# On the mathematical controllability in a simple growth tumors model by the internal localized action of inhibitors

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**Abstract**

We study a model of growth of tumors with a free boundary, delaying the tumor region. We take into account the presence of inhibitors and its interaction with nutrients. We study the approximate controllability of the internal distribution of density of cells, that is proportional to concentration of nutrients, injecting inhibitor in a small inner region $\omega_0$.

*Keywords:* Free boundary problem, Controllability, growth of tumors

# 1 The Model

In this paper we study the controllability of the growth of tumors by the internal localized action of inhibitors on a simplified mathematical model. The tumor, formed by life cells, is assumed to have a density proportional to the concentrations of nutrients $\widehat{\sigma}(x,t)$, $x=(x_1,x_2,x_3)$, mainly oxygen or glucose. We study the behavior of the tumor after *angiogenesis*, the formation of capillary sprouts from blood vessels, in response to a chemical stimuli produced by the tumor. Once the angiogenesis occurs, the tumor receives nutrient from the vessels (process named *vasculature*). We assume the tumor is a radially symmetric ball of $I\!R^3$ of radius $R(t)$, which is unknown (reason why is usually denoted as the free boundary of the problem). Denoting by $\sigma_B$ the constant nutrient concentration in the vasculature, $\widehat{r}_1$ the rate, per unit length, of nutrient transferred to the tissue, $\widehat{\sigma}$ satisfies the equation

$$\frac{\partial \widehat{\sigma}}{\partial t} - d_1\Delta\widehat{\sigma} - \widehat{r}_1(\sigma_B - \widehat{\sigma}) + \lambda_1\widehat{\sigma} + \lambda\widehat{\beta} = 0, \qquad |x| < R(t),\ t \in (0,T).$$

Here $d_1$ is the diffusion coefficient of the nutrient concentration and $\lambda_1\widehat{\sigma}$, $\lambda\widehat{\beta}$ represent the consume rate of nutrient and inhibitor, respectively.

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The density of the inhibitor $\widehat{\beta}(x,t)$ is assumed to satisfy a similar reaction - diffusion equation,

$$\frac{\partial \widehat{\beta}}{\partial t} - d_2 \Delta \widehat{\beta} - \widehat{r}_2(\beta_B - \widehat{\beta}) + \lambda_2 \widehat{\beta} = f\chi_{\omega_0}, \qquad |x| < R(t),\ t \in (0,T),$$

where $d_2$ is the diffusion coefficient, $\beta_B$ is the critical value of the inhibitor concentration for vasculature, $\widehat{r}_2$ the rate, per unit length, of inhibitor transferred to the tissue, and $\lambda_2 \beta$ is the inhibitor consumption rate. The permanent supply of inhibitors is assumed to be localized on a small domain $\omega_0$ with a rate given by $f$ (the control of the problem).

According the mass conservation principle, assuming the cell mass density constant, the tumor mass is proportional to the volume $\frac{4}{3}\pi R(t)^3$. The balance between birth and death cells is determinate by the concentration of nutrient and inhibitor. Denoting by $\widehat{S}$ the above balance, after normalizing we obtain the law

$$\frac{d}{dt}\left(\frac{4}{3}\pi R^3(t)\right) = \int_{\{|x|<R(t)\}} \widehat{S}(\widehat{\sigma}(x,t), \widehat{\beta}(x,t))dx, \qquad x \in I\!R^3.$$

According the inhibitor nature and the tumor tissue, the function $\widehat{S}$ has different representations. In any case we shall assume trough the paper that, $\widehat{S} \in W^{1,\infty}(I\!R^2)$.

For the sake of notation we shall assume that the diffusion coefficients are given by a unique positive constants, $d_1 = d_2 = d$. Thus by normalizing the unknown densities

$$\sigma := \widehat{\sigma} - \frac{\widehat{r}_1 \sigma_B(\widehat{r}_2 + \lambda_2) + \lambda \widehat{r}_2 \beta_B}{(\widehat{r}_1 + \lambda_1)(\widehat{r}_2 + \lambda_2)}, \qquad \beta := \widehat{\beta} - \frac{\widehat{r}_2 \beta_B}{\widehat{r}_2 + \lambda_2},$$

and denoting by

$$r_1 := \widehat{r}_1 + \lambda_1, \qquad r_2 := \widehat{r}_2 + \lambda_2, \qquad S(\sigma,\beta) := \frac{3}{4\pi}\widehat{S}(\widehat{\sigma}, \widehat{\beta}),$$

we arrive to the mathematical formulation of the model under consideration

$$\frac{\partial \sigma}{\partial t} - d\Delta\sigma + r_1\sigma + \lambda\beta = 0, \qquad |x| < R(t),\ t \in (0,T), \tag{1.1}$$

$$\frac{\partial \beta}{\partial t} - d\Delta\beta + r_2\beta = f\chi_{\omega_0}, \qquad |x| < R(t),\ t \in (0,T), \tag{1.2}$$

$$R(t)^2 \frac{dR(t)}{dt} = \int_{|x|<R(t)} S(\sigma,\beta)dx, \qquad R(0) = R_0,\ t \in (0,T), \tag{1.3}$$

$$\sigma(x,0) = \sigma_0(x), \qquad \beta(x,0) = \beta_0(x), \qquad |x| < R_0, \tag{1.4}$$

$$\sigma(x,t) = \overline{\overline{\sigma}},\ \beta(x,t) = \overline{\overline{\beta}}, \qquad |x| = R(t),\ t \in (0,T), \tag{1.5}$$

where $R_0 > 0$, the normalized nutrient and inhibitor densities at the exterior of the tumor $\overline{\overline{\sigma}}$, $\overline{\overline{\beta}}$ and the initial densities $(\sigma_0, \beta_0)$ are known. We shall assume that $(\sigma_0, \beta_0) \in W^{2,\infty}(B(R_0))$. Similar models have been studied by different authors (see Greenspan [12] , Byrne – Chaplain [3], Cui – Friedman [5] [6], Friedman – Reitich [11]). The main difference with previous models is the term $f\chi_{\omega_0}$, which models the supply of inhibitors localized on a small domain $\omega_0$, introduced in this work (see also Díaz – Tello [9]). The main results of this paper shows that this type of action by the inhibitor allows to control (in the usual weak sense typical of parabolic system) the tumor's cell density. This is formulated in the following terms:

**Theorem 1.1** *Given $T > 0$, $\omega_0 \subset B(R_0 exp\{-\|S\|_{L^\infty} T\})$, $\epsilon > 0$, and $\hat{\sigma}^d \in L^p_{loc}(I\!R^3)$, for some $p > 1$, there exists $f \in L^p((0,T) \times \omega_0)$ such that, if $(\sigma, \beta, R)$ is the solution to the problem (1.1)-(1.5), then*

$$\|\sigma(T) - \sigma^d\|_{L^p(B(R(T)))} \leq \epsilon, \tag{1.6}$$

*where $\sigma^d := \hat{\sigma}^d \chi_{B(R(T))}$.*

Due to some technical reasons, we shall prove the theorem firstly for $p \geq 5$, (necessary in the proof of Lemma 2.1) and then for all $p > 1$.

