# Nonlinear Stability of a Quasi-static Stefan Problem with Surface Tension: a Continuation Approach

by

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**Abstract**: We consider a one-phase quasi-steady Stefan free boundary problem with surface tension, when the initial position of the free boundary is close to the unit sphere in $\mathbf{R}^\nu$ ($\nu \geq 2$), and expressed in the form $r = 1 + \epsilon\lambda^0(\omega)$. It is proved that the problem has a unique global solution with free boundary which is analytic in $\epsilon$ and which converges exponentially fast, as $t \to \infty$, to a sphere whose center and radius can both be expressed as power series in $\epsilon$. The methods developed here clearly extend to a general class of free boundary problems.

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## §1. Introduction

In this paper, we develop a general method for the study of nonlinear stability of equilibrium solutions to free boundary problems. For definiteness we exemplify this approach by considering the classical Stefan problem. The $\nu$-dimensional quasi-static one phase Stefan model seeks to find a family of $\nu$-dimensional domains $\Omega(t)$ and a function $p(x,t)$, where $x=(x_1,\ldots,x_\nu)$ is a variable point in $\mathbf{R}^\nu$, such that at each time $t \geq 0$

$$\Delta p = 0 \text{ in } \Omega(t), \tag{1.1}$$

$$\frac{\partial p}{\partial n} = -V_n \text{ on } \partial\Omega(t), \tag{1.2}$$

$$p = \gamma\kappa \text{ on } \partial\Omega(t) \text{ (the Gibbs − Thomson condition)} \tag{1.3}$$

where $\Delta$ is the Laplace operator $\sum \partial^2/\partial x_i^2$, $n$ is the outward normal to $\partial\Omega(t)$, $\kappa$ is the mean curvature along $\partial\Omega(t)$, $V_n$ is the normal velocity of the boundary in direction $n$, $\gamma$ is a positive constant, and

$$\Omega(0) \text{ is given.} \tag{1.4}$$

The problem (1.1)-(1.4) is sometimes called also the Hele-Shaw model.

In case $\Omega(0)$ is the unit ball $B_1$, the solution is

$$p \equiv \gamma, \;\; \Omega(t) \equiv B_1. \tag{1.5}$$

In this paper we consider the nonlinear stability of this solution. Introducing polar coordinates $(r,\omega)$ where $\omega = (\omega_1,\ldots,\omega_{\nu-1})$, we shall prove that if

$$\Omega(0) = \{r < 1 + \epsilon\lambda^0(\omega)\} \tag{1.6}$$

where $\lambda^0(\omega)$ is a smooth function and $|\epsilon|$ is sufficiently small, then there exists a unique solution to (1.1)-(1.6) with the following properties:

$$\Omega(t) = \{r < 1 + \lambda(\omega,t,\epsilon)\}, \;\; \lambda(\omega,t,\epsilon) = \sum_{n=1}^{\infty} \lambda_n(\omega,t)\epsilon^n, \tag{1.7}$$

$$p(r,\omega,t,\epsilon) = \gamma + \sum_{n=1}^{\infty} p_n(r,\omega,t)\epsilon^n \tag{1.8}$$

where the series are absolutely uniformly convergent and define smooth functions in the variables $(x,t,\epsilon)$ for $x \in \overline{\Omega(t)}$, $|\epsilon| \leq \epsilon_0$, for some $\epsilon_0 > 0$. We shall also prove the following stability result:

There exist convergent series

$$x_\infty(\epsilon) \equiv \sum_{n=1}^{\infty} x_n \epsilon^n, \ \tilde{\lambda}_\infty(\epsilon) \equiv \sum_{n=1}^{\infty} \tilde{\lambda}_{n,\infty} \epsilon^n \quad (x_n = (a_{n1}, \ldots, a_{n\nu}))$$

for $|\epsilon| \leq \epsilon_0$ such that in terms of the polar coordinates $(\tilde{r}, \omega)$,

$$\tilde{r} = |x - x_\infty(\epsilon)|, \quad \omega \text{ as in (1.6)},$$

there holds:

$$\partial\Omega(t) = \{\tilde{r} = \tilde{\lambda}(\omega, t, \epsilon)\}$$

where $\tilde{\lambda}(\omega, t, \epsilon)$ is again a power series in $\epsilon$, as $\lambda(\omega, t, \epsilon)$, and

$$|\tilde{\lambda}(\omega, t, \epsilon) - \tilde{\lambda}_\infty(\epsilon)| \leq Ce^{-\beta t} \quad (C > 0, \beta > 0). \tag{1.9}$$

For $\nu = 2$, the existence, uniqueness and global stability for the corresponding two-phase problem was proved independently by Chen [3] and by Constantin and Pugh [7] (see also Appendix to [6]). Chen's approach is based on replacing the free boundary condition (1.2) by the more regularized condition

$$\frac{\partial p}{\partial n} = -V_n + \eta \frac{\partial^2 \kappa}{\partial s^2} \qquad (\text{s} = \text{arc length parameter}),$$

deriving estimates which are independent of $\eta$, and then letting $\eta \to 0$. The Constantin-Pugh approach is based on conformally mapping $\Omega(t)$ onto the unit disc and studying the evolution (in $t$) of the associated flow.

A different approach for $(\nu = 2)$ was earlier developed by Duchon and Robert [8] in the case where $\partial\Omega(0)$ is a curve $x_2 = f(x_1)$, $-\infty < x_1 < \infty$. They write the free boundary condition (1.2) in the form $\gamma T \kappa = V_n$ where $T$ is the Dirichlet-to-Neumann mapping, and use the decomposition

$$\frac{\partial}{\partial n} = H \frac{\partial}{\partial s} + R$$

where H is the Hilbert transform and $R$ is a compact operator.

We also mention that the (more regular) Stefan problem with the Gibbs-Thomson condition (1.3) and (1.1) replaced by $\partial_t p - \Delta p = 0$ was considered in the literature; see [13] [15] [5] and the references cited there.

Chen [3] assumes that

$$\partial\Omega(0) = \{(R_0(\theta)\cos\theta, R_0(\theta)\sin\theta), \ 0 \leq \theta \leq 2\pi\}$$

where

$$\|R_0 - 1\|_{C^1} \leq \epsilon \ , \ \|\nabla p \ |_{t=0} \ \|_{L^2} \leq \epsilon,$$

$\epsilon$ is positive and sufficiently small, and establishes global existence and uniqueness of a classical solution. Furthermore, he proves that

$$\text{dist}(\partial\Omega(t), \Gamma_\infty) \leq e^{-\beta t} \quad (C > 0, \beta > 0) \tag{1.10}$$

for some circle $\Gamma_\infty$.

Constantin and Pugh [7] assume that the initial boundary $\partial\Omega(0)$ is analytic, and they prove, in addition to (1.10), that the solution is analytic in the spatial variable.

In the case of dimension $\nu \geq 3$ local existence for (1.1)-(1.4) was proved in [4][2][9]. More recently Escher and Simonett [10] proved global existence for $\nu \geq 2$ in case $\Omega(0)$ is close to a ball, and they also established the stability result (1.10). Their method uses the tool of center manifolds.

The papers [2-4],[7-9] mentioned above deal with the two phase problem where $p$ is harmonic from both sides of $\partial\Omega(t)$ and on the left-hand side of (1.2) there appears the jump of $\partial p/\partial n$ across $\partial\Omega(t)$. For simplicity we consider in this paper just the one phase problem, but the two phase case can be treated in the same way.

Our approach has two advantages:

(i) It enables us to actually compute the limiting sphere to any order $\epsilon^n$.

(ii) It has a great flexibility in terms of regularity: basically we can show that the surfaces $\partial\Omega(t)$ are as smooth as $\partial\Omega(0)$; in particular, if $\partial\Omega(0)$ is analytic then so are the $\partial\Omega(t)$ (in fact, they are analytic jointly in $(\omega, \epsilon)$).

For the sake of clarity, we divide the paper into three parts. The first two deal with the case $\nu = 2$. In Part I we impose the constraint:

$$\lambda^0(\theta) = \sum_m \lambda_m^0 e^{i\ell_m\theta} \tag{1.11a}$$

where the $\ell_m$ are integers such that

$$\sum_m \eta_m \ell_m \neq \pm 1 \quad \forall \text{ integers } \eta_m. \tag{1.11b}$$

We then prove the results mentioned above with $X_\infty(\epsilon) \equiv 0$. In Part II we remove the restriction (1.11). Finally in Part III we extend the results to any dimension $\nu \geq 2$.

Our approach is entirely different from those of [3][7][10]. We shall briefly describe it in the case where $\nu = 2$ and (1.11) is satisfied. In this case we can derive, formally, recursive formulas for $\lambda_n, p_n$. However, as explained in our recent paper [11] (which dealt with a different problem), there is a serious difficulty in deriving estimates on $\lambda_n, p_n$ by simply taking bounds on absolute values in the recursive formulas. Indeed, such a procedure fails to take into account important subtle cancellations. We shall therefore use another approach based on first transforming the free boundary into a circle. The transformation we have in mind is

$$r' = \frac{r}{1 + \lambda(\theta, t, \epsilon)}. \tag{1.12}$$

However, by this transformation the equation $\Delta p = 0$ is transformed into an elliptic equation with coefficients, some of which are not smooth enough at the origin. To overcome this handicap we first consider the problem outside a small region $\{r > \delta\}$, imposing a "transparent" boundary condition on $p$ at $r = \delta$; this condition allows us to extend the solution, later on, from $\{\delta < r < 1+\lambda\}$ to $\{0 \leq r < 1+\lambda\}$. Correspondingly, instead of the change of variable (1.12) we shall take

$$r' = \frac{(r - \delta) + \delta(1 + \lambda - r)}{1 + \lambda - \delta}; \tag{1.13}$$

note that the free boundary $r = 1 + \lambda$ is mapped onto $r' = 1$ and the circle $r = \delta$ is mapped onto $r' = \delta$.

In Section 2 we reformulate the problem (1.1)-(1.4) after the change of variables (1.13). Letting

$$p'(r', \theta, t, \epsilon) = p(r, \theta, t, \epsilon) \tag{1.14}$$

we then wish to prove that (1.7) and

$$p'(r', \theta, t, \epsilon) = \gamma + \sum_{n=1}^{\infty} p'_n(r', \theta, t)\epsilon^n \tag{1.15}$$

hold, where the power series are convergent for $|\epsilon|$ small. This leads to a sequence of elliptic problems, with parameter $t$, for $(p'_n, \lambda_n)$, in terms of $(p'_m, \lambda_m)$, $1 \leq m < n$. Writing this system in the form

$$A(p'_n, \lambda_n) = F_n \tag{1.16}$$

where $F_n$ is assumed to be known inductively, we study this system in Section 4, and prove existence, uniqueness and bounds on some of the derivatives (in terms of $F_n$); an auxiliary result on ODE, needed in Section 4, is proved in Section 3.

In order to apply the results of Section 4 we also need to evaluate, in Sobolev norms, the product of functions that appear in $F_n$. This is done in a general way in Appendix A, by means of interpolation inequalites. We further need to estimate higher order derivatives of composite functions, and this is done in Appendix B.

Finally, in Section 5 we apply the results of Section 4 and the Appendices A,B in order to deduce from (1.16) estimates on $p'_n, \lambda_n$ which establish the asserted convergence of the series for $\lambda$ and $p'$.

The constraints in (1.11) are needed in order to eliminate the neutrally stable modes (wave numbers $\pm 1$) in the derivation of the bounds on $\lambda_n, p_n$. However, by choosing the origin at appropriate points

$$x_\infty^n \equiv \sum_{m=1}^{n} x_m \epsilon^m$$

we overcome (in Part II) this possible source of instability, and prove both global existence and the asymptotic estimate (1.9). The extension of our results to $\nu \geq 3$ (in Part III) proceeds as in the case $\nu = 2$ (Part II), but requires several additional properties regarding the mean curvature of a surface in $R^\nu$ and some norms defined in terms of spherical harmonics in $R^\nu$.

## Part I. Two dimensions; A Special Case

### §2. Reformulation of the problem

We shall later on use the fact that in two dimensions the boundary condition (1.2) can be written in the form

$$\frac{\partial p}{\partial r} - \frac{\lambda_\theta}{(1+\lambda)^2} \frac{\partial p}{\partial \theta} = -\frac{\partial \lambda}{\partial t}. \tag{2.1}$$

Indeed, the free boundary can be written as $r = 1 + \lambda \equiv \Lambda$ or, in Cartesian coordinates $x = (x_1, x_2)$, as $x(\theta, t) = \Lambda(\cos\theta, \sin\theta)$. The tangents are in the direction

$$x_\theta = \Lambda_\theta(\cos\theta, \sin\theta) + \Lambda(-\sin\theta, \cos\theta)$$

and the normal is

$$n = \frac{N}{|N|}$$

where

$$N(\theta, t) = (\Lambda_\theta \sin\theta + \Lambda\cos\theta,\ \Lambda\sin\theta - \Lambda_\theta\cos\theta).$$

Then

$$V_n = x_t \cdot n = \frac{\Lambda_t \Lambda}{(\Lambda^2 + \Lambda_\theta^2)^{1/2}}.$$

On the other hand,

$$\nabla p = (\cos\theta\ p_r - \frac{\sin\theta}{r} p_\theta,\ \sin\theta\ p_r + \frac{\cos\theta}{r} p_\theta)$$

so that

$$\nabla p \cdot n = \frac{\Lambda p_r - \Lambda_\theta p_\theta / \Lambda}{(\Lambda^2 + \Lambda_\theta^2)^{1/2}}.$$

Comparing this with the expression for $V_n$, the assertion (2.1) follows.

We shall later on use also the fact that the mean curvature $\kappa$ of $r = 1 + \lambda$ is given by

$$\kappa \equiv \kappa(1+\lambda) = \frac{2\lambda_\theta^2 - (1+\lambda)\lambda_{\theta\theta} + (1+\lambda)^2}{((1+\lambda)^2 + \lambda_\theta^2)^{3/2}}. \tag{2.2}$$

We anticipate that $\Omega(t) \supset \{r < \frac{3}{4}\}$ for all $t > 0$. Since $\Delta p = 0$ in $\Omega(t)$, we can then write

$$p(r, \theta, t) = \sum c_n(t) r^{|n|} e^{in\theta} \quad \text{if } r \le \frac{1}{2},\ t > 0 \tag{2.3}$$

where the series is uniformly convergent. In particular, if $\delta \in (0, \frac{1}{4})$,

$$p(\delta, \theta, t) = \sum c_n(t) \delta^{|n|} e^{in\theta}. \tag{2.4}$$

But then

$$\frac{\partial p}{\partial r}(\delta, \theta, t) = \frac{1}{\delta} H(\frac{\partial p}{\partial \theta}(\delta, \theta, t)) \tag{2.5}$$

where

$$H(\sum \gamma_n e^{in\theta}) = \sum (-i)(\text{sgn } (n)) \gamma_n e^{in\theta}$$

is the Hilbert transform; $H$ is an isometry in $H^s$. Set $B_\delta = \{r < \delta\}$.

