Large Deviations for Processes with Long-Range Dependence, with Queueing Applications

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Abstract

The main objective of this paper is to develop a large deviations theory for processes with long range dependence. Our starting point is the integral relationship between the standard Brownian motion, B, and the fractional Brownian motion, B_H (with H being the Hurst parameter). This integral is known to preserve self-similarity, and produce long-range dependence in B_H when $H \in (1/2, 1)$. To extend beyond the class of Gaussian processes, we replace the Brownian motion by other processes, in particular, a class of processes that we call *sample-path processes*, denoted X. We use the integral as a filter that takes as input a process X that has short-range dependence and outputs a process with long-range dependence, denoted Y. We develop a general theory for Y to satisfy a moderate deviations principle (MDP), based on on the MDP of X. Applying this new theory, we develop both transient and steady-state results, in terms of the MDP, regarding the asymptotic behavior of queues fed with the long-range dependent process Y. A key ingredient in our approach is a condition on X that guarantees a certain exponential equivalence relation. Applying martingale inequalities, we show this condition holds for a variety of processes of interest.

Key Words: sample path large deviations and moderate deviations, long-range dependence, queues, weak convergence, exponential equivalence, martingale inequalities.

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1 Introduction

It is well known, following Mandelbrot and van Ness [19], that the fractional Brownian motion (FBM), denoted B_H (with $B_H(0) = 0$), relates to the standard Brownian motion, B, through the following integral relation:

$$B_H(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \cdot \{ \int_0^t (t-s)^{H-1/2} dB(s) + \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] dB(s) \}, \quad (1)$$

where $0 < H < 1$ denotes the Hurst parameter and $\Gamma(\cdot)$ is the gamma function.

Recall that a stochastic process $\{X(t)\}$ is termed self-similar if the increment process, $\{X(t+\tau) - X(t)\}$, is equal in distribution to $\{a^{-H}[X(t+a\tau) - X(t)]\}$, for any t and any positive a and τ, where $H \geq 0$ is the Hurst parameter. It is well known (and easy to verify from (1)) that the FBM is a *self-similar* process. Note that when $H = 1/2$, the FBM is reduced to the standard Brownian motion. On the other hand, when $H \in (\frac{1}{2}, 1)$, the FBM is known to exhibit long-range dependence — for instance, the autocorrelation function is not summable. (Refer to, e.g., Cox [9] and Beran [3], for preliminaries regarding processes with long-range dependence, as well as short-range dependence.)

The relationship in (1) is the predominant mechanism in the literature for generating self-similar processes with long-range dependence. Indeed, (1) can be made more general through replacing the Brownian motion by an α-stable process, and replacing the power, $H - 1/2$, by $H - 1/\alpha$ (where α is the parameter that characterizes the stable process). For instance, suppose the Brownian motion is replaced by an α-stable Lévy motion, which retains the stationary independent increment property of the Brownian motion, but has a non-Gaussian marginal distribution — in particular, when $\alpha < 2$ the marginal distribution has an infinite variance. Then, (1) results in the so-called fractional Lévy motion. Refer to Samorodnitsky and Taqqu [24]. This generalization hence broadens the class of processes represented by (1) to processes outside of the Gaussian family.

Another advantage of the model in (1) lies in that fact that it is a continuous version of the autoregressive moving-average model. Therefore, it is a convenient mathematical device to study ARMA and ARIMA processes with non-Gaussian innovations. Refer to [24] (S7.12 and S7.13).

The main objective of this paper is to develop a large deviations theory for processes with long range dependence. Our idea is to use (1) as the basic model. To get beyond the class of Gaussian processes, we shall replace the Brownian motion in (1) by other random processes, in particular, a class of processes that we call *sample-path processes* (see Section 2). On the other hand, we cannot allow the power $H - 1/2$ in (1) to be replaced by the more general $H - 1/\alpha$ — in other words, we must have $\alpha = 2$ — as techniques of large deviations generally do not apply to processes with an infinite variance.

This paper is a sequel to our recent work, Chang, Yao, and Zajic [7]. (Also refer to [8], which is part of the proceedings of an international workshop on stochastic networks, where an early version of some of our results were previewed.) In [7], our starting point was an alternative FBM model, due to Lévy ([18] (also refer to [19]):

$$B_H(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \cdot \int_0^t (t-s)^{H-1/2} dB(s). \tag{2}$$

We used the integral relation in (2) as a basic "filter". (Indeed, in (2), the FBM can be viewed as the output of a time-invariant linear filter with the impulse response $h(t) = t^{H-1/2}$ subject to an input that is the standard Brownian motion. Similar types of linear filter are widely used in modeling stochastic systems; see, e.g., Wong and Hajek [30].) Specifically, with $H \in (\frac{1}{2}, 1)$ in (2), we replaced B by other stochastic processes with short-range dependence, denoted X; and we showed that the *moderate deviations principle* (MDP) of X can be carried over to the MDP of Y, the output of the filter.

The main advantage of working with (2) is that the integral is over a finite time interval, which leads to ready applications of the basic results and techniques in large deviations theory. The shortcoming, however, is that it does not preserve the stationary increment property; i.e., when X has stationary increments, Y need not have stationary increments. (Neither does the filter preserve self-similarity.) This creates certain difficulties in applications. For instance, the stationary increment property is the key that relates the workload process in queueing systems to the maximum of a random walk; and it is the monotonicity of the latter that leads to steady-state limiting results. Without this linkage, it is difficult to study the steady-state behavior of the workload process.

On the other hand, a filter based on (1), namely,

$$Y(t) = \int_0^t (t-s)^{H-1/2} dX(s) + \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] dX(s), \tag{3}$$

does preserve both stationary increments and self-similarity. That the integration takes place over an infinite interval, however, makes it more challenging to establish moderate deviations results.

The recent interests in studying queues with long-range dependent inputs are primarily motivated by studies of telecommunications traffic that reveal long-range dependence in the empirical data, typically accompanied by self-similarity. Refer to Leland *et al* [17] and Willinger [29], among others. Furthermore, it has been demonstrated that queues with long-range dependent inputs have subexponential tails (Norros [22], and Duffield and O'Connell [12]). This phenomenon is qualitatively different from the exponential tail distribution in queues with renewal or Markov arrival processes — inputs that have short-range dependence. The exponential tail behavior is usually established via large deviations techniques; refer to, e.g., Chang [5, 6], Glynn and Whitt [14, 15], Whitt [28], and the references there.

Our work is motivated in part by [12]. Assuming that the "netput" — input minus (potential) output — satisfies a large deviations principle with a speed that is slower than n or, more precisely, an MDP, Duffield and O'Connell showed that the maximum of a corresponding random walk, which is equal in distribution to the workload process of a single-server queue, also satisfies the MDP. They give conditions for the netput to satisfy the MDP and show, in particular, that the MDP holds with speed $n^{2(1-H)}$ when the netput is an FBM with Hurst parameter $H \in (\frac{1}{2}, 1)$ (See Section 4 for more details). Consequently, with the FBM as the netput, the steady-state queueing distribution has a subexponential tail; i.e., the exponential decay rate is $n^{2(1-H)}$ rather than n.

Hence, the starting point of [12] is the MDP of the Y process, in the parlance here. In contrast, our starting point is to relate Y, through the FBM-like filter in (3), to more basic processes X, and establish the MDP of Y based on the MDP of X. Specifically, we will develop (in Section 2) a criterion for the sequence of scaled X processes, $\{X^{(n)}(t) = X(nt)/n\}$ to satisfy the MDP and give several examples that satisfy this criterion, including a stationary Markov chain, a moving average process, and a ϕ-mixing (or, uniform mixing) sequence. Based on the MDP of $\{X^{(n)}\}$, we establish the MDP of $\{Y^{(n)}(t) = Y(nt)/n\}$ (in Section 3). Feeding the long-range dependent process Y into a single-server queue, we study (in Section 4) both steady-state and transient behaviors of the queue.

A key ingredient in our approach is a condition (see (5) below) which essentially makes it possible to work with a modification of the process Y obtained by truncating the second integral in (3). Using martingale inequalities, we show (in Section 5) that this condition is satisfied by the X processes highlighted as examples in Section 2.

2 Moderate Deviations for Sample-Path Processes

Let $\mathcal{D}$, $\mathcal{C}$, $\mathcal{BV}$ and $\mathcal{AC}$ denote the spaces of real-valued functions defined on $(-\infty, 1]$ that are, respectively, right continuous with left limits (*cadlag*), continuous, of bounded variation on finite intervals and absolutely continuous. These function spaces are equipped with the metric of uniform convergence on compact sets,

$$d_\infty(\phi, \psi) = \sum_{n=1}^{\infty} \frac{2^{-n} \sup_{t\in[-n,1]} |\phi(t) - \psi(t)|}{1 + \sup_{t\in[-n,1]} |\phi(t) - \psi(t)|} .$$

With $\mathcal{K}$ referring to any one of these spaces, we let $\mathcal{K}_0 = \{\phi \in \mathcal{K} : \ \phi(0) = 0\}$ and denote by $\mathcal{K}[a, b]$ the space obtained by restricting the elements of $\mathcal{K}$ to the interval $[a, b]$, equipped with the norm of uniform convergence.

We next recall the definition of the large deviations principle on a metric space $\mathcal{S}$ equipped with the topology generated by the metric and the corresponding Borel σ-field. (Refer to, e.g., Dembo and Zeitouni [11], and Schwartz and Weiss [26] for more details.)