We shall prove the result in several steps. For $n \in I\!N$, we start by assuming $R_n(t)$ prescribed and look for a control $f_n$ in $\omega_0$ such that the solution $(\sigma_n, \beta_n)$ to problem (1.1)-(1.5), satisfies (1.6). Then we obtain $R_{n+1}$ and $f_{n+1}$ from $(\sigma_n, \beta_n)$ which allow to find $(\sigma_{n+1}, \beta_{n+1})$. The proof of the theorem uses some methods introduced in the study of the approximate controllability (name attributed to conclusions as (1.6)) by different authors (see Lions [14] [15], Puel – Fabre – Zuazua [10], and Díaz – Ramos [7] [8]). Iterating the process we obtain a sequence $(R_n, f_n, \sigma_n, \beta_n)$, we show that there exists a subsequence such that converges to the searched control $f$ and the associate solution to (1.1)-(1.5).

## 2 Regularity and uniqueness of problem (1.1)-(1.5)

Although the existence of weak solutions to problem (1.1)-(1.5), was established by previous authors, (see Díaz – Tello [9]), we shall need some extra information which is collected in this section. Uniqueness of solutions is proved in a similar way that in Friedman – Cui [5].

In order to prove the regularity of the solutions we use the change of variables and unknowns,

$$\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \frac{x}{R(t)}, \qquad \tilde{t}(t) := \int_0^t R^{-2}(\rho) d\rho, \tag{2.1}$$

$$u(\tilde{x}, \tilde{t}) := \sigma(R(t(\tilde{t}))\tilde{x}, t(\tilde{t})) - \overline{\overline{\sigma}}, \qquad v(\tilde{x}, \tilde{t}) := \beta(R(t(\tilde{t}))\tilde{x}, t(\tilde{t})) - \overline{\overline{\beta}}. \tag{2.2}$$

(Notice that, since $R$ is a continuous function and $\frac{1}{R^2(t)} > 0$, we obtain $\tilde{t}(t) \in C^1([0, \tilde{T}])$ and by the theorem of implicit function, there exists the inverse function $t(\tilde{t}) \in C^1([0, T]))$.

Let $B = \{\tilde{x} \in I\!R^3, \quad |\tilde{x}| < 1\}$. Problem (1.1)-(1.5) can be equivalently formulated as

$$\frac{\partial u}{\partial \tilde{t}} - d\Delta u - R^2 \dot{R} \tilde{x} \cdot \nabla u + R^2 r_1 u = R^2 (r_1 \overline{\overline{\sigma}} + \lambda(v + \overline{\overline{\beta}})), \quad \tilde{x} \in B, \ \tilde{t} \in (0, \tilde{T}), \tag{2.3}$$

$$\frac{\partial v}{\partial \tilde{t}} - d\Delta v - R^2 \dot{R} \tilde{x} \cdot \nabla v + R^2 r_2 v = R^2 f \chi_{\tilde{\omega}_0} - R^2 r_2 \overline{\overline{\beta}}, \quad \tilde{x} \in B, \ \tilde{t} \in (0, \tilde{T}), \tag{2.4}$$

$$R(\tilde{t}) \frac{d}{d\tilde{t}} R(\tilde{t}) = \int_B S(u(\tilde{x}, \tilde{t}) + \overline{\overline{\sigma}}, v(\tilde{x}, \tilde{t}) + \overline{\overline{\beta}}) d\tilde{x}, \qquad R(0) = R_0, \tag{2.5}$$

$$u(\tilde{x}, \tilde{t}) = v(\tilde{x}, \tilde{t}) = 0, \qquad \tilde{x} \in \partial B, \ \tilde{t} \in (0, \tilde{T}), \tag{2.6}$$

$$u(\tilde{x},0)=u_0(\tilde{x})=\sigma_0(\tilde{x}R_0), \quad v(\tilde{x},0)=v_0(\tilde{x})=\beta_0(\tilde{x}R_0), \tag{2.7}$$

where $\tilde{T}=\tilde{t}(T)$ and $\tilde{\omega}_0^{\tilde{t}}=\{\tilde{x}\in B \text{ such that } R(t(\tilde{t}))\tilde{x}\in\omega_0\}$, for any $\tilde{t}\in[0,\tilde{T}]$.

**Lemma 2.1** *Under the assumptions of Theorem 1.1, for $p\geq 5$, the solution $(u,v,\ R)$ to (2.3)-(2.7) satisfies:*

$$u\in L^q(0,\tilde{T}:W^{2,q}(B))\cap W^{1,q}(0,\tilde{T}:L^q(B)),$$

*for all $1<q<\infty$ and*

$$v\in L^p(0,\tilde{T}:W^{2,p}(B))\cap W^{1,p}(0,\tilde{T}:L^p(B)).$$

**Proof.** By Díaz – Tello [9] (Theorem 1) we know that

$$(u,v,R)\in[L^2(0,\tilde{T}:H^1(B))]^2\times W^{1,\infty}(0,\tilde{T}).$$

Since $v_0\in H^2(B)$, $f\in L^p((0,T)\times B)$, we get

$$v\in W^{1,p}((0,\tilde{T})\times B)\cap L^p(0,\tilde{T}:W^{2,p}(B)),$$

(see e.g. Ladyzenkaya – Solonnikov – Uralceva [13], Theorem 9.1, Chap IV). Since $p>4$, $W^{1,p}((0,T)\times B)\subset L^\infty([0,\tilde{T}]\times B)$, and then

$$u\in W^{1,q}((0,T)\times B)\cap L^q(0,T:W^{2,q}(B)),$$

for $q\leq\infty$. Consequently, we obtain $R\in W^{2,p}(0,T)$. □

As a consequence of the lemma we obtain,

**Corollary 2.1** *By using that $W_0^{1,p}(B\times[0,\tilde{T}])\subset L^\infty(B\times[0,\tilde{T}])$, if $p>4$, then $u,v\in L^\infty(B\times[0,\tilde{T}])$.*