**Lemma 2.1**. *The problem (1.1)-(1.4) is equivalent to the problem consisting of*

$$\Delta p = 0 \text{ in } \Omega(t)\backslash B_\delta \tag{2.6}$$

*with the boundary condition (1.2), (1.3), (2.5) and the initial condition (1.4).*

**Proof**. If we solve the modified problem (2.6), (1.2), (1.3), (2.5), (1.4) and then extend $p(r,\theta,t)$ to $r \le \delta$ as a harmonic function $\tilde{p}(r,\theta,t)$ with boundary values $p(\delta,\theta,t)$ at $r=\delta$, then

$$\tilde{p}(r,\theta,t) = \sum \tilde{c}_n(t) r^{|n|} e^{in\theta}$$

holds for $r \le \delta$. As in (2.5) we have, at $r=\delta$,

$$\frac{\partial \tilde{p}}{\partial r} = \frac{1}{\delta} H(\tilde{p}) = \frac{1}{\delta} H(p) = \frac{\partial p}{\partial r}$$

where we used the fact that $\tilde{p} = p$ at $r=\delta$. It follows that $\tilde{p}$ is the harmonic extension of $p$ across $r=\delta$, thus providing a solution to the original problem (1.1)-(1.4). //

The boundary condition (2.5) is called *transparent*. The mapping

$$p \to \frac{1}{\delta} H(\frac{\partial p}{\partial \theta})$$

is called the Dirichlet-to-Neumann map.

We next proceed to transform the modified problem (1.2)-(1.4), (2.5), (2.6) by the change of variable (1.13). We can write

$$\begin{aligned} r' &= \frac{(1-\delta)r + \delta\lambda}{1+\lambda-\delta} \quad (\text{or } \delta - r' = \frac{(1-\delta)(\delta-r)}{1+\lambda-\delta}), \\ p(r,\theta,t) &= p'(r',\theta,t) = p'(\frac{(1-\delta)r+\delta\lambda}{1+\lambda-\delta},\theta,t). \end{aligned} \tag{2.7}$$

Then

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{(1-\delta)}{(1+\lambda-\delta)}\frac{\partial p'}{\partial r'}, \quad \frac{\partial^2 p}{\partial r^2} = \frac{(1-\delta)^2}{(1+\lambda-\delta)^2}\frac{\partial^2 p'}{\partial r'^2}, \\ \frac{\partial p}{\partial \theta} &= \frac{\partial p'}{\partial \theta} + \frac{(1-\delta)\lambda_\theta(\delta-r)}{(1+\lambda-\delta)^2}\frac{\partial p'}{\partial r'} = \frac{\partial p'}{\partial \theta} + \frac{\lambda_\theta(\delta-r')}{(1+\lambda-\delta)}\frac{\partial p'}{\partial r'}, \\ \frac{\partial^2 p}{\partial \theta^2} &= \frac{\partial^2 p'}{\partial \theta^2} + \frac{2(1-\delta)\lambda_\theta(\delta-r)}{(1+\lambda-\delta)^2}\frac{\partial^2 p'}{\partial r'\partial\theta} + \frac{\partial}{\partial\theta}(\frac{(1-\delta)\lambda_\theta(\delta-r)}{(1+\lambda-\delta)^2})\frac{\partial p'}{\partial r'} \\ &\quad + (\frac{(1-\delta)\lambda_\theta(\delta-r)}{(1+\lambda-\delta)^2})^2\frac{\partial^2 p'}{\partial r'^2} \\ &= \frac{\partial^2 p'}{\partial \theta^2} + \frac{2\lambda_\theta(\delta-r')}{(1+\lambda-\delta)}\frac{\partial^2 p'}{\partial\theta\partial r'} + (\frac{\lambda_\theta(\delta-r')}{1+\lambda-\delta})^2\frac{\partial^2 p'}{\partial r'^2} \\ &\quad + [\frac{\partial}{\partial\theta}(\frac{\lambda_\theta}{1+\lambda-\delta}) - (\frac{\lambda_\theta}{1+\lambda-\delta})^2](\delta-r')\frac{\partial p'}{\partial r'}, \end{aligned}$$

and

$$\frac{1}{r}\frac{\partial p}{\partial r} = \frac{(1-\delta)^2}{((1+\lambda-\delta)r' - \delta\lambda)(1+\lambda-\delta)}\frac{\partial p'}{\partial r'}.$$

The boundary condition (2.1) becomes

$$\begin{aligned}\frac{\partial p}{\partial r} - \frac{\lambda_\theta}{(1+\lambda)^2}\frac{\partial p}{\partial \theta} &= \frac{(1-\delta)}{(1+\lambda-\delta)}\frac{\partial p'}{\partial r'} - \frac{\lambda_\theta}{(1+\lambda)^2}\left(\frac{\partial p'}{\partial \theta} + \frac{\lambda_\theta(\delta-1)}{(1+\lambda-\delta)}\frac{\partial p'}{\partial r'}\right)\\ &= \frac{1-\delta}{(1+\lambda-\delta)(1+\lambda)^2}[(1+\lambda)^2 + \lambda_\theta^2]\frac{\partial p'}{\partial r'} - \frac{\lambda_\theta}{(1+\lambda)^2}\frac{\partial p'}{\partial \theta} = -\frac{\partial \lambda}{\partial t}.\end{aligned}$$

Dropping "$'$" both in $p'$ and $r'$, the system (1.1) (1.2) (or (2.1)), (1.3) and (2.5) becomes:

$$\Delta p = F^1, \quad \delta < r < 1, \tag{2.8}$$

$$\frac{\partial p}{\partial r} + \frac{\partial \lambda}{\partial t} = F^2, \quad r = 1, \tag{2.9}$$

$$p + \gamma((\lambda - 1) + \lambda_{\theta\theta}) = F^3, \quad r = 1, \tag{2.10}$$

$$\frac{\partial p}{\partial r} - \frac{1}{\delta}H\left(\frac{\partial p}{\partial \theta}\right) = F^4, \quad r = \delta \tag{2.11}$$

where

$$\begin{aligned}F^1 &= \left(1 - \frac{(1-\delta)^2}{(1+\lambda-\delta)^2}\right)\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\left(1 - \frac{(1-\delta)^2 r}{((1+\lambda-\delta)r - \delta\lambda)(1+\lambda-\delta)}\right)\frac{\partial p}{\partial r}\\ &+ \frac{(1-\delta)^2}{((1+\lambda-\delta)r - \delta\lambda)^2}\Big[-2(\delta - r)\frac{\lambda_\theta}{(1+\lambda-\delta)}\frac{\partial^2 p}{\partial r \partial \theta} - (\delta - r)\frac{\partial}{\partial \theta}\left(\frac{\lambda_\theta}{(1+\lambda-\delta)}\right)\frac{\partial p}{\partial r}\\ &- (\delta - r)^2\left(\frac{\lambda_\theta}{(1+\lambda-\delta)}\right)^2\frac{\partial^2 p}{\partial r^2} + (\delta - r)\left(\frac{\lambda_\theta}{(1+\lambda-\delta)}\right)^2\frac{\partial p}{\partial r}\Big],\end{aligned} \tag{2.12}$$

$$F^2 = \left[1 - \frac{(1-\delta)((1+\lambda)^2 + \lambda_\theta^2)}{(1+\lambda-\delta)(1+\lambda)^2}\right]\frac{\partial p}{\partial r} + \frac{\lambda_\theta}{(1+\lambda)^2}\frac{\partial p}{\partial \theta}, \tag{2.13}$$

$$F^3 = \gamma\left[\frac{2\lambda_\theta^2 - (1+\lambda)\lambda_{\theta\theta} + (1+\lambda)^2}{((1+\lambda)^2 + \lambda_\theta^2)^{3/2}} - 1 + \lambda_{\theta\theta} + \lambda\right], \tag{2.14}$$

$$F^4 = \frac{\lambda}{(1+\lambda-\delta)}\frac{\partial p}{\partial r}. \tag{2.15}$$

**Definition 2.1**. The problem of solving for $(p, \lambda)$ the system (2.8)-(2.15) with the initial condition $\lambda\,|_{t=0}= \epsilon\lambda^0(\theta)$ will be called Problem (A).

We intend to prove that Problem (A) has a unique solution of the form

$$p(r,\theta,t,\epsilon) = \gamma + \sum_{n\geq 1} P_n(r,\theta,t)\epsilon^n, \tag{2.16}$$

$$1+\lambda(\theta,t,\epsilon) = 1 + \sum_{n\geq 1} \lambda_n(\theta,t)\epsilon^n. \tag{2.17}$$

If we substitute these series into the system (2.8)-(2.15) we find that $(P_n, \lambda_n)$ satisfy the following equations

$$\Delta P_n = F_n^1(r,\theta,t), \quad \delta < r < 1,\ t > 0, \tag{2.18}$$

$$\frac{\partial P_n}{\partial r} + \frac{\partial \lambda_n}{\partial t} = F_n^2(\theta,t), \quad r = 1,\ t > 0, \tag{2.19}$$

$$P_n + \gamma(\frac{\partial^2 \lambda_n}{\partial \theta^2} + \lambda_n) = F_n^3(\theta,t), \quad r = 1,\ t > 0, \tag{2.20}$$

$$\frac{\partial P_n}{\partial r} - \frac{1}{\delta}H(\frac{\partial P_n}{\partial \theta}) = F_n^4(\theta,t), \quad r = \delta,\ t > 0 \tag{2.21}$$

where the $F_n^j$ depend on the $P_m, \lambda_m$ for $0 \leq m < n$; here $P_0 \equiv \gamma,\ \lambda_0 \equiv 1$.

In Section 4 we shall study this system and derive estimates, which depend on the $F_n^j$; these estimates will be used later on to establish the convergence of the series (2.16), (2.17).

## §3. An ODE Lemma

In this section, we prove:

**Lemma 3.1**. *Consider the initial value problem*

$$\dot{B}(t) + KB(t) = F(t), \quad t > 0 \tag{3.1}$$

$$B(0) = B_0 \tag{3.2}$$

*where $F \in L^2(0,T)$ for any $T > 0$. If $0 < \alpha < K$, then the following inequalities hold:*

$$\int_0^t e^{2\alpha s}(B(s))^2 ds \leq \frac{2}{(K-\alpha)^2}\int_0^t e^{2\alpha s}(F(s))^2 ds + \frac{B_0^2}{K-\alpha}, \tag{3.3}$$

$$\int_0^t e^{2\alpha s}(\dot{B}(s))^2 ds \le 2(\frac{2K^2}{(K-\alpha)^2}+1)\int_0^t e^{2\alpha s}(F(s))^2 ds + \frac{2K^2}{K-\alpha}B_0^2. \tag{3.4}$$

**Proof**. We have

$$B(t)=\int_0^t e^{-K(t-s)}F(s)ds + B_0 e^{-Kt}$$

or

$$B(t)e^{\alpha t}=\int_0^t e^{-(K-\alpha)(t-s)}e^{\alpha s}F(s)ds + B_0 e^{-(K-\alpha)t}.$$

Hence

$$\begin{aligned} B(t)^2 e^{2\alpha t} &\le 2(\int_0^t e^{-(K-\alpha)(t-s)}ds)\int_0^t e^{-(K-\alpha)(t-s)}(e^{\alpha s}F(s))^2 ds + 2B_0^2 e^{-2(K-\alpha)t} \\ &\le \frac{2}{K-\alpha}\int_0^t e^{-(K-\alpha)(t-s)}(e^{\alpha s}F(s))^2 ds + 2B_0^2 e^{-2(K-\alpha)t} \end{aligned}$$

and, by integration,

$$\begin{aligned} \int_0^t B(\tau)^2 e^{2\alpha\tau}d\tau &\le \frac{2}{K-\alpha}\int_0^t d\tau\int_0^\tau e^{-(K-\alpha)(\tau-s)}(e^{\alpha s}F(s))^2 ds + \frac{B_0^2}{K-\alpha} \\ &= \frac{2}{(K-\alpha)^2}\int_0^t e^{2\alpha s}(F(s))^2 ds + \frac{B_0^2}{K-\alpha}, \end{aligned}$$

which proves (3.3). The assertion (3.4) follows easily from (3.1) and (3.3). //

## §4. A Fundamental Lemma

In this section we consider a system for $(P,\Lambda)$ of the form

$$\Delta P = F^1(r,\theta,t) \text{ in } B=\{\delta<r<1\},\ t>0, \tag{4.1}$$

$$\frac{\partial P}{\partial r}+\frac{\partial\Lambda}{\partial t}=F^2(\theta,t) \text{ on } \partial_1 B=\{r=1\},\ t>0, \tag{4.2}$$

$$P+\gamma(\Lambda_{\theta\theta}+\Lambda)=F^3(\theta,t) \text{ on } \partial_1 B,\ t>0, \tag{4.3}$$

$$\frac{\partial P}{\partial r}-\frac{1}{\delta}H(\frac{\partial P}{\partial\theta})=F^4(\theta,t) \text{ on } \partial_\delta B=\{r=\delta\},\ t>0, \tag{4.4}$$

$$\Lambda\mid_{t=0}=\Lambda^0 \text{ on } \partial_1 B, \tag{4.5}$$

where the functions $F^j, \Lambda^0$ and the constant $\gamma$ are given, and $\gamma > 0$.

