

# Inverse Scattering with Partial Information on the Potential \*

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## Abstract

The one-dimensional Schrödinger equation is considered when the potential is real valued and integrable and has a finite first moment. The recovery of such a potential is analyzed in terms of the scattering data consisting of a reflection coefficient, all the bound-state energies, knowledge of the potential on a finite interval, and all of the bound-state norming constants except one. It is shown that a missing norming constant in the data can cause at most a double nonuniqueness in the recovery. In the particular case when the missing norming constant in the data corresponds to the lowest-energy bound state, the necessary and sufficient conditions are obtained for the nonuniqueness, and the two norming constants and the corresponding potentials are determined. Some explicit examples are provided to illustrate the nonuniqueness.

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# 1 Introduction

Consider the one-dimensional Schrödinger equation

$$-\psi''(k, x) + V(x) \psi(k, x) = k^2 \psi(k, x), \quad x \in \mathbf{R}, \quad (1.1)$$

where the potential  $V$  is real valued and belongs to  $L_1^1(\mathbf{R})$ , the set of measurable potentials such that  $\int_{-\infty}^{\infty} dx (1 + |x|) |V(x)|$  is finite. In our notation, the prime denotes the derivative with respect to the spatial variable  $x$ ,  $\mathbf{C}^+$  is the upper-half complex plane,  $\mathbf{R}^- := (-\infty, 0)$ , and  $\mathbf{R}^+ := (0, +\infty)$ . For a given subset  $J$  of  $\mathbf{R}$ , we will use  $V|_J$  to denote the fragment of  $V$  supported on  $J$ , i.e.

$$V|_J(x) := \begin{cases} V(x), & x \in J, \\ 0, & x \notin J. \end{cases}$$

Recall [1]-[6] that the scattering solutions of (1.1) are asymptotic to linear combinations of  $e^{\pm ikx}$  as  $x \rightarrow -\infty$  and  $x \rightarrow +\infty$ , and they occur for all  $k \in \mathbf{R} \setminus \{0\}$ . Among such solutions are the Jost solution from the left,  $f_l(k, x)$ , and the Jost solution from the right,  $f_r(k, x)$ , satisfying the boundary conditions

$$e^{-ikx} f_l(k, x) = 1 + o(1), \quad e^{-ikx} f_l'(k, x) = ik + o(1), \quad x \rightarrow +\infty,$$

$$e^{ikx} f_r(k, x) = 1 + o(1), \quad e^{ikx} f_r'(k, x) = -ik + o(1), \quad x \rightarrow -\infty.$$

From the spatial asymptotics

$$f_l(k, x) = \frac{e^{ikx}}{T(k)} + \frac{L(k) e^{-ikx}}{T(k)} + o(1), \quad x \rightarrow -\infty, \quad (1.2)$$

$$f_r(k, x) = \frac{e^{-ikx}}{T(k)} + \frac{R(k) e^{ikx}}{T(k)} + o(1), \quad x \rightarrow +\infty, \quad (1.3)$$

we obtain the scattering coefficients, namely, the transmission coefficient  $T$ , and the reflection coefficients  $L$  and  $R$ , from the left and right, respectively. We have

$$R(k) T(k)^* = -L(k)^* T(k), \quad k \in \mathbf{R}, \quad (1.4)$$

where the asterisk denotes complex conjugation.

The bound-state solutions decay exponentially as  $x \rightarrow \pm\infty$ , and they can occur only at certain  $k$  values on the positive imaginary axis in  $\mathbf{C}^+$ . The number of bound states is

finite, and we use  $N$  to denote that number. We let the bound states occur at  $k = i\kappa_j$  with  $0 < \kappa_1 < \dots < \kappa_N$ . Each bound state at  $k = i\kappa_j$  is simple, i.e. there is only one linearly independent bound-state solution of (1.1) at  $k = i\kappa_j$ . The bound-state norming constants  $c_{lj}$  from the left and  $c_{rj}$  from the right, respectively, are defined as

$$c_{lj} := \left[ \int_{-\infty}^{\infty} dx f_l(i\kappa_j, x)^2 \right]^{-1/2}, \quad c_{rj} := \left[ \int_{-\infty}^{\infty} dx f_r(i\kappa_j, x)^2 \right]^{-1/2},$$

and they are related to each other via the residues of  $T$  as

$$\text{Res}(T, i\kappa_j) = i c_{lj}^2 \gamma_j = i \frac{c_{rj}^2}{\gamma_j}, \quad (1.5)$$

where the  $\gamma_j$  are the dependency constants defined as

$$\gamma_j := \frac{f_l(i\kappa_j, x)}{f_r(i\kappa_j, x)}, \quad x \in \mathbf{R}. \quad (1.6)$$

Recall [2] that when  $\beta \in (0, +\infty)$ , the quantity  $1/T(i\beta)$  is real and continuous, has simple zeros at  $\beta = \kappa_j$ , and behaves like  $1 + O(1/\beta)$  as  $\beta \rightarrow +\infty$ . Thus,  $(-1)^N T(i\beta) > 0$  when  $\beta \in (0, \kappa_1)$ ,  $T(i\beta) > 0$  when  $\beta > \kappa_N$ , and  $(-1)^{N-j+1} T(i\beta) > 0$  when  $\beta \in (\kappa_{j-1}, \kappa_j)$  for  $j = 2, \dots, N$ . Hence, with the help of (1.5), it is seen that  $\gamma_j = (-1)^{N-j} c_{rj}/c_{lj}$ .

It is already known [1]-[6] that  $V$  is uniquely determined by either the left scattering data  $\{R, \{\kappa_j\}, \{c_{lj}\}\}$  or by the right scattering data  $\{L, \{\kappa_j\}, \{c_{rj}\}\}$ . Given  $\{|R|, \{\kappa_j\}\}$ , we can construct  $T$  explicitly (e.g. see Lemma 2.5 of [2]). Then, when  $\{R, \{\kappa_j\}\}$  is known, from (1.5) it follows that there is a one-to-one correspondence between each left norming constant  $c_{lj}$  and the dependency constant  $\gamma_j$ ; similarly, given  $\{L, \{\kappa_j\}\}$ , each  $c_{rj}$  determines  $\gamma_j$  uniquely and vice versa. Hence,  $V$  is uniquely determined by either  $\{R, \{\kappa_j\}, \{\gamma_j\}\}$  or  $\{L, \{\kappa_j\}, \{\gamma_j\}\}$ .

We are interested in the following inverse problem. Suppose we know one reflection coefficient, the bound-state energies  $-\kappa_j^2$ , and the potential  $V$  on a finite interval  $I$  of positive length. We may also know some but not all of the bound-state norming constants. In other words, we are missing the full information on the norming constants for a unique determination of the potential, and we would like to know if knowledge of  $V$  on  $I$  compensates for the missing information on the norming constants. We wish to find out if our data determines the potential uniquely on the entire line or if there are two or more potentials corresponding to our data. Clearly, if there are no bound states,  $V$  is uniquely determined by  $R$  or  $L$  even without knowing a fragment of the potential. Hence, our inverse problem has relevance only in the presence of bound states.

If the interval in which the potential is known is a half line, then we already know [7]-[13] the answer to our question; namely, if  $V|_{\mathbf{R}^+}$  and  $R$  are known, then  $V$  is uniquely determined without the bound-state information. In fact, the bound-state energies and norming constants are uniquely determined by  $\{R, V|_{\mathbf{R}^+}\}$ . On the other hand,  $V$  is not uniquely determined by  $\{L, V|_{\mathbf{R}^+}\}$  alone. However, if we know  $L$  and all the bound-state energies, then we can construct the corresponding  $R$  (e.g. see Lemma 2.5 of [2]). Thus,  $V$  is uniquely determined by  $\{L, V|_{\mathbf{R}^+}, \{\kappa_j\}\}$ . In a similar way,  $V$  is uniquely determined by  $\{L, V|_{\mathbf{R}^-}\}$  without the bound-state information, and in fact, the bound-state energies and norming constants are uniquely determined by  $\{L, V|_{\mathbf{R}^-}\}$ ; however,  $V$  is not uniquely determined by  $\{R, V|_{\mathbf{R}^-}\}$  alone, but is uniquely determined by  $\{R, V|_{\mathbf{R}^-}, \{\kappa_j\}\}$ . Therefore, our inverse problem needs to be studied only when our interval is finite.

In this paper we analyze and solve our inverse problem when only one norming constant is missing from the scattering data. If the data lacks two or more norming constants, the problem is still open; in that case, it would be desirable either to produce an example where infinitely many distinct potentials correspond to the same scattering data or to prove that there can only be a finite number of such potentials and to determine an upper bound for that number.