On the other hand, the continuous embedding

$$W^{1,q}((0,T)\times B)\cap L^q(0,T:W^{2,q}(B))\subset L^2(0,T:W^{1,\infty}(B)),$$

$$W^{1,p}((0,\tilde{T})\times B)\cap L^p(0,\tilde{T}:W^{2,p}(B))\subset L^2(0,T:W^{1,\infty}(B)),$$

and undoing the change of variables and unknown (2.1), (2.2), we obtain

**Corollary 2.2** *Under the assumptions of Theorem 1.1, we have*

$$\int_0^T(\|\sigma\|^2_{W^{1,\infty}(R(t))}+\|\beta\|^2_{W^{1,\infty}(R(t))})dt\leq k_0,$$

*for some $0<k_0<\infty$.*

The uniqueness of solutions is proved in the next proposition.

**Proposition 2.1** *Let $f \in L^p(\omega_0 \times (0,T))$ with $p \geq 5$, and $(\sigma_0 - \overline{\overline{\sigma}}, \beta_0 - \overline{\overline{\beta}}) \in W^{2,s}(B(R_0)) \cap H_0^1(B(R_0))$, for $s > 4$. Then, there exists a unique solution to (1.1)-(1.5).*

**Proof.** By contradiction, we assume there exist two solutions, $(\sigma_1, \beta_1, R_1)$ and $(\sigma_2, \beta_2, R_2)$. Let

$$R(t) = min\{R_1(t), R_2(t)\}, \qquad \sigma = \sigma_1 - \sigma_2, \qquad \beta = \beta_1 - \beta_2.$$

Then $(\sigma, \beta, R)$ satisfies the problem,

$$\frac{\partial \sigma}{\partial t} - d\Delta\sigma + r_1\sigma + \lambda\beta = 0, \qquad |x| < R(t),\ t \in (0,T), \tag{2.8}$$

$$\frac{\partial \beta}{\partial t} - d\Delta\beta + r_2\beta = 0, \qquad |x| < R(t),\ t \in (0,T), \tag{2.9}$$

$$\sigma(x,0) = 0, \quad \beta(x,0) = 0, \qquad |x| < R_0, \tag{2.10}$$

$$\sigma(x,t) = \sigma_1(x,t) - \sigma_2(x,t), \qquad |x| = R(t),\ t \in (0,T), \tag{2.11}$$

$$\beta(x,t) = \beta_1(x,t) - \beta_2(x,t), \qquad |x| = R(t),\ t \in (0,T). \tag{2.12}$$

We introduce a new unknown defined by

$$z = k_1\sigma - k_2\beta,$$

with

$$k_1 = 1, \qquad k_2 = \tfrac{\lambda}{r_1 - r_2}, \quad \text{if} \quad r_1 \neq r_2,$$

$$k_1 = \frac{1}{2}, \qquad k_2 = \tfrac{\lambda}{r_1 - 2r_2}, \quad \text{if} \quad r_1 = r_2 \neq 0,$$

and by $z = e^{-\lambda t}\sigma - \beta$ if $r_1 = r_2 = 0$. By construction we have

$$\begin{cases} \dfrac{\partial z}{\partial t} - d\Delta z + r_1 z = 0, & |x| < R(t),\ t \in (0,T), \\ z(x,0) = 0, & |x| < R_0, \\ z(x,t) = k_1\sigma(x,t) - k_2\beta(x,t), & |x| = R(t),\ t \in (0,T). \end{cases} \tag{2.13}$$

Now we prove a preliminary result:

**Lemma 2.2** *Let $z$ be the solution to (2.13) and $\beta$ the solution to the problem (2.9), (2.12), then $e^{r_1 t}z$ and $e^{r_2 t}\beta$ take their maximum and minimum on $|x| = R(t)$.*

**Proof.** Multiplying the equation (2.13) by $e^{r_1 t}$ we obtain that $e^{r_1 t}z$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}(e^{r_1 t}z) - d\Delta(e^{r_1 t}z) = 0, & |x| < R(t),\ t \in (0,T), \\ z(x,0) = 0, & |x| < R_0, \\ e^{r_1 t}z(x,t) = e^{r_1 t}(k_1\sigma(x,t) - k_2\beta(x,t)), & |x| = R(t),\ t \in (0,T). \end{cases} \tag{2.14}$$

In the same way, we obtain $e^{r_2 t}\beta$ satisfies

$$
(2.15) \qquad \begin{cases} \frac{\partial}{\partial t}(e^{r_2 t}\beta) - d\Delta(e^{r_2 t}\beta) = 0, & |x| < R(t),\ t \in (0,T), \\ \beta(x,0) = 0, & |x| < R_0, \\ e^{r_2 t}\beta(x,t) = e^{r_2 t}(\beta_1(x,t) - \beta_2(x,t)), & |x| = R(t),\ t \in (0,T). \end{cases}
$$

Applying Corollary 2.1 we obtain that $e^{r_1 t}z$ and $e^{r_2 t}\beta$ are bounded. Let

$$
z^{**} = max\{e^{r_1 t}z(x,t), t \in [0,T], x \in \partial B(R(t))\},
$$

$$
z_{**} = min\{e^{r_1 t}z(x,t), t \in [0,T], x \in \partial B(R(t))\},
$$

$$
\beta^{**} = max\{e^{r_2 t}\beta(x,t), t \in [0,T], x \in \partial B(R(t))\},
$$

$$
\beta_{**} = min\{e^{r_2 t}\beta(x,t), t \in [0,T], x \in \partial B(R(t))\}.
$$

Note that since $R(0) = R_0$, $z^{**}, \beta^{**} \geq 0$ and $z_{**}, \beta_{**} \leq 0$.

