a. We say that the subdivision scheme associated with the mask a and the dilation matrix M converges in the L_p norm if there exists a function $f \in L_p(\mathbb{R}^s)$ such that $\lim_{n\to\infty} \|Q_{a,M}^n \phi_0 - f\|_p = 0$. If this is the case, then the limit function f must be ϕ_a^M.

For $j = 1, \dots, s$, let e_j be the j-th coordinate unit vector in $\mathbb{R}^s$. For $j \in \mathbb{Z}^s$ and a sequence λ on $\mathbb{Z}^s$, we denote $[\nabla_j \lambda]_k := \lambda_k - \lambda_{k-j}, k \in \mathbb{Z}^s$ and $\|\lambda\|_p := (\sum_{k\in\mathbb{Z}^s} |\lambda_k|^p)^{1/p}, 1 \le p \le \infty$.

Given a finitely supported mask a, for $1 \le p \le \infty$ and $\ell \in \mathbb{N} \cup \{0\}$, we define

$$\rho_\ell(a, M; p) := \max\{\lim_{n\to\infty} \|\nabla_{e_1}^{i_1} \cdots \nabla_{e^s}^{i_s} S_{a,M}^n \delta\|_p^{1/n} \ : \ i_1 + \cdots + i_s = \ell, i_j \ge 0\ \}, \tag{2.14}$$

where δ is the Dirac sequence and $S_{a,M}$ is the **subdivision operator** associated with the mask a and dilation matrix M which is defined by

$$[S_{a,M}\lambda]_j := \sum_{k\in\mathbb{Z}^s} a_{j-Mk} \lambda_k, \qquad j \in \mathbb{Z}^s. \tag{2.15}$$

Such quantity $\rho_\ell(a, M; p)$ plays an important rule in the study of convergence of subdivision schemes and smoothness analysis of refinable functions. For example, it was proved in [7] that the subdivision scheme associated with mask a and dilation matrix M converges in the L_p norm if and only if $\rho_1(a, M; p) < |\det M|^{1/p-1}$.

In order to obtain a biorthogonal wavelet, we have to construct two refinable functions such that (2.2) holds. It is known that the biorthogonal relation in (2.2) holds if and only if a^d is a dual mask of a and the subdivision schemes associated with masks a and a^d and the dilation matrix M converge in the L_2 norm, respectively.

The following result generalizes [5, Lemma 4.2].

Theorem 2.2 *Let M be an $r \times r$ dilation matrix and N be an $s \times s$ dilation matrix. Let P be an $r \times s$ integer matrix such that $PN = MP$ and $P\mathbb{Z}^s = \mathbb{Z}^r$. Then for a mask a on $\mathbb{Z}^s$,*

$$|\det M|^{1-1/p}\rho_\ell(Pa, M; p) \le |\det N|^{1-1/p}\rho_\ell(a, N; p) \qquad \forall\, 1 \le p \le \infty, \ell \in \mathbb{N} \cup \{0\}, \quad (2.16)$$

where the mask Pa on $\mathbb{Z}^r$ is defined in (2.5). Consequently, if the subdivision scheme associated with mask a and the dilation matrix N converges in the L_p norm, then so does the subdivision scheme associated with the mask Pa and the dilation matrix M.

Proof: Since P is an $r \times s$ integer matrix and $P\mathbb{Z}^s = \mathbb{Z}^r$, there exist two integer matrices E and F such that $|\det E| = |\det F| = 1$ and $P = E[I_r\ 0]F$. Now the condition $PN = MP$ is equivalent to

$$FNF^{-1} = \begin{bmatrix} E^{-1}ME & 0 \\ * & * \end{bmatrix}, \qquad (2.17)$$

where $*$ denotes some integer matrices. Since E and F are integer matrices such that $|\det E| = |\det F| = 1$, it is easy to prove that for all $1 \le p \le \infty$ and $\ell \in \mathbb{N}$,