Let us say a few words about the existence aspect of our inverse problem. Clearly, we must expect a severe restriction on the fragment of the potential contained in our data. We cannot specify that fragment arbitrarily, and in general a solution to our inverse problem does not exist. In our paper, we are only interested in the uniqueness aspect of our inverse problem; namely, we assume that there exists at least one potential corresponding to our data and we would like to find out if there are more than one.

Our paper is organized as follows. In Section 2 we consider the scattering data consisting of one reflection coefficient, knowledge of the potential on a finite interval  $I$ , all the bound-state energies, and all the norming constants except one; we show that a missing norming constant in our data can cause at most a double nonuniqueness in the determination of the potential. In Section 3 we analyze in detail the uniqueness and nonuniqueness when our data lacks only the norming constant corresponding to the lowest-energy bound state; we give the necessary and sufficient conditions on the scattering data so that it corresponds to two distinct potentials, and we also determine the corresponding dependency constants

and those two potentials. In Section 4 we illustrate, with explicit examples, various cases of uniqueness and nonuniqueness.

## 2 Nonuniqueness in the General Case

When exactly one norming constant is missing from our data, as the next theorem shows, there can be at most two distinct potentials corresponding to that data.

**THEOREM 2.1.** *Let  $V$  be a real-valued potential belonging to  $L^1_1(\mathbf{R})$ , and consider the scattering data consisting of one reflection coefficient, all the  $N$  bound-state energies, knowledge of  $V$  on a finite interval  $I$  of positive length, and  $N - 1$  of the bound-state norming constants. Then, besides  $V$ , there can be at most one other potential corresponding to this scattering data.*

PROOF: Without loss of generality it is enough to give the proof when our data consists of  $R(k)$  for  $k \in \mathbf{R}$ , the  $N$  constants  $\kappa_j$ , knowledge of  $V|_I$ , and  $N - 1$  norming constants. Let us use the superscript  $[0]$  to indicate the quantities associated with the potential  $V^{[0]}$  that is obtained from  $V$  by removing all the  $N$  bound states. Note [2] that  $V^{[0]} \in L^1_1(\mathbf{R})$  and it can uniquely be constructed from the corresponding right reflection coefficient  $R^{[0]}$  that is given by

$$R^{[0]}(k) = (-1)^N \left( \prod_{j=1}^N \frac{k - i\kappa_j}{k + i\kappa_j} \right) R(k).$$

The transmission coefficient  $T^{[0]}$  corresponding to  $V^{[0]}$  can be uniquely constructed [2] from  $|R(k)|$  alone, and we have

$$T^{[0]}(k) = \left( \prod_{j=1}^N \frac{k - i\kappa_j}{k + i\kappa_j} \right) T(k). \quad (2.1)$$

Define

$$\alpha_j := \frac{2(-1)^{j-1}\kappa_j}{c_{1j}^2} \left( \prod_{m \neq j}^N \frac{\kappa_m + \kappa_j}{\kappa_m - \kappa_j} \right) T^{[0]}(i\kappa_j), \quad j = 1, 2, \dots, N. \quad (2.2)$$

From (1.5), (2.1), and (2.2), it follows that

$$\alpha_j = |\gamma_j| = (-1)^{N-j} \gamma_j = \frac{(-1)^{N-j} \operatorname{Res}(T, i\kappa_j)}{i c_{1j}^2} = \frac{(-1)^{N-j} i c_{rj}^2}{\operatorname{Res}(T, i\kappa_j)}, \quad j = 1, \dots, N. \quad (2.3)$$

Thus, in our data, for each fixed  $j$ , we can use any one of  $c_{lj}$ ,  $c_{rj}$ ,  $\alpha_j$ , and  $\gamma_j$  interchangeably. So, instead of the  $j$ th norming constant, we can simply use  $\alpha_j$  in our analysis. In fact, each  $\alpha_j$  acts as a dependency constant (cf. (1.6)) in the sense that

$$\alpha_j = \frac{f_1^{[j]}(i\kappa_j, x)}{f_r^{[j]}(i\kappa_j, x)}, \quad x \in \mathbf{R}, \quad (2.4)$$

where  $f_1^{[j]}$  and  $f_r^{[j]}$  are the Jost solutions from the left and from the right, respectively, for the potential  $V^{[j]}$ ; here  $\{V^{[j]}\}_{j=0}^N$  is the sequence of potentials given in Theorem 3.6 of [2]. Note that  $V^{[N]} = V$  and that  $V^{[j]}$  is obtained from  $V^{[j-1]}$  by adding the bound state at  $k = i\kappa_j$ .

Define

$$\omega_j(x) := (-1)^{j+1} f_1^{[0]}(i\kappa_j, x) + \alpha_j f_r^{[0]}(i\kappa_j, x), \quad j = 1, \dots, N, \quad (2.5)$$

and let  $\Gamma(k, x)$  be the  $N \times N$  matrix whose entries are defined as

$$\Gamma_{2j-1,s}(k, x) := \kappa_s^{2j-2} \omega_s(x), \quad \Gamma_{2j,s}(k, x) := \kappa_s^{2j-2} \omega'_s(x).$$

Let us use the absolute value bars to denote the matrix determinant. We have [2]

$$V(x) = V^{[0]}(x) - 2 \frac{d}{dx} \left[ \frac{|\Gamma(x)|'}{|\Gamma(x)|} \right], \quad x \in \mathbf{R}. \quad (2.6)$$

Now let us assume that our data contains all the dependency constants except one, namely  $\alpha_m$ . In view of (2.5) and (2.6), we need to determine how many distinct positive  $\alpha_m$  values can be found in such a way that

$$V(x) = V^{[0]}(x) - 2 \frac{d}{dx} \left[ \frac{|\Gamma(x)|'}{|\Gamma(x)|} \right], \quad x \in I. \quad (2.7)$$

When we replace  $\omega_m(x)$  by  $f_r^{[0]}(i\kappa_m, x)$  in  $\Gamma(x)$ , let us denote the resulting matrix by  $A(x)$ . Similarly, let  $B(x)$  denote the matrix obtained from  $\Gamma(x)$  by replacing  $\omega_m(x)$  by  $(-1)^{m+1} f_1^{[0]}(i\kappa_m, x)$ . As seen from (2.5) we have

$$\Gamma(x) = \alpha_m A(x) + B(x). \quad (2.8)$$

The elements in the first row of  $A(x)$  are eigenfunctions of the Schrödinger equation with distinct eigenvalues, and hence they are linearly independent on  $\mathbf{R}$ . The same property holds also for the elements in the first row of  $B(x)$  and in the first row of  $\Gamma(x)$ . Consequently, none of  $|A(x)|$ ,  $|B(x)|$ , and  $|\Gamma(x)|$  can vanish identically on the interval  $I$ . In fact, because  $V^{[0]}$

has no bound states, for any  $1 \leq j \leq N$  we have  $f_1^{[0]}(i\kappa_j, x) > 0$  and  $f_r^{[0]}(i\kappa_j, x) > 0$  on  $\mathbf{R}$ ; by using induction (cf. pp. 179–180 of [2]) one can prove that these three determinants are strictly positive for  $x \in \mathbf{R}$ .

By letting

$$G(x) := -\frac{1}{2} [V(x) - V^{[0]}(x)],$$

we write (2.7) as

$$G(x) |\Gamma(x)|^2 = |\Gamma(x)|'' - (|\Gamma(x)|')^2, \quad x \in I. \quad (2.9)$$

Let us suppress the  $x$ -dependence in the rest of the proof. From (2.5) and (2.6), we see that  $|A|'$  is equal to the determinant of the matrix that is obtained by taking the derivative of only the last row of  $A$ ; similar remarks also apply to  $|A|''$  and to  $|B|'$ ,  $|\Gamma|'$ , and their derivatives. With the help of (2.8), we can write (2.9) as

$$\begin{aligned} (G|A|^2 - |A|''|A| + (|A|')^2) \alpha_m^2 + (2G|A||B| + 2|A|'|B|' - |A|''|B| - |A||B|'') \alpha_m \\ + (G|B|^2 - |B|''|B| + (|B|')^2) = 0, \quad x \in I. \end{aligned} \quad (2.10)$$

Let us analyze (2.10) as a quadratic equation in  $\alpha_m$  where the coefficients depend on  $x \in I$ . It has at most two distinct positive solutions (that are independent of  $x \in I$ ) unless the coefficients are identically zero for all  $x \in I$ , in which case (2.7) would hold for all  $\alpha_m > 0$  and there would be infinitely many distinct potentials corresponding to our data. We prove below that this is not possible.