Let $(\cdot)^+$ be the continuous function defined by $(s)^+ = s$, if $s > 0$ and 0 otherwise, and $(s)^- = (-s)^+$. Taking $(e^{r_1 t}z - z^{**})^+$ as test function in (2.14), integrating by parts in $B(R(t))$, by Leibnitz Theorem$'$s, after some manipulations, we arrive to

$$
\frac{d}{dt}\int_{B(R(t))} [(e^{r_1 t}z - z^{**})^+]^2 dx \leq 0,
$$

and we deduce that $e^{r_1 t}z$ takes his maximum on $|x| = R(t)$. In the same way, taking $-(e^{r_1 t}z - z_{**})^-$ as test function we obtain

$$
(2.16) \qquad z_{**} \leq e^{r_1 t}z \leq z^{**}.
$$

The proof of

$$
(2.17) \qquad \beta_{**} \leq e^{r_2 t}\beta \leq \beta^{**},
$$

is analogous. □

**End of the proof of Proposition 2.1**. Given $t \in [0,T]$, we can assume, without lost of generality, that $R_1(t) \leq R_2(t)$. Using that

$$
R_1^2(t)\dot{R}_1(t) - R_2^2(t)\dot{R}_2(t) = \int_{B(R(t))} (S(\sigma_1,\beta_1) - S(\sigma_2,\beta_2))dx -
$$

$$
\int_{R_1(t)<|x|<R_2(t)} S(\sigma_2,\beta_2)dx.
$$

Since $S$ is bounded, then

$$
|\int_{R_1(t)<|x|<R_2(t)} S(\sigma_2,\beta_2)dx| \leq N|R_1^3(t) - R_2^3(t)| \leq M|R_1(t) - R_2(t)|,
$$

where $M$ depends only of $|S|_{L^\infty}$. Since $S$ is Lipschitz continuous, integrating in time, it results

$$
\int_0^T \int_{B(R(t))} |S(\sigma_1,\beta_1) - S(\sigma_2,\beta_2)|dxdt \leq
$$

$$\int_0^T \int_{B(R(t))} |S|_{W^{1,\infty}(I\!R^2)}(sup|\sigma| + sup|\beta|)dxdt \leq$$

$$\int_0^T \int_{B(R(t))} k_0(\frac{1}{k_1} sup|z + k_2\beta| + sup|\beta|)dxdt \leq$$

$$\int_0^T \int_{B(R(t))} C(sup|z| + sup|\beta|)dxdt \leq$$

$$\int_0^T \int_{B(R(t))} C(sup|e^{-r_1 t} e^{r_1 t} z| + sup|e^{-r_2 t} e^{r_2 t}\beta|)dxdt \leq$$

$$\int_0^T \int_{B(R(t))} C(e^{|r_1|T} sup|e^{r_1 t} z| + e^{|r_2|T} sup|e^{r_2 t}\beta|)dxdt \leq$$

$$\int_0^T \int_{B(R(t))} k_3(sup|e^{r_1 t} z| + sup|e^{r_2 t}\beta|)dxdt.$$

From Lemma 2.2, we know

$$\int_0^T \int_{B(R(t))} \sup |e^{r_1 t} z(x,t)| dxdt \leq e^{r_1 T} \frac{3\pi}{4} \int_0^T R^3(t) \sup_{|x|=R(t)} |z(x,t)| dt.$$

By Corollary 2.2, we deduce that

$$\int_0^T (\|\sigma_2\|^2_{W^{1,\infty}(B(R(t))} + \|\beta_2\|^2_{W^{1,\infty}(B(R(t))})dt \leq K_0,$$

and consequently,

$$\int_0^T \|z\|^2_{W^{1,\infty}(B(R(t)))} dt \leq K.$$

Since

$$e^{r_1 t} z(x,t) = e^{r_1 t}(k_1(\sigma_2(x,t) - \overline{\overline{\sigma}}) - k_2(\beta_2(x,t) - \overline{\overline{\beta}})), \text{ on } |x| = R(t),$$

it results

$$e^{r_1 T} \frac{3\pi}{4} \int_0^T R^3(t) \sup_{|x|=R(t)} |z(x,t)| dt \leq$$

$$k_4 \int_0^T \|\sigma_2\|_{W^{1,\infty}(B(R_2(t)))} + \|\beta_2\|_{W^{1,\infty}(B(R_2(t)))} |R_1(t) - R_2(t)| dt \leq$$

$$k_4 \sup_{0<t<T} |R_1(t) - R_2(t)| T^{\frac{1}{2}} \int_0^T (\|\sigma_2\|^2_{W^{1,\infty}(B(R_2(t)))} + \|\sigma_2\|^2_{W^{1,\infty}(B(R_2(t)))})dt \leq$$

$$k \sup_{0<t<T} |R_1(t) - R_2(t)| T^{\frac{1}{2}}.$$

In the same way,

$$\int_0^T \int_{B(R(t))} k_3 sup|\beta| dxdt \leq k \sup_{0<t<T} |R_1(t) - R_2(t)| T^{\frac{1}{2}}.$$

Then

$$\int_0^T |R_1^2(t)\dot{R}_1(t) - R_2^2(t)\dot{R}_2(t)| dt \leq C_0 \sup_{0<t<T} |R_1(t) - R_2(t)| (T + T^{\frac{1}{2}}). \tag{2.18}$$

Let $\delta = \max_{t\in[0,T]}\{R_1(t) - R_2(t)\}$, then $|R_1^3(t) - R_2^3(t)| \leq 3C_0\delta(T + T^{\frac{1}{2}})$, since

$$|R_1^3(t) - R_2^3(t)| \geq 3R_0^2|R_1(t) - R_2(t)|,$$

it results $\delta \leq k_0\delta(T + T^{\frac{1}{2}})$. If $T < T_1 = \min\{\frac{1}{4k_0^2}, 1\}$, then $R_1(t) = R_2(t)$. Since $e^{r_1 t}z$ and $e^{r_2 t}\beta$ take his maximum and minimum on $R(t) = R_1(t) = R_2(t)$ and it is zero, then $\beta = 0$ and $z = 0$ and we deduce $\sigma = 0$.

Using the same argument, now from $T_1$ we conclude the uniqueness of solutions for any $T > 0$. □

# 3 Approximate controllability: Proof of Theorem 1.1

The next proposition shows the conclusion of Theorem 1.1 (the so called approximate controllability in $L^p$) under some particular assumptions (mainly when $R(t)$ is a priori prescribed). After that, using a fixed point argument we proof the Theorem 1.1.