$$\rho_\ell(Pa, M; p) = \rho_\ell(P_1(Fa), E^{-1}ME; p), \qquad \rho_\ell(a, N; p) = \rho_\ell(Fa, FNF^{-1}; p),$$

where $P_1 = [I_r\ 0]$ and $\widetilde{Fa}(\xi) = \widetilde{a}(F^*\xi)$. Therefore, it suffices to prove the claim under the assumption that

$$P = [I_r\ 0], \qquad N = \begin{bmatrix} M & 0 \\ * & L \end{bmatrix}.$$

It is easy to check that $\widetilde{S^n_{a,N}\delta}(\xi) = \Pi_{j=0}^{n-1}\widetilde{a}((N^*)^j\xi)$ by $\widetilde{S_{a,N}\lambda}(\xi) = \widetilde{a}(\xi)\widetilde{\lambda}(N^*\xi)$. Since $\widetilde{Pa}(\xi) = \widetilde{a}(\xi, 0)$ and $N^*P^* = P^*M^*$, we have $\widetilde{S^n_{Pa,M}\delta}(\xi) = \widetilde{S^n_{a,N}\delta}(\xi, 0)$ for all $\xi \in \mathbb{R}^r$. That is,

$$[S^n_{Pa,M}\delta]_j = \sum_{k\in\mathbb{Z}^{s-r}} [S^n_{a,N}\delta]_{(j,k)}, \qquad j \in \mathbb{Z}^r.$$

By a simple computation, the above equality yields that

$$[\nabla^{i_1}_{e_1}\cdots\nabla^{i_r}_{e_r}S^n_{Pa,M}\delta]_j = \sum_{k\in\mathbb{Z}^{s-r}} [\nabla^{i_1}_{e_1}\cdots\nabla^{i_r}_{e_r}S^n_{a,N}\delta]_{(j,k)}, \qquad j \in \mathbb{Z}^r, \qquad (2.18)$$

where e_j is the jth coordinate unit vector in $\mathbb{R}^r$. Since a is finitely supported, it is not difficult to see that the support of the sequence $\nabla_{e_1}^{i_1} \cdots \nabla_{e_r}^{i_r} S_{a,N}^n \delta$ is contained in the set $N^n\{x \in \mathbb{R}^s \ : \ \|x\| \le C\}$ for some constant C depending only on the support of a, the dilation matrix N and the integers $i_1, \dots, i_r$. Since $N^n = \begin{bmatrix} M^n & 0 \\ * & L^n \end{bmatrix}$, for a fixed integer $j \in \mathbb{Z}^r$, it is evident that the number of nonzero terms in the summation of (2.18) is at most $|\det L^n|(2C)^{s-r}$. Applying the Hölder inequality to the right side of (2.18), we have

$$\left|[\nabla_{e_1}^{i_1} \cdots \nabla_{e_r}^{i_r} S_{Pa,M}^n \delta]_j\right|^p \le \left[|\det L^n|(2C)^{s-r}\right]^{p/q} \sum_{k \in \mathbb{Z}^{s-r}} \left|[\nabla_{e_1}^{i_1} \cdots \nabla_{e_r}^{i_r} S_{a,N}^n \delta]_{(j,k)}\right|^p, \qquad j \in \mathbb{Z}^r,$$

where $1/p + 1/q = 1$. Since $\det L = \det N / \det M$, we have

$$\begin{aligned} \|\nabla_{e_1}^{i_1} \cdots \nabla_{e_r}^{i_r} S_{Pa,M}^n \delta\|_p^p &= \sum_{j \in \mathbb{Z}^r} \left|[\nabla_{e_1}^{i_1} \cdots \nabla_{e_r}^{i_r} S_{Pa,M}^n \delta]_j\right|^p \\ &\le \left[|\det L^n|(2C)^{s-r}\right]^{p/q} \sum_{j \in \mathbb{Z}^r} \sum_{k \in \mathbb{Z}^{s-r}} \left|[\nabla_{e_1}^{i_1} \cdots \nabla_{e_r}^{i_r} S_{a,N}^n \delta]_{(j,k)}\right|^p \\ &= |\det N / \det M|^{np/q} (2C)^{(s-r)p/q} \|\nabla_{e_1}^{i_1} \cdots \nabla_{e_r}^{i_r} S_{a,N}^n \delta\|_p^p. \end{aligned}$$

Therefore, (2.16) holds.