If the coefficients in (2.10) vanished for all  $x \in I$ , since  $|A(x)| > 0$ ,  $|B(x)| > 0$ , and  $|\Gamma(x)| > 0$ , we would have for  $x \in I$

$$\begin{aligned} G &= \frac{|A|''}{|A|} - \left(\frac{|A|'}{|A|}\right)^2 = \left(\frac{|A|'}{|A|}\right)', \\ 2G &= \frac{|A|''}{|A|} + \frac{|B|''}{|B|} - 2\frac{|A|'|B|'}{|A||B|} = \left(\frac{|A|'}{|A|}\right)' + \left(\frac{|B|'}{|B|}\right)' + \left(\frac{|A|'}{|A|} - \frac{|B|'}{|B|}\right)^2, \\ G &= \frac{|B|''}{|B|} - \left(\frac{|B|'}{|B|}\right)^2 = \left(\frac{|B|'}{|B|}\right)', \end{aligned}$$

which, with the help of (2.9), would imply that

$$G = \left(\frac{|\Gamma|'}{|\Gamma|}\right)' = \left(\frac{|A|'}{|A|}\right)' = \left(\frac{|B|'}{|B|}\right)', \quad \frac{|A|'}{|A|} = \frac{|B|'}{|B|}, \quad x \in I. \quad (2.11)$$

The second equation in (2.11) would imply that  $|A|$  and  $|B|$  are linearly dependent on  $I$ , and this would mean that, for some constant  $c$ , the matrix  $A - cB$  would have zero determinant for all  $x \in I$ . However, this is impossible because the entries in the first row of  $A - cB$  are eigenvectors of the Schrödinger operator with distinct eigenvalues and hence are linearly independent on  $I$ . Thus, (2.10) can at most have two distinct positive roots. ■

### 3 Characterization of the Nonuniqueness

Let us use the notation introduced below (2.4); namely,  $f_1^{[j]}$  and  $f_r^{[j]}$  denote the Jost solutions for the potential  $V^{[j]}$  obtained from  $V^{[j+1]}$  by removing the bound state at  $k = i\kappa_j$ , where  $V^{[N]} := V$ . In terms of the  $\alpha_j$  appearing in (2.2)-(2.4), let us define

$$g^{[j]}(x) := f_1^{[j-1]}(i\kappa_j, x) + \alpha_j f_r^{[j-1]}(i\kappa_j, x), \quad j = 1, \dots, N. \quad (3.1)$$

Note that  $g^{[j]}(x) > 0$  on  $\mathbf{R}$  for any  $\alpha_j \geq 0$  because it is known [2] that  $f_1^{[j-1]}(i\kappa_j, x) > 0$  and  $f_r^{[j-1]}(i\kappa_j, x) > 0$  on  $\mathbf{R}$ . According to the Darboux transformation formulas [2], [5] we have

$$V^{[j]}(x) = -V^{[j-1]}(x) - 2\kappa_j^2 + 2 \left( \frac{g^{[j]'}(x)}{g^{[j]}(x)} \right)^2, \quad j = 1, \dots, N, \quad x \in \mathbf{R}. \quad (3.2)$$

Thus, we can unambiguously define the (nonnegative and real-valued) quantities

$$\Lambda^{[j]}(x) := \frac{1}{\sqrt{2}} \sqrt{V^{[j]}(x) + V^{[j-1]}(x) + 2\kappa_j^2}, \quad j = 1, \dots, N. \quad (3.3)$$

In Theorem 2.1 we have seen that a missing norming constant in our data can cause at most a double nonuniqueness. We will now analyze how this happens when the norming constant missing in our data corresponds to the lowest-energy bound state.

**THEOREM 3.1.** *Assume  $V$  is real valued, belongs to  $L_1^1(\mathbf{R})$ , and has  $N$  bound states at  $k = i\kappa_j$  with  $0 < \kappa_1 < \dots < \kappa_N$ . Consider the scattering data consisting of one reflection coefficient, knowledge of  $V$  on a finite interval  $I$  of positive length, all the  $N$  bound-state energies, and all the  $N$  norming constants except for the one corresponding to the  $N$ th bound state. Then, either  $V$  is the only potential on  $\mathbf{R}$  corresponding to this data, or there is exactly one other such potential; the latter happens if and only if all the following four conditions are satisfied.*



(i)  $V|_I \equiv V^{[N-1]}|_I$ .

(ii)  $V|_I(x)$  is constant.

(iii)  $V|_I(x) > -\kappa_N^2$ .

(iv) The quantities  $\alpha_{N;\pm}$  given in (3.10) are finite and strictly positive at any one particular  $x$  value in  $I$ .

When (i)-(iv) are satisfied, the two potentials corresponding to the aforementioned scattering data are given by

$$V(x; \alpha_{N;\pm}) = -V^{[N-1]}(x) - 2\kappa_N^2 + 2 \left( \frac{f_1^{[N-1]'}(i\kappa_N, x) + \alpha_{N;\pm} f_{\mathbb{r}}^{[N-1]'}(i\kappa_N, x)}{f_1^{[N-1]}(i\kappa_N, x) + \alpha_{N;\pm} f_{\mathbb{r}}^{[N-1]}(i\kappa_N, x)} \right)^2, \quad x \in \mathbf{R}. \quad (3.4)$$

PROOF: Our data uniquely determines (see e.g. Theorems 3.3 and 3.6 of [2]) the potentials  $V^{[j]}$  on  $\mathbf{R}$  for  $0 \leq j \leq N-1$ . Because of (2.3), we can use  $\alpha_N$  as the missing dependency constant in our data. There can be at most two distinct potentials corresponding to the two possible distinct positive values of  $\alpha_N$ , say  $\alpha_{N;\pm}$ , and let us denote the corresponding potentials by  $V^{[N;\pm]}$ . Thus,  $V^{[N;\pm]} \equiv V|_I$  on  $I$  even though  $V^{[N;+]} \not\equiv V^{[N;-]}$  on  $\mathbf{R}$ . Then, as seen from (3.3), our data uniquely determines  $\Lambda^{[N]}$  on  $I$ , even though there can be two distinct  $\Lambda^{[N]}$  on  $\mathbf{R}$ .

Using (3.1) and (3.3), from (3.2) we get

$$[\Lambda^{[N]}(x)]^2 = \left( \frac{f_1^{[N-1]'}(i\kappa_N, x) + \alpha_N f_{\mathbb{r}}^{[N-1]'}(i\kappa_N, x)}{f_1^{[N-1]}(i\kappa_N, x) + \alpha_N f_{\mathbb{r}}^{[N-1]}(i\kappa_N, x)} \right)^2, \quad x \in I. \quad (3.5)$$

Note that (3.5) is a quadratic equation in  $\alpha_N$  with the  $x$ -dependent coefficients. We can write the two solutions  $\alpha_{N;\pm}$  of (3.5) as

$$\alpha_{N;\pm} = \frac{f_1^{[N-1]'}(i\kappa_N, x) \mp \Lambda^{[N]}(x) f_1^{[N-1]}(i\kappa_N, x)}{-f_{\mathbb{r}}^{[N-1]'}(i\kappa_N, x) \pm \Lambda^{[N]}(x) f_{\mathbb{r}}^{[N-1]}(i\kappa_N, x)}, \quad x \in I. \quad (3.6)$$

For the nonuniqueness we must have  $\alpha_{N;\pm}$  finite, positive, distinct, and independent of  $x$ . Differentiating the right-hand side of (3.6) with respect to  $x$  and setting the result to zero, after some simplifications, we obtain

$$\left[ V^{[N-1]}(x) + \kappa_N^2 - \Lambda^{[N]}(x)^2 \mp \Lambda^{[N]'}(x) \right] [f_{\mathbb{r}}^{[N-1]}(i\kappa_N, x); f_1^{[N-1]}(i\kappa_N, x)] = 0, \quad x \in I, \quad (3.7)$$

where  $[F; G] := FG' - F'G$  denotes the Wronskian. Note that the Wronskian in (3.7) is equal (see e.g. [2], [4], [5]) to  $-2\kappa_N/T^{[N-1]}(i\kappa_N)$ , which is a negative constant. Thus, from (3.7) we get

$$\Lambda^{[N]}(x)^2 - V^{[N-1]}(x) - \kappa_N^2 = \mp \Lambda^{[N]'}(x), \quad x \in I. \quad (3.8)$$