**Proposition 3.1** *Let $\omega_0 \subset B(R_0 exp\{-\|S\|_{L^\infty}T\}$, and $\sigma_0 = \beta_0 = \overline{\overline{\sigma}} = \overline{\overline{\beta}} = 0$. Let $R \in W^{1,\infty}(0,T)$ a given function such that $R(0) = R_0$, $|\dot{R}| \leq \|S\|_{L^\infty}R_0\ exp\ \{|S|_{L^\infty}T\}$. Then, given $\widehat{\sigma}^d \in L^2_{loc}(I\!R^3)$, there exists $f \in L^p(\omega_0 \times (0,T))$, with $p \geq 5$, such that, if $(\sigma, \beta)$ is the solution to problem (1.1), (1.2), (1.4) and (1.5), with $R(t)$ prescribed, then*

$$\|\sigma(T) - \sigma^d\|_{L^p(B(R(T)))} \leq \epsilon,$$

*where $\sigma^d = \widehat{\sigma}^d|_{B(R(T))}$.*

**Proof.** Let $p' = \frac{p}{p-1}$, we consider the functional $J : L^{p'}(B(R(T))) \longrightarrow I\!R$ defined by

$$J(\varphi^0) = \frac{1}{p'}\int_0^T\int_{\omega_0} |\psi(x,t)|^{p'}dxdt + \epsilon\|\varphi^0\|_{L^{p'}(B(R(T)))} - \int_{B(R(T))} \sigma^d\varphi^0 dx,$$

where $\varphi_0 \in L^{p'}(B(R(T)))$ and $(\varphi, \psi)$ is the solution to the adjoint problem

$$-\frac{\partial\varphi}{\partial t} - d\Delta\varphi + r_1\varphi = 0, \qquad |x| < R(t),\ t \in (0,T), \tag{3.1}$$

$$-\frac{\partial\psi}{\partial t} - d\Delta\psi + r_2\psi + \lambda\varphi = 0, \qquad |x| < R(t),\ t \in (0,T), \tag{3.2}$$

$$\varphi(x,T) = \varphi_0(x), \quad \psi(x,T) = 0, \qquad |x| < R(T), \tag{3.3}$$

$$\varphi(x,t) = 0, \qquad \psi(x,t) = 0, \qquad |x| = R(t), \qquad t \in (0,T). \tag{3.4}$$

We point out that the existence of a weak solutions of (3.1)-(3.4), $(\varphi, \psi)$ can be obtained as in section 2, doing the change (2.1), (2.2).

In order to prove the uniqueness of solutions by contradiction, we assume there exists two solutions, $(\varphi_1, \psi_1)$, $(\varphi_2, \psi_2)$. Then $\varphi := \varphi_1 - \varphi_2$ satisfies (3.1), taking $|\varphi|^{p'-2}\varphi$ as test function and integrating by parts, it results,

$$-\frac{d}{dt}\int_{B(R(t))} |\varphi|^{p'}dx \leq r_1\int_{B(R(t))} |\varphi|^{p'}dx,$$

by Gronwall's Lemma we obtain $\varphi = \varphi_1 - \varphi_2 = 0$. Once proved $\varphi \equiv 0$, in the same way we obtain $\psi_1 - \psi_2 = 0$ and uniqueness is proved.

Let us assume that $J$ is convex, continuous and coercive (in the sense that $\liminf J \to \infty$ if $\|\varphi^0\|_{L^{p'}(B(R_0))} \to \infty$). Then $J$ takes a minimum $\varphi_0$ (see, e.g., Brezis [2], Corollary III.20). Moreover if $(\xi, \zeta)$ is the solution to (3.1) – (3.4) with initial datum $(\xi^0, 0)$, we have

$$\begin{aligned} &\int_0^T \int_{\omega_0} |\psi|^{p'-2}\psi\zeta dxdt - \int_{B(R(T))} \sigma^d \xi^0 dx + \\ &\epsilon\|\varphi^0\|^{1-p'}_{L^{p'}(B(R(T)))} \int_{B(R(T))} |\varphi^0|^{p'-2}\varphi^0\xi^0 dx = 0. \end{aligned} \tag{3.5}$$

Multiplying (1.1), (1.2) by $(\xi, \zeta)$, integrating by parts and applying Leibnitz Theorem, we arrive to

$$-\int_0^T < \sigma, \frac{\partial \xi}{\partial t} > dt - d\int_0^T < \sigma, \Delta\xi > dt + \int_0^T \int_{B(R(t))} r_1 \sigma\xi dxdt +$$

$$\int_0^T \int_{B(R(t))} \lambda\beta\xi dxdt - \int_0^T < \beta, \frac{\partial \zeta}{\partial t} > dt - d\int_0^T < \beta, \Delta\zeta > dt +$$

$$\int_0^T \int_{B(R(t))} r_2\beta\zeta dxdt - \int_0^T \int_{\omega_0} f\zeta dxdt + \int_{B(R(t))} \sigma\xi dx]_0^T + \int_{B(R(t))} \beta\zeta dx]_0^T = 0,$$

where $< \, , \, >$ is the duality product $W_0^{1,p'}(B(R(t)) \times W_0^{-1,p'}(B(R(t))$. From the choice of $(\xi, \zeta)$ and since $\sigma(0,x) = \beta(0,x) = 0$ we obtain

$$-\int_0^T \int_{\omega_0} f\zeta dxdt + \int_{B(R(T))} \sigma(T)\xi^0 dx = 0. \tag{3.6}$$

Now, let us take $f$,

$$f := |\psi|^{p'-2}\psi.$$

Substituting it in (3.6) and using (3.5) it results

$$\int_{B(R(T))} (\sigma(T) - \sigma^d)\xi^0 dx + \epsilon\|\varphi^0\|^{1-p'}_{L^{p'}(B(R(T)))} \int_{B(R(T))} |\varphi^0|^{p'-2}\varphi^0\xi^0 dx = 0,$$

for all $\xi^0 \in L^{p'}(B(R(T)))$. Taking

$$\xi^0 = (\sigma(T) - \sigma^d)^{\frac{1}{p'-1}} \in L^{p'}(B(R(T)))$$

since $p = 1 + \frac{1}{p'-1}$, we obtain

$$\|\sigma(T) - \sigma^d\|^p_{L^p(B(R(T)))} =$$

$$\epsilon\|\varphi^0\|^{1-p'}_{L^{p'}(B(R(T)))} \int_{B(R(T))} |\varphi^0|^{p'-2}\varphi^0 |\sigma(T) - \sigma^d|^{\frac{1}{p'-1}-1}(\sigma(T) - \sigma^d) dx.$$