Definition 2.1 A sequence of probability measures $\mu^{(n)}$ on a metric space $\mathcal{S}$ satisfies the large deviations principle, with speed $\zeta_n \uparrow \infty$ and good rate function $I(\cdot) : \mathcal{S} \to [0, \infty]$ if

(i) the level sets $\{x \in \mathcal{S} : I(x) \leq \alpha\}$, $\alpha \in I\!R$, are compact,

(ii) for any closed set F,

$$\limsup_{n\to\infty} \frac{1}{\zeta_n} \log \mu^{(n)}(F) \leq - \inf_{x\in F} I(x),$$

(iii) for any open set G,

$$\liminf_{n\to\infty} \frac{1}{\zeta_n} \log \mu^{(n)}(G) \geq - \inf_{x\in G} I(x).$$

A sequence of random variables is said to satisfy the large deviations principle if the corresponding sequence of distributions does. Furthermore, when the space $\mathcal{S}$ is realized as a space of functions, such as $\mathcal{D}$ or $\mathcal{D}[a, b]$, we refer to the large deviations principle as a *sample-path* large deviations principle.

We say that a sequence $\mu^{(n)}$ satisfies a *moderate* deviations principle (MDP) if $\mu^{(n)}$ satisfies the LDP with speed γ_n^2, where γ_n satisfies

$$\lim_{n\to\infty} \frac{\gamma_n}{\sqrt{n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \gamma_n = \infty. \tag{4}$$

(Refer to Mogulskii [21], and Wu [31].) In case $\mu^{(n)}$ satisfies a sample-path large deviation principle with speed γ_n^2 satisfying (4), we say $\mu^{(n)}$ satisfies a sample-path moderate deviations principle, denoted as MDP(sp). Allowing that the definition of $\mu^{(n)}$ may depend on γ_n, we say $\mu^{(n)}$ satisfies the *complete* MDP (resp. MDP(sp)) if $\mu^{(n)}$ satisfies the MDP (resp. MDP(sp)) for every sequence satisfying (4).

Our starting point in the following sections will be that a scaled version of some process X, which is the input to the filter in (3), satisfies a MDP(sp) in $\mathcal{D}$. In the remainder of this section we demonstrate that in fact the class of such processes is far from trivial.

With $\mathbb{N}$ denoting the set of all integers, $\{0, \pm 1, \pm 2, \ldots\}$, consider a sequence of real-valued random variables, $\{a(t), t \in \mathbb{N}\}$. Let $A(t_1, t_2) = \sum_{t=t_1+1}^{t_2} a(t)$; and define a sequence of stochastic processes (indexed by n) as follows:

$$A^{(n)}(t) = \begin{cases} \frac{1}{n} A(0, [nt]), & t \in [0, 1] \\ -\frac{1}{n} A(\lfloor nt \rfloor, 0), & t \in (-\infty, 0) \end{cases} .$$

Below, we shall refer to such processes as *sample-path processes.*

Theorem 2.2 Let $\{a(t), t \in \mathbb{N}\}$ be a stationary sequence of bounded, mean zero, real-valued random variables. For a fixed positive integer m and $t_0 < t_1 < \cdots < t_m \leq 1$, let

$$Z_n = \frac{\sqrt{n}}{\gamma_n} (A^{(n)}(t_1) - A^{(n)}(t_0), A^{(n)}(t_2) - A^{(n)}(t_1), \ldots, A^{(n)}(t_m) - A^{(n)}(t_{m-1})).$$

Suppose that for each positive integer m and partition $\{t_i\}_{i=1}^m$, $\{Z_n\}$ satisfies the complete MDP in $\mathbb{R}^m$ with rate function

$$I_m(z) = \sum_{i=1}^m \frac{1}{2\sigma^2} \frac{z_i^2}{(t_i - t_{i-1})} ,$$

for some $\sigma^2 > 0$, where $z = (z_1, \ldots, z_m)$. In addition, assume

$$\limsup_{n \to \infty} \frac{1}{\gamma_n^2} \log \mathsf{E}[\exp(\frac{\gamma_n \sum_{i=1}^n a(i)}{\sqrt{n}}) + \exp(\frac{-\gamma_n \sum_{i=1}^n a(i)}{\sqrt{n}})] < \infty$$

for every sequence γ_n satisfying (4). Denote by $\nu_A^{(n)}$ the distribution of $\{\frac{\sqrt{n}}{\gamma_n} A^{(n)}(t),\ -\infty < t \leq 1\}$. Then the sequence $\nu_A^{(n)}$ satisfies the complete MDP(sp) in $\mathcal{D}$, with a good rate function $I(\cdot) : \mathcal{D} \to \mathbb{R}$,

$$I(\phi) = \begin{cases} \int_{-\infty}^1 \frac{1}{2\sigma^2} (\phi'(t))^2 dt & \text{if } \phi \in \mathcal{AC}_0 \\ \infty & \text{otherwise} . \end{cases}$$

Proof. For any positive integer m, let $\nu_{A,m}^{(n)}$ denote the distribution of $\{\frac{\sqrt{n}}{\gamma_n}A^{(n)}(t),\ -m \le t \le 1\}$. A straightforward modification of the proof of [7, Theorem 2.2] yields that $\nu_{A,m}^{(n)}$ satisfies the complete MDP(sp) on $\mathcal{D}[-m,1]$ with a good rate function

$$I_m(\phi) = \begin{cases} \int_{-m}^{1} \frac{1}{2\sigma^2}(\phi'(t))^2 dt & \text{if } \phi \in \mathcal{AC}_0[-m,1] \\ \infty & \text{otherwise} . \end{cases}$$

Since we may view the space $\mathcal{D}(-\infty,1]$ as the projective limit of the spaces $\mathcal{D}[-m,1]$, $m = 1,2,\ldots$, the proof now follows from an application of [11, Theorem 4.6.1]. ∎

Remark. The proof above extends readily to yield, under the assumptions of the theorem, the complete MDP(sp) for the sequence $\{\frac{\sqrt{n}}{\gamma_n}A^{(n)}(t), -\infty < t < \infty\}$ on the space of real-valued cadlag functions on $I\!R$ equipped with the topology of uniform convergence on compact sets. Here, for $t > 1$, $A^{(n)}(t) = \frac{1}{n}A(0,[nt])$.

We now present three classes of stochastic processes that satisfy the hypotheses in Theorem 2.2. The verification that they do indeed satisfy the hypotheses is similar to the verification of their counterparts in [7], and hence omitted.

Example 1. Let $\{b(t), t \in I\!N\}$ be a stationary Markov chain taking values in a Polish space and satisfying the conditions in Theorem 2.1 of Wu [31, 32]. Let $a(t) = f(b(t)) - \mathsf{E}_\mu[f(b(t))]$ for any bounded real-valued function f, where μ is the stationary distribution of $\{b(t)\}$. Here $\sigma^2 = \mathsf{E}_\mu[a^2(0)] + 2\sum_{k=1}^{\infty}\mathsf{E}_\mu[a(k)a(0)]$, which is required to be positive.

Example 2. Let $\{\xi(t), t \in I\!N\}$ be a sequence of bounded, mean zero, i.i.d. random variables and $\{b(t), t \in I\!N\}$, a sequence of real numbers which is absolutely summable. Let the sequence $\{a(t), t \in I\!N\}$ be the infinite moving-average sequence defined as $a(t) = \sum_{i=-\infty}^{\infty} b(i+t)\xi(i)$. Here $\sigma^2 = (\sum_{i=-\infty}^{\infty} b_i)^2\mathsf{E}[\xi^2(0)]$ is assumed to be positive.

Example 3. Let $\{a(t), t \in I\!N\}$ be a stationary sequence of bounded, mean zero random variables. For n a positive integer, denote

$$\phi(n) = \sup\{\,|P(A|B) - P(B)|;\ A \in \mathcal{F}_k \text{ with } P(A) > 0, B \in \mathcal{F}^{k+n}, k \in I\!N\,\},$$

where, for $k \in I\!N$,

$$\mathcal{F}_k = \sigma\{a(i) : i \le k\} \quad \text{and} \quad \mathcal{F}^k = \sigma\{a(i) : i \ge k\}.$$

Recall when $\lim_{n\to\infty}\phi(n) = 0$, the sequence is said to be ϕ-mixing or uniform-mixing (cf. Bradley [4]). We further assume that $\phi(n) = o(e^{-\alpha n})$ for some $\alpha > 0$, and that $\sigma^2 = \lim_{n\to\infty}\frac{1}{n}\mathsf{E}[\sum_{i=1}^{n} a(i)]^2$ is positive.

3 MDP for the Filtered Process

To start with, we make precise the processes that we shall allow as possible input, X, to the filter in (3). First, we allow X to be a standard Brownian motion, so that our results will apply to FBM (output of the filter). Beyond this, we shall restrict X to be a sample-path process (i.e., X is the process $A^{(1)}$ of Section 2).

Letting

$$X^{(n)}(t) = X(nt)/n \quad \text{and} \quad Y^{(n)}(t) = Y(nt)/n,$$

where Y relates to X through (3), we have, for $t \in [0,1]$,

$$\begin{aligned} Y^{(n)}(t) &= \int_0^t (t-s)^{H-1/2} d(n^{H-1/2}X^{(n)}(s)) \\ &\quad + \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] d(n^{H-1/2}X^{(n)}(s)) . \end{aligned}$$

Whereas $Y^{(n)}$ is the FBM when X is the standard Brownian motion, when X is a sample-path process, we need to *define* the above integral relationship. To this end, let $Y_a^{(n)}$ denote the process $Y^{(n)}$ modified so that the second integral above starts at a instead of $-\infty$. We assume the input process X is such that, for each $n = 1, 2, \ldots$ fixed, the sequence $\{Y_{-j/n}^{(n)}\}_{j=1}^{\infty}$ is a Cauchy sequence in $L^2[\mathcal{C}[0,1]]$. We then take $Y^{(n)}$ to be the limit of this sequence.

