On the level-lines and geometry of vector-valued images *

Do Hyun Chung and Guillermo Sapiro†

Abstract

In this note we extend the concept of level-lines of scalar images to vector-valued data. Consistent with the scalar case, we define the level-lines of vector-valued images as the integral curves of the directions of minimal vectorial change. This direction, and the magnitude of the change, are computed using classical Riemannian geometry. As an example of the use of this new concept, we show how to visualize the basic geometry of vector-valued images with a scalar image.

Index Terms: Level-lines, vector-valued images, Riemannian geometry.

EDICS: SPL.IP.1.9 or SPL.IP.1.7

1 Introduction

As argued by the *Mathematical Morphology* school [1, 5], basic operations on images are invariant with respect to contrast changes, i.e., homomorphic transformations. As a consequence, it follows that the basic information (and geometry) of an image is contained in the family of its binary shadows or *level-sets* (see below for the exact definition).[1] A considerable amount of the research in image processing is based on assuming that regions with (almost) equal grey-values, which are topologically connected, belong to the same physical object in the 3D world. Following this, it is natural to assume then that the "shapes" in an given image are represented by its level-sets.

*This work was partially supported by a grant from the Office of Naval Research ONR-N00014-97-1-0509, the Office of Naval Research Young Investigator Award, the Presidential Early Career Awards for Scientists and Engineers (PECASE), a National Science Foundation CAREER Award, and the National Science Foundation Learning and Intelligent Systems Program (LIS). M. Bertalmio and V. Caselles provided useful comments. Original images courtesy of Prof. R. Kimmel and NASA.

†The authors are with the Electrical and Computer Engineering Department, University of Minnesota, Minneapolis, MN 55455 (guille@ece.umn.edu).

[1] Observe that, under fairly general conditions, an image can be reconstructed from its level-sets.

Moreover, techniques based on level-sets and their connected components have provided state-of-the-art results in image processing problems like image denoising, contrast enhancement, and registration. It is the goal of this note to extend this concept to vector-valued images.

2 Gradients of vector-valued images

We now present the definition of gradients in vector-valued images based on classical Riemannian geometry [4]. Let $I(x_1, x_2) : I\!R^2 \to I\!R^n$ be a vector-valued image with components $I_i(x_1, x_2) : I\!R^2 \to I\!R$, $i = 1, 2, ..., n$. The value of the image at a given pixel is a vector in $I\!R^n$, and the difference of image values at two points P and Q is given by $\Delta I = I(P) - I(Q)$. When the (Euclidean) distance between P and Q tends to zero, the difference becomes the arc element $dI = \sum_{i=1}^{2} \frac{\partial I}{\partial x_i} dx_i$, and its squared norm dI^2 is a quadratic form called the *first fundamental form*. Let us denote $g_{ij} := \frac{\partial I}{\partial x_i} \cdot \frac{\partial I}{\partial x_j}$, then $dI^2 = \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}^T \begin{bmatrix} g_{11}\, g_{12} \\ g_{21}\, g_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$. For a unit vector $(\cos\theta, \sin\theta)$, dI^2 is a measure of the rate of change of the image in the θ direction. The extrema of this quadratic form are obtained in the directions of the eigenvectors of the matrix $[g_{ij}]$, and the values attained there are the corresponding eigenvalues. Simple algebra shows that $\lambda_\pm = \frac{g_{11}+g_{22}\pm\sqrt{(g_{11}-g_{22})^2+4g_{12}^2}}{2}$, $\theta_+ = \frac{1}{2}\arctan\frac{2g_{12}}{g_{11}-g_{22}}$, and $\theta_- = \theta_+ + \pi/2$, where $\lambda_\pm$ are the eigenvalues and $(\cos\theta_\pm, \sin\theta_\pm)$ are the eigenvectors. Thus, the eigenvectors provide the direction of maximal and minimal changes at a given point in the image, and the eigenvalues are the corresponding rates of change. We call θ_+ the *direction of maximal change* or *gradient* and λ_+ the *maximal rate of change*. Similarly, θ_- and λ_- are the *direction of minimal change* and the *minimal rate of change* respectively. Note that for $n = 1$, $\lambda_+ \equiv \| \nabla I \|^2$, $\lambda_- \equiv 0$, and $(\cos\theta_+, \sin\theta_+) = \nabla I / \|\nabla I\|$.

3 Level-lines of vector valued images

Level-sets provide a fundamental concept and representation for scalar images. The basic idea is that a scalar image $I(x,y) : I\!R^2 \to I\!R$ is represented as a collection of sets of the form $\Lambda_h := \{(x,y) : I(x,y) = h\}$, or $\hat{\Lambda}_h := \{(x,y) : I(x,y) \leq h\}$. Level-lines are then curves of constant gray-values.

One straightforward possibility to extend this concept to vector-valued data is to consider the collection of classical level-sets for each one of the image components $I_i(x,y) : I\!R^2 \to I\!R$, $1 \leq i \leq n$. Although this is an interesting approach, it has a number of caveats and it is not entirely analogue to the scalar level-sets. For example, it is not clear how to combine the level-sets from the different components. In contrast with the scalar case, a point on the plane belongs to more than one level-set $\Lambda_i(x, y)$, and therefore it might belong to more than one connected component ("objects").