By Hölder inequality, we obtain

$$\|\varphi^0\|^{1-p'}_{L^{p'}(B(R(T)))} \int_{B(R(T))} |\varphi^0|^{p'-2}\varphi^0 |\sigma(T) - \sigma^d|^{\frac{1}{p'-1}-1}(\sigma(T) - \sigma^d) dx \leq$$

$$\|\sigma(T)-\sigma^d\|_{L^p(B(R(T)))}^{p-1},$$

which leads to

$$\|\sigma(T)-\sigma^d\|_{L^p(B(R(T)))}\le\epsilon$$

and the conclusion holds.
So, it only remains to check the mentioned properties of $J$:
**J is convex.** We can write $J$ as addition of the functionals,

$$J_1(\varphi^0):=-\int_{B(R(T))}\sigma^d\varphi^0dx,\qquad J_2(\varphi^0):=\epsilon\|\varphi^0\|_{L^{p'}(B(R(T)))},$$

$$J_3(\varphi^0):=\frac{1}{p'}\int_0^T\int_{B(R(t))}|\psi|^{p'}dxdt.$$

First we shall see that $J_3$ is convex. Let $(\varphi_1,\psi_1)$ and $(\varphi_2,\psi_2)$ be solutions to the problem (3.1)–(3.4) with initial datum $\varphi_1^0,\ \varphi_2^0\in L^p(B(R(T)))$ and let $\alpha\in(0,1)$. Then, since the system is linear we get

$$J_3(\alpha\varphi_1^0+(1-\alpha)\varphi_2^0)=\frac{1}{p'}\int_0^T\int_{B(R(t))}(|\alpha\psi_1+(1-\alpha)\psi_2|^{p'}dxdt,$$

and then

$$J_3(\alpha\varphi_1^0+(1-\alpha)\varphi_2^0)-\alpha J_3(\varphi_1^0)-(1-\alpha)J_3(\varphi_2^0)=$$
$$=\frac{1}{p'}\int_0^T\int_{B(R(t))}(|\alpha\psi_1+(1-\alpha)\psi_2|^{p'}-\alpha|\psi_1|^{p'}-(1-\alpha)|\psi_2|^{p'})dxdt.$$

Since $p'>1$ we obtain

$$|\alpha\psi_1+(1-\alpha)\psi_2|^{p'}-\alpha|\psi_1|^{p'}-(1-\alpha)|\psi_2|^{p'}\le 0,$$

and integrating it results

$$\frac{1}{p'}\int_0^T\int_{B(R(t))}(|\alpha\psi_1+(1-\alpha)\psi_2|^{p'}-\alpha|\psi_1|^{p'}-(1-\alpha)|\psi_2|^{p'})dxdt\le 0,$$

which proves the convexity of $J_3$. Finally $J_1$ is linear and so convex and since the norm $\|\cdot\|_{L^{p'}(B(R(T))}$ is convex, $J_2$ is also convex.
**J is continuous.** By construction, $J_1$ and $J_2$ are continuous. Now we shall prove that $J_3$ is also continuous. Let $\varphi_n^0\in L^{p'}(B(R(T)))$ such that $\varphi_n^0\longrightarrow\varphi^0$ and let $(\varphi_n,\psi_n)$, $(\varphi,\psi)$ be the solutions to (3.1)-(3.4) with initial datum $\varphi_n^0$ and $\varphi^0$. Subtracting both systems and taking

$$(p'|\varphi-\varphi_n|^{p'-2}(\varphi-\varphi_n),p'|\psi-\psi_n|^{p'-2}(\psi-\psi_n)),$$

as test function, using the integration by parts formula (see e.g. Alt – Luckhaus [1]) and Young inequality, we arrive to

$$-\frac{\partial}{\partial t}\int_{B(R(t))}[(\varphi-\varphi_n)^{p'}+(\psi-\psi_n)^{p'}]dx+$$

$$\int_{B(R(t))} (r_1 p' - |\lambda|)|\varphi - \varphi_n|^{p'} dx + \int_{B(R(t))} (r_2 p' - |\lambda|)|\psi - \psi_n|^{p'} dx \leq 0.$$

Denoting by

$$X_n(t) = \|\varphi - \varphi_n\|^{p'}_{L^{p'}(B(R(t)))} + \|\psi - \psi_n\|^{p'}_{L^{p'}(B(R(t)))},$$

we obtain the differential inequality

$$-X_n'(t) \leq C X_n(t), \quad t \in (0, T),$$

$$X_n(T) = \|\varphi_n^0 - \varphi^0\|^{p'}_{L^{p'}(B(R(T))}),$$

where

$$C = max\{-r_1 p' + |\lambda|, -r_2 p' + |\lambda|\}.$$

Thus we obtain

$$0 \leq X_n(t) \leq |X_n(T)| e^{-C(t-T)}.$$

Since

$$0 \leq \int_{\omega_0} |\psi - \psi_n|^{p'} dx \leq X_n(t),$$

integrating on $[0, T]$ and taking limits we conclude that

$$\int_0^T \int_{\omega_0} |\psi - \psi_n|^{p'} dx dt \leq \int_0^T X_n(t) dt \longrightarrow 0,$$

which shows the continuity of $J_3$.

**J is coercive.** Let $\varphi_n^0 \in L^{p'}(B(R(T)))$ such that $\|\varphi_n^0\|_{L^{p'}(B(R(T)))} \longrightarrow \infty$, as $n \longrightarrow \infty$. Now, we shall see

$$\liminf_{n \longrightarrow \infty} \frac{J(\varphi_n^0)}{\|\varphi_n^0\|_{L^{p'}(B(R(T)))}} \geq \epsilon.$$

Let

$$I := \liminf_{n \longrightarrow \infty} \frac{J(\varphi_n^0)}{\|\varphi_n^0\|_{L^{p'}(B(R(T)))}} \geq -\|\sigma^d\|_{L^p(B(R(T)))}.$$

Then, there exists a minimizing subsequence, (which we denote again by $\varphi_n^0$) such that

$$\lim_{n \to \infty} \frac{J(\varphi_n^0)}{\|\varphi_n^0\|_{L^{p'}(B(R(T)))}} = I.$$

We define

$$\bar{\varphi}_n^0 := \frac{\varphi_n^0}{\|\varphi_n^0\|_{L^{p'}(B(R(T)))}},$$

and denote by $(\bar{\varphi}_n, \bar{\psi}_n)$ the solution to the problem (3.1)-(3.4) with initial datum $(\bar{\varphi}_n^0, 0)$. Since the system is linear we have

$$(\bar{\varphi}_n, \bar{\psi}_n) = \frac{1}{\|\varphi_n^0\|_{L^{p'}}} (\varphi_n, \psi_n).$$