We shall also require the following condition, which essentially ensures that $Y_m^{(n)}$ is an exponentially good approximation of $Y^{(n)}$ (see [11, Definition 4.2.14]):

$$\begin{aligned} \lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n^{2(1-H)}}\log \mathsf{P}\{\sup_{t\in[0,1]} | \int_{-\infty}^{-m} [(t-s)^{H-1/2} - (-s)^{H-1/2}] \\ d(n^{H-1/2}X^{(n)}(s))| \geq \eta\} = -\infty, \quad \text{for any } \eta > 0. \end{aligned} \tag{5}$$

In Section 5 we shall verify the above assumptions. Specifically, we will show that,

- the condition in (5) holds when X is either the standard Brownian motion or the sample-path processes such as those in the examples of Section 2 (with additional assumptions);
- and, in the case of sample-path processes, for each n, the sequence $\{Y_{-j/n}^{(n)}\}_{j=1}^{\infty}$ is Cauchy in $L^2[\mathcal{C}[0,1]]$.

Finally, before stating the results of this section, we remark that by Schilder's theorem (cf. [11] and [25]), when X is the standard Brownian motion, the sequence $\{n^{H-\frac{1}{2}}X^{(n)}(t), t \in (-\infty, 1]\}$ satisfies the MDP in $\mathcal{D}$ with speed $n^{2(1-H)}$ and a good rate function as in the statement of Theorem 2.2, with $\sigma^2 = 1$.

Theorem 3.1 Denote by $\mu_X^{(n)}$ the probability law of $\{n^{H-\frac{1}{2}}X^{(n)}(t), t \in (-\infty, 1]\}$, for some $\frac{1}{2} < H < 1$. Suppose $\mu_X^{(n)}$ satisfies the MDP(sp) on $\mathcal{D}$ with speed $n^{2(1-H)}$ and a good rate function $I_X(\cdot) : \mathcal{D} \mapsto I\!R$,

$$I_X(\phi) = \begin{cases} \int_{-\infty}^{1} \frac{1}{2\sigma^2}(\phi'(t))^2 dt & \text{if } \phi \in \mathcal{AC}_0 \\ \infty & \text{otherwise} \end{cases}.$$

Suppose the condition in (5) holds. Then, the sequence $\{Y^{(n)}(t), t \in [0, 1]\}$ satisfies the MDP(sp) in $\mathcal{C}[0, 1]$ with speed $n^{2(1-H)}$ and a good rate function $I_Y(\cdot) : \mathcal{C}[0, 1] \mapsto I\!R$, given by

$$\begin{aligned} I_Y(\psi) = \inf\{I_X(\phi) : \psi(t) = & \int_0^t (t-s)^{H-1/2}\phi'(s)ds \\ & + \int_{-\infty}^{0} [(t-s)^{H-1/2} - (-s)^{H-1/2}]\phi'(s)ds,\ t \in [0, 1]\}. \end{aligned}$$

Corollary 3.2 Suppose the conditions in Theorem 3.1 hold. Then the sequence $\{Y(n)/n, n \geq 0\}$ satisfies the MDP(sp) with speed $n^{2(1-H)}$ and a good rate function $\Lambda_Y^*(\cdot) : I\!R \mapsto I\!R$,

$$\Lambda_Y^*(\gamma) = \frac{\tilde{H}}{\sigma^2}\gamma^2,$$

where

$$\tilde{H} = \frac{1}{2}\Big[\frac{1}{2H} + \int_{-\infty}^{0} [(1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]^2 ds\Big]^{-1}. \tag{6}$$

Remark. It is known (e.g., [19]) that the variance of the FBM is $\mathsf{E}[B_H^2(t)] = t^{2H}V_H$, where $V_H = 1/(2\tilde{H})$ (ignoring the constant $\Gamma(H + 1/2)$ in (1)), with $\tilde{H}$ following (6) above.

Note that, here and below, δ_ϕ denotes the distribution defined as follows: for any Borel set A, $\delta_\phi(A) = 1\{\phi \in A\}$, with $1\{\cdot\}$ being the indicator function.

Theorem 3.3 Suppose the conditions in Theorem 3.1 are in force.

(i) Let $\tilde{\mu}_X^{(n)}$ denote the law of $\{n^{H-\frac{1}{2}}X^{(n)}|Y^{(n)}(1) \geq \gamma\}$ for some $\gamma \geq 0$. Then, $\tilde{\mu}_X^{(n)}$ converges in distribution on $\mathcal{D}$ to δ_{ϕ_γ}, where

$$\phi_\gamma(t) = \begin{cases} \gamma\frac{2\tilde{H}}{H+\frac{1}{2}}(1-(1-t)^{H+\frac{1}{2}}) & t \in [0,1] \\ \gamma\frac{2\tilde{H}}{H+\frac{1}{2}}(1-(1-t)^{H+\frac{1}{2}} + (-t)^{H+1/2}) & t \in (-\infty, 0] \end{cases} .$$

(ii) Let $\tilde{\mu}_Y^{(n)}$ denote the law of $\{Y^{(n)}(\cdot)|Y^{(n)}(1) \geq \gamma\}$. Then $\tilde{\mu}_Y^{(n)}$ converges in distribution on $\mathcal{C}[0,1]$ to δ_{ψ_γ}, where

$$\psi_\gamma(t) = \int_0^t (t-s)^{H-1/2} d\phi_\gamma(s) + \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] d\phi_\gamma(s), \quad t \in [0,1].$$

Proof. [Theorem 3.1]

Consider first the case when X is a sample-path process. Define the map $F^m : \mathcal{BV} \to \mathcal{C}[0,1]$ as follows:

$$F^m(\phi)(t) = \int_0^t (t-s)^{H-1/2} d\phi(s) + \int_{-m}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] d\phi(s) \ , \tag{7}$$

and a second map, $\tilde{F}^m : \mathcal{D} \to \mathcal{C}[0,1]$, as

$$\begin{aligned} \tilde{F}^m(\phi)(t) &= \int_0^t (H-1/2)(t-s)^{H-3/2}\phi(s)ds \\ &\quad + \int_{-m}^0 (H-1/2)[(t-s)^{H-3/2} - (-s)^{H-3/2}]\phi(s)ds \\ &\quad - [(t+m)^{(H-1/2)} - m^{(H-1/2)}]\phi(-m) \ . \end{aligned} \tag{8}$$

Clearly, $\tilde{F}^m$ is continuous. Also, integration by parts verifies that $\tilde{F}^m(\phi) = F^m(\phi)$ for all $\phi \in \mathcal{BV}$ with $\phi(0) = 0$.

Thus, it follows from the contraction principle (see, for example, [11, Theorem 4.2.1]) that the following sequence

$$\begin{aligned} Y_m^{(n)}(t) &= \int_0^t (t-s)^{H-1/2} d(n^{H-1/2}X^{(n)}(s)) \\ &\quad + \int_{-m}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}] d(n^{H-1/2}X^{(n)}(s)), \quad t \in [0,1], \end{aligned}$$

satisfies the MDP(sp), with a good rate function given by

$$\begin{aligned} I_Y^m(\psi) = \inf\{I_X(\phi) : \psi(t) = &\int_0^t (t-s)^{H-1/2}\phi'(s)ds \\ &+ \int_{-m}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}]\phi'(s)ds, \ t \in [0,1]\}. \end{aligned}$$

For $t \in [0,1]$ and $\phi \in \mathcal{BV}$,

$$F^m(\phi)(t) - F^\infty(\phi)(t) = \int_{-\infty}^{-m} [(t-s)^{H-1/2} - (-s)^{H-1/2}] d\phi(s). \tag{9}$$

Thus, by (5) $Y_m^{(n)}$ are exponentially good approximations of $Y^{(n)}$ (see [11, Definition 4.2.14]). Moreover, for any $\alpha < \infty$, making use of the Cauchy-Schwartz inequality and the definition of $I_X(\cdot)$ in Theorem 3.1, we have

$$\begin{aligned}
& \sup_{\{\phi: I_X(\phi)\le\alpha\}} \sup_{0\le t\le 1} |F^m(\phi)(t) - F^\infty(\phi)(t)| \\
= & \sup_{\{\phi: I_X(\phi)\le\alpha\}} \sup_{0\le t\le 1} |\int_{-\infty}^{-m} [(t-s)^{H-1/2} - (-s)^{H-1/2}]\phi'(s)ds| \\
\le & \sup_{\{\phi: I_X(\phi)\le\alpha\}} \sup_{0\le t\le 1} \Big(\int_{-\infty}^{-m} [(t-s)^{H-1/2} - (-s)^{H-1/2}]^2 ds \int_{-\infty}^{-m} (\phi'(s))^2 ds\Big)^{1/2} \\
\le & \sqrt{2\alpha}\sigma\Big(\int_{-\infty}^{-m} [(1-s)^{H-1/2} - (-s)^{H-1/2}]^2 ds\Big)^{1/2}.
\end{aligned}$$

Therefore,

$$\limsup_{m\to\infty} \sup_{\{\phi: I_X(\phi)\le\alpha\}} \sup_{0\le t\le 1} |F^m(\phi)(t) - F^\infty((\phi)(t)| = 0. \tag{10}$$

It then follows from [11, Theorem 4.2.23] that $\{Y^{(n)}(t), t\in[0,1]\}$ satisfies the desired MDP(sp).
■

Proof. [Corollary 3.2]

Recall that $Y^{(n)}(1) = Y(n)/n$. Since the mapping $f: \psi \mapsto \psi(1)$ from $\mathcal{C}[0,1]$ to $I\!R$ is continuous, it follows from the contraction principle and the MDP(sp) of Theorem 3.1 that the distribution of $Y^{(n)}(1)$ satisfies the MDP with speed $n^{2(1-H)}$ and a good rate function $\Lambda_Y^*(\cdot): I\!R \to I\!R$,

$$\begin{aligned}
& \Lambda_Y^*(\gamma) \\
= & \inf\{I_Y(\psi) : \psi(1) = \gamma\} \\
= & \inf\{I_X(\phi) : \int_0^1 (1-s)^{H-\frac{1}{2}}\phi'(s)ds + \int_{-\infty}^0 [(1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]\phi'(s)ds = \gamma\}.
\end{aligned}$$

From the definition of $I_X(\cdot)$, the infimum above amounts to solving the following minimization problem:

$$\begin{aligned}
\min_{\phi\in\mathcal{AC}_0} \quad & \int_{-\infty}^1 \frac{1}{2\sigma^2}(\phi'(s))^2 ds \\
\text{s.t.} \quad & \int_0^1 (1-s)^{H-\frac{1}{2}}\phi'(s)ds + \int_{-\infty}^0 [(1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]\phi'(s)ds = \gamma.
\end{aligned}$$

It can be directly verified that

$$\phi'_\gamma(s) = \begin{cases} \gamma(2\tilde{H})(1-s)^{H-\frac{1}{2}} & s \in [0,1] \\ \gamma(2\tilde{H})[(1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] & s \in (-\infty, 0] \end{cases}$$

uniquely solves the above minimization problem. Furthermore, the achieved mimimum value is $\tilde{H}\gamma^2/\sigma^2$, as desired.