We introduce the norms

$$\|F\|_{s,B} = \{\int_0^\infty e^{2\alpha t}[\|F(\cdot,t)\|^2_{H^s(B)} + \|F_t(\cdot,t)\|^2_{H^{s-3}(B)} + \|F_{tt}(\cdot,t)\|^2_{H^{s-6}(B)}]dt\}^{1/2}, \tag{4.6}$$

$$\|f\|_{s,\partial B} = \{\int_0^\infty e^{2\alpha t}[\|f(\cdot,t)\|^2_{H^s(\partial B)} + \|f_t(\cdot,t)\|^2_{H^{s-3}(\partial B)} + \|f_{tt}(\cdot,t)\|^2_{H^{s-6}(\partial B)}]dt\}^{1/2} \tag{4.7}$$

for $s \geq 6$ where $F = F(x,t)$, $x = (x_1, x_2)$ and $f = f(\theta, t)$, and $\partial B$ is the interval $0 \leq \theta \leq 2\pi$; $f(\theta,t)$ is assumed to be periodic in $\theta$. The function $f$ will be viewed as defined either on $\partial_1 B$ or on $\partial_\delta B$, depending on the context. We also restrict $\alpha$ to

$$0 < \alpha < 6\gamma. \tag{4.8}$$

When $s$ is not an integer then the $H^s(B)$ norm can be defined either by Fourier transform or by interpolation [1]. On the other hand the $H^s(\partial B)$ norm can simply be defined in terms of Fourier series: If

$$f(\theta) = \sum_{-\infty<n<\infty} f_n e^{in\theta}$$

then

$$\|f\|_{H^s(\partial B)} = (\sum_{-\infty<n<\infty} |f_n|^2(1+|n|^2)^s)^{1/2}.$$

In the sequel we shall use the norm (4.6) only for $s$ integer and the norm (4.7) only for $s + \frac{1}{2}$ integer. In Lemma 4.1 below we shall use the expansions

$$F^1(r,\theta,t) = \sum_{-\infty<n<\infty} F^1_n(r,\theta)e^{in\theta}, \tag{4.9}$$

$$F^j(\theta,t) = \sum_{-\infty<n<\infty} F^j_n(t)e^{in\theta} \quad (j=2,3,4), \tag{4.10}$$

and

$$\Lambda^0(\theta) = \sum_{-\infty<n<\infty} \lambda^0_n e^{in\theta} \tag{4.11}$$

and seek a solution to (4.1)-(4.5) in the form

$$P(r,\theta,t) = \sum_{-\infty<n<\infty} P_n(r,t)e^{in\theta}, \tag{4.12}$$

$$\Lambda(\theta,t)=\sum_{-\infty<n<\infty}\lambda_n(t)e^{in\theta}. \tag{4.13}$$

The following important assumption will be needed:

$$\lambda^0_{\pm1}+\int_0^\infty[F^2_{\pm1}(t)-F^3_{\pm1}(t)-\delta^2F^4_{\pm1}(t)-\int_\delta^1\rho^2F^1_{\pm}(\rho,t)d\rho]dt=0. \tag{4.14}$$

**Lemma 4.1**. *Assume that*

$$\|\Lambda^0\|_{H^8(\partial B)},\ \|F^1\|_{6,B},\ \|F^2\|_{6\frac12,\partial B},\ \|F^3-F^3_\infty\|_{7\frac12,\partial B},\ \|F^4\|_{6\frac12,\partial B} \tag{4.15}$$

*are finite, where $F^3_\infty$ is some constant. Then there exists a unique solution $P,\Lambda$ of (4.1)-(4.5) satisfying the following estimates:*

$$\begin{aligned}&\|P-P_\infty\|_{8,B}+\|\Lambda-\Lambda_\infty\|_{9\frac12,\partial B}+\|\Lambda_t\|_{6\frac12,\partial B}\\&\quad\le C\{\|F^1\|_{6,B}+\|F^2\|_{6\frac12,\partial B}+\|F^3-F^3_\infty\|_{7\frac12,\partial B}+\|F^4\|_{6\frac12,\partial B}+\|\Lambda^0\|_{H^8(\partial B)}\}\end{aligned} \tag{4.16}$$

*where*

$$\Lambda_\infty=\lambda^0_0+\int_0^\infty[F^2_0(t)-\delta F^4_0(t)-\int_\delta^1 rF^1_0(r,t)dr]dt, \tag{4.17}$$

$$P_\infty=F^3_\infty-\gamma\Lambda_\infty, \tag{4.18}$$

*and the constant $C$ depends only on $\delta,\gamma$ and $\alpha$.*

Note that from (4.17) we get the (very crude) estimate

$$|\Lambda_\infty|\le C[\|\Lambda^0\|_{L^2(\partial B)}+\|F^1\|_{6,B}+\|F^2\|_{6\frac12,\partial B}+\|F^4\|_{6\frac12,\partial B}]. \tag{4.19}$$

**Proof**. Substituting (4.12), (4.13) into the system (4.1)-(4.5) we obtain the following system for the $P_n,\lambda_n$:

$$P_{n,rr}+\frac1r P_{n,r}-\frac{n^2}{r^2}P_n=F^1_n(r,t),\quad \delta<r<1, \tag{4.20}$$

$$P_{n,r}(1,t)+\lambda_{n,t}(t)=F^2_n(t), \tag{4.21}$$

$$P_n(1,t)+\gamma(1-n^2)\lambda_n(t)=F^3_n(t), \tag{4.22}$$

$$P_{n,r}(\delta,t)-\frac1\delta|n|P_n(\delta,t)=F^4_n(t), \tag{4.23}$$

$$\lambda_n \mid_{t=0} = \lambda_n^0. \tag{4.24}$$

Consider first the case $n \neq 0, \pm 1$. The general solution of (4.20) is

$$\begin{aligned} P_n(r,t) = &-\frac{r^{|n|}}{2|n|}\int_r^1 \rho^{-|n|+1}F_n^1(\rho,t)d\rho - \frac{r^{-|n|}}{2|n|}\int_\delta^r \rho^{|n|+1}F_n^1(\rho,t)d\rho \\ &+ c_1 r^{-|n|} + c_2 r^{|n|} \equiv \tilde{P}_n(r,t) + c_1 r^{-|n|} + c_2 r^{|n|}, \end{aligned} \tag{4.25}$$

where $c_1 = c_1(t),\ c_2 = c_2(t)$ are independent of $r$; we also have

$$\begin{aligned} P_{n,r} = &\frac{1}{2}r^{|n|-1}\int_1^r \rho^{-|n|+1}F_n^1(\rho,t)d\rho + \frac{1}{2}r^{-|n|-1}\int_\delta^r \rho^{|n|+1}F_n^1(\rho,t)d\rho \\ &- |n|c_1 r^{-|n|-1} + c_2|n|r^{|n|-1}. \end{aligned} \tag{4.26}$$

We claim that

$$\|\tilde{P}_n(\cdot,t)\|_{L^2(B)} \leq \frac{C}{|n|(|n|+1)}\|F_n^1(\cdot,t)\|_{L^2(B)}. \tag{4.27}$$

To prove this, let

$$-2|n|\tilde{P}_n = P_{n1} + P_{n2}$$

where

$$P_{n1} = r^{|n|}\int_r^1 \rho^{-|n|+1}F_n^1(\rho,t)d\rho,$$

$$P_{n2} = r^{-|n|}\int_\delta^r \rho^{|n|+1}F_n^1(\rho,t)d\rho.$$

Then

$$\begin{aligned} (\int_\delta^1 (P_{n1})^2 r dr)^{1/2} &= \left(\int_\delta^1 \left(\int_r^1 r s^{|n|-1}F_n^1(\frac{r}{s},t)\frac{r}{s^2}ds\right)^2 r dr\right)^{1/2} \qquad (\rho = \frac{r}{s}) \\ &\leq \int_\delta^1 ds \left(\int_\delta^s r^4 (F_n^1(\frac{r}{s},t))^2 r dr\right)^{1/2} s^{|n|-3} \text{ (by Minkowski's inequality)} \\ &= \int_\delta^1 s^{|n|-3}\left(\int_{\delta/s}^1 s^4\rho^4(F^1(\rho,t))^2 s\rho s d\rho\right)^{1/2} ds \quad (r = \rho s) \\ &= \int_\delta^1 s^{|n|}\left(\int_{\delta/s}^1 \rho^4 (F_n^1(\rho,t))^2\rho d\rho\right)^{1/2} ds \end{aligned}$$

so that

$$\|P_{n1}(\cdot,t)\|_{L^2(B)} \leq \frac{1}{(|n|+1)}\|F_n^1(\cdot,t)\|_{L^2(B)}.$$

Similarly

$$\Big(\int_\delta^1 (P_{n2})^2 r dr\Big)^{1/2} = \left(\int_\delta^1 \left(\int_{\delta/r}^1 r\ s^{|n|+1} F_n^1(rs,t) r ds\right)^2 r dr\right)^{1/2} \quad (\rho = rs)$$

$$\leq \int_\delta^1 ds \left(\int_{\delta/s}^1 r^5 (F_n^1(rs,t))^2 dr\right)^{1/2} s^{|n|+1} \quad \text{(by Minkowski's inequality)}$$

$$= \int_\delta^1 s^{|n|+1} \left(\int_s^1 \frac{\rho^5}{s^6} (F_n^1(\rho,t))^2 d\rho\right)^{1/2} \quad (r = \rho/s)$$

so that

$$\|P_{n2}(\cdot,t)\|_{L^2(B)} \leq \frac{C}{|n|+1} \|F_n^1(\cdot,t)\|_{L^2(B)}.$$

Combining the $L^2$ estimates on $P_{n1},\ P_{n2}$, the assertion (4.27) follows.

Similarly one can prove that

$$\|\frac{\partial}{\partial r}\tilde{P}_n(\cdot,t)\|_{L^2(B)} \leq \frac{C}{|n|+1}\|F_n^1(\cdot,t)\|_{L^2(B)}$$

and, therefore, from (4.20),

$$\|\frac{\partial^2}{\partial r^2}\tilde{P}_n(\cdot,t)\|_{L^2(B)} \leq C\|F_n^1(\cdot,t)\|_{L^2(B)}.$$

Inductively it can also be established that

$$\begin{aligned} &\sum_{j=0}^{s+2}(1+|n|^{2(s+2-j)}) \int_\delta^1 |D_r^j \tilde{P}_n(r,t)|^2 r dr \\ &\qquad \leq C\sum_{j=0}^{s}(1+|n|^{2(s-j)}) \int_\delta^1 |D_r^j F_n^1(r,t)|^2 r dr. \end{aligned} \tag{4.28}$$

We next insert (4.25), (4.26) into the boundary conditions (4.21)-(4.23) and obtain

$$\frac{1}{2}\int_\delta^1 \rho^{|n|+1} F_n^1(\rho,t) d\rho - |n|c_1 + |n|c_2 + \lambda_{n,t} = F_n^2(t), \tag{4.29}$$

$$-\frac{1}{2|n|}\int_\delta^1 \rho^{|n|+1} F_n^1(\rho,t) d\rho + c_1 + c_2 + \gamma(1-n^2)\lambda_n = F_n^3(t), \tag{4.30}$$

and (after some cancellations)

$$c_1(t) = -\frac{1}{2|n|}\delta^{|n|+1} F_n^4(t). \tag{4.31}$$

If we multiply both sides of (4.30) by $|n|$ and subtract from (4.29), we get

$$\int_\delta^1 \rho^{|n|+1} F_n^1(\rho,t)d\rho - 2|n|c_1 + \lambda_{n,t} + \gamma|n|(n^2-1)\lambda_n = F_n^2 - |n|F_n^3.$$

Substituting $c_1$ from (4.31), we arrive at the differential equation for $\lambda_n$:

$$\begin{aligned}\lambda_{n,t} + \gamma|n|(n^2-1)\lambda_n &= F_n^2 - |n|F_n^3 - \delta^{|n|+1}F_n^4 - \int_\delta^1 \rho^{|n|+1}F_n^1(\rho,t)d\rho \\ &\equiv G_n(t)\end{aligned} \tag{4.32}$$

with initial condition (4.24).

After solving for $\lambda_n$, we can compute $c_2$ from (4.30) and (4.31):

$$\begin{aligned}c_2(t) = F_n^3(t) + \frac{\delta^{|n|+1}}{2|n|}F_n^4(t) + \frac{1}{2|n|}\int_\delta^1 \rho^{|n|+1}F_n^1(\rho,t)d\rho \\ + \gamma(n^2-1)\lambda_n.\end{aligned} \tag{4.33}$$

We next apply Lemma 3.1 to (4.32), (4.24) (noting that $0 < \alpha < 6\gamma \le \gamma|n|(n^2-1)$ if $n \neq 0, \pm 1$). We obtain

$$\begin{aligned}\int_0^T e^{2\alpha s}(\lambda_n(s))^2 ds \le \frac{2}{(\gamma|n|(n^2-1)-\alpha)^2}\int_0^T e^{2\alpha s}(G_n(s))^2 ds \\ + \frac{(\lambda_n^0)^2}{\gamma|n|(n^2-1)-\alpha},\end{aligned} \tag{4.34}$$

$$\begin{aligned}\int_0^T e^{2\alpha s}(\lambda_{n,s}(s))^2 ds \le 2\Big(\frac{2(\gamma|n|(n^2-1))^2}{(\gamma|n|(n^2-1)-\alpha)^2} + 1\Big)\int_0^T e^{2\alpha s}(G_n(s))^2 ds \\ + \frac{2(\gamma|n|n^2-1))^2}{\gamma|n|(n^2-1)-\alpha}(\lambda_n^0)^2.\end{aligned} \tag{4.35}$$

Note that

$$\begin{aligned}\frac{1}{8}|n|^m \int_0^T e^{2\alpha s}(G_n(s))^2 ds \le |n|^m \int_0^T e^{2\alpha s}(F_n^2(s))^2 ds + |n|^{m+2}\int_0^T e^{2\alpha s}(F_n^3(s))^2 ds \\ + \delta^{|n|+1}|n|^m \int_0^T e^{2\alpha s}(F_n^4(s))^2 ds \\ + |n|^m \int_0^T e^{2\alpha s}\Big(\int_\delta^1 \rho^{|n|+1}F_n^1(\rho,s)d\rho\Big)^2 ds\end{aligned} \tag{4.36}$$

for any $m \ge 0$, and

$$\begin{aligned}\int_0^T e^{2\alpha s}\Big(\int_\delta^1 \rho^{|n|+1}F_n^1(\rho,s)d\rho\Big)^2 ds &\le \int_0^T e^{2\alpha s}\Big(\int_\delta^1 \rho^{2|n|+1}d\rho\Big)\Big(\int_\delta^1 (F_n^1(\rho,s))^2\rho d\rho\Big)ds \\ &\le \frac{1}{2(|n|+1)}\int_0^T e^{2\alpha s}\int_\delta^1 (F_n^1(\rho,s))^2 \rho d\rho ds.\end{aligned} \tag{4.37}$$

If we differentiate (4.32) in $t$ and again apply Lemma 3.1 we also get, analogously to (4.35),

$$\begin{aligned}\int_0^T e^{2\alpha s}(\lambda_{n,ss}(s))^2 ds \le 2\Big(&\frac{2(\gamma|n|(n^2-1))^2}{(\gamma|n|(n^2-1)-\alpha)^2}+1\Big)\int_0^T e^{2\alpha s}(G_n'(s))^2 ds \\ &+\frac{2(\gamma|n|(n^2-1))^2}{\gamma|n|(n^2-1)-\alpha}(\lambda_{n,t}\,|_{t=0})^2 \\ \le\, & C\int_0^T c^{2\alpha s}(G_n'(s))^2 ds + C\, n^3(\lambda_{n,t}\,|_{t=0})^2\end{aligned} \tag{4.38}$$

and the integral on the right-hand side can be estimated as in (4.36), (4.37) (with $F_n^j$ replaced by $F_{n,s}^j$).