Using (3.8) we can write (3.8) as

$$\frac{1}{2}[V^{[N]}(x) - V^{[N-1]}(x)] = \mp \Lambda^{[N]'}(x), \quad x \in I. \quad (3.9)$$

Note that  $\kappa_N$ ,  $\Lambda^{[N]}$  on  $I$ , and  $V^{[N-1]}(x)$  on  $\mathbf{R}$  are all uniquely determined by our data, and hence the left-hand side in (3.9) is uniquely specified. Thus, the two equations in (3.8) specified by the sign  $\mp$  cannot simultaneously be satisfied unless  $\Lambda^{[N]'}(x) \equiv 0$  on  $I$ . However, in that case (3.9) implies that  $V^{[N]} \equiv V^{[N-1]}$  on  $I$ . Thus, (i) is proved. There are now two possibilities: either  $\Lambda_I^{[N]}(x)$  is identically zero or a positive constant. In either case, we see from (3.8) and (i) that (ii) is also proved. From (3.6) we get  $\alpha_{N,+} = \alpha_{N,-}$  if and only if  $\Lambda_I^{[N]}(x) [f_{\mathbf{r}}^{[N-1]}(i\kappa_N, x); f_{\mathbf{l}}^{[N-1]}(i\kappa_N, x)] = 0$ ; however, the Wronskian on the left-hand side is a negative constant, and hence  $\alpha_{N,+} = \alpha_{N,-}$  if and only if  $\Lambda_I^{[N]}(x) \equiv 0$ . Thus, for the nonuniqueness to occur,  $\Lambda_I^{[N]}$  cannot be zero and instead it must be a positive constant. Then, however, with the help of (i), (ii), and (3.8), we see that (iii) must hold for the nonuniqueness. Thus, the nonuniqueness occurs when  $V|_I + \kappa_N^2$  is a positive constant on  $I$ , in which case from (3.6) we get

$$\alpha_{N;\pm} = \frac{f_{\mathbf{l}}^{[N-1]'}(i\kappa_N, x) \mp \sqrt{V|_I + \kappa_N^2} f_{\mathbf{l}}^{[N-1]}(i\kappa_N, x)}{-f_{\mathbf{r}}^{[N-1]'}(i\kappa_N, x) \pm \sqrt{V|_I + \kappa_N^2} f_{\mathbf{r}}^{[N-1]}(i\kappa_N, x)}, \quad x \in I, \quad (3.10)$$

where the right-hand side is independent of  $x$  and can be evaluated at any  $x \in I$ . Note that we do not have to require  $\alpha_{N,+} \neq \alpha_{N,-}$  because this is automatically satisfied when (i)-(iii) are satisfied. Finally, (3.4) is obtained from (3.1) and (3.2) with  $j = N$  by replacing  $\alpha_N$  with  $\alpha_{N;\pm}$ , which are the constants in (3.10). ■

**PROPOSITION 3.2.** *Under the assumptions of Theorem 3.1, suppose (i)-(iv) are satisfied so that there are two distinct potentials corresponding to the scattering data. Assume further that  $V^{[N-1]}$  is evenly symmetric with respect to the midpoint  $a$  of the interval  $I$ . Then we have the following:*

(i) *The  $\alpha_{N;\pm}$  given in (3.10) satisfy  $\alpha_{N,+} \alpha_{N,-} = e^{-4\kappa_N a}$ .*

(ii) *The two potentials  $V(x; \alpha_{N;\pm})$  corresponding to the scattering data stated in Theorem 3.1 satisfy  $V(x; \alpha_{N;+}) = V(2a - x; \alpha_{N;-})$  for all  $x \in \mathbf{R}$ .*

PROOF: From (3.10) we get

$$\alpha_{N;+} \alpha_{N;-} = \frac{[f_1^{[N-1]'}(i\kappa_N, x)]^2 - (V|_I + k_N^2) [f_1^{[N-1]}(i\kappa_N, x)]^2}{[f_r^{[N-1]'}(i\kappa_N, x)]^2 - (V|_I + k_N^2) [f_r^{[N-1]}(i\kappa_N, x)]^2}, \quad x \in I. \quad (3.11)$$

When  $V^{[N-1]}(x) = V^{[N-1]}(2a - x)$  for  $x \in \mathbf{R}$ , we have

$$f_1^{[N-1]}(k, x) = e^{2ika} f_r^{[N-1]}(k, 2a - x), \quad f_1^{[N-1]'}(k, x) = -e^{2ika} f_r^{[N-1]'}(k, 2a - x), \quad x \in \mathbf{R}, \quad (3.12)$$

and hence, in particular

$$f_1^{[N-1]}(k, a) = e^{2ika} f_r^{[N-1]}(k, a), \quad f_1^{[N-1]'}(k, a) = -e^{2ika} f_r^{[N-1]'}(k, a), \quad x \in \mathbf{R}. \quad (3.13)$$

Since the right-hand side of (3.11) is independent of  $x$  on  $I$ , we can evaluate it at  $x = a$  and use (3.13) to see that (i) holds. Next, using  $\alpha_{N;-} = e^{-4\kappa_N a} / \alpha_{N;+}$  and the first equality in (3.12), for  $x \in \mathbf{R}$  we get

$$\begin{aligned} f_1^{[N-1]}(k, 2a - x) + \alpha_{N;-} f_r^{[N-1]}(k, 2a - x) \\ = \alpha_{N;-} e^{-2ika} [f_1^{[N-1]}(k, x) + \alpha_{N;+} e^{4(\kappa_N + ik)a} f_r^{[N-1]}(k, x)]. \end{aligned} \quad (3.14)$$

Using (3.14) in (3.1) and (3.2) with  $j = N$ , we get (ii). ■

In the rest of this section, we will express  $\alpha_{N;\pm}$  in (3.10) in other equivalent forms, which will be useful in the analysis of our inverse problem.

Without any loss of generality, we may choose our interval  $I$  as  $(0, 1)$ , and in the rest of this section we will do so. Let us fragment the potential  $V^{[j]}$  (see below (2.4) and above (3.1)) as  $V^{[j]} = V_1^{[j]} + V_2^{[j]} + V_3^{[j]}$ , where we have defined

$$V_1^{[j]} := V^{[j]}|_{(-\infty, 0)}, \quad V_0^{[j]} := V^{[j]}|_{(0, 1)}, \quad V_2^{[j]} := V^{[j]}|_{(1, +\infty)}, \quad (3.15)$$

and let  $f_{l_s}^{[j]}$  and  $f_{r_s}^{[j]}$  denote the Jost solutions from the left and from the right, respectively, corresponding to the fragment  $V_s^{[j]}$  for  $s = 0, 1, 2$ . Similarly, let us use  $T_s^{[j]}$ ,  $R_s^{[j]}$ , and  $L_s^{[j]}$  for the scattering coefficients corresponding to the fragment  $V_s^{[j]}$ ; in the same way, let  $T^{[j]}$ ,  $R^{[j]}$ , and  $L^{[j]}$  denote the scattering coefficients corresponding to  $V^{[j]}$ . Since we have  $V^{[N]} := V$ , we

will use  $V_0 := V_0^{[N]}$ . In case of the nonuniqueness, as seen from Theorem 3.1,  $V_0$  is a constant strictly greater than  $-\kappa_N^2$ , and  $V_0^{[N-1]} \equiv V_0$ . In that case, we have

$$f_1^{[N-1]}(k, x) = \begin{cases} f_{12}^{[N-1]}(k, x), & x \geq 1, \\ E_+(k) e^{i\Delta x} + E_-(k) e^{-i\Delta x}, & 0 \leq x \leq 1, \\ \frac{f_{r1}^{[N-1]}(-k, x)}{T^{[N-1]}(k)} + \frac{L^{[N-1]}(k) f_{r1}(k, x)}{T^{[N-1]}(k)}, & x \leq 0, \end{cases} \quad (3.16)$$

$$f_r^{[N-1]}(k, x) = \begin{cases} \frac{f_{12}^{[N-1]}(-k, x)}{T^{[N-1]}(k)} + \frac{R^{[N-1]}(k) f_{12}^{[N-1]}(k, x)}{T^{[N-1]}(k)}, & x \geq 1, \\ D_+(k) e^{i\Delta x} + D_-(k) e^{-i\Delta x}, & 0 \leq x \leq 1, \\ f_{r1}^{[N-1]}(k, x), & x \leq 0, \end{cases} \quad (3.17)$$

where we have defined  $\Delta := \sqrt{k^2 - V_0}$ . Using the boundary conditions at  $x = 0$  and  $x = 1$  resulting from the continuity of  $f_1^{[N-1]}$ ,  $f_r^{[N-1]}$ ,  $f_1^{[N-1]}'$ , and  $f_r^{[N-1]}'$ , we get