If the subdivision scheme associated with the mask a and the dilation matrix N converges in the L_p norm, then by [7, Theorem 3.2], we have $\rho_1(a, N; p) < |\det N|^{1/p-1}$. It follows from (2.16) that $\rho_1(Pa, M; p) < |\det M|^{1/p-1}$. By [7, Theorem 3.2] again, the subdivision scheme associated with mask Pa and dilation matrix M converges in the L_p norm as well. ■

As demonstrated by the following example, the conditions $PN = MP$ and $P\mathbb{Z}^s = \mathbb{Z}^r$ in Theorem 2.2 can not be replaced by the weaker conditions that $PN = MP$ and $(j + M\mathbb{Z}^r) \cap P\mathbb{Z}^s \ne \emptyset$ for all $j \in \mathbb{Z}^r$, which is similar to (2.8). Let $N = M = [2]$ and $P = [3]$. Evidently, $PN = MP$ and $(j + M\mathbb{Z}) \cap P\mathbb{Z} \ne \emptyset$ for all $j \in \mathbb{Z}$. Take $\widetilde{a}(\xi) = (e^{-i\xi} + 2 + e^{i\xi})/4$. Then $\widetilde{Pa}(\xi) = (e^{-i3\xi} + 2 + e^{i3\xi})/4$. It is known that the subdivision scheme associated with a converges in any $L_p (1 \le p \le \infty)$ norm while the subdivision scheme associated with Pa diverges in any L_p norm.

In the following let us investigate the smoothness property of a function under projection. We shall use the generalized Lipschitz space to measure smoothness of a given function. For $y \in \mathbb{R}^s$ and $f \in L_p(\mathbb{R}^s)$, define

$$\nabla_y^k f := \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^j f(\cdot - jy), \qquad k \in \mathbb{N}.$$

For a positive real number $\nu > 0$, let k be an integer greater than ν. The **generalized Lipschitz space** $Lip^*\big(\nu, L_p(\mathbb{R}^s)\big)$ consists of those functions f in $L_p(\mathbb{R}^s)$ for which

$$\sup_{|y| \le h} \|\nabla_y^k f\|_p \le C h^\nu \qquad \forall\, h > 0, \tag{2.19}$$

where C is a constant independent of h. The L_p smoothness of a function $f \in L_p(\mathbb{R}^s)$ in the L_p norm sense is described by its L_p **critical exponent** $\nu_p(f)$ defined by

$$\nu_p(f) := \sup\left\{ \nu \;:\; f \in Lip^*\big(\nu, L_p(\mathbb{R}^s)\big) \right\}. \tag{2.20}$$

Lemma 2.3 *Let P be an $r \times s$ real-valued matrix of full rank r and $r \le s$. Let ϕ be a compactly supported distribution defined on $\mathbb{R}^s$. Define η by $\widehat{\eta}(\xi) = \widehat{\phi}(P^*\xi), \xi \in \mathbb{R}^r$, where $\widehat{\phi}$ is understood to be a continuous function. If $\phi \in L_p(\mathbb{R}^s)$, then $\eta \in L_p(\mathbb{R}^r)$ and*

$$\eta(x) = \frac{1}{\sqrt{\det(PP^*)}} \int_{\{t \in \mathbb{R}^s \,:\, Pt = x\}} \phi \, d\sigma, \tag{2.21}$$

where $d\sigma$ denotes the unit surface element on the superplane $\{t \in \mathbb{R}^s : Pt = 0\}$. Moreover, $\nu_p(\phi) \le \nu_p(\eta)$ for all $1 \le p \le \infty$.