Similarly, if we differentiate (4.32) twice in $t$ and apply Lemma 3.1, we get

$$\begin{aligned}\int_0^T e^{2\alpha s}(\lambda_{n,sss}(s))^2 ds \le\, & C\int_0^T e^{2\alpha s}(G_n''(s))^2 ds \\ &+ C\, n^3(\lambda_{n,tt}\,|_{t=0})^2.\end{aligned} \tag{4.39}$$

Consider next the case $n=0$. One easily finds that the solution $p_0$ has the form

$$p_0 = \int_0^r sF_0^1(s,t)ds\cdot\log r + \int_r^1 sF_0^1(s,t)\log s\; ds + c_1\log r + c_2,$$

and

$$p_{0,r} = \frac{1}{r}\int_0^r sF_0^1(s,t)ds + \frac{c_1}{r}.$$

The boundary conditions then reduce to

$$\begin{aligned}\int_0^1 sF_0^1(s,t)ds + c_1 + \lambda_{0,t} &= F_0^2, \\ c_2+\gamma\lambda_0 &= F_0^3, \\ \int_0^\delta sF_0^1(s,t)ds + c_1 &= F_0^4\delta,\end{aligned}$$

so that

$$\lambda_{0,t} = F_0^2 - \delta F_0^4 - \int_\delta^1 sF_0^1 ds, \tag{4.40}$$

$$c_1 = \delta F_0^4 - \int_0^\delta sF_0^1 ds, \tag{4.41}$$

$$c_2 = F_0^3 - \gamma\lambda_0. \tag{4.42}$$

It follows from (4.15) and (4.40)-(4.42) that

$$
(4.43) \qquad
\begin{aligned}
&(a) \quad \lambda_0(t) - \lambda_0(\infty) = -\int_t^\infty \lambda_{0,t} dt, \\
&(b) \quad \lambda_0(\infty) = \lambda_0(0) + \int_0^\infty \lambda_{0,t} dt = \Lambda_\infty, \\
&(c) \quad |\lambda_0(\infty)| \le |\lambda_0(0)| + \frac{1}{\alpha}\Big(\int_0^\infty e^{2\alpha t}|\lambda_{0,t}|^2\Big)^{1/2}, \\
&(d) \quad p_0(r,\infty) = F_0^3(\infty) - \gamma\lambda_0(\infty) = F_\infty^3 - \gamma\Lambda_\infty = P_\infty \ (= \text{const.}),
\end{aligned}
$$

and

$$
(4.44) \qquad
\begin{aligned}
\Big(\int_0^T e^{2\alpha t}(\lambda_0 - \lambda_0(\infty))^2 dt\Big)^{1/2} &= \Big(\int_0^T e^{2\alpha t}\Big(\int_t^\infty \lambda_{0,t}(s)ds\Big)^2 dt\Big)^{1/2} \\
&= \Big(\int_0^T dt\Big(\int_t^\infty e^{\alpha(t-s)}e^{\alpha s}\lambda_{0,t}(s)ds\Big)^2\Big)^{1/2} \\
&= \Big(\int_0^T dt\Big(\int_0^\infty e^{-\alpha u}e^{\alpha(t+u)}\lambda_{0,t}(t+u)du\Big)^2\Big)^{1/2} \quad (s = t+u) \\
&\le \int_0^\infty e^{-\alpha u}du\Big(\int_0^T (e^{\alpha(t+u)}\lambda_{0,t}(t+u))^2 dt\Big)^{1/2} \\
&\qquad \text{(by Minkowski's inequality)} \\
&= \int_0^\infty e^{-\alpha u}du\Big(\int_u^{T+u} e^{2\alpha t}(\lambda_{0,t}(t))^2 dt\Big)^{1/2} \quad (t+u \to t).
\end{aligned}
$$

Consider finally the case $|n| = 1$. By (4.32),

$$
(4.45) \qquad \lambda_{\pm1,t} = F_{\pm1}^2 - F_{\pm1}^3 - \delta^2 F_{\pm1}^4 - \int_\delta^1 \rho^2 F_{\pm1}^1(\delta,t)d\delta \equiv G_{\pm1}(t)
$$

so that

$$
\lambda_{\pm1}(\infty) = \lambda_{\pm1}^0 + \int_0^\infty G_{\pm1}(t)dt = 0 \quad \text{by (4.14)}.
$$

We then obtain, as in (4.43),

$$
(4.46) \qquad \Big(\int_0^T e^{2\alpha t}(\lambda_{\pm1}(t))^2 dt\Big)^{1/2} \le \int_0^\infty e^{-\alpha u}du\Big(\int_u^{T+u} e^{2\alpha t}(\lambda_{\pm1,t}(t))^2 dt\Big)^{1/2}.
$$

We now multiply (4.34) by $|n|^{19}$, (4.35) by $|n|^{13}$, (4.38) by $|n|^7$ and (4.39) by $|n|$, add the three inequalities and sum over $n$, $|n| \ge 2$. For $|n| = 1$ we use instead the inequality (4.46) and the estimates on $\lambda_{\pm1,t}$, $\lambda_{\pm,tt}$ and $\lambda_{\pm1,ttt}$ that follow from (4.45). Finally for $n = 0$ we use (4.43) and (4.40). We then arrive at the inequality (4.16) with

$$
P_\infty = F_0^3(\infty) - \gamma\lambda_0(\infty), \quad \Lambda_\infty = \lambda_0(\infty)
$$

and with the additional terms

$$\|\Lambda_t \mid_{t=0} \|_{H^5(\partial B)} + \|\Lambda_{tt} \mid_{t=0} \|_{H^2(\partial B)} \tag{4.47}$$

on the right-hand side.

Setting

$$\begin{aligned} \Lambda_1^0 = \|F^1 \mid_{t=0} \|_{H^{9/2}(B)} + \|F_t^1 \mid_{t=0} \|_{H^{3/2}(B)} + \|F^2 \mid_{t=0} \|_{H^5(\partial B)} \\ + \|F_t^2 \mid_{t=0} \|_{H^2(\partial B)} + \|F^3 \mid_{t=0} \|_{H^6(\partial B)} + \|F_t^3 \mid_{t=0} \|_{H^3(\partial B)} \\ + \|F^4 \mid_{t=0} \|_{L^2(\partial B)} + \|F_t^4 \mid_{t=0} \|_{L^2(\partial B)}, \end{aligned} \tag{4.48}$$

we shall prove that

$$\|\Lambda_t \mid_{t=0} \|_{H^5(\partial B)} \le C[\|\Lambda^0\|_{H^8(\partial B)} + \Lambda_1^0], \tag{4.49}$$

$$\|\Lambda_{tt} \mid_{t=0} \|_{H^2(\partial B)} \le C[\|\Lambda^0\|_{H^8(\partial B)} + \Lambda_1^0] \tag{4.50}$$

and that

$$\Lambda_1^0 \text{ is bounded by the right} - \text{hand side of (4.16).} \tag{4.51}$$

It then follows that the expression in (4.47) can be dropped from the right-hand side of (4.16). This completes the estimate (4.16) for $\Lambda - \Lambda_\infty$ and $\Lambda_t$. Using this bound to estimate $c_2$ (in (4.33)) and using also (4.31) and (4.41)-(4.43), we get the bound on $P - P_\infty$ as asserted in (4.16), and the proof of the lemma is thus complete.

To prove (4.49) we note by (4.3) that

$$\|P \mid_{t=0} \|_{H^6(\partial B)} \le C\bar{\Lambda} \text{ where } \bar{\Lambda} = \|\Lambda^0\|_{H^8(\partial B)} + \Lambda_1^0.$$

Applying $L^2$ elliptic estimates to (4.1), (4.3), (4.4) at $t = 0$ we then get

$$\|P \mid_{t=0} \|_{H^{6\frac{1}{2}}(B^*)} \le C\bar{\Lambda}$$

when $B^* = \{2\delta < r < 1\}$ (we take $\delta < \frac{1}{2}$). Consequently

$$\|\frac{\partial P}{\partial r} \mid_{t=0} \|_{H^5(\partial B)} \le C\bar{\Lambda}$$

and then, by (4.2), the inequality (4.49) follows. Similarly, by differentiating the system (4.1)-(4.4) in $t$ and using the same arguments, as well as the bound (4.49), we derive the inequality (4.50).

To prove (4.51) we shall first prove that

$$\|F^2 \mid_{t=0} \|_{H^5(\partial B)} \le C\|F^2\|_{6\frac{1}{2},\partial B} \ . \tag{4.52}$$

Let $\mu(\theta, t)$ be a function defined for $0 \leq \theta \leq 2\pi$, $-\infty < t < \infty$, which coincides with $F^2$ for $0 \leq t \leq 1$, and which vanishes for $t \leq -1$ and $t \geq 2$, such that

$$M \equiv \int_0^2 \int_{\partial B} [|D_\theta^{13/2}\mu|^2 + |D_\theta^{7/2} D_t \mu|^2 + |D_\theta^{1/2}\mu_{tt}|^2] \leq C(\|F^2\|_{6\frac{1}{2},\partial B})^2.$$

To prove (4.52) it suffices to show that

$$\|\mu(\theta, 0)\|^2_{H^5(\partial B)} \leq CM. \tag{4.53}$$

By the trace theorem [1] the left-hand side of (4.53) is bounded by

$$C \int_0^2 \int_{\partial B} [|D_\theta^5 \mu(\theta, t)|^2 + |\nabla^{1/2} D_\theta^5 \mu(\theta, t)|^2]$$

when $\nabla = (\frac{\partial}{\partial\theta}, \frac{\partial}{\partial t})$. Hence taking the Fourier transform of $\mu$ we see (as in Appendix A) that (4.53) would follow from the inequality

$$(|\xi|^2 + |\tau|^2)^{1/2}|\xi|^{10} \leq C(|\xi|^{13} + |\xi|^7|\tau|^2 + |\xi||\tau|^4) \tag{4.54}$$

for $|\xi| > 1$, $|\tau| > 1$. But since

$$|\tau||\xi|^{10} \leq |\tau||\xi|^{7/2}|\xi|^{13/2} \leq |\tau|^2|\xi|^7 + |\xi|^{13}.$$

(4.54) is indeed true.

Having proved (4.53), we can similarly prove that

$$\|F_t^2 \,|_{t=0}\, \|_{H^2(\partial B)} \leq C\|F^2\|_{6\frac{1}{2},\partial B}.$$

Here, in the Fourier transform variables we have to show that

$$(|\xi|^2 + |\tau|^2)^{1/2}|\tau|^2|\xi|^4 \leq C(|\xi|^{13} + |\xi|^7|\tau|^2 + |\xi||\tau|^4)$$

which easily follows from the relations

$$\begin{aligned} |\tau|^3|\xi|^4 &= (|\tau|^3|\xi|^{3/4})|\xi|^{13/4} \\ &\leq (|\tau|^3|\xi|^{3/4})^{4/3} + |\xi|^{(13/4)4} = |\xi||\tau|^4 + |\xi|^{13}. \end{aligned}$$

The other terms in $\Lambda^0$, can be estimated in the same way by the corresponding terms on the right-hand side of (4.16).

## §5. Convergence

In this section we apply Lemma 4.1 in order to estimate inductively the solutions $(P_n, \lambda_n)$ of (2.18)-(2.21) with initial conditions

$$\lambda_n \,|_{t=0} = \begin{cases} \lambda^0 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases} \tag{5.1}$$

**Lemma 5.1**. *Assume that $\lambda^0 \in H^8(\partial B)$ and (1.11) holds. Then the system (2.18)-(2.21), (5.1) has a unique solution satisfying the bounds*

$$\|P_n - P_n^\infty\|_{8,B},\ \|\lambda_n - \lambda_n^\infty\|_{9\frac{1}{2},\partial B},\ \|\lambda_{n,t}\|_{6\frac{1}{2},\partial B} \le C_0 \frac{H^{n-1}}{n^2} \tag{5.2}$$

$$|P_n^\infty|,\ |\lambda_n^\infty| \le C_0 \frac{H^{n-1}}{n^2} \tag{5.3}$$

*for all $n \ge 1$ where $C_0, H$ are positive constants; furthermore, the modes of the function $P_n, \lambda_n$ are of the form*

$$\sum_m \eta_m \ell_m,\ \eta_m \text{ integers.} \tag{5.4}$$

**Proof**. Note that from (2.12)-(2.15) we have $F_1^j = 0$ for $1 \le j \le 4$. Thus, by (1.11), the assertion (5.4) holds for $n = 1$. Also, by Lemma 4.1, the estimates (5.2), (5.3) are valid for $n = 1$ with $H = 1$ and some constant $C_0$. From now on $C_0$ is fixed, and we proceed inductively from $n-1$ to $n$, $n \ge 2$; the constant $H$ will be determined in the inductive process (independently of $n$).

In order to express the $F_n^j$ which appear in (2.18)-(2.21) in a convenient way, we introduce auxiliary functions:

$$\log(1+\lambda) = \sum_{k\ge1} L_k(\theta,t)\epsilon^k, \quad \frac{\lambda_\theta}{1+\lambda} = \sum_{k\ge1} \partial_\theta L_k(\theta,t)\epsilon^k,$$

$$\log(1+\lambda-\delta) = \sum_{k\ge1} \tilde{L}_k(\theta,t)\epsilon^k + \log(1-\delta),$$

$$\frac{\lambda_\theta}{(1+\lambda)^2} = \sum_{k\ge1} U_k(\theta,t)\epsilon^k, \quad \frac{\lambda}{1+\lambda-\delta} = \sum_{k\ge1} R_k(\theta,t)\epsilon^k,$$

$$(\frac{\lambda_\theta}{1+\lambda})^2 = \sum_{k\ge2} W_k(\theta,t)\epsilon^k, \quad \frac{1}{1+\lambda-\delta} = \frac{1}{1-\delta} + \sum_{k\ge1} G_k(\theta,t)\epsilon^k,$$

$$\frac{1}{(1+(\frac{\lambda_\theta}{1+\lambda})^2)^{3/2}} = 1 + \sum_{k\ge2} Q_k^1(\theta,t)\epsilon^k,$$

$$\frac{(\frac{\lambda_\theta}{1+\lambda})^2}{(1+(\frac{\lambda_\theta}{1+\lambda})^2)^{3/2}} = \sum_{k\ge2} Q_k^2(\theta,t)\epsilon^k,$$

$$\frac{1}{(1+\lambda)^m} = 1 + \sum_{k\ge1} J_k^m(\theta,t)\epsilon^k \quad (m=1,2), \quad \frac{\lambda}{1+\lambda} = \sum_{k\ge1} J_k^3(\theta,t)\epsilon^k$$

and

$$\frac{1}{(1+\lambda)^2}\frac{1}{(1+(\frac{\lambda_\theta}{1+\lambda})^2)^{3/2}} = 1 + \sum_{k\ge1} I_k(\theta,t)\epsilon^k.$$

By Theorem A.4, Lemma B.2 (see Appendices A and B) and the inductive assumption, for $1 \le k \le n-1$, there holds:

$$
(5.5) \quad \begin{aligned}
&(a) \quad \|L_k - L_k^\infty\|_{9\frac{1}{2},\partial B},\ \|\tilde{L}_k - \tilde{L}_k^\infty\|_{9\frac{1}{2},\partial B},\ \|R_k - R_k^\infty\|_{9\frac{1}{2},\partial B}, +\|G_k - G_k^\infty\|_{9\frac{1}{2},\partial B} \le 2CC_0\frac{H^{k-1}}{k^2},\\
&(b) \quad |L_k^\infty|,\ |\tilde{L}_k^\infty|,\ |R_k^\infty|,\ |G_k^\infty| \le 2CC_0\frac{H^{k-1}}{k^2},
\end{aligned}
$$

and, for $i = 1, 2, 3$,

$$
(5.6) \qquad \begin{aligned}
&(a) \quad \|J_k^i - J_k^{i,\infty}\|_{9\frac{1}{2},\partial B} \le 2CC_0\frac{H^{k-1}}{k^2},\\
&(b) \quad |J_k^{i,\infty}| \le 2CC_0\frac{H^{k-1}}{k^2};
\end{aligned}
$$

here and in the sequel $C$ is a generic constant $> (1-\delta)^{-1}$ which depends only on $C_0$, and $H$ is sufficiently large (depending only on $C_0, C$). Since $\lambda_\theta/(1+\lambda)^2 = -\partial_\theta(1+\lambda)^{-1}$, we also have

$$
(5.7) \qquad \|U_k\|_{8\frac{1}{2},\partial B} \le 2CC_0\frac{H^{k-1}}{k^2} \quad (1 \le k \le n-1).
$$