$$E_+(k) = \frac{e^{-i\Delta}}{2} \left[ f_{12}^{[N-1]}(k, 1) - i f_{12}^{[N-1]'}(k, 1)/\Delta \right] = \frac{1}{2} \left[ f_1^{[N-1]}(k, 0) - i f_1^{[N-1]'}(k, 0)/\Delta \right], \quad (3.18)$$

$$E_-(k) = \frac{e^{i\Delta}}{2} \left[ f_{12}^{[N-1]}(k, 1) + i f_{12}^{[N-1]'}(k, 1)/\Delta \right] = \frac{1}{2} \left[ f_1^{[N-1]}(k, 0) + i f_1^{[N-1]'}(k, 0)/\Delta \right], \quad (3.19)$$

$$D_+(k) = \frac{1}{2} \left[ f_{r1}^{[N-1]}(k, 0) - i f_{r1}^{[N-1]'}(k, 0)/\Delta \right] = \frac{e^{-i\Delta}}{2} \left[ f_r^{[N-1]}(k, 1) - i f_r^{[N-1]'}(k, 1)/\Delta \right], \quad (3.20)$$

$$D_-(k) = \frac{1}{2} \left[ f_{r1}^{[N-1]}(k, 0) + i f_{r1}^{[N-1]'}(k, 0)/\Delta \right] = \frac{e^{i\Delta}}{2} \left[ f_r^{[N-1]}(k, 1) + i f_r^{[N-1]'}(k, 1)/\Delta \right]. \quad (3.21)$$

In this case, using (3.18)-(3.21) we can write  $\alpha_{N;\pm}$  given in (3.10) in various forms such as

$$\alpha_{N;\pm} = -\frac{E_{\pm}(i\kappa_N)}{D_{\pm}(i\kappa_N)}, \quad (3.22)$$

$$\alpha_{N;\pm} = \frac{e^{\pm\sqrt{\kappa_N^2 + V_0}} \left[ f_{12}^{[N-1]}(i\kappa_N, 1) \mp f_{12}^{[N-1]'}(i\kappa_N, 1)/\sqrt{\kappa_N^2 + V_0} \right]}{-f_{r1}^{[N-1]}(i\kappa_N, 0) \pm f_{r1}^{[N-1]'}(i\kappa_N, 0)/\sqrt{\kappa_N^2 + V_0}}, \quad (3.23)$$

$$\alpha_{N;\pm} = \frac{e^{\mp\sqrt{\kappa_N^2 + V_0}} \left[ f_1^{[N-1]}(i\kappa_N, 0) \mp f_1^{[N-1]'}(i\kappa_N, 0)/\sqrt{\kappa_N^2 + V_0} \right]}{-f_r^{[N-1]}(i\kappa_N, 1) \pm f_r^{[N-1]'}(i\kappa_N, 1)/\sqrt{\kappa_N^2 + V_0}}.$$

When (i)-(iv) of Theorem 3.1 are satisfied, from (3.16) and (3.17), for  $x \in [0, 1]$  we get

$$\begin{aligned} f_1^{[N-1]}(i\kappa_N, x) + \alpha_{N;\pm} f_r^{[N-1]}(i\kappa_N, x) &= [E_+(i\kappa_N) + \alpha_{N;\pm} D_+(i\kappa_N)] e^{-\sqrt{\kappa_N^2 + V_0}} \\ &\quad + [E_-(i\kappa_N) + \alpha_{N;\pm} D_-(i\kappa_N)] e^{\sqrt{\kappa_N^2 + V_0}}, \end{aligned}$$

which, with the help of (3.22), simplifies to

$$f_1^{[N-1]}(i\kappa_N, x) + \alpha_{N;\pm} f_r^{[N-1]}(i\kappa_N, x) = [E_{\mp}(i\kappa_N) + \alpha_{N;\pm} D_{\mp}(i\kappa_N)] e^{\pm\sqrt{\kappa_N^2 + V_0}}, \quad x \in [0, 1].$$

Thus, in (3.4) we get

$$\frac{f_1^{[N-1]'}(i\kappa_N, x) + \alpha_{N;\pm} f_r^{[N-1]'}(i\kappa_N, x)}{f_1^{[N-1]}(i\kappa_N, x) + \alpha_{N;\pm} f_r^{[N-1]}(i\kappa_N, x)} = \pm\sqrt{\kappa_N^2 + V_0},$$

and we see why the two distinct  $\alpha_{N;\pm}$  lead to the same potential on  $[0, 1]$ .

At the end of the proof of Theorem 3.1 we have indicated that  $\alpha_{N;+} \neq \alpha_{N;-}$  when (i)-(iii) in Theorem 3.1 are satisfied. Next we elaborate on this point by using the representations of  $\alpha_{N;\pm}$  given in (3.22).

**PROPOSITION 3.3.** *Assume  $V$  is real valued, belongs to  $L_1^1(\mathbf{R})$ , and has  $N$  bound states at  $k = i\kappa_j$  with  $0 < \kappa_1 < \dots < \kappa_N$ . Consider the scattering data consisting of one reflection coefficient, knowledge of  $V|_{(0,1)}$ , all the  $N$  bound-state energies, and all the  $N$  norming constants except for the one corresponding to the  $N$ th bound state. Moreover, suppose that  $V|_{(0,1)}$  and  $V^{[N-1]}|_{(0,1)}$  are both equal to the same constant  $V_0$  that is greater than  $-\kappa_N^2$ . Then, we have the following.*

- (i) *The quantity  $E_+(i\kappa_N) D_-(i\kappa_N) - E_-(i\kappa_N) D_+(i\kappa_N)$  is nonzero, and hence  $D_+(i\kappa_N)$  and  $D_-(i\kappa_N)$  cannot simultaneously be zero.*
- (ii) *If either of  $D_+(i\kappa_N)$  and  $D_-(i\kappa_N)$  is zero, then the scattering data uniquely determines  $V$  on  $\mathbf{R}$ .*
- (iii) *If both  $D_+(i\kappa_N)$  and  $D_-(i\kappa_N)$  are nonzero, then  $\alpha_{N;+}$  and  $\alpha_{N;-}$  given in (3.22) are necessarily distinct.*

PROOF: Using (3.18)-(3.21), we see that

$$\begin{aligned} E_+(k) D_-(k) - E_-(k) D_+(k) \\ = \frac{i}{2\sqrt{k^2 - V_0}} [f_1^{[N-1]}(k, 0) f_{r1}^{[N-1]'}(k, 0) - f_1^{[N-1]'}(k, 0) f_{r1}^{[N-1]}(k, 0)]. \end{aligned} \quad (3.24)$$

Note that the quantity in the brackets on the right-hand side in (3.24) is nothing but the Wronskian  $[f_1(k, x); f_{r1}(k, x)]$  at  $x = 0$ . This Wronskian is independent of  $x$  when  $x \in \mathbf{R}^-$  because both  $f_1(k, x)$  and  $f_{r1}(k, x)$  are solutions of (1.1) for  $x \in \mathbf{R}^-$ . In fact, by using

$$f_{r1}^{[N-1]}(k, 0) = f_r^{[N-1]}(k, 0), \quad f_{r1}^{[N-1]'}(k, 0) = f_r^{[N-1]'}(k, 0),$$

$$\frac{2ik}{T^{[N-1]}(k)} = [f_r^{[N-1]}(k, x); f_1^{[N-1]}(k, x)],$$

we can write (3.24) in the form

$$E_+(k) D_-(k) - E_-(k) D_+(k) = \frac{k}{T^{[N-1]}(k) \sqrt{k^2 - V_0}},$$

which implies that

$$E_+(i\kappa_N) D_-(i\kappa_N) - E_-(i\kappa_N) D_+(i\kappa_N) = \frac{\kappa_N}{T^{[N-1]}(i\kappa_N) \sqrt{\kappa_N^2 + V_0}}.$$

By using an argument similar to that given below (1.6) we get  $T^{[N-1]}(i\kappa_N) > 0$  and hence (i) holds. From (i) it follows that  $E_+(i\kappa_N)$  and  $D_+(i\kappa_N)$  cannot simultaneously vanish; similarly,  $E_-(i\kappa_N)$  and  $D_-(i\kappa_N)$  cannot simultaneously vanish. Hence,  $|\alpha_{N;\pm}| = +\infty$  if and only if  $D_{\pm}(i\kappa_N) = 0$ . However, there cannot be a nonuniqueness unless  $\alpha_{N;\pm}$  are both finite and positive, and thus, we have proved (ii). Finally, from (3.22) we see that  $\alpha_{N;+} = \alpha_{N;-}$  would be possible either when  $E_+(i\kappa_N) D_-(i\kappa_N) - E_-(i\kappa_N) D_+(i\kappa_N) = 0$  or  $D_-(i\kappa_N) = D_+(i\kappa_N) = 0$ , but this is not possible because of (i). ■

Next, when the assumptions in Proposition 3.3 are satisfied, we will write the numerators and the denominators in (3.23) by using the scattering coefficients of  $V_1^{[N-1]}$  and  $V_2^{[N-1]}$ , and this will enable us to study the signs of  $\alpha_{N;\pm}$  when those scattering coefficients are known. Then, in view of Theorem 3.1(iv), we already know that our scattering data uniquely determines  $V$  unless  $\alpha_{N;\pm}$  are both finite and positive.