When $X \equiv B$, the standard Brownian motion, simply replace ϕ by B and treat the integrals as Itô integrals. In particular, we still have $F^m(B) = \tilde{F}^m(B)$, and the above argument still applies. ■

Proof. [Theorem 3.3]

(i) It suffices to show that, for $\epsilon > 0$ arbitrary, we have

$$\lim_{n\to\infty} \frac{\tilde{\mu}_X^{(n)}(B_\epsilon \cap \Gamma)}{\tilde{\mu}_X^{(n)}(\Gamma)} = 0 , \tag{11}$$

where

$$B_\epsilon = \{\phi \in \mathcal{D} : d_\infty(\phi, \phi_\gamma) \geq \epsilon\}$$

and

$$\Gamma = \{\phi \in \mathcal{AC}_0 : \int_0^1 (1-s)^{H-\frac{1}{2}}\phi'(s)ds + \int_{-\infty}^0 [(1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]\phi'(s)ds \geq \gamma\}.$$

We have from the proof of Corollary 3.2 that

$$\inf_{\phi\in\Gamma} \int_{-\infty}^1 \frac{1}{2\sigma^2}(\phi'(s))^2 ds = \int_{-\infty}^1 \frac{1}{2\sigma^2}(\phi'_\gamma(s))^2 ds = \Lambda_Y^*(\gamma) .$$

Furthermore, observe that the set $B_\epsilon \cap \Gamma \cap \{\phi \ : \ I_X(\phi) \leq 2\Lambda_Y^*(\gamma)\}$ is closed. An application of Lemma 2.8 of [26] yields

$$\inf_{\phi\in B_\epsilon\cap\Gamma} \int_{-\infty}^1 \frac{1}{2\sigma^2}(\phi'(s))^2 ds > \Lambda_Y^*(\gamma) .$$

Applying Theorem 3.1 yields (11).

(ii) Since the map $\tilde{F}^m : \mathcal{D} \to \mathcal{C}[0,1]$ defined in (8) is continuous, the probability law of

$$\{\tilde{F}^m(n^{H-1/2}X^{(n)}(\cdot))|Y^{(n)}(1) \geq \gamma\}$$

converges in distribution to $\delta_{\tilde{F}^m(\phi_\gamma)}$. Also, since $\tilde{F}^m(\phi) = F^m(\phi)$ for all $\phi \in \mathcal{BV}$ and $\tilde{F}^m(B) = F^m(B)$, where B denotes Brownian motion and F^m is defined in (7), we conclude that the probability law of

$$\{F^m(n^{H-1/2}X^{(n)}(\cdot))|Y^{(n)}(1) \geq \gamma\}$$

converges in distribution to $\delta_{F^m(\phi_\gamma)}$. Suppressing the argument $t \in [0,1]$, we have

$$\begin{aligned}
|Y^{(n)} - F^\infty(\phi_\gamma)| &= |F^\infty(n^{H-1/2}X^{(n)}) - F^\infty(\phi_\gamma)| \\
&\leq |F^\infty(n^{H-1/2}X^{(n)}) - F^m(n^{H-1/2}X^{(n)})| + |F^m(n^{H-1/2}X^{(n)}) - F^m(\phi_\gamma)| \\
&\quad + |F^m(\phi_\gamma) - F^\infty(\phi_\gamma)|.
\end{aligned}$$

As in (10), it follows that

$$\lim_{m\to\infty} \sup_{0\leq t\leq 1} |F^m(\phi_\gamma)(t) - F^\infty(\phi_\gamma)(t)| = 0,$$

since $I_X(\phi_\gamma) < \infty$. Using (5) then allows us to conclude

$$\lim_{n\to\infty} \mathsf{P}\Big[\sup_{0\leq t\leq 1} |Y^{(n)}(t) - F^\infty(\phi_\gamma)(t)| > \epsilon \,\Big|\, Y^{(n)}(1) \geq \gamma\Big] = 0,$$

for all $\epsilon > 0$, which is the desired result, observing that $F^\infty(\phi_\gamma) = \psi_\gamma$. ■

4 Queueing Applications

Consider a single-server queue with an input process Y, which, in turn, is the output of the filter in (3) subject to an input process X. As noted before, for $H \in (1/2, 1)$ the input process to the queue, Y, typically has long-range dependence.

Denote

$$V(t) = \sup_{0\leq s\leq t} (Y(t) - Y(s) - c(t-s)). \tag{12}$$

Then, V is the *state* process of a queue with input Y and a deterministic service mechanism that depletes work at rate $c > 0$. (Alternatively, $Y(t)$ can be viewed as the mean-zero part of the "netput" — input minus output, with $-ct$ as the deterministic "negative drift" part of the netput. The negative drift is crucial to stability.) In the queueing literature, V is often referred to as the virtual waiting time process, or the workload/inventory process. See, for instance, [5] and [12].

The steady-state limit, $V(\infty)$, is typically obtained through the limit of another process, $M(t) = \sup_{0\leq s\leq t}(-Y(-s) - cs)$, which has the interpretation of the maximum of a random walk. The key linkage between the two is the stationary increment property of the process Y, which guarantees that for each t,

$$V(t) \;=_d\; \sup_{0\leq s\leq t} (Y(t-t) - Y(s-t) - c(t-s))$$

$$
\begin{aligned}
&= \sup_{0 \le s \le t} (-Y(s-t) - c(t-s)) \\
&= M(t).
\end{aligned}
$$

That is, for each t, $V(t)$ and $M(t)$ are equal in distribution, and hence so are $V(\infty)$ and $M(\infty)$. This fact plays a fundamental role in characterizing the steady-state limit, since whereas $M(t)$ is an increasing process (and hence approaches a limit), $V(t)$ is not.

Below, we begin by considering the tail behavior of $M(\infty)$ in Theorem 4.1, based on which the tail behavior of $V(\infty)$ readily follows (Theorem 4.5). Finally, in Theorem 4.6, we present the transient behavior of the workload process, in terms of conditional limits. Throughout, $H \in (1/2, 1)$ and $\tilde{H}$ follows (6).

Theorem 4.1 Let $M(t) = \sup_{0 \le s \le t}(-Y(-s) - cs)$ for some $c > 0$. Then, under the conditions of Theorem 3.1, $M(t)$ converges almost surely to a finite random variable $M(\infty)$, which has a subexponential tail distribution. Specifically,

$$
\lim_{x \to \infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[M(\infty) > x] = -\frac{\tilde{H}}{H} \theta^*, \tag{13}
$$

where

$$
\theta^* = \frac{H}{\sigma^2} \frac{c^{2H}}{H^{2H}(1-H)^{2(1-H)}}.
$$

To prove Theorem 4.1, we require three lemmas, presented below. The proof of the first lemma follows closely the proofs of Theorem 3.1 and Corollary 3.2 (the only difference being $t \in [-1, 0]$ instead of $t \in [0, 1]$), and is hence omitted.

Lemma 4.2 Suppose the conditions in Theorem 3.1 hold. Then the distribution of $\{Y^{(n)}(t), t \in [-1, 0]\}$, denoted $\mu_Y^{(n)}$, satisfies the MDP(sp) in $\mathcal{C}[-1, 0]$ with speed $n^{2(1-H)}$ and a good rate function $\tilde{I}_Y(\cdot) : \mathcal{C}[-1, 0] \mapsto I\!\!R$,

$$
\begin{aligned}
\tilde{I}_Y(\psi) = \inf\{I_X(\phi) : \psi(t) = &\int_{-\infty}^{t} (t-s)^{H-1/2} \phi'(s) ds \\
&+ \int_{-\infty}^{0} -(-s)^{H-1/2} \phi'(s) ds, \ t \in [-1, 0]\}.
\end{aligned}
$$

In addition, the sequence $\{Y(-n)/n, n \ge 0\}$ satisfies the MDP with speed $n^{2(1-H)}$ and a good rate function $\tilde{\Lambda}_Y^*(\cdot) : I\!\!R \mapsto I\!\!R$,

$$
\tilde{\Lambda}^*{}_Y(\gamma) = \frac{\tilde{H}}{\sigma^2} \gamma^2 .
$$

Lemma 4.3 For all $\gamma \geq 0$, under the conditions of Theorem 3.1,

$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[M(n)/n > \gamma] = -\tilde{g}(\gamma),$$

where

$$\tilde{g}(\gamma) = \begin{cases} \frac{\tilde{H}}{\sigma^2}(c+\gamma)^2 & \text{if } \gamma \geq \frac{c(1-H)}{H} \\ \frac{\tilde{H}}{\sigma^2}\frac{c^{2H}}{H^{2H}(1-H)^{2(1-H)}}\gamma^{2(1-H)} & \text{if } \gamma < \frac{c(1-H)}{H} \end{cases}.$$