By Theorem A.4, Lemma B.1 ($N = 2$) and (5.5(a)) we also deduce that

$$
(5.8) \qquad \|W_k\|_{8\frac{1}{2},\partial B} \le A_0 M_0 (2CC_0)^2\frac{H^{k-2}}{k^2} \quad (2 \le k \le n-1),
$$

and then, by Lemma B.2,

$$
(5.9) \qquad \|Q_k^i\|_{8\frac{1}{2},\partial B} \le 2A_0 M_0 (2CC_0)^2\frac{H^{k-2}}{k^2}, \quad 2 \le k \le n-1 \ (i = 1, 2).
$$

Since

$$
I_k(\theta, t) = Q_k^1 + J_k^2 + \sum_{m=2}^{k-1} Q_m^1 J_{k-m}^2,
$$

we have

$$
\begin{aligned}
\|I_k - I_k^\infty\|_{8\frac{1}{2},\partial B} &\le \|Q_k^1\|_{8\frac{1}{2},\partial B} + \|J_k^2 - J_k^{2,\infty}\|_{8\frac{1}{2},\partial B}\\
&\quad + M_0 \sum_{m=2}^{k-1} \|Q_m^1\|_{8\frac{1}{2},\partial B}[|J_{k-m}^{2,\infty}| + \|J_{k-m}^2 - J_{k-m}^{2,\infty}\|_{8\frac{1}{2},\partial B}]\\
&\le 2CC_0\frac{H^{k-1}}{k^2}\{\frac{4C_0 C A_0 M_0}{H} + 1 + 2M_0 \sum_{m=2}^{k-1} k^2 (8A_0 M_0 C^2 C_0^2)\frac{H^{m-2}}{m^2}\frac{H^{-m}}{(k-m)^2}\}.
\end{aligned}
$$

Hence

$$
(5.10) \qquad \|I_k - I_k^\infty\|_{8\frac{1}{2},\partial B} \le 3CC_0\frac{H^{k-1}}{k^2}
$$

if $H$ is large enough.

Consider now $F_n^2$. Comparing (2.13) with (2.19) we find that

$$F_n^2(\theta,t)=\sum_{k=1}^{n-1}[U_{n-k}\frac{\partial P_k}{\partial\theta}(1,\theta,t)+V_{n-k}\frac{\partial P_k}{\partial r}(1,\theta,t)]$$

where

$$\sum_{k\geq 1}V_k\epsilon^k=1-\frac{(1-\delta)[(1+\lambda)^2+\lambda_\theta^2]}{(1+\lambda)^2(1+\lambda-\delta)}=\frac{\lambda}{1+\lambda-\delta}-(1-\delta)\frac{\lambda_\theta}{(1+\lambda)^2}\frac{\lambda_\theta}{(1+\lambda-\delta)}$$

$$=\sum_{k\geq 1}\{R_k-(1-\delta)\sum_{m=1}^{k-1}[U_m\cdot\partial_\theta\tilde{L}_{k-m}]\}\epsilon^k.$$

Applying Theorem A.4 and (5.5), (5.7) we get

$$\|V_k-V_k^\infty\|_{8\frac{1}{2},\partial B}\leq\|R_k-R_k^\infty\|_{8\frac{1}{2},\partial B}+(1-\delta)\sum_{m=1}^{k-1}M_0(2CC_0)^2\frac{H^{m-1}}{m^2}\frac{H^{k-m-1}}{(k-m)^2}$$

$$\leq 2CC_0\frac{H^{k-1}}{k^2}[1+(1-\delta)\frac{2CC_0A_0M_0}{H}],$$

so that

$$\|V_k-V_k^\infty\|_{8\frac{1}{2},\partial B}\leq 3CC_0\frac{H^{k-1}}{k^2}.$$

We also have, by (5.5)(b),

$$|V_k^\infty|=|R_k^\infty|\leq 2CC_0\frac{H^{k-1}}{k^2}.$$

Next by the trace theorem and the inductive assumptions,

$$\|\frac{\partial P_k}{\partial\theta}(1,\theta,t)\|_{6\frac{1}{2},\partial B},\ \|\frac{\partial P_k}{\partial r}(1,\theta,t)\|_{6\frac{1}{2},\partial B}\leq CC_0\frac{H^{k-1}}{k^2}$$

if $k\leq n-1$. Combining these estimates with (5.7) and using Theorem A.4, we get

$$\begin{aligned}\|F_n^2\|_{6\frac{1}{2},\partial B}&\leq\sum_{k=1}^{n-1}M_0[2CC_0\frac{H^{n-k-1}}{(n-k)^2}CC_0\frac{H^{k-1}}{k^2}+5CC_0\frac{H^{n-k-1}}{(n-k)^2}\frac{CC_0H^{k-1}}{k^2}]\\&\leq 7M_0A_0C^2C_0^2\frac{H^{n-2}}{n^2}.\end{aligned}\tag{5.11}$$

Similarly

$$F_n^4=\sum_{k=1}^{n-1}R_{n-k}\frac{\partial P_k}{\partial r}(\delta,\theta,t),$$

so that

$$
\begin{aligned}
\|F_n^4\|_{6\frac{1}{2},\partial B} &\le M_0 \sum_{k=1}^{n-1} [\|R_{n-k} - R_{n-k}^\infty\|_{6\frac{1}{2},\partial B} \cdot C\|P_k - P_k^\infty\|_{8,B} \\
&\quad + |R_{n-k}^\infty| C \|P_k - P_k^\infty\|_{8,B}] \\
&\le M_0 \sum_{k=1}^{n-1} 2CC_0 \frac{H^{n-k-1}}{(n-k)^2} CC_0 \frac{H^{k-1}}{k^2} \le 2C^2 C_0^2 M_0 A_0 \frac{H^{n-2}}{n^2}.
\end{aligned} \tag{5.12}
$$

Next we consider $F_n^3$. Writing $F^3$ (in (2.14)) in the form

$$
\begin{aligned}
F^3 = \gamma\{ &\frac{(\frac{\lambda_\theta}{1+\lambda})^2}{(1+(\frac{\lambda_\theta}{1+\lambda})^2)^{3/2}} \frac{2}{1+\lambda} - [\frac{1}{(1+(\frac{\lambda_\theta}{1+\lambda})^2)^{3/2}} \frac{1}{(1+\lambda)^2} - 1]\lambda_{\theta\theta} \\
&+ [\frac{1}{(1+(\frac{\lambda_\theta}{1+\lambda})^2)^{3/2}} - 1] \frac{1}{1+\lambda} + \frac{\lambda^2}{1+\lambda}\},
\end{aligned}
$$

we see that

$$
\begin{aligned}
F_n^3 = \gamma\{ &2\sum_{m=2}^{n-1} Q_m^2 J_{n-m}^1 + 2Q_n^2 - \sum_{m=1}^{n-1} I_m \lambda_{n-m,\theta\theta} \\
&+ \sum_{m=2}^{n-1} Q_m^1 J_{n-m}^1 + Q_n^1 + \sum_{m=1}^{n-1} J_m^3 \lambda_{n-m}\}.
\end{aligned}
$$

Hence, by Theorem A.4, (5.6), (5.9) and the inductive assumption on $\lambda_{n-m}$,

$$
\|F_n^3 - F_n^{3,\infty}\|_{6\frac{1}{2},\partial B} \le C_0 C_1 \frac{H^{n-2}}{n^2}, \tag{5.13}
$$

where $C_1$ is a constant independent of $n$ and $H$.

We finally consider $F_n^1$ and write it in the form

$$
F_n^1(r,\theta,t) = \sum_{k=1}^{n-1} [A_{n-k}(r,\theta,t) \frac{\partial^2 P_k}{\partial r^2} + B_{n-k}(r,\theta,t) \frac{\partial^2 P_k}{\partial r \partial \theta} + D_{n-k}(r,\theta,t) \frac{\partial P_k}{\partial r}]. \tag{5.14}
$$

Then

$$
\begin{aligned}
A_k &= \frac{1}{k!} \partial_\epsilon^k \mid_{\epsilon=0} \{1 - \frac{(1-\delta)^2}{(1+\lambda-\delta)^2} - \frac{(1-\delta)^2}{((1+\lambda-\delta)r - \delta\lambda)^2} (\frac{\lambda_\theta}{1+\lambda-\delta})^2 (\delta - r)^2\} \\
&= \alpha_k(\theta,t) - \frac{1}{k!} \partial_\epsilon^k \mid_{\epsilon=0} \{\sum_{\ell \ge 0} (\ell+1) \frac{(1-\delta)^2(\delta-r)^2}{r^2 r^\ell} \frac{1}{(1+\lambda-\delta)^2} \frac{(\delta\lambda)^\ell}{(1+\lambda-\delta)^\ell} (\frac{\lambda_\theta}{1+\lambda-\delta})^2\}
\end{aligned} \tag{5.15}
$$

where

$$
\|\alpha_k - \alpha_k^\infty\|_{9\frac{1}{2},\partial B} \le 2CC_0 \frac{H^{k-1}}{k^2}.
$$

¿From the expansion of $\lambda/(1+\lambda-\delta)$ and the estimates on the $R_k$ (in (5.5)) we deduce, using Lemma B.1, that

$$(\frac{\delta\lambda}{1+\lambda-\delta})^\ell = \sum_{k\geq m} R_k^\ell \epsilon^k$$

where

$$\|R_k^\ell - R_k^{\ell,\infty}\|_{9\frac{1}{2},\partial B} \leq M_0^{\ell-1}(2CC_0\delta)^\ell A_0^{\ell-1}\frac{H^{k-\ell}}{k^2}.$$

Also

$$\frac{\lambda_\theta}{(1+\lambda-\delta)^2} = -\sum_{k\geq 1}(\partial_\theta G_k)\epsilon^k$$

where, by (5.5),

$$\|\partial_\theta G_k\|_{8\frac{1}{2},\partial B} \leq 2CC_0\frac{H^{k-1}}{k^2}.$$

Hence, by Lemma B.1 ($N=2$)

$$(\frac{\lambda_\theta}{(1+\lambda-\delta)^2})^2 = \sum_{k\geq 2}\tilde{G}_k\epsilon^k$$

where

$$\|\tilde{G}_k\|_{8\frac{1}{2},\partial B} \leq 4C^2C_0^2A_0M_0\frac{H^{k-2}}{k^2}.$$

We can now write

$$(\frac{\lambda_\theta}{(1+\lambda-\delta)^2})^2(\frac{\delta\lambda}{1+\lambda-\delta})^\ell = \sum_{k\geq \ell+2}\alpha_k^\ell\epsilon^k$$

and deduce, using Theorem A.4, that

$$\begin{aligned}\|\alpha_k^\ell\|_{8\frac{1}{2},\partial B} &\leq M_0\sum_{m=2}^{k-\ell}4C^2C_0^2A_0M_0\frac{H^{m-2}}{m^2}M_0^{\ell-1}(2CC_0\delta)^\ell A_0^{\ell-1}\frac{H^{k-m-\ell}}{(k-m)^2}\\ &\leq C^2C_0^2A_0M_0(2CC_0\delta A_0M_0)^\ell\frac{H^{k-\ell-2}}{k^2}.\end{aligned}$$

Since

$$A_k = \alpha_k - \sum_{\ell\geq 0}(\ell+1)\frac{(1-\delta)^2(\delta-r)^2}{r^{\ell+2}}\alpha_k^\ell(\theta,t),$$

it follows, by Theorem A.2 and the inductive assumptions, that

$$
\begin{aligned}
\|\sum_{k=1}^{n-1} A_{n-k}(r,\theta,t)\frac{\partial^2 P_k}{\partial r^2}\|_{6,B} &\le M_0 \sum_{k=1}^{n-1}[\|A_{n-k}-A^\infty_{n-k}\|_{6,B}\|P_k-P^\infty_k\|_{8,B}+|A^\infty_{n-k}|\|P_k-P^\infty_k\|_{8,B}] \\
&\le M_0 C\sum_{k=1}^{n-1}[\|\alpha_{n-k}-\alpha^\infty_{n-k}\|_{6\frac{1}{2},\partial B} \\
&\quad + |\alpha^\infty_{n-k}| + \sum_{\ell\ge 0}\|\alpha^\ell_{n-k}\|_{6\frac{1}{2},\partial B}(1+\ell)^7]\|P_k-P^\infty_k\|_{8,B} \\
&\le M_0 C\sum_{k=1}^{n-1} 4CC_0\frac{H^{n-k-1}}{(n-k)^2}C_0\frac{H^{k-1}}{k^2} \\
&\quad + M_0 C\sum_{k=1}^{n-1}\sum_{\ell\ge 0}(1+\ell)^7 C^2 C_0^2 A_0 M_0 (2CC_0\delta A_0 M_0)^\ell \frac{H^{n-k-\ell-2}}{(n-k)^2}C_0\frac{H^{k-1}}{k^2} \\
&\le C_0 C_2\frac{H^{n-2}}{n^2}
\end{aligned}
$$

where the constant $C_2$ is independent of $H$; notice that a factor of $(1+\ell)^6$ comes from the six derivatives of $1/r^{\ell+2}$.

The coefficients $B_k(r,\theta,t)$ and $C_k(r,\theta,t)$ in $F^1_n$ have a similar structure to $A_k(r,\theta,t)$ and can be estimated in the same way. We thus conclude that

$$
\|F^1_n\|_{6,B} \le C_3 C_0 \frac{H^{n-2}}{n^2} \tag{5.16}
$$

where the constant $C_3$ is independent of $H$.

¿From the structure of $A_{n-k}$ in $F^1_n$ (see (5.14), (5.15)) it is clear that it is a sum of products of functions $\lambda_j, D_\theta\lambda_j, D^2_\theta\lambda_j$ and their powers. Since, by the inductive assumptions, each of these functions has only modes of the form (5.4), the same is then true for $A_{n-k}$. Similarly, $A_{n-k}\partial^2 P_k/\partial r^2$ has only modes of the form (5.4), and the same is true also for the other terms of $F^1_n$ and, similarly, for the other $F^j_n$. Recalling (5.1) and (1.11)(b) we deduce the condition (4.14) for the system $(P_n,\lambda_n)$ is trivially satisfied (each of the terms in this condition is zero). Thus Lemma 4.1 can be applied.

The proof of Lemma 4.1 uses Lemma 3.1 and shows that the solution $P_n,\lambda_n$ will only have modes of the form (5.4). Next, using the estimates (5.11)-(5.13), (5.16) we conclude that (5.2), (5.3) hold with the right-hand sides

$$
C_0 C^* \frac{H^{n-2}}{n^2}
$$

when $C^*$ depends on $C_0$ but not on $H$. Hence, choosing $H > C^*$, the proof of (5.2), (5.3) is complete.