Proof: For any Schwartz function φ on $\mathbb{R}^r$, by the Plancherel Theorem, we have (see [1])

$$\begin{aligned}
\langle \eta, \varphi \rangle &= (2\pi)^{-r} \int_{\mathbb{R}^r} \widehat{\eta}(\xi)\widehat{\varphi}(\xi)\, d\xi = (2\pi)^{-r} \int_{\mathbb{R}^r} \widehat{\phi}(P^*\xi)\widehat{\varphi}(\xi)\, d\xi \\
&= \int_{\mathbb{R}^s} \phi(t)\varphi(Pt)\, dt = \int_{y \in (\ker P)^\perp} \int_{z \in \ker P} \phi(y+z)\varphi\big(P(y+z)\big)\, dz dy \\
&= \int_{y \in (\ker P)^\perp} \varphi(Py) \int_{z \in \ker P} \phi(y+z)\, dz\, dy.
\end{aligned}$$

Let $x = Py$. Then $y + \ker P = \{t \in \mathbb{R}^s : Pt = x\}$ and $dx = \sqrt{\det(PP^*)}dy$. Since P is of full rank,

$$\langle \eta, \varphi \rangle = \frac{1}{\sqrt{\det(PP^*)}} \int_{\mathbb{R}^r} \varphi(x) \int_{\{t \in \mathbb{R}^s \,:\, Pt = x\}} \phi \, d\sigma,$$

which is equivalent to (2.21). For any $y \in \mathbb{R}^r$ and $\ell \in \mathbb{N}$, it follows from (2.21) that

$$\nabla_y^\ell \eta(x) = \frac{1}{\sqrt{\det(PP^*)}} \int_{\{t \in \mathbb{R}^s \,:\, Pt = 0\}} \nabla_{P^{-1}y}^\ell \phi_a^N(\cdot + P^{-1}x)\, d\sigma,$$

where P^{-1} is a fixed generalized inverse to P such that $PP^{-1} = I_r$.

Since ϕ is compactly supported, applying the Minkowski inequality to the right side of the above equality, we have

$$\|\nabla_y^\ell \eta\|_{L_p(\mathbb{R}^r)} \le C \|\nabla_{P^{-1}y}^k \phi\|_{L_p(\mathbb{R}^s)},$$

where C is a constant depending only on ϕ, p and the matrix P. Hence, $\nu_p(\phi) \le \nu_p(\eta)$. ∎

The main result in this section is as follows.

Theorem 2.4 *Let M be an $r \times r$ dilation matrix and N be an $s \times s$ dilation matrix. Let P be an $r \times s$ integer matrix such that $PN = MP$ and $P\mathbb{Z}^s = \mathbb{Z}^r$. Let ϕ_a^N and $\phi_{a^d}^N$ be two refinable functions in $L_2(\mathbb{R}^s)$ such that the biorthogonal relation in (2.2) holds. Let Pa and Pa^d be the masks which are defined in (2.5). If a and a^d satisfy the sum rules of order ℓ_1 and ℓ_2 with respect to the lattice $N\mathbb{Z}^s$ respectively, then Pa and Pa^d must satisfy the sum rules of order at least ℓ_1 and ℓ_2 with respect to the lattice $M\mathbb{Z}^r$ respectively. Suppose that the mask a satisfies the condition in (2.8). Then Pa^d is a dual mask of Pa with respect to the lattice $M\mathbb{Z}^r$. In addition, ϕ_{Pa}^M and $\phi_{Pa^d}^M$ are refinable functions in $L_2(\mathbb{R}^r)$ and satisfy the biorthogonal relation in (2.2) as well. Moreover, if $\phi_a^N \in L_p(\mathbb{R}^s)$ and $\phi_{a^d}^N \in L_q(\mathbb{R}^s)$ for some $1 \le p, q \le \infty$, then $\phi_{Pa}^M \in L_p(\mathbb{R}^r)$, $\phi_{Pa^d}^M \in L_q(\mathbb{R}^r)$, and*

$$\nu_p(\phi_a^N) \le \nu_p(\phi_{Pa}^M), \qquad \nu_q(\phi_{a^d}^N) \le \nu_q(\phi_{Pa^d}^M).$$