Proof. By the contraction principle and Lemma 4.2, the sequence $\{M(n)/n, n \geq 0\}$ satisfies the MDP(sp) with speed $n^{2(1-H)}$ and we may write

$$\limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[M(n)/n \geq \gamma] \leq -\inf_{-1\leq s\leq 0} \{ \inf_{\tilde{F}_s} \int_{-\infty}^{0} \frac{1}{2\sigma^2}(\phi'(\tau))^2 d\tau \} \tag{14}$$

where

$$\begin{aligned} \tilde{F}_s &= \{\phi \in \mathcal{AC}_0 : -[\int_{-\infty}^{s} (s-\tau)^{H-1/2}\phi'(\tau)d\tau + \int_{-\infty}^{0} -(-\tau)^{H-1/2}\phi'(\tau)d\tau] + cs \geq \gamma\} \\ &= \{\phi \in \mathcal{AC}_0 : -[\int_{-\infty}^{s} [(s-\tau)^{H-1/2} - (-\tau)^{H-1/2}]\phi'(\tau)d\tau \\ &\qquad + \int_{s}^{0} -(-\tau)^{H-1/2}\phi'(\tau)d\tau] \geq (\gamma - cs)\} . \end{aligned}$$

It can be directly verified (using the Cauchy-Schwartz inequality) that the solution to the minimization problem within braces in (14), ϕ, satisfies the following:

$$\phi'(t) = \begin{cases} [2\tilde{H}(\gamma - cs)/|s|^{2H}] \cdot (-t)^{H-1/2} & t \in [s, 0] \\ -[2\tilde{H}(\gamma - cs)/|s|^{2H}] \cdot [(s-t)^{H-1/2} - (-t)^{H-1/2}] & t \in (-\infty, s] \end{cases}.$$

Furthermore,

$$\begin{aligned} & \inf_{\tilde{F}_s} \int_{-\infty}^{0} \frac{1}{2\sigma^2}(\phi'(\tau))^2 d\tau \\ = \;& (\gamma - cs)^2/[2\sigma^2(\int_{-\infty}^{s} [(s-\tau)^{(H-1/2)} - (-\tau)^{(H-1/2)}]^2 d\tau + \int_{s}^{0} (-(-\tau)^{(H-1/2)})^2 d\tau)]. \end{aligned}$$

By a change of variables, $\tau \to |s|\tau$, the denominator above becomes

$$\begin{aligned} & 2\sigma^2|s|^{2H}(\int_{-\infty}^{-1} [(-1-\tau)^{(H-1/2)} - (-\tau)^{(H-1/2)}]^2 d\tau + \int_{-1}^{0} (-(-\tau)^{(H-1/2)})^2 d\tau) \\ = \;& 2\sigma^2|s|^{2H}\Big[\int_{-\infty}^{0} [(-\tau)^{(H-1/2)} - (1-\tau)^{(H-1/2)}]^2 d\tau + \frac{1}{2H}\Big] \\ = \;& \sigma^2|s|^{2H}/\tilde{H}. \end{aligned}$$

Therefore,

$$\inf_{\tilde{F}_s} \int_{-\infty}^{0} \frac{1}{2\sigma^2}(\phi'(\tau))^2 d\tau = \frac{\tilde{H}(\gamma - cs)^2}{\sigma^2 |s|^{2H}} .$$

The right hand side above is minimized at

$$s^* = \begin{cases} -H\gamma/[c(1-H)] & \text{if } \gamma < c(1-H)/H \\ -1 & \text{otherwise} \end{cases} ,$$

with the minimum following $\tilde{g}(\gamma)$ as stated. A lower bound is obtained similarly and the lemma follows by the continuity of $\tilde{g}$. ■

Lemma 4.4 Let $Z(n) = \sup_{n-1<s\leq n}(-Y(-s) - cs)$, $n = 1, 2, \ldots$. For all $\gamma \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[Z(n)/n > \gamma] = -\frac{\tilde{H}}{\sigma^2}(c+\gamma)^2.$$

Proof. We establish the limit by showing that the liminf and the limsup are dominated from below and from above by the same bound. Note that $Z(n) \geq -Y(-n) - cn$. Hence, the lower bound follows from the liminf part of the MDP for $Y(-n)$ in Lemma 4.2.

To establish the upper bound, we have

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[Z(n)/n \geq \gamma] &= \limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[\sup_{1-1/n<s\leq 1} (-Y^{(n)}(-s) - cs) \geq \gamma] \\ &\leq \limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[\sup_{1-\delta\leq s\leq 1} (-Y^{(n)}(-s) - cs) \geq \gamma], \end{aligned}$$

for any $0 < \delta < 1$. Following an argument similar to the proof of Lemma 4.3, we have

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[\sup_{1-\delta\leq s\leq 1} (-Y^{(n)}(-s) - cs) \geq \gamma] &\leq -\inf_{1-\delta\leq s\leq 1} \frac{\tilde{H}}{\sigma^2} \frac{(\gamma + cs)^2}{s^{2H}} \\ &= -g_\delta(1-\gamma), \end{aligned}$$

where

$$g_\delta(\gamma) = \begin{cases} \frac{\tilde{H}}{\sigma^2}(c+\gamma)^2 & \text{if } \gamma \geq \frac{c(1-H)}{H} \\ \frac{\tilde{H}}{\sigma^2} \frac{c^{2H}}{H^{2H}(1-H)^{2(1-H)}} \gamma^{2(1-H)} & \text{if } \frac{c(1-H)}{H}\delta < \gamma < \frac{c(1-H)}{H} \\ \frac{\tilde{H}}{\sigma^2} \frac{(c\delta+\gamma)^2}{\delta^{2H}} & \text{if } \frac{c(1-H)}{H}\delta \geq \gamma \end{cases} .$$

Since δ is arbitrary, letting $\delta \to 1$ yields the upper bound, noting that

$$\lim_{\delta\to 1} g_\delta(\gamma) = \frac{\tilde{H}}{\sigma^2}(c+\gamma)^2.$$

■

Proof. [Theorem 4.1]

Since $M(t)$ is an increasing sequence, it converges almost surely to a random variable $M(\infty)$.

Again, we prove the limit (13) by arguing that the liminf and the limsup are dominated from below and from above by the same bound.

To establish the lower bound, we proceed as in Chang et al [7]. For all $\gamma > 0$, we have

$$\begin{aligned}\liminf_{x\to\infty}\frac{1}{x^{2(1-H)}}\log\mathsf{P}[M(\infty)>x] &\geq \liminf_{n\to\infty}\frac{1}{(\gamma n)^{2(1-H)}}\log\mathsf{P}[-Y(-n)-cn>\gamma n]\\ &\geq -\frac{\tilde{H}}{\sigma^2}\frac{(\gamma+c)^2}{\gamma^{2(1-H)}},\end{aligned}$$

where we have used Lemma 4.2. Thus,

$$\begin{aligned}\liminf_{x\to\infty}\frac{1}{x^{2(1-H)}}\log\mathsf{P}[M(\infty)>x] &\geq -\inf_{\gamma>0}\frac{\tilde{H}}{\sigma^2}\frac{(\gamma+c)^2}{\gamma^{2(1-H)}}\\ &= -\frac{\tilde{H}}{H}\theta^*.\end{aligned}$$

To establish the upper bound, note that for all $\gamma > 0$

$$\limsup_{x\to\infty}\frac{1}{x^{2(1-H)}}\log\mathsf{P}[M(\infty)\geq x] = \limsup_{n\to\infty}\frac{1}{(\gamma n)^{2(1-H)}}\log\mathsf{P}[M(\infty)\geq\gamma n],$$

and write

$$\begin{aligned}\mathsf{P}[M(\infty)\geq\gamma n] &= \mathsf{P}[M(n)\geq\gamma n]+\mathsf{P}[M(n)<\gamma n,\ \sup_{s>n}(-Y(-s)-cs)\geq\gamma n]\\ &\leq \mathsf{P}[M(n)\geq\gamma n]+\mathsf{P}[\sup_{m\geq n+1}Z(m)\geq 0],\end{aligned} \tag{15}$$

where $Z(n)$ follows the definition in Lemma 4.4. If $\gamma < \frac{c(1-H)}{H}$ then, from Lemma 4.3,

$$\limsup_{n\to\infty}\frac{1}{(\gamma n)^{2(1-H)}}\log\mathsf{P}[M(n)\geq\gamma n] = -\frac{\tilde{H}}{H}\theta^*. \tag{16}$$

Next, in view of the upper bound for $\{Z(n)/n, n\geq 0\}$ in Lemma 4.4, for every $\epsilon > 0$ there exists an n_0 such that for all $n \geq n_0$ we have

$$\mathsf{P}(Z(n)/n\geq 0)\leq\exp\Big(-n^{2(1-H)}(\frac{\tilde{H}}{\sigma^2}c^2-\epsilon)\Big).$$

Choosing ϵ such that $0 < \epsilon < \frac{\tilde{H}}{2\sigma^2}c^2$ and then γ small enough so that

$$\gamma^{2(1-H)}\theta^*\frac{\tilde{H}}{H}+\epsilon<\frac{\tilde{H}}{\sigma^2}c^2-\epsilon \quad\text{and}\quad \gamma<\frac{c(1-H)}{H}$$

we therefore have, for $n \geq n_0$,

$$\begin{aligned}
\mathsf{P}[\sup_{m \geq n+1} Z(m) \geq 0] &\leq \sum_{m=n+1}^{\infty} \mathsf{P}[Z(m)/m \geq 0] \\
&\leq \sum_{m=n+1}^{\infty} \exp\Big(-n^{2(1-H)}(\gamma^{2(1-H)}\frac{\tilde{H}}{H}\theta^* + \epsilon)\Big). \qquad (17)
\end{aligned}$$