¿From Lemma 5.1 we obtain the following:

**Theorem 5.2**. *If $\lambda^0 \in H^8(\partial B)$ and it satisfies (1.11), then there exists a unique solution $(p,\lambda)$ to problem (A) of the form (2.16), (2.17) where the series are uniformly convergent for $|\epsilon| < 1/(2H)$, and*

$$
\|p - p_\infty\|_{8,B} < \infty, \tag{5.17}
$$

$$(5.18) \qquad \|\lambda - \lambda_\infty\|_{9\frac{1}{2},\partial B}, \ \|\lambda_t\|_{6\frac{1}{2},\partial B} < \infty$$

*where $p_\infty, \lambda_\infty$ are constants having the power series expansions*

$$(5.19) \qquad p_\infty = \gamma + \sum_{n\geq 1} p_{n,\infty}\epsilon^n, \ \lambda_\infty = \sum_{n\geq 1} \lambda_{n,\infty}\epsilon^n \quad (|\epsilon| < \frac{1}{2H}).$$

Note that the uniform convergence of the series expansions (2.16), (2.17) follows from the estimates of Lemma 5.1 and Lemmas A.1, A.3 (which imply that the $L^\infty$-norms of $P - P_\infty$ and $\lambda - \lambda_\infty$ are bounded by the norms in (5.17), (5.18), respectively).

¿From (5.18) we also deduce that

$$(5.20) \qquad |\lambda(\theta, t, \epsilon) - \lambda_\infty| \leq Ce^{-\alpha t} \qquad \forall\ 0 < \alpha < 6\gamma.$$

Reversing the transformation (1.13) and using Theorem 5.2 and Lemma 2.1, we have:

**Theorem 5.3**. *If $\lambda^0 \in H^8(\partial B)$ and it satisfies (1.11), then there exists a unique solution to the problem (1.1)-(1.3), (1.6) with free boundary of the form (2.17), where the series is uniformly convergent for $|\epsilon| < 1/(2H)$, and (5.18) holds.*

**Proof**. We only need to prove uniqueness. Note first that if we apply the transformation (1.13) to a solution $(p, \lambda)$ of (1.1)-(1.3), (1.6) with $\lambda$ analytic in $\epsilon$, we obtain an elliptic problem for $p'$ (the transform of $p$ under the change of variables) with analytic coefficients. We can then estimate

$$D^n_\epsilon p' \mid_{\epsilon=0}$$

inductively, as in Lemma 5.1, but actually much more simply since $\lambda$ is already analytic in $\epsilon$. We find that $p'$ must be also analtyic in $\epsilon$, so that we can write a convergent power series expansion for $p'$, with coefficients, say, $P_n$. But then the system $(P_n, \lambda_n)$ must coincide with the system in Lemma 5.1, and the uniqueness for $p', \lambda$ (and then also for $p, \lambda$) follows.

//

## Part II. Two Dimensions; The General Case

### §6. Reformulation of the problem

Part II is devoted to extending Theorem 5.2 (for $\nu = 2$) to the case where the condition (1.11) is dropped. In that case, we expect that the global solution will exist and that the free boundary will converge to a circle with center $P_0$ which depends on $\epsilon$. We write, in polar coordinates,

$$P_0 = (\rho_0(\epsilon),\ \theta_0(\epsilon)) \equiv (\rho_0, \theta_0)$$

and prove that the global solution is analytic in $\epsilon$ and, furthermore,

$$(6.1) \qquad \begin{aligned} \rho_0 &= \sum_{m\geq 1} \rho_{0m}\epsilon^m, \\ \theta_0 &= \theta_{00} + \sum_{m\geq 1} \theta_{0m}\epsilon^m. \end{aligned}$$

As in Part I, we modify the original problem by considering it in a region $\{\delta < r < 1+\lambda\}$ with the additional boundary condition (2.5).

It is natural to introduce a new coordinate system centered about $(\rho_0, \theta_0)$ with polar coordinates $(\rho, \psi)$. Then

$$\rho^2 = \rho_0^2 + r^2 - 2\rho_0 r\cos(\theta - \theta_0) \tag{6.2}$$

where $(r, \theta)$ are the original polar coordinates. One can easily verify that

$$\frac{\sin(\theta - \psi)}{\rho_0} = \frac{\sin(\theta_0 - \theta)}{\rho},$$

from which we deduce the relation

$$\psi = \theta - \sin^{-1}[\frac{\rho_0 \sin(\theta_0 - \theta)}{[r^2 + \rho_0^2 - 2r\rho_0\cos(\theta - \theta_0)]^{1/2}}]. \tag{6.3}$$

For $r > \frac{1}{2}\delta$ and $\rho_0$ small, the function on the right-hand side of (6.3), written in the form $\psi = \psi(\theta, r, \theta_0, \rho_0)$, is analytic in all its variables and, for fixed $(r, \theta_0, \rho_0)$, it has an inverse $\theta = \theta(\psi, r, \theta_0, \rho_0)$.

Later on we shall find it more convenient to work mostly with the independent variables $(\rho, \theta)$ rather than with the polar coordinates $(r, \theta)$, or $(\rho, \psi)$.

Along the free boundary

$$\rho = [\rho_0^2 + (1+\lambda)^2 - 2\rho_0(1+\lambda)\cos(\theta - \theta_0)]^{1/2} \equiv N(\theta - \theta_0, \rho_0, \lambda) \tag{6.4}$$

where $\lambda = \lambda(\theta, t)$. The function $N(\theta - \theta_0, \rho_0, \lambda)$ is analytic in the variables $(\theta - \theta_0, \rho_0, \lambda)$ (for $\rho_0$ small). It will be convenient to use the notation

$$N(\theta - \theta_0, \rho_0, \lambda(\theta, t)) \equiv \hat{N}(\theta, t) \equiv \tilde{N}(\psi, t) \tag{6.5}$$

where

$$\psi = \theta - \sin^{-1}\frac{\rho_0 \sin(\theta_0 - \theta)}{[(1+\lambda(\theta,t))^2 + \rho_0^2 - 2(1+\lambda(\theta,t))\rho_0\cos(\theta - \theta_0)]^{1/2}} \tag{6.6}$$

is the mapping $\theta \leftrightarrow \psi$ along the free boundary.

It will also be useful to express $\lambda$ in terms of $N$ from (6.4): Since

$$(1+\lambda)^2 - 2\rho_0(1+\lambda)\cos(\theta - \theta_0) - (N^2 - \rho_0^2) = 0$$

we get

$$1 + \lambda = N + \rho_0\cos(\theta - \theta_0) + N\{[1 - \frac{\rho_0^2}{N^2}\sin^2(\theta - \theta_0)]^{1/2} - 1]\} \tag{6.7}$$

where the last two terms on the right-hand side are small if $\rho_0$ is small. We shall find it convenient to rewrite (6.7) in the form

$$1 + \lambda = N + \rho_0\cos(\theta - \theta_0) + W(\theta - \theta_0, \rho_0, N), \quad W(\theta - \theta_0, \rho_0, N) = O(\rho_0^2) \tag{6.8}$$

where $W(\theta-\theta_0, \rho_0, N)$ is analytic in all its variables for $\rho_0$ and $|N-1|$ small. On the free boundary, the independent variable $N$, in (6.8), is to be substituted by $\hat{N}(\theta,t)$.

The circle $r=\delta$ can be written as

$$\rho = [\rho_0^2 + \delta^2 - 2\rho_0\delta\cos(\theta-\theta_0)]^{1/2} \equiv M(\theta-\theta_0, \rho_0) \equiv \hat{M}(\theta). \tag{6.9}$$

We next introduce another change of variables $\rho \to \rho'$, by

$$\rho' = \frac{(1-\delta)\rho + \hat{N}(\theta,t)\delta - \hat{M}(\theta)}{\hat{N}(\theta,t) - \hat{M}(\theta)} \tag{6.10}$$

which maps

$$\begin{aligned} \rho &= \hat{N}(\theta,t) \quad \text{onto } \rho' = 1, \text{ and} \\ \rho &= \hat{M}(\theta) \quad \text{onto } \rho' = \delta. \end{aligned}$$

¿From (6.2), (6.8)-(6.10) we see that

$$\begin{aligned} r &= \Phi(\rho', \theta-\theta_0, \rho_0, \hat{N}), \\ \rho' &= \Psi(r, \theta-\theta_0, \rho_0, \hat{N}) \end{aligned} \tag{6.11}$$

where $\Phi, \Psi$ are analytic jointly in all their variables provided $\rho_0$ and $|\hat{N}-1|$ are small and $r \geq \delta/2$, $\rho' \geq \delta/2$.

We introduce the function

$$p'(\rho', \theta, t) = p(r, \theta, t)$$

and, as in Part I, we want to transform the problem for $(p,\lambda)$ in $\{\delta < r < 1+\lambda\}$ into a problem for $(p', N)$ in $\{\delta < \rho' < 1\}$, in the variables $(\rho', \theta)$.

We first compute

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{\partial \rho'}{\partial r}\frac{\partial p'}{\partial \rho'}, \quad \frac{\partial p}{\partial \theta} = \frac{\partial \rho'}{\partial \theta}\frac{\partial p'}{\partial \rho'} + \frac{\partial p'}{\partial \theta}, \\ \frac{\partial^2 p}{\partial r^2} &= \frac{\partial^2 \rho'}{\partial r^2}\frac{\partial p'}{\partial \rho'} + (\frac{\partial \rho'}{\partial r})^2 \frac{\partial^2 p'}{\partial \rho'^2}, \\ \frac{\partial^2 p}{\partial \theta^2} &= \frac{\partial}{\partial \theta}(\frac{\partial \rho'}{\partial \theta}\frac{\partial p'}{\partial \rho'} + \frac{\partial p'}{\partial \theta}) \\ &= \frac{\partial^2 \rho'}{\partial \theta^2}\frac{\partial p'}{\partial \rho'} + \frac{\partial \rho'}{\partial \theta}(\frac{\partial \rho'}{\partial \theta}\frac{\partial}{\partial \rho'} + \frac{\partial}{\partial \theta})\frac{\partial p'}{\partial \rho'} + (\frac{\partial \rho'}{\partial \theta}\frac{\partial}{\partial \rho'} + \frac{\partial}{\partial \theta})\frac{\partial \rho'}{\partial \theta}, \\ &= \frac{\partial^2 \rho'}{\partial \theta^2}\frac{\partial p'}{\partial \rho'} + (\frac{\partial \rho'}{\partial \theta})^2 \frac{\partial^2 p'}{\partial \rho'^2} + 2\frac{\partial \rho'}{\partial \theta}\frac{\partial^2 p'}{\partial \rho' \partial \theta} + \frac{\partial^2 p'}{\partial \theta^2}. \end{aligned} \tag{6.12}$$

The boundary condition (2.1) becomes

$$\frac{\partial p'}{\partial \rho'} + (\frac{\partial \rho'}{\partial r} - 1)\frac{\partial p'}{\partial \rho'} - \frac{\lambda_\theta}{(1+\lambda)^2}\frac{\partial p'}{\partial \theta} - \frac{\lambda_\theta}{(1+\lambda)^2}\frac{\partial \rho'}{\partial \theta}\frac{\partial p'}{\partial \rho'} = -\lambda_t \tag{6.13}$$

and we shall later on express $\lambda, \lambda_\theta$ in terms of $\hat{N}, \hat{N}_\theta$.

The boundary condition $p = \gamma\kappa$ becomes $p' = \gamma\kappa$. Since the free boundary in polar coordinates $(\rho, \psi)$ is given by $\rho = \tilde{N}(\psi, t)$, we then have

$$p' = \gamma \frac{2\tilde{N}_\psi^2 - \tilde{N}\tilde{N}_{\psi\psi} + \tilde{N}^2}{(\tilde{N}^2 + \tilde{N}_\psi^2)^{3/2}}, \tag{6.14}$$

and later on we shall express $\tilde{N}(\psi, t)$ and its derivatives in terms of $\hat{N}(\theta, t)$ and its derivatives.

Finally, the boundary condition (2.5) becomes

$$\frac{\partial p'}{\partial \rho'}\frac{\partial \rho'}{\partial r} = \frac{1}{\delta} H\left(\frac{\partial p'}{\partial \theta} + \frac{\partial \rho'}{\partial \theta}\frac{\partial p'}{\partial \rho'}\right) \quad \text{at } \rho' = \delta, \tag{6.15}$$

and, by (6.10), at $\rho = \hat{M}$ $(\rho' = \delta)$

$$\frac{\partial \rho'}{\partial \theta} = \frac{1}{(N - M)^2}\{(\delta\hat{N}_\theta - \hat{M}_\theta)(\hat{N} - \hat{M}) - (\hat{N}_\theta - \hat{M}_\theta)[(1-\delta)\hat{M} + \delta\hat{N} - \hat{M}]\}.$$

Since the expression in brackets is equal to $\delta(\hat{N} - \hat{M})$, we obtain

$$\frac{\partial \rho'}{\partial \theta} = \frac{(\delta - 1)\hat{M}_\theta}{\hat{N} - \hat{M}} \text{ at } \rho' = \delta. \tag{6.16}$$

Later on we shall use the relations

$$\frac{\lambda_\theta}{(1+\lambda)^2} = \frac{\hat{N}_\theta - \rho_0 \sin(\theta - \theta_0) + \hat{W}_\theta}{(\hat{N} + \rho_0\cos(\theta - \theta_0) + \hat{W})^2}, \quad \lambda_t = \hat{N}_t + \hat{W}_t \tag{6.17}$$

where

$$\hat{W} = \hat{W}(\theta, t) \equiv W(\theta - \theta_0, \rho_0, \hat{N}(\theta, t)) \tag{6.18}$$

and $\hat{N}$ is defined as in (6.5). Note that all the derivatives of $\hat{W}$ in (6.17) are taken as total derivatives.

We shall also need the following relations along the free boundary:

$$\begin{aligned} \tilde{N}_\psi &= \hat{N}_\theta\theta_\psi, \ (\tilde{N}_\psi)^2 = \hat{N}_\theta^2 + \hat{N}_\theta^2[(\theta_\psi)^2 - 1], \\ \tilde{N}_{\psi\psi} &= \frac{\partial}{\partial\psi}(\theta_\psi\hat{N}_\theta) = \hat{N}_\theta\theta_{\psi\psi} + (\theta_\psi)^2\hat{N}_{\theta\theta} \\ &= \hat{N}_{\theta\theta} + \{\hat{N}_\theta\theta_{\psi\psi} + \hat{N}_{\theta\theta}[(\theta_\psi)^2 - 1]\}; \end{aligned} \tag{6.19}$$

$|1 - \theta_\psi|$ and $|\theta_{\psi\psi}|$ are small if $\rho_0$ is small (see (6.6)).