Proof: Since ϕ_a^N and $\phi_{a^d}^N$ satisfy the biorthogonal relation in (2.2), then a^d is a dual mask of a and the subdivision schemes associated with the masks a and a^d converge in the L_2 norm. Therefore, it follows from Lemma 2.1 and Theorem 2.2 that Pa^d is a dual mask of Pa and the subdivision schemes associated with Pa and Pa^d converge in the L_2 norm as well. Therefore, ϕ_a^M and $\phi_{a^d}^M$ satisfy the biorthogonal relation in (2.2) as well.

Note that $\widehat{\phi}_{Pa}^M(\xi) = \widehat{\phi}_a^N(P^*\xi)$ and $\widehat{\phi}_{Pa^d}^M(\xi) = \widehat{\phi}_{a^d}^N(P^*\xi)$. The rest of the claims follows directly from Lemma 2.1 and Lemma 2.3. ∎

From the proof of Theorem (2.2), we observe that the conditions $PN = MP$ and $P\mathbb{Z}^s = \mathbb{Z}^r$ will force the dilation matrix N to be separable in the sense of (2.17). However, when N^ℓ is separable in the sense of (2.17) for some positive integer ℓ, the analysis in this section still applies due to the fact that $\phi_a^N = \phi_{S_{a,N}^\ell \delta}^{N^\ell}$ since $\phi_a^N = \sum_{k \in \mathbb{Z}^s} [S_{a,N}^\ell \delta]_k \phi_a^N(N^\ell \cdot - k)$.

On the other hand, the approach is this section can be easily generalized to the multivariate multiwavelets. When a mask a is symmetric, we can consider a more general class of projections P. Let $U(\mathbb{Z}^s)$ denote all the $s \times s$ integer matrices E such that $|\det E| = 1$. As in [6], we define

$$G_a^N := \{E \in G_a \ : \ N^j E N^{-j} \in G_a \ \forall\ j \in \mathbb{N}\},$$

where $G_a := \{E \in U(\mathbb{Z}^s) \ : \ a_{Ek} = a_k \ \forall\ k \in \mathbb{Z}^s\}$. Then all the results in this section still hold if the condition $PN = MP$ is replaced by $PNE = MP$ as long as $E \in G_a^N$. The key point here is that when $PNE = MP$, we still have the relations $\widehat{S_{Pa,M}^n \delta}(\xi) = \widehat{S_{a,N}^n \delta}(P^*\xi)$ and $\widehat{\phi}_{Pa}^M(\xi) = \widehat{\phi}_a^N(P^*\xi)$. For discussion on symmetry of masks and their associated refinable functions with a general dilation matrix, see [6].

3 Projectable Multivariate Wavelets

In this section, we shall discuss projectable wavelets for the commonly used dilation matrix $2I_s$. For simplicity, the dilation matrices in this section are assumed to be $2I_s$ for some $s \in \mathbb{N}$.