Let us now pursue a different approach. Basically, we redefine the level-set lines of a scalar image $I(x, y) : I\!R^2 \rightarrow I\!R$ $(n = 1)$ as the integral curves of the directions of minimal change θ_- as defined above. In other words, we select a given pixel (x, y) and travel the image plane $I\!R^2$ always in the direction of minimal change. Scalar images have the particular property that the minimal change is zero, and therefore, the integral curves of the directions of minimal change are exactly the classical level-lines defined above. The advantage of this definition is that it is dimension independent, and as we have seen above, minimal change and direction of minimal change can also be defined for vector-valued images following classical Riemannian geometry (λ_- and θ_-). We have then obtained a definition of *level-lines for vector-valued images* as well. As in the scalar case, there will be a unique set of level-lines, in contrast with the case when we treat each component separately. Moreover, under regularity conditions, it is easy to show that these vectorial level lines do not cross each other and are closed curves, as in the scalar case. Figure 1 shows some of these vector-valued level-lines for a synthetic color image first, followed by a real image. The synthetic image was built to have a color-edge which is not detectable from its intensity image. Note the comparison with scalar level-lines and also how the level-lines follow our intuition, being located at the color boundaries.

3.1 Scalar representations of vector-valued images

We can now follow the scalar literature and consider these level-lines of vector-valued images as containing the basic geometry of the vector-valued image. We can therefore attempt to reconstruct a scalar image that contains this geometry. In other words, we can reconstruct a scalar image whose level-lines (and therefore its basic geometry) coincide with the level-lines of the vector-valued images as defined above. We will then obtain a scalar image that represents the basic geometry of the the vector-valued data (see [2] for a different approach to this and a review of the literature).

We can formulate this as a variational problem. That is, the vector-valued level-lines give a direction, θ_- (and θ_+), and a value, $f(\lambda_+, \lambda_-)$ (a function of the eigenvalues), for the level-lines and gradient of the unknown single-valued image. In other words, we search for a single valued image $\tilde{I} : I\!R^2 \rightarrow I\!R$ such that $\nabla\tilde{I}$ is as close as possible to $f(\lambda_+, \lambda_-)\theta_+$ (or $\frac{\nabla\tilde{I}}{||\nabla\tilde{I}||}$ is as close as possible to θ_+). This can be formulated as a variational problem, and we can for example search for the minimizer of $\int \parallel \nabla\tilde{I} - f(\lambda_+, \lambda_-)\theta_+ \parallel^2$. Recall that the unknown is $\tilde{I}$, the single-valued image, while θ_+, λ_+, and λ_- are obtained from the given vector-valued image. The gradient descent flow equation corresponding to this energy is a Poisson equation, and existence and uniqueness results can be obtained (similar results can be obtained if we chose to work only with the direction of the gradient θ_+, ignoring its magnitude). (See also [3].) Yet there is a technical problem, since θ_+ is defined only between $-\pi/2$ and $+\pi/2$, which can be resolved in a number of different ways. We used the lexicographic order to determine the order of the vector valued pixels, which in turn provides a consistent way to assign the direction of the gradient vectors, completing the 360 degrees. Figure 2 gives examples of color images and multispectral data and their corresponding scalar representations. The first row shows three color images, and the second row their scalar counterpart. The last row show 4 out of 7 bands of a multispectral image (LANDSAT image of Amsterdam), followed by the corresponding scalar representation. Note how the scalar reconstructions captures the basic geometry of the vector-valued images.

4 Concluding remarks

In this note we have introduced the concept of level-lines of vector-valued images and exemplified the use of this new concept for visualizing vector-valued images. The next steps to follow are to define connected components and use these for applications like contrast enhancement, segmentation, and registration, as was done for the scalar case. This will be reported elsewhere.

References

[1] L. Alvarez, F. Guichard, P. L. Lions, and J. M. Morel, "Axioms and fundamental equations of image processing," *Arch. Rational Mechanics* **123**, pp. 199-257, 1993.

[2] G. Harikumar and Y. Bresler, "Feature extraction techniques for exploratory visualization of vector-valued imagery," *IEEE Trans. Image Processing*, pp. 1324–1334, Sept. 1996.

[3] C. Kenney and J. Langan. "A new image processing primitive: reconstructing images from modified flow fields," *University of California Santa Barbara Preprint, 1999.*

[4] E. Kreyszig, *Differential Geometry*, University of Toronto Press, Toronto, 1959.

[5] J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press, New York, 1982.

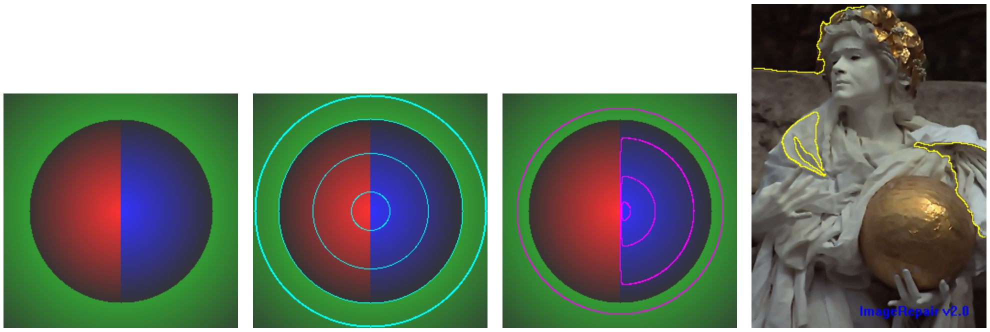

Figure 1: *Level-lines of a synthetic image (original, gray level-lines (cyan), color level-lines (magenta)) and a real image.*

Figure 2: *Scalar representation of vector valued images. See text for details.*