For simplicity we shall henceforth drop the " $'$ " in $p'$. Collecting the previous formulas, we can now reformulate the problem in the variables $(\rho', \theta)$ as follows:

$$\Delta p = F^1 \quad \text{in} \quad \delta < \rho' < 1, \tag{6.20$a$}$$

(6.20b)
$$F^1 = [1-(\frac{\partial \rho'}{\partial r})^2]\frac{\partial^2 p}{\partial \rho'^2} - (\frac{\partial^2 \rho'}{\partial r^2})\frac{\partial p}{\partial \rho'} + \frac{1}{r}(1-\frac{\partial \rho'}{\partial r})\frac{\partial p}{\partial \rho'}$$
$$- \frac{1}{r^2}[\frac{\partial^2 \rho'}{\partial \theta^2}\frac{\partial p}{\partial \rho'} + (\frac{\partial \rho'}{\partial \theta})^2\frac{\partial^2 p}{\partial \rho'^2} + 2\frac{\partial \rho'}{\partial \theta}\frac{\partial^2 p}{\partial \theta \partial \rho'}],$$

(6.21a)
$$\frac{\partial p}{\partial \rho'} + \hat{N}_t = F^2 \text{ at } \rho' = 1,$$

(6.21b)
$$F^2 = -\hat{W}_t + (1-\frac{\partial \rho'}{\partial r})\frac{\partial p}{\partial \rho'} + \frac{(\hat{N}_\theta - \rho_0\sin(\theta-\theta_0) + \hat{W}_\theta)}{(\hat{N} + \rho_0\cos(\theta-\theta_0) + \hat{W})^2}\frac{\partial p}{\partial \theta} + \frac{(\hat{N}_\theta + \hat{W}_\theta)}{(\hat{N}+\hat{W})^2}\frac{\partial \rho'}{\partial \theta}\frac{\partial p}{\partial \rho'},$$

(6.22a)
$$p + \gamma(\hat{N}_{\theta\theta} + \hat{N} - 1) = F^3 \quad \text{at } \rho' = 1,$$

(6.22b)
$$F^3 = \gamma\{\frac{2\hat{N}_\theta^2 - \hat{N}\hat{N}_{\theta\theta} + N^2 + 2((\theta_\psi)^2-1)\hat{N}_\theta^2 - \hat{N}(\hat{N}_\theta\theta_{\psi\psi} + \hat{N}_{\theta\theta}((\theta_\psi)^2-1))}{[\hat{N}^2 + \hat{N}_\theta^2 + \hat{N}_\theta^2((\theta_\psi)^2-1)]^{3/2}}$$
$$+ \hat{N}_{\theta\theta} + \hat{N} - 1\},$$

(6.23a)
$$\frac{\partial p}{\partial \rho'} - \frac{1}{\delta}H(\frac{\partial p'}{\partial \theta}) = F^4 \quad \text{at } \rho' = \delta,$$

(6.23b)
$$F^4 = (1-\frac{\partial \rho'}{\partial r})\frac{\partial p}{\partial \rho'} + \frac{1}{\delta}H(\frac{(\delta-1)\hat{M}_\theta}{\hat{N}-\hat{M}}\frac{\partial p}{\partial \rho'}).$$

The inhomogeneous terms $F^j$ will be treated, in the sequel, as small perturbations of the linear system for $p, N$.

We finally write down the initial condition for the free boundary, in terms of $N$:

$$\begin{aligned}\hat{N}\mid_{t=0} &= [\rho_0^2 + (1+\epsilon\lambda^0)^2 - 2\rho_0(1+\epsilon\lambda^0)\cos(\theta-\theta_0)]^{1/2}\\ &= [((1+\epsilon\lambda^0) - \rho_0\cos(\theta-\theta_0))^2 + \rho_0^2\sin^2(\theta-\theta_0)]^{1/2},\end{aligned}$$

or

(6.24)
$$\hat{N}\mid_{t=0} = [1+\epsilon\lambda^0 - \rho_0\frac{e^{-i\theta_0}}{2}e^{i\theta} - \rho_0\frac{e^{i\theta_0}}{2}e^{-i\theta}]$$
$$\times [1 + \frac{\rho_0^2\sin^2(\theta-\theta_0)}{(1+\epsilon\lambda^0 - \rho_0\cos(\theta-\theta_0))^2}]^{1/2} \equiv N^0.$$

## §7. Convergence

Analogously to the procedure in Part I we substitute

$$p = \gamma + \sum_{n\geq 1} P_n(\rho', \theta, t)\epsilon^n, \tag{7.1}$$

$$\hat{N} = 1 + \sum_{n\geq 1} N_n(\theta, t)\epsilon^n \tag{7.2}$$

and $\rho_0, \theta_0$ from (6.1) into the system (6.20)-(6.23), and equate the coefficients of $\epsilon^n$ to obtain a system similar to (2.18)-(2.21):

$$\Delta P_n = F_n^1, \quad \delta < \rho' < 1, \tag{7.3}$$

$$\frac{\partial P_n}{\partial \rho'} + \frac{\partial N_n}{\partial t} = F_n^2, \quad \rho' = 1, \tag{7.4}$$

$$P_n + \gamma(\frac{\partial^2 N_n}{\partial \theta^2} + N_n) = F_n^3, \quad \rho' = 1, \tag{7.5}$$

$$\frac{\partial P_n}{\partial \rho'} - \frac{1}{\delta} H(\frac{\partial P_n}{\partial \theta}) = F_n^4, \quad \rho' = \delta, \tag{7.6}$$

with initial condition

$$N_n \mid_{t=0} = N_n^0(\theta) \tag{7.7}$$

where

$$N^0(\theta) = 1 + \sum_{n\geq 1} N_n^0(\theta)\epsilon^n, \quad N^0 \text{ as in (6.24)}. \tag{7.8}$$

Set

$$\begin{aligned}
F_n^j &= \sum_{-\infty<m<\infty} F_{n,m}^j e^{im\theta}, \\
P_n &= \sum_{-\infty<m<\infty} P_{n,m}(\rho', t) e^{im\theta}, \\
N_n &= \sum_{-\infty<m<\infty} N_{n,m}(t) e^{im\theta}, \\
N_n^0 &= \sum_{-\infty<m<\infty} N_{n,m}^0 e^{im\theta}.
\end{aligned}$$

We aim at choosing the coefficients of $\theta_0, \rho_0$ in the expansion (6.1) in such a way that

$$
\begin{aligned}
(7.9)\qquad N^0_{n,\pm 1} + \int_0^\infty [F^2_{n,\pm 1}(t) - F^3_{n,\pm 1}(t) - \delta^2 F^4_{n,\pm 1}(t) \\
- \int_\delta^1 (\rho')^2 F^1_{n,\pm 1}(\rho', t) d\rho'] dt = 0.
\end{aligned}
$$

This condition, which is analogous to (4.14), will enable us to apply Lemma 4.1 to the system (7.3)-(7.7).

For clarity let us first consider the case $n = 1$. It is easy to check that $F^1_j \equiv 0$ so that (7.9) reduces to

$$
\begin{aligned}
(7.10)\qquad -\rho_{01}\frac{e^{-i\theta_{00}}}{2} + \lambda^0_1 = 0, \\
-\rho_{01}\frac{e^{i\theta_{00}}}{2} + \lambda^0_{-1} = 0;
\end{aligned}
$$

$\lambda^0_{-1}$ is actually the complex conjugate of $\lambda^0_1$. Consequently, if

$$
(7.11)\qquad |\lambda^0_1|^2 + |\lambda^0_{-1}|^2 > 0
$$

then (7.10) has a unique solution given by

$$
(7.12)\qquad \rho_{01} = 2|\lambda^0_{\pm 1}|, \quad \theta_{00} = \pm\arg(\lambda^0_{\mp 1}).
$$

If, on the other hand, (7.11) is not satisfied then (7.10) is satisfied with $\rho_{01} = 0$, and this leaves $\theta_{00}$ undetermined.

We have a similar situation when $n = 2$, whereby if $\rho_{01} = 0$ then either $(\rho_{02}, \theta_{00})$ is uniquely determined, or else $\rho_{02} = 0$, etc. For simplicity we shall first consider the case where (7.11) is satisfied.

**Lemma 7.1**. *If $\lambda^0 \in H^8(\partial B)$ and (7.11) is satisfied then, for all $n \geq 1$, $(\rho_{0n}, \theta_{0,n-1})$ can be uniquely chosen such that the system (7.3)-(7.7) has a unique solution $P_n, N_n$ satisfying the following estimates:*

$$
(7.13)\qquad \|P_n - P^\infty_n\|_{8,B},\ \|N_n - N^\infty_n\|_{9\frac{1}{2},\partial B},\ \|N_{n,t}\|_{6\frac{1}{2},\partial B} \leq C_0 \frac{H^{n-1}}{n^2},
$$

$$
(7.14)\qquad |P^\infty_n|,\ |N^\infty_n|,\ |\rho_{0n}|,\ |\theta_{0,n-1}| \leq C_0 \frac{H^{n-1}}{n^2},
$$

*where $C_0$, $H$ are positive constants.*

**Proof**. We shall first prove existence. We proceed inductively from $n-1$ to $n$, $n \geq 2$. We need to estimate the various terms that appear in $F^j_n$. We begin with the term $\partial\rho'/\partial r$ in $F^1$. We expand the analytic function (see (6.11))

$$
r = \Phi(\rho', \theta - \theta_0, \rho_0, \hat{N})
$$

about $(\theta-\theta_{00},0,1)$, keeping $\rho'$ fixed. We obtain

$$r=\Phi(\rho',\theta-\theta_0,\rho_0,\hat N)=\rho'+\sum_{|\alpha|>0}\frac{D^\alpha\Phi(\rho',\theta-\theta_{00},0,1)}{\alpha!}(\theta_{00}-\theta_0)^{\alpha_2}\rho_0^{\alpha_3}(\hat N-1)^{\alpha_4}. \tag{7.15}$$

Similarly, we expand the analytic function

$$\frac{\partial\rho'}{\partial r}=\Psi_r(r,\theta-\theta_0,\rho_0,\hat N)$$

about $(\rho',\theta-\theta_{00},0,1)$ to get

$$\begin{aligned}\frac{\partial\rho'}{\partial r}&=\Psi_r(r,\theta-\theta_0,\rho_0,\hat N)\\&=1+\sum_{|\alpha|>0}\frac{D^\alpha\Psi_r(\rho',\theta-\theta_{00},0,1)}{\alpha!}(r-\rho')^{\alpha_1}(\theta_{00}-\theta_0)^{\alpha_2}\rho_0^{\alpha_3}(\hat N-1)^{\alpha_4}.\end{aligned} \tag{7.16}$$

To compute $\partial\rho'/\partial r$ we shall substitute $r-\rho'$ from (7.15) into the right-hand side of (7.16).

We now observe that if we substitute the power series in $\epsilon$ for $\theta_0,\rho_0$ and for $\hat N-1$ into the right-hand side of (7.15), we get

$$r=\rho'+\sum_{k\geq 1}X_k(\rho',\theta,t)\epsilon^k \tag{7.17}$$

where, as in §5, by the inductive assumptions and Appendices A,B,

$$\|X_k\|_{6,B}\leq C_1C_0\frac{H^{k-2}}{k^2}\quad\text{if } 1\leq k\leq n \tag{7.18}$$

where $C_1$ is a generic constant which depends on $C_0$ but is independent of $H$; here we use the convention $H^{k-2}=1$ if $k=1$.

Next we substitute the series expansions for $\theta_0,\rho_0$, $\hat N-1$ and $r-\rho'$ (from (7.17)) into the right-hand side of (7.16) and again use the inductive assumptions (as well as (7.18)) and Appendices A,B to conclude that

$$\frac{\partial\rho'}{\partial r}=1+\sum_{k\geq 1}Y_k(\rho',\theta,t)\epsilon^k$$

where

$$\|Y_k\|_{6,B}\leq C_1C_0\frac{H^{k-2}}{k^2}\quad\text{if } 1\leq k\leq n.$$

Another application of Appendices A,B yields

$$\left(\frac{\partial\rho'}{\partial r}\right)^2-1=\sum_{k\geq 1}Z_k(\rho',\theta,t)\epsilon^k \tag{7.19}$$

with

$$\|Z_k\|_{6,B} \le C_1 C_0 \frac{H^{k-2}}{k^2}. \tag{7.20}$$

The other terms in $F_n^1$ can be treated in the same way, with the result that

$$\|F_n^1\|_{6,B} \le C_1 C_0 \frac{H^{n-2}}{n^2}.$$

We next consider $F_n^2$. Using the relations (6.17),(6.18), $W(\theta-\theta_0,0,1)=0$, and the inductive assumptions, we can proceed, as in the case of $F_n^1$, to apply Appendices A and B and conclude that

$$\|F_n^2\|_{6\frac{1}{2},\partial B} \le C_1 C_0 \frac{H^{n-2}}{n^2}.$$

In the same way, but more easily, we can derive the estimate

$$\|F_n^4\|_{6\frac{1}{2},\partial B} \le C_1 C_0 \frac{H^{n-2}}{n^2}.$$

¿From (6.6) and (6.7), (6.8) we see that, on the free boundary,

$$\theta - \psi = L(\theta-\theta_0, \rho_0, \hat{N})$$

where

$$L = \sum_{|\alpha|>0} \frac{D^\alpha L(\theta-\theta_{00},0,1)}{\alpha!}(\theta_{00}-\theta_0)^{\alpha_2}\rho_0^{\alpha_3}(\hat{N}-1)^{\alpha_4}.$$

Hence $|\theta_\psi - 1|$ is small and $\theta_\psi$ can be treated similarly to $\partial\rho'/\partial r$. Similarly $\theta_{\psi\psi}$ is "small" in the sense that the corresponding power series in $\epsilon$ vanishes at $\epsilon=0$. Using the inductive assumptions and applying Appendices A and B, we derive the estimate

$$\|F_n^3 - F_\infty^3\|_{7\frac{1}{2},\partial B} \le C_1 C_0 \frac{H^{n-2}}{n^2}$$

for some constant $F_\infty^3$ which is bounded by $C_1 C_0 H^{n-2}/n^2$; this constant arises from the constants $N_k^\infty$ in the inductive assumptions.

In order to be able to apply Lemma 4.1 we need to satisfy the condition (7.9). We begin by expanding (6.24) in powers of $\epsilon$, dropping all the coefficients of $\theta_{0k}, \rho_{0k}$ with $k>n$, and collecting all the coefficients of $\epsilon^n e^{i\theta}$. Substituting into (7.9) $(m=1)$, we find that

$$\rho_{0n}\frac{e^{-i\theta_{00}}}{2} + \rho_{01}(-i)\frac{e^{-i\theta_{00}}}{2}\theta_{0,n-1} = A_n \tag{7.21}$$

where, by the inductive assumptions and Appendix B,

$$|A_n| \le C_1 C_0 \frac{H^{n-2}}{n^2}.$$

Similarly, collecting all the coefficients of $\epsilon^n e^{-i\theta}$ we arrive at an equation

$$\rho_{0n}\frac{e^{i\theta_{00}}}{2}+\rho_{01}i\frac{e^{i\theta_{00}}}{2}\theta_{0,n-1}=B_n \tag{7.22}$$

where

$$|B_n|\le C_1C_0\frac{H^{n-2}}{n^2}.$$

The system (7.21), (7.22) has a unique solution $(\rho_{0n},\theta_{0,n-1})$ and, clearly,

$$|\rho_{0n}|+|\theta_{0,n-1}|\le C_1C_0\frac{H^{n-2}}{n^2}.$$

For this choice of $(\rho_{0n},\theta_{0,n-1})$ the condition (7.9) is satisfied. We can therefore apply Lemma 4.1 to the system (7.3)-(7.7) and conclude that the left-hand sides in (7.13)-(7.14) are bounded by $C_1C_0H^{n-2}/n^2$. Choosing $H>C_1$ the proof of (7.13), (7.14) is complete.