Given a sequence a on $\mathbb{Z}^s$, we say that a is **projectable onto the first coordinate** if

$$\widetilde{a}(\xi_1, j\pi) = 0 \qquad \forall\, \xi_1 \in \mathbb{R}, j \in \{0,1\}^{s-1}\backslash\{0\}, \tag{3.1}$$

or equivalently,

$$\sum_{k\in\mathbb{Z}^{s-1}} a_{(i,2k+j)} = \sum_{k\in\mathbb{Z}^{s-1}} a_{(i,2k)} \qquad \forall\, i \in \mathbb{Z}, j \in \{0,1\}^{s-1}. \tag{3.2}$$

In particular, when $s = 2$, (3.1) is equivalent to saying that $\widetilde{a}(\xi_1, \xi_2)$ contains the factor $(1 + e^{-i\xi_2})$. Let $N = 2I_s, M = [2]$ and $P = [1, 0, \dots, 0]$. Then it is easy to see that $PN = MP, P\mathbb{Z}^s = \mathbb{Z}$ and in this case, the condition in (3.1) is equivalent to the condition in (2.8).

If a mask a is projectable onto every coordinate, then a is called a **projectable mask**. The interest to investigate projectable masks is due to the following result which is a special case of Theorem 2.4 for $N = 2I_s, M = [2]$ and $P = [1, 0, \dots, 0]$.

Corollary 3.1 *Let ϕ_a and ϕ_{a^d} be two refinable functions with finitely supported masks a and a^d such that the biorthogonal relation in* (2.2) *holds with the dilation matrix $2I_s$. Suppose that a is projectable onto the first coordinate. Define a univariate sequence Pa by*

$$[Pa]_{k_1} := \sum_{k_2\in\mathbb{Z}} \cdots \sum_{k_s\in\mathbb{Z}} a_{(k_1,k_2,\dots,k_s)}, \qquad k_1 \in \mathbb{Z} \tag{3.3}$$

and define Pa^d similarly. Then $\phi_{Pa}, \phi_{Pa^d} \in L_2(\mathbb{R})$ and they satisfy the biorthogonal relation

$$\int_{\mathbb{R}} \overline{\phi_{Pa}(x+k)}\phi_{Pa^d}(x)\, dx = \delta_k \qquad \forall\, k \in \mathbb{Z}.$$

Moreover, Pa (or Pa^d) satisfies the same order of sum rules as a (or a^d) does. If $\phi_a \in L_p$ and $\phi_{a^d} \in L_q$ for some $1 \le p, q \le \infty$, then

$$\nu_p(\phi_a) \le \nu_p(\phi_{Pa}) \quad \textit{and} \quad \nu_q(\phi_{a^d}) \le \nu_q(\phi_{Pa^d}).$$

A biorthogonal wavelet is **projectable** if it is derived from a primal mask which is projectable onto every coordinate. Corollary 3.1 demonstrates that projectable biorthogonal wavelets essentially carry the tensor product structure even though themselves may not be tensor product biorthogonal wavelets. In a sense, a projectable wavelet can be easily replaced by a tensor product wavelet such that the tensor product wavelet function has no larger support (measured in size of squares), no worse L_p smoothness for all $1 \le p \le \infty$, and no less order of vanishing moments (or approximation order) than those of the given projectable wavelet function.

A mask a is an **interpolatory mask** (with respect to the lattice $2\mathbb{Z}^s$) if $a_k = 0$ for all $k \in 2\mathbb{Z}^s \backslash \{0\}$. It turns out that many known primal masks in multiple dimensions in the literature are projectable.

Theorem 3.2 *Let a be a mask on $\mathbb{Z}^s$. Then a is projectable onto the first coordinate if any one of the following statements holds:*

(1) $\widetilde{a}(\xi) = \widetilde{b}(\xi)\widetilde{c}(\xi), \xi \in \mathbb{R}^s$, *where b is projectable onto the first coordinate and c is a finitely supported sequence on $\mathbb{Z}^s$;*

(2) $\widetilde{a}(\xi_1, \xi_2) = \widetilde{b}(\xi_1)\widetilde{c}(\xi_1, \xi_2), \xi_1 \in \mathbb{R}^{s-1}, \xi_2 \in \mathbb{R}$, *where b is a finitely supported sequence on $\mathbb{Z}^{s-1}$ such that $\widetilde{b}(j\pi) = 0$ for all $j \in \{0,1\}^{s-1}\backslash\{0\}$, and c is a finitely supported sequence on $\mathbb{Z}^s$;*