Since  $V_1^{[N-1]}$  is supported in  $\mathbf{R}^-$ ,  $R_1^{[N-1]}$  has (see e.g. [7]-[13]) a meromorphic extension from  $k \in \mathbf{R}$  to  $k \in \mathbf{C}^+$  having its poles exactly coinciding with the poles of  $T_1^{[N-1]}$  there. Similarly, since  $V_2^{[N-1]}$  is supported in  $(1, +\infty)$ ,  $L_2^{[N-1]}$  has a meromorphic extension to  $\mathbf{C}^+$  having its poles coinciding with the poles of  $T_2^{[N-1]}$  there. Thus, we have

$$f_{r1}^{[N-1]}(i\kappa_N, 0) = \frac{1 + R_1^{[N-1]}(i\kappa_N)}{T_1^{[N-1]}(i\kappa_N)}, \quad f_{r1}^{[N-1]'}(i\kappa_N, 0) = \kappa_N \frac{1 - R_1^{[N-1]}(i\kappa_N)}{T_1^{[N-1]}(i\kappa_N)},$$

$$f_{12}^{[N-1]}(i\kappa_N, 1) = \frac{e^{-\kappa_N} + L_2^{[N-1]}(i\kappa_N) e^{\kappa_N}}{T_2^{[N-1]}(i\kappa_N)}, \quad f_{12}^{[N-1]'}(i\kappa_N, 1) = -\kappa_N \frac{e^{-\kappa_N} - L_2^{[N-1]}(i\kappa_N) e^{\kappa_N}}{T_2^{[N-1]}(i\kappa_N)}.$$

Let us define

$$\epsilon_{\pm} := \sqrt{\kappa_N^2 + V_0} \pm \kappa_N, \quad (3.25)$$

$$q_{1;\pm} := \epsilon_{\pm} R_1^{[N-1]}(i\kappa_N) + \epsilon_{\mp}, \quad q_{2;\pm} := \epsilon_{\mp} L_2^{[N-1]}(i\kappa_N) + \epsilon_{\pm} e^{-2\kappa_N}. \quad (3.26)$$

Then, from (3.23) we get

$$\alpha_{N;\pm} = -e^{\pm\epsilon_{\pm}} \frac{T_1^{[N-1]}(i\kappa_N)}{T_2^{[N-1]}(i\kappa_N)} \frac{q_{2;\pm}}{q_{1;\pm}}. \quad (3.27)$$

## 4 Examples

In this section we illustrate the uniqueness and nonuniqueness stated in Theorem 3.1 with some examples. Without any loss of generality we use the interval  $(0, 1)$  as  $I$ . In a typical example we use the fragmentation given in (3.15). As in (i)-(iii) of Theorem 3.1, we assume that  $V_0$  and  $V_0^{[N-1]}$  are equal to each other and equal to a constant greater than  $-\kappa_N^2$ . Since the scattering data specified in Theorem 3.1 is equivalent to  $\{V_0, N, V_1^{[N-1]}, V_2^{[N-1]}, \kappa_N\}$ , which is also equivalent to  $\{V_0, N, R_1^{[N-1]}, L_2^{[N-1]}, \kappa_N\}$ , we use either of these two sets as our scattering data in our examples. We then use (3.27) to determine the signs of  $\alpha_{N;\pm}$ . As in (iv) of Theorem 3.1, the nonuniqueness occurs when both  $\alpha_{N;\pm}$  are finite and positive. Otherwise, our scattering data uniquely determines  $V$ .

For the sake of exact analytical calculations, our  $V_1^{[N-1]}$  and  $V_2^{[N-1]}$  consist of constant steps or pieces of Bargmann potentials (recall [5] that a potential is called a Bargmann potential if the corresponding scattering matrix is a rational function of  $k$ ). Such choices allow us to explicitly and uniquely evaluate  $V_1^{[N-1]}$  and  $T_1^{[N-1]}$  when  $R_1^{[N-1]}$  is known, and conversely evaluate  $R_1^{[N-1]}$  and  $T_1^{[N-1]}$  when  $V_1^{[N-1]}$  is known. Similarly, we are able to explicitly and uniquely evaluate  $V_2^{[N-1]}$  and  $T_2^{[N-1]}$  when  $L_2^{[N-1]}$  is known, and evaluate  $L_2^{[N-1]}$  and  $T_2^{[N-1]}$  when  $V_2^{[N-1]}$  is known. In our examples, we choose  $R_1^{[N-1]}$  and  $L_2^{[N-1]}$  in such a way that the existence in  $L_1^1$  of  $V_1^{[N-1]}$  and  $V_2^{[N-1]}$ , respectively, is assured by the characterization results [1]-[6] or by the results in Section XVII.3.2 of [5].

When  $V_0$  is supported on  $(0, 1)$  and equal to a constant, the corresponding transmission coefficient  $T_0$  and the left reflection coefficient  $L_0$  are obtained by using (1.2), (1.3), (3.16),

and (3.17), and we get

$$\frac{1}{T_0(k)} = e^{ik} \left[ \cos \left( \sqrt{k^2 - V_0} \right) + \frac{2k^2 - V_0}{2ik\sqrt{k^2 - V_0}} \sin \left( \sqrt{k^2 - V_0} \right) \right], \quad (4.1)$$

$$\frac{L_0(k)}{T_0(k)} = \frac{V_0 e^{ik}}{2ik\sqrt{k^2 - V_0}} \sin \left( \sqrt{k^2 - V_0} \right). \quad (4.2)$$

When  $V_0 \geq 0$  we have  $N_0 = 0$ , where  $N_0$  denotes the number of bound states of  $V_0$ . When  $V_0 < 0$  we have  $N_0 \geq 1$ , and recall [14] that  $N_0$  is evaluated as the unique integer satisfying

$$(N_0 - 1)\pi < \sqrt{-V_0} \leq N_0\pi. \quad (4.3)$$

Recall also that under the shift  $V(x) \mapsto \tilde{V}(x) := V(x - b)$ , the corresponding scattering coefficients undergo the changes  $\tilde{T}(k) = T(k)$ ,  $\tilde{L}(k) = e^{2ikb}L(k)$ , and  $\tilde{R}(k) = e^{-2ikb}R(k)$ . This observation will help us to evaluate  $R_1^{[N-1]}$  and  $L_2^{[N-1]}$  in some of our examples when  $V_1^{[N-1]}$  and  $V_2^{[N-1]}$  are square-well potentials.

In order to know  $N$  in our examples, we count the number of zeros of  $1/T^{[N-1]}(i\beta)$  for  $\beta \in (0, +\infty)$ , which is equal to  $N - 1$ . For this purpose, we use the following formula, see e.g. [15], in order to express  $T^{[N-1]}(k)$  in terms of the appropriate scattering coefficients for the pieces  $V_1^{[N-1]}$ ,  $V_0$ , and  $V_2^{[N-1]}$ :

$$\begin{aligned} \frac{1}{T^{[N-1]}(k)} &= \frac{1}{T_1^{[N-1]}(k)} \frac{1}{T_0(k)} \frac{1}{T_2^{[N-1]}(k)} - \frac{R_1^{[N-1]}(k) L_0(k)}{T_1^{[N-1]}(k) T_0(k)} \frac{1}{T_2^{[N-1]}(k)} \\ &+ \frac{1}{T_1^{[N-1]}(k)} \frac{L_0(-k)}{T_0(-k)} \frac{L_2^{[N-1]}(k)}{T_2^{[N-1]}(k)} - \frac{R_1^{[N-1]}(k)}{T_1^{[N-1]}(k)} \frac{1}{T_0(-k)} \frac{L_2^{[N-1]}(k)}{T_2^{[N-1]}(k)}. \end{aligned} \quad (4.4)$$

Recall that we know the set  $\{T_1^{[N-1]}, R_1^{[N-1]}, L_0, T_0, L_2^{[N-1]}, T_2^{[N-1]}\}$  when  $\{V_1^{[N-1]}, V_0, V_2^{[N-1]}\}$  is known.