Then

$$\frac{J(\varphi_n^0)}{\|\varphi_n^0\|_{L^{p'}(B(R(T)))}} = \|\varphi_n^0\|^{p'-1} \int_0^T \int_{\omega_0} \bar{\psi}_n^{p'} dx dt - \int_{B(R(T)} \sigma^d \bar{\varphi}_n^0 dx + \epsilon.$$

Now, it is clear that if

$$\liminf_{n\to\infty}\int_0^T\int_{\omega_0}\bar{\psi}_n^{p'}dx \ge \alpha_0, \tag{3.7}$$

for some positive $\alpha_0$, then

$$\frac{J(\varphi_n^0)}{\|\varphi_n^0\|_{L^{p'}(B(R(T)))}} \ge \alpha_0\|\varphi_n^0\|^{p'-1}_{L^{p'}(B(R(T)))} + \epsilon - \|\sigma^d\|_{L^p(B(R((T)))} \longrightarrow \infty$$

as $n\to\infty$, which proves the property. Let us assume now that $lim\ inf \int_0^T\int_{\omega_0}\bar{\psi}_n^{p'}dx$ $=0$. Then there exists a subsequence $\bar{\psi}_{n_i}$ such that

$$\int_0^T\int_{\omega_0}|\bar{\psi}_{n_i}|^{p'}dxdt \longrightarrow 0,$$

therefore $\bar{\psi}_{n_i}\to 0$ in $L^{p'}(\omega_0\times[0,T])$. Taking $(0,\zeta)$ as test function in (3.2), where $\zeta\in C_c^2((0,T)\times\omega_0)$, we obtain

$$\int_0^T\int_{\omega_0}\bar{\psi}_{n_i}\frac{\partial\zeta}{\partial t}dxdt - \int_0^T\int_{\omega_0}\bar{\psi}_{n_i}\Delta\zeta dxdt -$$

$$r_2\int_0^T\int_{\omega_0}\bar{\psi}_{n_i}\zeta dxdt + \lambda\int_0^T\int_{\omega_0}\bar{\varphi}_{n_i}\zeta dxdt = 0.$$

Now, taking limits, it results that

$$\int_0^T\int_{\omega_0}\bar{\varphi}_{n_i}\zeta dxdt \longrightarrow 0, \tag{3.8}$$

where $\bar{\varphi}_{n_i}$ is the solution to the problem

$$\begin{cases} -\dfrac{\partial\bar{\varphi}_{n_i}}{\partial t} - d\Delta\bar{\varphi}_{n_i} + r_1\bar{\varphi}_{n_i} = 0, & |x|<R(t),\ t\in(0,T), \\ \bar{\varphi}_{n_i}(T,x) = \bar{\varphi}^0. \end{cases} \tag{3.9}$$

Doing the change (2.1) and introducing the unknown

$$\bar{u}_{n_i}(\tilde{x},\tilde{t}) := \bar{\varphi}_{n_i}(R(t(\tilde{t}))\tilde{x}, t(\tilde{t})),$$

we obtain

$$\begin{cases} -\dfrac{\partial\bar{u}_{n_i}}{\partial\tilde{t}} - d\Delta\bar{u}_{n_i} - R^2R'\tilde{x}\cdot\nabla\bar{u}_{n_i} + R^2r_1\bar{u}_{n_i} = 0, & \tilde{x}\in B,\ \tilde{t}\in(0,\tilde{T}), \\ \bar{u}_{n_i}(\tilde{x},\tilde{t}) = 0, & \tilde{x}\in\partial B, \tilde{t}\in(0,\tilde{T}), \\ \bar{u}_{n_i}(T,\tilde{x}) = \bar{u}^0_{ni}(\tilde{x}) = \bar{\varphi}^0_{n_i}(\tilde{x}R_0), & |\tilde{x}|<1. \end{cases} \tag{3.10}$$

Since $\bar{u}^0_{n_i}\rightharpoonup\bar{u}_0$ in $L^{p'}(B)$, it results that $\bar{u}_{n_i}\rightharpoonup\bar{u}$ (the solution to (3.10) with $\bar{u}_0=\bar{\varphi}^0$). By (3.8), $\bar{u}_{n_i}\longrightarrow 0$ weakly in $L^{p'}(B(\widehat{\omega}_0))$, where $\widehat{\omega}_0$ is an open subset of $B$, such that

$\widehat{\omega}_0 \subset \tilde{\omega}_0$. Consequently $\bar{u} \equiv 0$ on $\tilde{\omega}_0$ for all $0 \leq \tilde{t} \leq \tilde{T}$. By the unique continuation of the solution for the equation (3.10) (see Chi-Cheung Poon [4], Theorem 1.1′) we deduce that $\bar{u} = 0$ in $B \times (0, \tilde{T})$, and by uniqueness of (3.10), it result $\bar{u}_0 \equiv 0$ and $\bar{\varphi}^0 \equiv 0$. Furthermore

$$-\int_{B(R(T))} \sigma^d \bar{\varphi}^0 dx = 0,$$

and $I = \epsilon$, which proves that $J$ is coercive. □

**Proof of the Theorem 1.1.**

We consider the sequence $\{R_n(t)\}$, where $R_n$ is defined as the solution of the problem

$$R_n^2(t)\dot{R}_n(t) = \int_{B(R_{n-1}(t))} S(\sigma_{n-1} + \sigma_{n-1}^s, \beta_{n-1} + \beta_{n-1}^s)dx, \quad R_n(0) = R_0,$$

for $n > 1$, where $(\sigma_{n-1}^s, \beta_{n-1}^s)$ is the solution to the problem (1.1), (1.2), (1.4) and (1.5), with $f \equiv 0$, and initial data $\sigma_{n-1}^s(x,0) = \sigma_0(x)$, $\beta_{n-1}^s(x,0) = \beta_0(x)$, and $R(t) = R_{n-1}(t)$, and $(\sigma_{n-1}, \beta_{n-1})$ is the solution mentioned in Proposition 3.1. We start the process by taking, e.g. $R_1(t) = R_0$. Since $S$ is bounded, $R_n \in W^{1,\infty}(0,T)$ and we deduce there exists a subsequence of functions $R_{n_i}$ such that converges weakly to $R(t)$ in $W^{1,q}(0,T)$, for all $q \in (1, \infty)$. By Proposition 3.1, for each $R_n$ there exists $\varphi_n^0$ such that minimize the functional

$$J_n(\varphi_n^0) := \int_0^T \int_{\omega_0} |\psi_n|^{p'} dxdt + \epsilon \|\varphi_n^0\|_{L^{p'}(B(R_n(T)))} - \int_{B(R_n(T))} \sigma_n^d \varphi_0^n dx,$$

where $\sigma_n^d = \widehat{\sigma}^d \chi_{B(R_n(T))}$. We show, by contradiction, that the sequence $\|\varphi_n^0\|_{L^{p'}(B(R(T)))}$ is uniformly bounded . We assume $\|\varphi_n^0\|_{L^{p'}(B(R_n(T)))} \longrightarrow \infty$, then, since $J_n(\varphi_n^0) \leq 0$, we get

$$\frac{J_n(\varphi_n^0)}{\|\varphi_n^0\|_{L^{p'}}} = \|\varphi_n^0\|_{L'(B(R_n(T)))}^{p'-1} \int_0^T \int_{\omega_0} \bar{\psi}_n^p dxdt + \epsilon - \int_{B(R_n(T))} \sigma_n^d \bar{\varphi}_n^0 dx \leq 0. \tag{3.11}$$