(3) *a is an interpolatory mask on $\mathbb{Z}^s$ such that a satisfies the sum rules of order 2ℓ and $a_{(j,k)} = 0$ for all $j \in \mathbb{Z}\backslash[1-2\ell, \dots, 2\ell-1]$ and $k \in \mathbb{Z}^{s-1}$ for some positive integer ℓ;*

(4) *a is a mask on $\mathbb{Z}^s$ such that a satisfies the sum rules of order 2ℓ and $a_{(j,k)} = 0$ for all $j \in \mathbb{Z}\backslash[-\ell, \dots, \ell]$ and $k \in \mathbb{Z}^{s-1}$ for some positive integer ℓ;*

(5) *a is an orthogonal mask on $\mathbb{Z}^s$ such that a satisfies the sum rules of order ℓ and $a_{(j,k)} = 0$ for all $j \in \mathbb{Z}\backslash[0, \dots, 2\ell-1]$ and $k \in \mathbb{Z}^{s-1}$ for some positive integer ℓ.*

Proof: The statements in (1) and (2) are trivial by (3.1). The statements in (3) and (4) can be proved using a similar proof as in Lemma 4.1 in [5]. The statement in (5) is a direct consequence of the statement in (3). ∎

As a consequence of Theorem 3.2, if a is an interpolatory mask on $\mathbb{Z}^s$ such that a satisfies the sum rules of order 2ℓ and the support of a is contained in $[1-2\ell, 2\ell-1]^s$ for some positive integer ℓ, then a is a projectable mask. By Theorem 3.2, all the tensor product masks, many interpolatory masks, and many masks of most Box spline refinable functions in [1] are projectable masks.

It is known that a finitely support univariate mask a on $\mathbb{Z}$ satisfies the sum rules of order ℓ if and only if $\widetilde{a}(\xi)$ contains the factor $(1 + e^{-i\xi})^\ell$. Therefore, in higher dimensions, one may require that the symbol of a finitely supported mask on $\mathbb{Z}^s$ contains the factor $\Pi_{j=1}^s (1 + e^{-i\xi_j})$. By Theorem 3.2, such a mask must be projectable onto every coordinate.

In the rest of this section, we shall demonstrate that when a multivariate mask is a projectable mask, the above results about projectable wavelets provide some useful information to help us understand and design multivariate wavelets.

Proposition 3.3 *Suppose that a is an interpolatory mask such that a is supported on $[1-2\ell, 2\ell-1]^s$ and a satisfies the sum rules of order 2ℓ for some positive integer ℓ. If a^d is a dual mask of a such that a^d is supported on $[-k, k]^s$ for some positive integer k, then a^d can satisfy the sum rules of order at most $2\lfloor \frac{k}{2} \rfloor - 2\ell + 2$, where $\lfloor \cdot \rfloor$ is the floor function.*

Proof: By Theorem 3.2, a is projectable onto the first coordinate. Therefore, Pa^d is a dual mask of Pa by Corollary 3.1. As proved in [5, Lemma 4.1], Pa must be the interpolatory mask b_ℓ in [4] which is supported on $[1-2\ell, 2\ell-1]$. By the CBC (coset by coset) algorithm and Theorem 2.2 in [5], it is not difficult to prove that if the dual mask Pa^d of b_ℓ is supported on $[-k, k]$, then Pa^d can satisfy the sum rules of order at most $2\lfloor \frac{k}{2} \rfloor - 2\ell + 2$. Consequently, a^d can satisfy the sum rules of order at most $2\lfloor \frac{k}{2} \rfloor - 2\ell + 2$ as well. ∎

Let a be an interpolatory mask such that a is supported on $[1-2\ell, 2\ell-1]^s$ and a satisfies the sum rules of order 2ℓ for some positive integer ℓ. As a consequence of Proposition 3.3, all the dual masks of a, which are constructed via the TCBC (Triangle Coset By Coset) algorithm in [5], satisfy the optimal orders of sum rules with respect to their support. For discussion on optimal multivariate wavelets, the reader is referred to [5].