For our examples, it is also helpful to remember, see e.g. [16], that if one fragment has  $n_1$  bound states and the other has  $n_2$  bound states, then the combination of these two fragments has either  $n_1 + n_2$  or  $n_1 + n_2 - 1$  bound states. This helps us to estimate the number of bound states of the whole potential in terms of the number of bound states of its fragments. In particular, if none of the fragments have any bound states, then the total potential does not have any bound states either.

**EXAMPLE 4.1.** Note that in the special case  $V_0 \equiv 0$ , from (3.25) we get  $\epsilon_+ = 2\kappa_N$  and  $\epsilon_- = 0$ , and hence (3.27) leads to

$$\alpha_{N;+} = -\frac{T_1^{[N-1]}(i\kappa_N)}{R_1^{[N-1]}(i\kappa_N)} \frac{1}{T_2^{[N-1]}(i\kappa_N)}, \quad \alpha_{N;-} = -T_1^{[N-1]}(i\kappa_N) \frac{L_2^{[N-1]}(i\kappa_N)}{T_2^{[N-1]}(i\kappa_N)}. \quad (4.5)$$



Let us choose

$$V_1^{[N-1]}(x) = \begin{cases} v_1, & x \in (-1, 0), \\ 0, & x \notin (-1, 0), \end{cases} \quad V_2^{[N-1]}(x) = \begin{cases} v_2, & x \in (1, 2), \\ 0, & x \notin (1, 2), \end{cases}$$

where  $v_1$  and  $v_2$  are adjustable constant parameters. When  $v_1 > 0$  and  $v_2 > 0$ , we have  $N = 1$ , and with the help of (4.1), (4.2), and the observation below (4.3), for any  $\kappa_1 > 0$  we see from (4.5) that  $\alpha_{1;\pm} > 0$  and hence the nonuniqueness follows. When at least one of  $v_1$  and  $v_2$  is zero, we see from (4.5) that one of  $\alpha_{N;\pm}$  is either zero or infinite, leading to the uniqueness. On the other hand, when at least one of  $v_1$  and  $v_2$  is negative, we mostly get the uniqueness.

**EXAMPLE 4.2.** In the special case  $V_1^{[N-1]} \equiv 0$  and  $V_2^{[N-1]} \equiv 0$ , we have  $R_1^{[N-1]} = L_2^{[N-1]} \equiv 0$  and  $T_1^{[N-1]} = T_2^{[N-1]} \equiv 1$ ; hence, from (3.25)-(3.27) we see that  $\alpha_{N;\pm} = -\epsilon_{\pm} e^{\pm\epsilon_{\mp}} / \epsilon_{\mp}$ . Thus,  $\alpha_{N;\pm} > 0$  if and only if  $-\kappa_N^2 < V_0 < 0$ . As indicated in Proposition 3.2(ii), the two potentials corresponding to our data satisfy the symmetry  $V(x; \alpha_{N;+}) = V(1-x; \alpha_{N;-})$  for  $x \in \mathbf{R}$ . For example, with  $V_0 = -1$ , with the help of (4.3) we get  $N = 2$ . Using (4.1), we obtain  $\kappa_1 = 0.43513\bar{1}$ , where we use the overline on the last digit to indicate roundoff. Then, as long as  $\kappa_2 > 1$ , there are two distinct potentials corresponding to our data. With  $\kappa_2 = \sqrt{2}$  so that  $\alpha_{2;\pm} = (\sqrt{2} \pm 1)^2 e^{-\sqrt{2} \pm 1}$ , the graphs of the two resulting potentials are given in Figures 1 and 2, from which the symmetry mentioned above is observed.

**EXAMPLE 4.3.** Let

$$R_1^{[N-1]}(k) = e^{-2ik} L_2^{[N-1]}(k) = \frac{2}{(k+i)(k+2i)}, \quad T_1^{[N-1]}(k) = T_2^{[N-1]}(k) = \frac{k(k+\sqrt{5}i)}{(k+i)(k+2i)}, \quad (4.6)$$

so that  $V_1^{[N-1]}$  is supported on  $(-\infty, 0)$  and  $V_2^{[N-1]}$  on  $(1, +\infty)$ . Neither  $V_1^{[N-1]}$  nor  $V_2^{[N-1]}$  has any bound states, and  $V_1^{[N-1]}(x) = V_2^{[N-1]}(1-x)$  for  $x \in \mathbf{R}$ . Thus, when  $V_0$  is constant, the potential  $V^{[N-1]}$  has even symmetry with respect to the point  $x = 1/2$ . Due to that symmetry, when there is a nonuniqueness, Proposition 3.2 applies and we have  $\alpha_{N;+} = e^{-2\kappa_N} / \alpha_{N;-}$ , and  $V(x; \alpha_{N;+}) = V(1-x; \alpha_{N;-})$  for  $x \in \mathbf{R}$ . Using (4.6) in (3.27), we get

$$\alpha_{N;\pm} = e^{\pm\epsilon_{\mp}} \frac{V_0(\kappa_N + 1)(\kappa_N + 2) - 2(\sqrt{\kappa_N^2 + V_0} \mp \kappa_N)^2}{2V_0 - (\kappa_N + 1)(\kappa_N + 2)(\sqrt{\kappa_N^2 + V_0} \mp \kappa_N)^2}, \quad (4.7)$$

where  $\epsilon_{\pm}$  are the constants defined in (3.25). With the help of (4.4), we can get  $N - 1$  by counting the number of zeros of  $1/T^{[N-1]}$  on the positive imaginary axis. We find that

$N = 1$  when  $V_0 \geq -2.16681\bar{5}$ ,  $N = 2$  when  $-15.54\bar{8} \leq V_0 < -2.16681\bar{5}$ ,  $N = 3$  when  $-44.5877\bar{5} \leq V_0 < -15.54\bar{8}$ , etc. As indicated in Theorem 3.1(iii), for the nonuniqueness we need  $V_0 + \kappa_N^2 > 0$ . From (4.7) we see that  $\alpha_{N;\pm} > 0$  when  $V_0 < 0$ . When  $V_0 = 0$ , we see that  $\alpha_{1;-} = 0$  while  $\alpha_{1;+}$  is not finite, and hence we have the uniqueness. When  $V_0 > 0$ , we may or may not have the nonuniqueness depending on whether  $\alpha_{1;\pm} > 0$  or not. For example, for  $V_0 = 1$  we get the nonuniqueness for any  $\kappa_1 > 0$ . When  $V_0 = 5$ , we get  $\alpha_{1;\pm} < 0$  for  $\kappa_1 \in (0, \beta_0)$  and  $\alpha_{1;\pm} > 0$  for  $\kappa_1 > \beta_0$ , where  $\beta_0 = 4.2659\bar{7}$ ; for  $\kappa_1 = \beta_0$  we get  $\alpha_{1;-} = 0$  while  $\alpha_{1;+}$  is not finite. On the other hand, for  $V_0 = 10$ , we get  $\alpha_{\pm} < 0$  for any  $\kappa_1 > 0$ . When  $V_0 = 1$  and  $\kappa_1 = 1$ , we show the potentials  $V^{[0]}(x)$ ,  $V(x; \alpha_{1;+})$ , and  $V(x; \alpha_{1;-})$  in Figures 3, 4, 5, respectively.