Simple algebra yields

$$\limsup_{n\to\infty} \frac{1}{(\gamma n)^{2(1-H)}} \log \sum_{m=n+1}^{\infty} \exp\Big(-n^{2(1-H)}(\gamma^{2(1-H)}\frac{\tilde{H}}{H}\theta^* + \epsilon)\Big) \leq -\frac{\tilde{H}}{H}\theta^*. \qquad (18)$$

(Note that for $\alpha > 0, z > 0$, $n > (\frac{2}{\alpha z})^{1/\alpha}$

$$ne^{-n^\alpha z} = \int_n^\infty (\alpha z s^\alpha - 1)e^{-s^\alpha z} ds \geq \int_n^\infty e^{-s^\alpha z} ds.$$

Choose $\alpha = 2(1-H)$, $z = \gamma^{2(1-H)}\tilde{H}\theta^*/H + \epsilon$, with $\epsilon > 0$ arbitrarily small.) Therefore, combining (15), (16), (17) and (18), we have established the desired upper bound. ∎

As noted earlier, the stationary increment property of Y ensures that $V(t) =_d M(t)$, for each $t \in [0, \infty)$. This is certainly the case when $X \equiv B$, the standard Brownian motion. When X is a sample-path process, we can directly verify that $V(n) =_d M(n)$, for each integer n. To see this, it suffices to observe that, for any $u \in [0,1]$, we have

$$\begin{aligned}
n^{3/2-H}[Y^{(n)}(1) - Y^{(n)}(u)] &= \sum_{j=-\infty}^{n} (1-j/n)^{H-1/2}a(j) - \sum_{j=-\infty}^{[nu]} (u-j/n)^{H-1/2}a(j) \\
&=_d \sum_{j=-\infty}^{n} (1-j/n)^{H-1/2}a(j+n) - \sum_{j=-\infty}^{[nu]} (u-j/n)^{H-1/2}a(j+n) \\
&= \sum_{j=-\infty}^{0} (-j/n)^{H-1/2}a(j) - \sum_{j=-\infty}^{[n(u-1)]} [(u-1)-j/n]^{H-1/2}a(j) \\
&= n^{3/2-H}[Y^{(n)}(0) - Y^{(n)}(u-1)]\,.
\end{aligned}$$

(The equality $=_d$ above follows from the stationarity of $\{a(t)\}$.)

The following theorem is a direct consequence of Lemma 4.3 and Theorem 4.1. (Given the above discussion, part (ii) of the Theorem should be interpreted in the discrete time setting when X is a sample-path process.)

Theorem 4.5 Suppose that the conditions in Theorem 4.1 hold.

(i) For all $\gamma \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}[V^{(n)}(1) > \gamma] = -\tilde{g}(\gamma) .$$

(ii) The random variable $V(t)$ converges in distribution to a finite random variable $V(\infty)$, which has a subexponential tail distribution. Specifically,

$$\lim_{x\to\infty} \frac{1}{x^{2(1-H)}} \log \mathsf{P}[V(\infty) > x] = -\frac{\tilde{H}}{H}\theta^*.$$

Remark. When Y is the FBM, the result in Theorem 4.5 is consistent with the result in Duffield and O'Connell [12] for FBM. To see this, observe that when $X \equiv \sqrt{2\tilde{H}}B$, with B being the standard Brownian motion, $\mathsf{E}[Y^2(t)] = t^{2H}$ for $t \in [0,\infty)$, which is consistent with relation (57) of [12]. In addition, X satisfies the conditions in Theorem 3.1 with $\sigma^2 = 2\tilde{H}$. It then suffices to note that the minimization problem in (60) of [12] achieves the value θ^*.

Theorem 4.6 (i) Let $\hat{\mu}_X^{(n)}$ denote the law of $\{n^{H-\frac{1}{2}}X^{(n)}(\cdot)|V^{(n)}(1) \geq \gamma\}$ for some $\gamma \geq 0$. Then, $\hat{\mu}_X^{(n)}$ converges in distribution on $\mathcal{D}$ to $\delta_{\hat{\phi}_\gamma}$, where $\hat{\phi}_\gamma(t) = \phi_\gamma(t-1) - \phi_\gamma(-1)$ for $t \in (-\infty, 1]$, with ϕ_γ given in the proof of Lemma 4.3.
(ii) Let $\hat{\mu}_V^{(n)}$ denote the law of $\{V^{(n)}(\cdot)|V^{(n)}(1) \geq \gamma\}$. Then $\hat{\mu}_V^{(n)}$ converges in distribution on $\mathcal{C}[0,1]$ to $\delta_{\hat{\psi}_\gamma}$, where, for $t \in [0,1]$,

$$\hat{\psi}_\gamma(t) = \sup_{0\leq s\leq t} [\int_0^t (t-\tau)^{H-1/2} d\hat{\phi}_\gamma(\tau) + \int_{-\infty}^0 [(t-\tau)^{H-1/2} - (-\tau)^{H-1/2}] d\hat{\phi}_\gamma(\tau)$$
$$- \int_0^s (s-\tau)^{H-1/2} d\hat{\phi}_\gamma(\tau) - \int_{-\infty}^0 [(s-\tau)^{H-1/2} - (-\tau)^{H-1/2}] d\hat{\phi}_\gamma(\tau) - c(t-s)].$$

Remark. The most likely path to a large build up, $\hat{\psi}_\gamma$, has been studied in the case of input with short-range dependence by Anantharam [2], Chang [6] and Dembo and Zajic [10], where it has been found to be linear. Here, as in [7], the most likely path is *nonlinear*. In Figure 1, below, we have plotted the function $\hat{\psi}_\gamma$ for different values of H.

Proof. [Theorem 4.6] As in the proof of Theorem 3.3 (i), it suffices to show that

$$\inf_{\phi\in F\cap B_\epsilon} \int_{-\infty}^1 (\phi'(\tau))^2 d\tau > \Lambda_V^*(\gamma) = \tilde{g}(\gamma) ,$$

where $B_\epsilon = \{\phi : \sup_{t\in[0,1]} |\phi(t) - \hat{\phi}_\gamma(t)| \geq \epsilon\}$, $F = \cup_{s\in[0,1]} F_s$, and

$$F_s = \{\phi \in \mathcal{AC}_0 : \int_{-\infty}^1 (1-\tau)^{(H-1/2)}\phi'(\tau)d\tau - \int_{-\infty}^s (s-\tau)^{(H-1/2)}\phi'(\tau)d\tau - c(1-s) \geq \gamma\} .$$

Letting $\hat{\phi}(\cdot) = \phi(\cdot\ +1)$, we have

$$F_s = \{\hat{\phi} \in \mathcal{AC}_0 : \int_{-\infty}^{0} (-\tau)^{(H-1/2)} \hat{\phi}'(\tau) d\tau - \int_{-\infty}^{s-1} (s-1-\tau)^{(H-1/2)} \hat{\phi}'(\tau) d\tau - c(1-s) \geq \gamma\} .$$

If $\phi(\cdot) \in \tilde{F}_s$ (where $\tilde{F}_s$ was defined in the proof of Lemma 4.3), then $\hat{\phi}(\cdot) = \phi(\cdot\ -1) \in F_{1+s}$, and

$$\int_{-\infty}^{0} (\phi'(\tau))^2 d\tau = \int_{-\infty}^{1} (\hat{\phi}'(\tau))^2 d\tau . \tag{19}$$

Likewise, if $\hat{\phi}(\cdot) \in F_s$, then $\phi(\cdot) = \hat{\phi}(\cdot\ +1) \in \tilde{F}_{s-1}$ and (19) holds. From this it is clear that

$$\inf_{\phi \in F} \int_{-\infty}^{1} (\phi'(\tau))^2 d\tau = \int_{-\infty}^{1} (\hat{\phi}_\gamma'(\tau))^2 d\tau = \tilde{g}(\gamma) .$$

That

$$\inf_{\phi \in F \cap B_\epsilon} \int_{-\infty}^{1} (\phi'(\tau))^2 d\tau > \Lambda_V^*(\gamma)$$

holds true may be shown as in the proof of Theorem 3.3 (i).

For part (ii), make use of the continuity of the mapping as in the proof of Theorem 3.3 (ii).

■

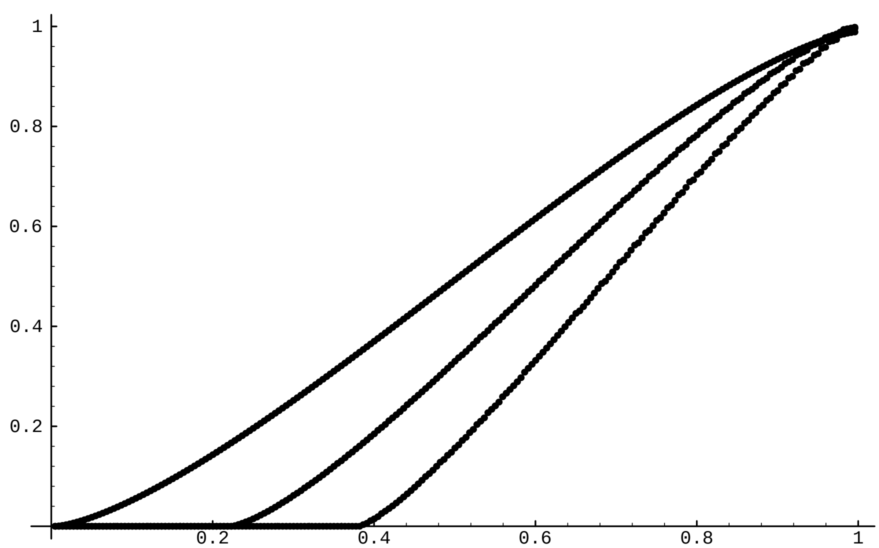

Figure 1. $\hat{\psi}_\gamma$ with $c = 3$, $\gamma = 1$ and, from left to right, $H = .75$, $H = .70$ and $H = .65$.