Example 3.4 *Let a be a mask on $\mathbb{Z}^s$ such that a satisfies the sum rules of order 4 and $a_{(j,k)} = 0$ for all $j \in \mathbb{Z}\backslash\{-2, -1, 0, 1, 2\}$ and $k \in \mathbb{Z}^{s-1}$. Then there does not exist a compactly supported dual refinable function in $L_2(\mathbb{R}^s)$ of ϕ_a such that the support of the dual refinable function is contained in $[-4, 4] \times \mathbb{R}^{s-1}$.*

Proof: By Theorem 3.2, we observe that a must be projectable onto the first coordinate. Suppose that there is a dual refinable function ϕ_{a^d} in $L_2(\mathbb{R}^s)$ of ϕ_a such that the support of a^d is contained in $[-4, 4] \times \mathbb{Z}^{s-1}$. Therefore, by Corollary 3.1, ϕ_{Pa^d} is a dual refinable function of ϕ_{Pa}. Note that $\widetilde{Pa}(\xi) = (e^{-i\xi} + 2 + e^{i\xi})^2/16$ since Pa is supported on $[-2, 2]$ and satisfies the sum rules of order 4. Define $b_k := ([Pa^d]_k + [Pa^d]_{-k})/2, k \in \mathbb{Z}$. Then an easy argument demonstrates that ϕ_b is also a dual refinable function of ϕ_{Pa}. Moreover, b must satisfy the sum rules of order at least 2. However, it is easy to prove that there exists a unique dual mask c of Pa such that c is supported on $[-4, 4]$ and c satisfies the sum rules of order 2. In fact, c is given by $\widetilde{c}(\xi) = 5/4 + 5/16 \cos(\xi) - 3/4 \cos(2\xi) + 3/16 \cos(3\xi)$. Hence, $b = c$. But $\phi_c \notin L_2$ which is a contradiction to the fact that $\phi_b \in L_2$. Therefore, no compactly supported dual refinable function of ϕ_a can be supported on $[-4, 4] \times \mathbb{R}^{s-1}$. ∎

In the following, we consider some masks of Box spline functions (see [1]). Let a mask a on $\mathbb{Z}^s$ be given by

$$\widetilde{a}(\xi_1, \ldots, \xi_s) = 4^{-s-1}(e^{-i(\xi_1+\cdots+\xi_s)} + 2 + e^{i(\xi_1+\cdots+\xi_s)})\Pi_{j=1}^{s}(e^{-i\xi_j} + 2 + e^{i\xi_j}), \qquad (3.4)$$

or

$$\widetilde{a}(\xi_1, \ldots, \xi_s) = 16^{-s}\Pi_{j=1}^{s}(e^{-i\xi_j} + 2 + e^{i\xi_j})^2.$$

Then it is easy to check that a satisfies all the conditions in Example 3.4. Therefore, there is no compactly supported dual refinable function in $L_2(\mathbb{R}^s)$ of ϕ_a such that the dual refinable function is supported in $[-4,4]^s$. However, it is not difficult to see that there are infinitely many dual masks of a such that they are supported on $[-4,4]^s$ and satisfies the sum rules of order at least 1. Let us illustrate this for the mask a in (3.4) with $s=2$. By computation, we found that there are 33 free parameters in the parametric form of all the dual masks of the mask a such that the dual masks are supported on $[-4,4]^2$ and satisfy the sum rules of order 1.

In conclusion, when a primal mask is projectable, the univariate wavelets provide upper bounds of the approximation order and $L_p (1 \le p \le \infty)$ smoothness exponents of its dual refinable functions whose supports are contained in a given square.

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