**EXAMPLE 4.4.** Let  $V_2^{[N-1]} \equiv 0$ ,  $V_0 = (t^2 - 1) \kappa_N^2$ , where  $t$  is a positive parameter and  $N$  and  $\kappa_N$  are to be specified. Note that (iii) of Theorem 3.1 is assured because  $t > 0$ . Let us choose

$$R_1^{[N-1]}(k) = \frac{-2(k - it\kappa_N)}{(k + i)(k + 2i)(k + it\kappa_N)}, \quad T_1^{[N-1]}(k) = \frac{k(k + \sqrt{5}i)}{(k + i)(k + 2i)},$$

so that  $V_1^{[N-1]}$  is supported on  $\mathbf{R}^-$  and has no bound states. Thus,  $T_1^{[N-1]}(i\kappa_N) > 0$ . Using (3.25)-(3.27) we get

$$\alpha_{N;+} = \frac{(\kappa_N + \sqrt{5})e^{(\sqrt{1+t}-1)\kappa_N}}{(1-t)(\kappa_N + 3)}, \quad \alpha_{N;-} = \frac{(1-t)(1+t)\kappa_N(\kappa_N + \sqrt{5})e^{-(\sqrt{1+t}+1)\kappa_N}}{(1+t)^2(\kappa_N + 1)(\kappa_N + 2) - 2(1-t)^2}, \quad (4.8)$$

and hence  $\alpha_{N;\pm} > 0$  only if  $t < 1$ . Therefore, in order to have the nonuniqueness, we must impose the restriction  $t \in (0, 1)$ . By analyzing the zeros (on the positive imaginary axis) of  $1/T_1^{[N-1]}$  given in (4.4), we are able to get  $N$  for any specific value of  $\kappa_N$  in our data. For example, if  $\kappa_N = 1$ , we find that  $N = 1$  for  $t \in [t_1, 1)$  and  $N = 2$  for  $t \in (0, t_1)$ , where  $t_1 = 0.7639\bar{8}$ . For the particular values  $\kappa_2 = 1$  and  $t = 0.5$ , from (4.8) we obtain  $\alpha_{2;+} = 2.0257\bar{8}$  and  $\alpha_{2;-} = 0.020180\bar{9}$ .

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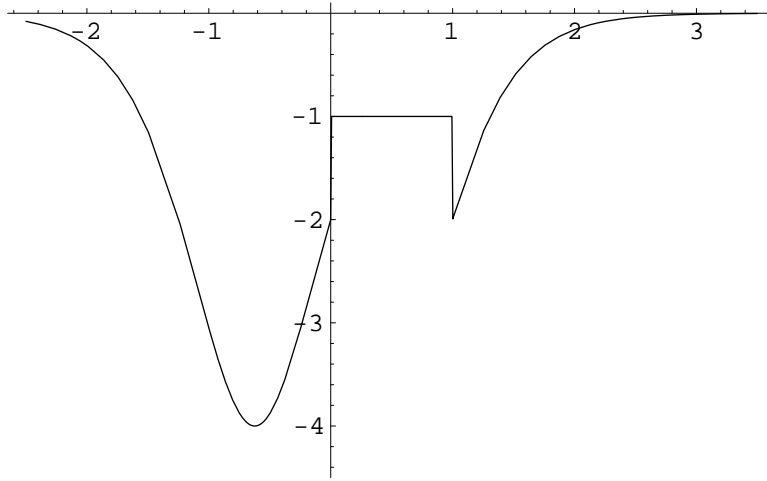


Figure 1: The potential  $V(x; \alpha_{2,+})$  in Example 4.2 with  $\kappa_2 = \sqrt{2}$  and  $\alpha_{2,+} = (\sqrt{2} + 1)^2 e^{-\sqrt{2}+1}$

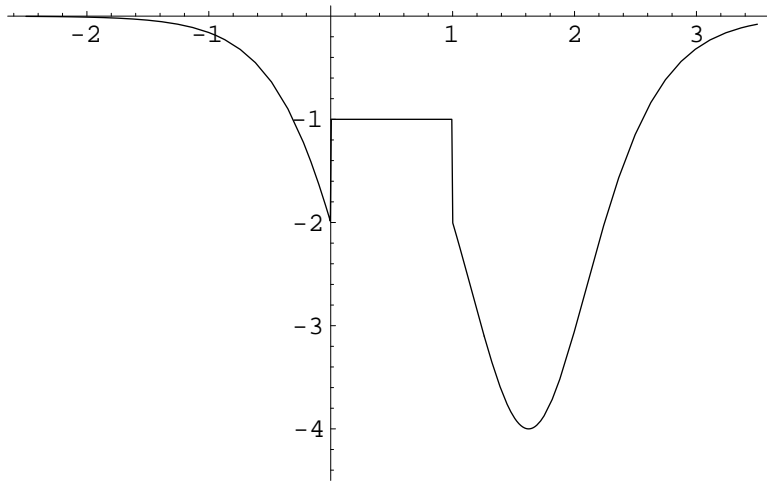


Figure 2: The potential  $V(x; \alpha_{2,-})$  in Example 4.2 with  $\kappa_2 = \sqrt{2}$  and  $\alpha_{2,-} = (\sqrt{2} - 1)^2 e^{-\sqrt{2}-1}$

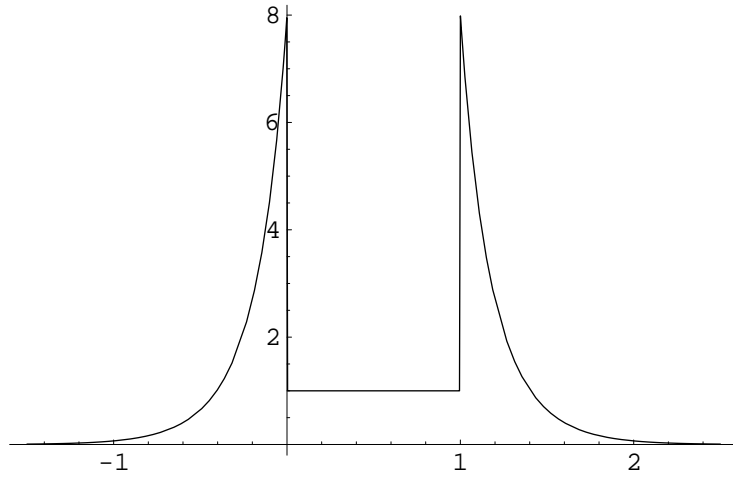


Figure 3: The potential  $V^{[0]}(x)$  in Example 4.3 with  $V_0 = 1$ .

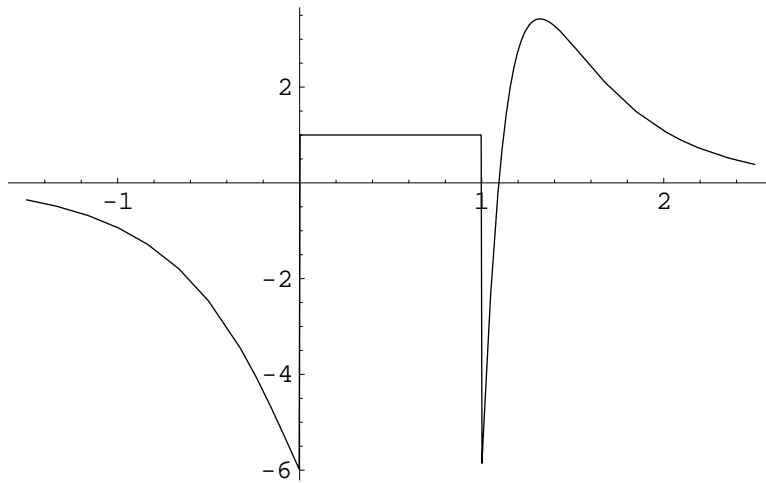


Figure 4: The potential  $V(x; \alpha_{1,+})$  in Example 4.3 with  $V_0 = 1$ ,  $\kappa_1 = 1$ , and  $\alpha_{1,+} = 8.8194\bar{6}$ .

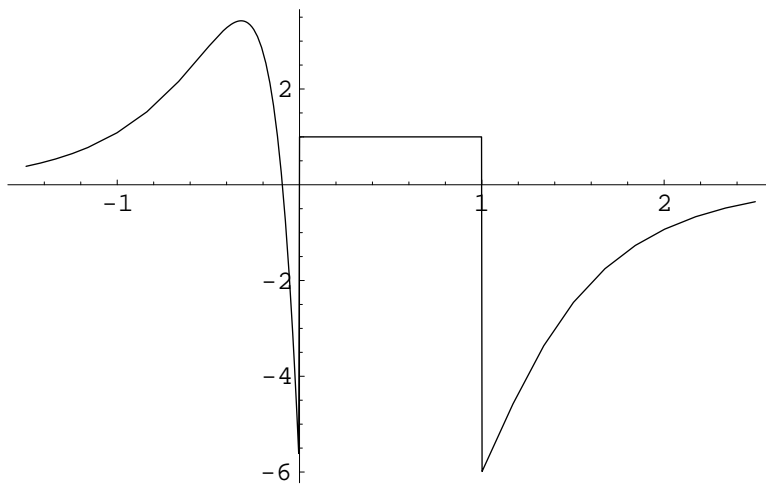


Figure 5: The potential  $V(x; \alpha_{1;-})$  in Example 4.3 with  $V_0 = 1$ ,  $\kappa_1 = 1$ , and  $\alpha_{1;-} = 0.015345\bar{1}$ .