5 Verifying the Key Assumptions

Here we verify the assumptions outlined in Section 3 (preceeding Theorem 3.1). First we show that the condition in (5) holds when X is the standard Brownian motion (Proposition 5.1). Next, when X is a sample-path process constructed from the sequence $\{a(t)\}$, we establish the condition, assuming that $\{a(t)\}$ admits an appropriate representation involving a martingale difference sequence (Lemma 5.2). With this representation, we also verify that, for each n, $\{Y^{(n)}_{-j/n}\}_{j=1}^{\infty}$ is a Cauchy sequence in $L^2[\mathcal{C}[0,1]]$ (Proposition 5.4). Finally, in the last three propositions we revisit the examples of Section 2.

Proposition 5.1 When $X = B$, the standard Brownian motion, (5) holds for any $H \in (1/2, 1)$ and $\eta > 0$.

Proof. Since $B(nt)/\sqrt{n}$ is also a (standard) Brownian motion, we have

$$\begin{aligned}&\int_{-\infty}^{-m}[(t-s)^{H-1/2}-(-s)^{H-1/2}]n^{H-1}d(\frac{B(ns)}{\sqrt{n}})\\ =_d\ &\int_{-\infty}^{-m}[(t-s)^{H-1/2}-(-s)^{H-1/2}]n^{H-1}dB(s)\,.\end{aligned}$$

By a result of Marcus and Shepp [20] (see also page 43 of Adler [1]) regarding the tail behavior of the distribution of the supremum of a Gaussian process, we have

$$\lim_{n\to\infty}\frac{1}{n^{2(1-H)}}\log \mathsf{P}(\sup_{t\in[0,1]}|\int_{-\infty}^{-m}[(t-s)^{H-1/2}-(-s)^{H-1/2}]n^{H-1}dB(s)|\geq\eta) = -(2\sigma_m^2)^{-1}\ ,$$

where

$$\begin{aligned}\sigma_m^2 &= \sup_{t\in[0,1]} E[\int_{-\infty}^{-m}[(t-s)^{H-1/2}-(-s)^{H-1/2}]dB(s)]^2\\ &= \sup_{t\in[0,1]}\int_{-\infty}^{-m}[(t-s)^{H-1/2}-(-s)^{H-1/2}]^2 ds\ .\end{aligned}$$

Since $\sigma_m \to 0$, the desired result follows. ■

Next we consider the case of a sample-path process, i.e. $X = A^{(1)}$ in the notation of Section 2, which is constructed from a stationary sequence of bounded, mean zero, real-valued random variables, $\{a(t), t \in \mathbb{N}\}$.

By Taylor's theorem, we have, for any positive integers m and h,

$$\begin{aligned}
&\mathsf{P}(\sup_{t\in[0,1]}|\int_{-h}^{-m}[(t-s)^{H-1/2}-(-s)^{H-1/2}]d(n^{H-1/2}X^{(n)}(s))|\geq\eta)\\
&\leq\mathsf{P}\Big(\sup_{t\in[0,1]}|\int_{-h}^{-m}(H-1/2)t(-s)^{H-3/2}d(n^{H-1/2}X^{(n)}(s))|\\
&\quad+\sup_{t\in[0,1]}|\int_{-h}^{-m}\frac{1}{2}(H-1/2)(H-3/2)(t-s)^{H-5/2}d(n^{H-1/2}X^{(n)}(s))|\geq\eta\Big)\\
&\leq\mathsf{P}\Big(\frac{1}{2}|\int_{-h}^{-m}(-s)^{H-3/2}d(n^{H-1/2}X^{(n)}(s))|\geq\eta/2\Big)\\
&\quad+\mathsf{P}\Big(\frac{1}{8}\sup_{t\in[0,1]}|\int_{-h}^{-m}(t-s)^{H-5/2}d(n^{H-1/2}X^{(n)}(s))|\geq\eta/2\Big)\,.
\end{aligned}$$

Hence, to establish the condition in (5), it suffices to show the following limits, for any $H\in(1/2,1)$ and $\eta>0$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\lim_{h\to\infty}\frac{1}{n^{2(1-H)}}\log\mathsf{P}(|\int_{-h}^{-m}(-s)^{H-3/2}d(n^{H-1/2}X^{(n)}(s))|\geq\eta)=-\infty \tag{20}$$

and

$$\lim_{m\to\infty}\limsup_{n\to\infty}\lim_{h\to\infty}\frac{1}{n^{2(1-H)}}\log\mathsf{P}(\sup_{t\in[0,1]}|\int_{-h}^{-m}(t-s)^{H-5/2}d(n^{H-1/2}X^{(n)}(s))|\geq\eta)=-\infty\,. \tag{21}$$

To establish the limits in (20) and (21), we further assume that $a(-k)$ can be expressed in the following form:

$$a(-k)=D_k+Z_k-Z_{k+1}, \tag{22}$$

for $k=0,1,\ldots$, where $\{Z_k\}$ is a bounded sequence, and $\{D_k\}$ is a bounded martingale difference sequence. That is, D_k is $\mathcal{F}_j$-measurable for $k\leq j$, where $\{\mathcal{F}_j\}$ is some filtration, $\mathsf{E}(D_j|\mathcal{F}_{j-1})=0$ a.s., and $|D_j|\leq C$ for some constant $C<\infty$.

We shall find useful the following inequality for martingale differences, due originally to Azuma (following Rhee and Talagrand [23], where Stout [27] is referred to for a proof). In our context, the inequality takes the following form:

Lemma 5.2 Suppose $\{D_j;\, j\geq 0\}$ is a martingale difference sequence, and $|D_j|\leq d_j$ for each j, where d_j is a constant. Then, for any $z>0$, we have

$$\mathsf{P}(|\sum_{j=0}^{n}D_j|\geq z)\leq 2\exp(-\frac{z^2}{2\sum_{j=0}^{n}d_j^2}).$$

Lemma 5.3 Suppose $a(-k) = D_k + Z_k - Z_{k+1}$, for $k = 0, 1, \ldots$, with $\{Z_k\}$ a bounded sequence, and $\{D_k\}$ a bounded martingale difference sequence. Then, (20) and (21) hold for any $H \in (1/2, 1)$ and $\eta > 0$.

Proof. We first establish the limit in (20). With $h = \ell/n + m$, $\ell \geq nm$ a positive integer, we have

$$\begin{aligned}
&\int_{-h}^{-m} (-s)^{H-3/2} d(n^{H-1/2} X^{(n)}(s)) \\
= &\sum_{j=nm}^{\ell+nm} j^{H-3/2} a(-j) \\
= &\sum_{j=nm}^{\ell+nm} j^{H-3/2} D_j + \sum_{j=nm}^{\ell+nm} [(j+1)^{H-3/2} - j^{H-3/2})] Z_{j+1} \\
&\quad + (nm)^{H-3/2} Z_{nm} - (\ell + nm + 1)^{H-3/2} Z_{\ell+nm+1} \, .
\end{aligned}$$

Since Z_k are bounded, the second, third and fourth terms in the last expression above can be made less than $\eta/2$ by choosing m large enough. It suffices, therefore, to consider only the first term. Making use of Lemma 5.2, with $\{j^{H-3/2} D_j\}$ the martingale difference, and $d_j = j^{H-3/2} C$ (recall $C \geq |D_j|$), we have

$$\begin{aligned}
\mathsf{P}(|\sum_{j=nm}^{l} j^{H-3/2} D_j| \geq \eta/2) &\leq 2\exp\Big[-\frac{\eta^2}{8C^2 \sum_{j=nm}^{\ell+nm} j^{2H-3}}\Big] \\
&\leq 2\exp\Big[-\frac{n^{2-2H}\eta^2}{C_1 m^{2H-2}}\Big],
\end{aligned}$$

where C_1 is some positive constant. Therefore

$$\limsup_{n\to\infty} \lim_{h\to\infty} \frac{1}{n^{2(1-H)}} \log \mathsf{P}(|\int_{-h}^{-m} (-s)^{H-3/2} d(n^{H-1/2} X^{(n)}(s))| \geq \eta) \leq -\frac{\eta^2}{C_1} m^{2-2H} \, . \tag{23}$$

Letting $m \to \infty$ yields the desired limit in (20).