Since

$$\int_{B(R_n(T))} \sigma_n^d \frac{\varphi_n^0}{\|\varphi_n^0\|_{L^{p'}(B(R(T)))}} dx \leq \|\sigma_n^d\|_{L^p(B(R_n(T)))} \leq \|\widehat{\sigma}^d\|_{L^p(B(R_0 exp\{\|S\|_{L^\infty} T\}))},$$

it results, by (3.11),

$$\int_0^T \int_{\omega_0} \bar{\psi}_n^{p'} dxdt \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Using the same argument that in the proof of coercivety of $J$, we obtain

$$\bar{\varphi}_0^n \rightharpoonup 0 \text{ in } L^{p'}(B(R(T)))$$

and

$$\liminf_{n \to \infty} \frac{J_n(\varphi_n^0)}{\|\varphi_n^0\|} \geq \epsilon,$$

which contradicts (3.11). Consequently $\|\varphi_n^0\|_{L^{p'}(B(R_n(T)))}$ is uniformly bounded and so $\|\varphi_n\|_{L^{p'}(B(R_n(T)))}$ is uniformly bounded, furthermore

$$\|f_n\|_{L^p(0,T:L^p(\omega_0))} \leq C, \tag{3.12}$$

for some $C$ independent of $n$.
Doing the change of variable (2.1), (2.2), applying Lemma 2.1, we obtain that $(u_n, v_n, R_n)$ is uniformly bounded in $(W^{1,p}(B\times(0,\tilde{T}))^2, H^2(0,T))$. Then, there exists a subsequence $(u_{ni},\ v_{ni},\ R_{ni})$ such that converges strongly in $(C^\alpha((0,T]\times B)^2, C^1([0,T]))$ to $(u,v,R)$ for $\alpha=\frac{1}{6}$, where $(u_{ni}, v_{ni})$ satisfies

$$(3.13)\quad \begin{cases} \dfrac{\partial u_{ni}}{\partial t}-\dfrac{d}{R_{ni}^2}\Delta u_{ni}-\dfrac{R'_{ni}}{R_{ni}}\tilde{x}\cdot\nabla u_{ni}+r_1u_{ni}+\lambda v_{ni}=0, & \text{in } B\times(0,T),\\ \dfrac{\partial v_{ni}}{\partial t}-\dfrac{d}{R_{ni}^2}\Delta v_{ni}-\dfrac{R'_{ni}}{R_{ni}}\tilde{x}\cdot\nabla v_{ni}+r_2v_{ni}=f_n\chi_{\tilde{\omega}_0}, & \text{in } B\times(0,T),\\ u_{ni}(\tilde{x},t)=\ v_{ni}(\tilde{x},t)=0, & \text{on } \partial B\times(0,T),\\ u_{ni}(\tilde{x},0)=u_{ni}^0(\tilde{x}),\ v_{ni}(\tilde{x},0)=v_{ni}^0(\tilde{x}), & \text{in } B, \end{cases}$$

and $(u,v,R)$ is the solution of (2.3)-(2.7). In particular

$$(3.14)\qquad \|u(T)-u_n(T)\|_{L^p(B)}^p\longrightarrow 0, \qquad \text{as } n_i\to+\infty.$$

Moreover

$$\|\sigma(T)-\sigma^d\|_{L^p(B(R(T)))}=\|\sigma(T)-\sigma_n(T)\|_{L^p(B(\min\{R(T),R_n(T)\}))}+$$
$$\|\sigma_n(T)-\sigma^d\|_{L^p(B(\min\{R(T),R_n(T)\}))}+\|\sigma-\sigma^d\|_{L^p(B_n^*(T))},$$

where

$$B_n^*(T)=\begin{cases} B(R(T))\cap B^c(B(R_n(T))), & \text{if } R(T)>R_n(T),\\ \emptyset, & \text{if } R(T)\le R_n(T). \end{cases}$$

Doing the change (2.1) and since

$$\|\sigma_n(T)-\sigma^d\|_{L^p(B(\min\{R(T),R_n(T)\}))}\le\epsilon,$$

we obtain

$$\|\sigma(T)-\sigma^d\|_{L^p(B(R(T)))}\le\|u(T)-u_n(T)\|_{L^p(B)}+\|\sigma-\sigma^d\|_{L^p(B_n^*(T))}+\epsilon.$$

Since $\mu(B_n^*(T))\longrightarrow 0$, by the Lebesgue dominated convergence theorem we obtain that

$$\lim_{n\to\infty}\|\sigma-\sigma^d\|_{L^p(B_n^*(T))}=0.$$

Taking limits as $n\to\infty$, it results

$$\|\sigma(T)-\sigma^d\|_{L^p(B(R(T)))}\le\epsilon,$$

and the theorem is thereby proved in the case $p\ge 5$.

In the case $p<5$, we consider the control $f$ for $p=5$, then

$$\|\sigma(T)-\sigma^d\|_{L^p(B(R(T)))}\le\frac{3\pi}{4}B(R(T))\|\sigma(T)-\sigma^d\|_{L^5(B(R(T)))}\le\frac{3\pi}{4}exp\{T\|S\|_{L^\infty}\}\epsilon,$$

taking $\epsilon=\epsilon'(\frac{3\pi}{4}exp\{T\|S\|_{L^\infty}\})^{-1}$ we conclude the Theorem. □

**Remark 3.1** *Note that the final observation is made on the density $\sigma(T,\cdot)$ and that once we chose the control in order to have (1.6) the free boundary, $R(t)$, and the inhibitor density $\beta(T,\cdot)$ are univocally determined.*

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