Next, we prove (21). As before, with $h = \ell/n + m$, we have

$$\begin{aligned}
&\int_{-h}^{-m} (t-s)^{H-5/2} d(n^{H-1/2} X^{(n)}(s)) \\
= \; &n \sum_{j=nm}^{\ell+nm} (j+tn)^{H-5/2} a(-j) \\
= \; &n \sum_{j=nm}^{\ell+nm} (j+tn)^{H-5/2} D_j + n \sum_{j=nm}^{\ell+nm} [(j+tn+1)^{H-5/2} - (j+tn)^{H-5/2}] Z_{j+1} \\
&\quad + n(nm+tn)^{H-5/2} Z_{nm} - n(\ell+nm+1+tn)^{H-5/2} Z_{\ell+nm+1} \, .
\end{aligned}$$

Again, we only need to bound the first term above, which we denote as

$$S_n(t) = n\sum_{j=nm}^{\ell+nm}(j+tn)^{H-5/2}D_j.$$

Note that

$$\begin{aligned}
\mathsf{P}(\sup_{0\le t\le 1}|S_n(t)|\ge\eta) &\le \sum_{i=1}^{n}\mathsf{P}(\sup_{\frac{i-1}{n}\le t\le\frac{i}{n}}|S_n(t)|\ge\eta)\\
&\le n\max_{1\le i\le n}\mathsf{P}(\sup_{\frac{i-1}{n}\le t\le\frac{i}{n}}|S_n(t)|\ge\eta). \qquad (24)
\end{aligned}$$

Furthermore,

$$\sup_{\frac{i-1}{n}\le t\le\frac{i}{n}}|S_n(t)| \le |S_n(\frac{i-1}{n})| + \sup_{\frac{i-1}{n}\le t\le\frac{i}{n}}|S_n(t)-S_n(\frac{i-1}{n})|, \qquad (25)$$

while,

$$\begin{aligned}
|S_n(t)-S_n(\frac{i-1}{n})| &= n|\sum_{j=nm}^{\ell+nm}[(j+tn)^{H-5/2}-(j+i-1)^{H-5/2})D_j|\\
&\le nC\sum_{j=nm}^{\ell+nm}[(j+i-1)^{H-5/2}-(j+i)^{H-5/2}]\\
&\le nC\sum_{j=nm}^{\infty}(5/2-H)(j+i)^{H-7/2}\\
&\le nC_2(nm)^{H-5/2},
\end{aligned}$$

for some positive constant C_2. Hence, choosing m sufficiently large, the second term on the right hand side of (25) can be bounded by $\eta/2$. Applying Lemma 5.2 once more, we have

$$\begin{aligned}
\mathsf{P}(\sup_{\frac{i-1}{n}\le t\le\frac{i}{n}}|S_n(t)|\ge\eta) &\le \mathsf{P}[|S_n(\frac{i-1}{n})|\ge\eta/2]\\
&= \mathsf{P}[\sum_{j=nm}^{\ell+nm}(j+i-1)^{H-5/2}D_j\ge\eta/(2n)]\\
&\le 2\exp\Big[-\frac{\eta^2}{8n^2C^2\sum_{j=nm}^{\ell+nm}(j+i-1)^{2H-5}}\Big]\\
&\le 2\exp\Big[-\frac{\eta^2}{C_3n^2(nm)^{2H-4}}\Big].
\end{aligned}$$

for some positive constant C_3. Putting the above together with (24), we have

$$\begin{aligned}
\limsup_{n\to\infty}\lim_{h\to\infty}\frac{1}{n^{2(1-H)}}\log\mathsf{P}(\sup_{0\le t\le 1}|S_n(t)|\ge\eta) &\le \limsup_{n\to\infty}[\frac{\log n}{n^{2(1-H)}}]-\frac{\eta^2}{C_3m^{2H-4}}\\
&= -\frac{\eta^2}{C_3}m^{4-2H}.
\end{aligned}$$

Letting $m \to \infty$ completes the argument. ■

Proposition 5.4 For each positive integer n, the sequence $\{Y^{(n)}_{-j/n}\}_{j=1}^{\infty}$ is a Cauchy sequence in $L^2[\mathcal{C}[0,1]]$.

Proof. For each n, we must show the following:

$$\begin{aligned}
&\lim_{\ell,m\to\infty} \mathsf{E}(\sup_{t\in[0,1]} |Y^{(n)}_{-\ell/n}(t) - Y^{(n)}_{-m/n}(t)|^2) \\
= &\lim_{\ell,m\to\infty} \mathsf{E}(\sup_{t\in[0,1]} |\int_{-\ell/n}^{-m/n} [(t-s)^{H-1/2} - (-s)^{H-1/2}] d(n^{H-1/2} X^{(n)}(s))|^2) = 0 \,.
\end{aligned}$$

Applying Taylor's theorem as we did at the beginning of this section, we see that it is sufficient to show (keeping in mind that n is fixed):

$$\lim_{\ell,m\to\infty} \mathsf{E}(|\int_{-\ell/n}^{-m/n} (-s)^{H-3/2} d(X^{(n)}(s))|^2) = 0, \tag{26}$$

and

$$\lim_{\ell,m\to\infty} \mathsf{E}(\sup_{t\in[0,1]} |\int_{-\ell/n}^{-m/n} (t-s)^{H-5/2} d(X^{(n)}(s))|^2) = 0 \,. \tag{27}$$

To see that (27) holds, observe that

$$\begin{aligned}
\int_{-\ell/n}^{-m/n} (t-s)^{H-5/2} d(X^{(n)}(s)) &= \frac{1}{n} \sum_{j=-\ell}^{-m} (t - j/n)^{H-5/2} a(j) \\
&= \frac{1}{n} \sum_{j=\ell}^{m} (t + j/n)^{H-5/2} a(-j).
\end{aligned}$$

Since $\{a(t), t \in \mathbb{N}\}$ is bounded and n is fixed, the limit in (27) is indeed zero.

To see that (26) holds, write

$$\mathsf{E}(|\int_{-\ell/n}^{-m/n} (-s)^{H-3/2} d(X^{(n)}(s))|^2) = \frac{1}{n} \mathsf{E}(|\sum_{j=-\ell}^{-m} (-j/n)^{H-3/2} (D_j + Z_j - Z_{j+1})|^2) \,.$$

As in the proof of Lemma 5.3, we may ignore the Z terms. For the remaining term, since $\{D_j\}$ is a martingale difference sequence, we have

$$\mathsf{E}(|\sum_{j=-\ell}^{-m} (-j/n)^{H-3/2} D_j|^2) = \sum_{j=-\ell}^{-m} (-j/n)^{2H-3} \mathsf{E}[D_j^2] \,.$$

From this, (26) readily follows (taking into account that D_j is bounded). ■

Proposition 5.5 Let $\{a(t), t \in \mathbb{N}\}$ be the sequence defined in Example 1 of Section 2. Suppose that the reversed process, $\{b(-t),\ t = 0, 1, 2, \ldots\}$, corresponds to a Markov chain with kernel P such that the Poisson equation, $Ph - h = f$, admits a bounded solution h. (Without loss of generality, assume $\mathsf{E}[f(b(t))] = 0$.) Then, (5) holds for any $H \in (1/2, 1)$ and $\eta > 0$.

Proof. Write

$$\begin{aligned} f(b(-t)) &= Ph(b(-t)) - h(b(-t)) \\ &= [Ph(b(-t)) - h(b(-t-1))] + [-h(b(-t))] - [-h(b(-t-1))]. \end{aligned} \tag{28}$$

Note that $\{Ph(b(-t)) - h(b(-t-1))\}$ is a martingale difference sequence with respect to the filtration $\mathcal{F}_t = \sigma\{b(-t), b(-t+1), \ldots\}$. We may now identify D_k and Z_k with $\{Ph(b(-k)) - h(b(-k-1))\}$ and $-h(b(-k))$, respectively. ∎

Proposition 5.6 Let $\{a(t), t \in \mathbb{N}\}$ be the moving-average sequence defined in Example 2 of Section 2. Assume that

$$\sum_{k=1}^{\infty}\{\sum_{j=-\infty}^{-k} |b(j)| + \sum_{j=k}^{\infty} |b(j)|\} < \infty\ . \tag{29}$$

Then, (5) holds for any $H \in (1/2, 1)$ and $\eta > 0$.

Proof. Let $\mathcal{F}_k = \sigma\{\xi(j),\ j = -k, -k+1, \ldots\}$. Then, it can be directly verified that the given condition on $b(j)$ leads to the sequences

$$D_k = \sum_{i=-\infty}^{+\infty} \{\mathsf{E}[a(-i)|\mathcal{F}_k] - \mathsf{E}[a(-i)|\mathcal{F}_{k-1}]\},\ \ k = 0, 1, \ldots,$$

and

$$Z_k = \sum_{i=0}^{+\infty} \mathsf{E}[a(-i)|\mathcal{F}_{k-1}] - \sum_{i=-\infty}^{-1} [a(-i-k) - \mathsf{E}(a(-i)|\mathcal{F}_{k-1})],\ \ k = 0, 1, \ldots,$$

being well defined and bounded. Furthermore, it readily follows that D_k is a martingale difference sequence with respect to the filtration $\mathcal{F}_k$ and that the representation (22) holds. The desired conclusion then follows from Lemma 5.3. ∎

Remark. Note that the condition in (29) also plays a key role in [16] to imply that the infinite moving-average sequence satisfies the functional central limit theorem and law of iterated logarithm.

Proposition 5.7 Let $\{a(t), t \in \mathbb{N}\}$ be as in Example 3 of Section 2. Assume that the corresponding reversed sequence is ϕ-mixing and $\sum_{k=1}^{\infty} \phi(k) < \infty$. We have that (5) holds for any $H \in (1/2, 1)$ and $\eta > 0$.

Proof. Let $\mathcal{F}_k = \sigma\{a(k),\ j = -k, -k+1, \ldots\}$. It can be directly verified (cf. [13]), under the assumed conditions, that

$$D_k = \sum_{i=0}^{+\infty} \{\mathsf{E}[a(-i-k)|\mathcal{F}_k] - \mathsf{E}[a(-i-k)|\mathcal{F}_{k-1}]\},\ \ k = 0, 1, \ldots,$$

is a bounded martingale difference sequence,

$$Z_k = \sum_{i=0}^{+\infty} \mathsf{E}[a(-i-k)|\mathcal{F}_{k-1}],\ \ k = 0, 1, \ldots,$$

is bounded, and that the relation in (22) is satisfied. That (5) holds then follows from Lemma 5.3. ■

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