To prove uniqueness note that the assertion (7.13) implies that any mode $N_{n,m},\ m\neq 0$, goes to zero as $t\to\infty$. In particular,

$$N_{n,\pm 1}(t)\to 0 \text{ as } t\to\infty,$$

and this condition is equivalent to the condition (7.9) which, in turn, was used in determining $(\rho_{0n},\theta_{0,n-1})$ uniquely. Thus $(\rho_{0n},\theta_{0,n-1})$ is unique, and the uniqueness of $(P_n,N_n)$ follows from the above existence proof. //

In Lemma 7.1 we have made the assumption that (7.11) holds. If this is not the case then $\rho_{01}=0$ and $\theta_{00}$ is undetermined at the step $n=1$. Going to $n=2$, the condition (7.9) takes the form

$$\begin{aligned}\rho_{02}\frac{e^{-i\theta_{00}}}{2}+K_1&=0,\\ \rho_{02}\frac{e^{i\theta_{00}}}{2}+K_2&=0,\end{aligned} \tag{7.23}$$

where the $K_i$ depend on $\lambda^0$ and the solution $P_1,N_1$; here again $K_2$ is the complex conjugate of $K_1$. If $|K_1|^2+|K_2|^2>0$ then the system (7.23) has a unique solution. We can then proceed as in Lemma 7.1, the only difference being that at the level $\epsilon^n$ we determine $(\rho_{0n},\theta_{0,n-2})$ instead of $(\rho_{0n},\theta_{0,n-1})$.

If, on the other hand, $K_1=K_2=0$, then $\rho_{02}=0$ whereas $\theta_{00}$ is yet undertermined. Proceeding to $n=3$ we are again in a similar situation as before where either $(\rho_{03},\theta_{00})$ is uniquely determined by the condition (7.9), or else $\rho_{03}=0$ and $\theta_{00}$ is undetermined. In the first case we proceed as in Lemma 7.1, determining $(\rho_{0n},\theta_{n-3})$ at the level $\epsilon^n$. In general we either run into a situation where $\rho_{0n}\neq 0$ for a first $n$, and then we proceed analogously to Lemma 7.1, as already explained above, or $\rho_{0n}=0$ for all $n$. The second alternative means that the condition (4.14) is satisfied for all $n$, so that we do not need to move the origin; we simply proceed as in Lemma 5.1 and adopt the convention $\theta_0\equiv 0$ in case $\rho_0\equiv 0$.

Thus we have:

**Theorem 7.2**. *If $\lambda^0 \in H^8(\partial B)$ then there exist unique $(\rho_0, \theta_0)$ and $(p, \lambda)$ with the following properties: (i) $\rho_0, \theta_0$ have power series expansions as in (6.1), for $|\epsilon| < 1/(2H)$; (ii) $(p, \lambda)$ is a solution to Problem (A) having the form (2.16), (2.17) where the series are uniformly convergent for $|\epsilon| < 1/(2H)$; (iii) $\|p - p_\infty\|_{8,B} < \infty$; (iv) denoting by $\rho$ the distance from a point $(r, \theta)$ to $(\rho_0, \theta_0)$ and, writing*

$$r = 1 + \lambda(\theta, t) \text{ as } \rho = \hat{N}(\theta, t), \tag{7.24}$$

*there holds:*

$$\|\hat{N} - N_\infty\|_{9\frac{1}{2},\partial B}, \ \|\hat{N}_t\|_{6\frac{1}{2},\partial B} < \infty, \tag{7.25}$$

*and (v) $p_\infty$ and $N_\infty$ are constants having power series expansions*

$$p_\infty = \gamma + \sum_{n\geq 1} p_{n,\infty}\epsilon^n, \quad N_\infty = 1 + \sum_{n\geq 1} N_{n,\infty}\epsilon^n \quad (|\epsilon| < \frac{1}{2H}). \tag{7.26}$$

Reversing the maps $r \to \rho \to \rho'$ we obtain from Theorem 7.2 the following extension of Theorem 5.3.

**Theorem 7.3**. *If $\lambda^0 \in H^8(\partial B)$ then there exists a unique solution of the problems (1.1)-(1.4), (1.6) with free boundary $r = 1 + \lambda$ having the following properties: (i) $1 + \lambda(\theta, t, \epsilon)$ has a uniformly convergent power series expansion as in (2.17), for $|\epsilon| < 1/(2H)$; (ii) There exist convergent power series (6.1), for $|\epsilon| < 1/(2H)$, such that with the representation (7.24), where $\rho$ is the distance from $(r, \theta)$ to $(\rho_0, \theta_0)$, the estimates in (7.25) hold.*

We summarize a part of Theorem 7.3 in simpler words: The quasi-steady Stefan problem with surface tension has a unique solution with free boundary which is analytic in $\epsilon$ and which converges exponentially in $t$ to a circle whose center and radius are also analytic in $\epsilon$.

**Remark 7.1**. If $D_\theta^m \lambda^0 \in H^8(\partial B)$ for $m \leq \ell$ where $\ell$ is any positive integer, then we can apply $D_\theta^m$ successively to the system (7.3)-(7.8) and conclude that the solution to the problem has additional $\ell$ $\theta$-derivatives. From the differential equation $\Delta p = 0$ we then deduce that $p$ also has additional $\ell$ $(r, \theta)$-mixed derivatives and in fact, also $\ell$ $x$-derivatives ($x = (x_1, x_2)$); cf. [7]. Similarly, if $\lambda^0(\theta)$ is analytic in $\theta$ then we can establish by induction on $n$ that

$$\begin{aligned}\|\frac{1}{k!}D_\theta^k(P_n - P_n^\infty)\|_{8,B}, \ \|\frac{1}{k!}D_\theta^k(N_n - N_n^\infty)\|_{9\frac{1}{2},\partial B}, \\ \|\frac{1}{k!}D_\theta^k N_{n,t}\|_{6\frac{1}{2},\partial B} \leq C_0\frac{A^{k-1}H^{n-1}}{(k+n)^2}\end{aligned}$$

for all $k \geq 0$ and some constants $A, H$ with $A/H << 1$. This shows that the solution is analytic in $(\theta, \epsilon)$. Using the differential equation $\Delta p = 0$ we can then also deduce analyticity in $(x, \epsilon)$ for $|x| \leq 1 + \delta_0$, for $|\epsilon| \leq 1/(2H)$. Thus, if $\lambda^0(\theta)$ is analytic in $\theta$ then the free boundary is analytic in $(\theta, \epsilon)$ $(0 \leq \theta \leq 2\pi, \ |\epsilon| \leq \epsilon_0)$ and $p(x, t, \epsilon)$ is analytic in $(x, t)$ $(|x| \leq 1 + \delta_0, \ |\epsilon| \leq \epsilon_0)$ for some $\epsilon_0 > 0$, $\delta_0 > 0$.

## Part III. General dimension $\nu$

In this part we extend Theorem 7.3 to $\nu$-dimensional domains for any $\nu \geq 2$. We introduce spherical coordinates $(r, \omega)$, where $\omega = (\omega_1, \dots, \omega_{\nu-1})$, by

$$\begin{aligned} x_1 &= \cos\omega_1 \\ x_2 &= \sin\omega_1 \cos\omega_2 \\ x_3 &= \sin\omega_1 \sin\omega_2 \cos\omega_3 \\ &\dots \\ x_{\nu-1} &= \sin\omega_1 \sin\omega_2 \dots \sin\omega_{\nu-2} \cos\omega_{\nu-1} \\ x_\nu &= \sin\omega_1 \sin\omega_2 \dots \sin\omega_{\nu-2} \sin\omega_{\nu-1}. \end{aligned}$$

The Laplace operator can be written in the form [14]

$$\Delta p = p_{rr} + \frac{\nu-1}{r} p_r + \frac{1}{r^2} \Delta_\omega p \tag{8.1}$$

where $\Delta_\omega$ is a second order elliptic operator in $\omega$; for $\nu = 3$, $\omega = (\theta, \varphi)$ and

$$\Delta_\omega p = \frac{1}{\sin^2\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial p}{\partial\theta}) + \frac{1}{\sin^2\theta} \frac{\partial^2 p}{\partial\varphi^2}.$$

Consider a surface

$$S_\epsilon : r = 1 + \epsilon f(\omega)$$

with $|f|_{C^2} < \infty$, and denote by $\kappa$ the mean curvature of $S_\epsilon$.

**Theorem 8.1**. *There holds:*

$$\kappa(\omega) = 1 - \epsilon(f + \frac{1}{\nu-1} \Delta_\omega f) + O(\epsilon^2) \tag{8.2}$$

*as* $\epsilon \to 0$.

**Proof**. From the direct definition of $\kappa$ one deduces the structure

$$\kappa(\omega) = 1 + \epsilon M(\omega) + O(\epsilon^2)$$

where $M(\omega)$ is a nonlinear function in the derivatives $D^\alpha f(\omega)$, $|\alpha| \leq 2$. In order to compute $M(\omega)$ we may, without loss of generality, assume that $f \in C^\infty$.

It suffices to prove (8.2) at, say $\omega = 0$. Denote by $d = d(r, \omega) = d(r, \omega, \epsilon)$ the distance from $(r, \omega)$ to $S_\epsilon$. The normals $N(\omega) = N(\omega, \epsilon)$ to $S_\epsilon$ at $(1 + \epsilon f(\omega), \omega)$ for $\omega$ near 0 cover in 1-1 way a region $\{|\omega| \leq \eta_0,\ 0 \leq d \leq \eta_0\}$ $(\eta_0 > 0)$ where $d = d(r, \omega, \epsilon)$, $r < 1 + \epsilon f(\omega)$. Denote by $\omega_0 = \omega_0(r, \omega, \epsilon)$ the angular coordinate for which $(1 + \epsilon f(\omega_0), \omega_0)$ is nearest to $(r, \omega)$ on $S_\epsilon$. Thus $\omega_0$ is determined by the relation

$$d(r, \omega, \epsilon) = |(1 + \epsilon f(\omega_0)) \tilde{N}(\omega_0) - r\tilde{N}(\omega)| \tag{8.3}$$

where $\tilde{N}(\hat{\omega})$ is the unit vector in the direction from $(1, \hat{\omega})$ to the origin.

Note that the normals $N(\omega)$ vary smoothly (i.e., $C^\infty$) in $(\omega, \epsilon)$ and, for any $\delta \geq 0$ and small, each point on the surface

$$S_{\epsilon,\delta} : (\tilde{r}, \tilde{\omega}) = (1 + \epsilon f(\omega), \omega) + \delta N(\omega)$$

has distance $\delta$ to $S_\epsilon$, attained at $(1 + \epsilon f(\omega), \omega)$. The mapping $\omega \to \tilde{\omega}$ is $C^\infty$ jointly in $\omega$ and the parameters $(\delta, \epsilon)$. The inverse function is then also $C^\infty$ jointly in $\tilde{\omega}$ and the parameters $(\delta, \epsilon)$, and we write it as

$$\tilde{\omega} = W(\omega, d, \epsilon).$$

Clearly

$$\omega_0(r, \omega, \epsilon) = W(\omega, \delta, \epsilon) \text{ if } r = \tilde{r},\ \delta = d(\tilde{r}, \omega).$$

Since, furthermore, the mapping $\tilde{r} \to d(\tilde{r}, \omega, \epsilon)$ is in $C^\infty$ in $\tilde{r}$ and the parameters $(\omega, \epsilon)$ and has an inverse, we can write

$$\omega_0 = \omega_0(r, \omega, \epsilon) = \hat{\omega}_0(d, \omega, \epsilon) \quad (d = d(r, \omega, \epsilon))$$

where both $\omega_0$ and $\hat{\omega}_0$ are $C^\infty$ in all their variables.

By Taylor's theorem we can then write

$$\omega_0(r, \omega, \epsilon) = \omega_0(r, \omega, 0) + \epsilon A_0(d, \omega, \epsilon)$$

where $A_0(d, \omega, \epsilon)$ is in $C^\infty$. But if $\epsilon = 0$ then $S_\epsilon$ is a sphere, so that $\omega_0(r, \omega, 0) = \omega$; consequently,

$$\omega_0(r, \omega, \epsilon) = \omega + \epsilon A_0(d, \omega, \epsilon).$$

We can also write

$$A_0(d, \omega, \epsilon) = A_0(0, \omega, \epsilon) + dA_1(d, \omega, \epsilon)$$

where $A_1(d, \omega, \epsilon)$ is $C^\infty$. But if $d = 0$ then $\omega_0 \equiv \omega$ and therefore $A_0(0, \omega, \epsilon) = 0$. We thus conclude that

$$\omega_0(r, \omega, \epsilon) = \omega + \epsilon d A(r, \omega, \epsilon) \tag{8.4}$$

where $A(r, \omega, t)$ is $C^\infty$ in all its variables.

Note that

$$\begin{aligned} 1 + \epsilon f(\omega_0) - r &= (1 + \epsilon f(\omega) - r) + \epsilon(f(\omega_0) - f(\omega)) \\ &\geq d + \epsilon(f(\omega_0) - f(\omega)) \geq d - C\epsilon^2 d \end{aligned}$$

since $\operatorname{dist}((r, \omega), (1 + \epsilon f(\omega), \omega)) \geq d$. Using (8.3) and (8.4) we can also derive the estimate

$$1 + \epsilon f(\omega_0) - r \leq d + C\epsilon^2 d.$$

Hence

$$\frac{1}{2} d < |1 + \epsilon f(\omega_0) - r| < 2d. \tag{8.5}$$

We want to estimate the first two derivatives of $d$. From (8.3) we get

$$
\begin{aligned}
d &= [(1+\epsilon f(\omega_0))^2 + r^2 - 2r(1+\epsilon f(\omega_0)\tilde{N}(\omega_0)\cdot\tilde{N}(\omega)]^{1/2} \\
&= [(1+\epsilon f(\omega_0) - r)^2 + 2r(1+\epsilon f(\omega_0)(1-\tilde{N}(\omega_0)\cdot\tilde{N}(\omega)]^{1/2} \\
&= (1+\epsilon f(\omega_0) - r)[1 + \frac{2r(1+\epsilon f(\omega_0))}{(1+\epsilon f(\omega_0)-r)^2}(1-\tilde{N}(\omega_0)\cdot\tilde{N}(\omega))]^{1/2} \\
&= (1+\epsilon f(\omega_0) - r) + (1+\epsilon f(\omega_0) - r)\cdot G
\end{aligned}
\tag{8.6}
$$

where

$$G = \{[1 + \frac{2r(1+\epsilon f(\omega_0))}{(1+\epsilon f(\omega_0)-r)^2}(1-\tilde{N}(\omega_0)\cdot\tilde{N}(\omega))]^{1/2} - 1\}.$$

Consider the function

$$g(\omega_0,\omega) = 1 - \tilde{N}(\omega_0)\cdot\tilde{N}(\omega).$$

Clearly

$$g(\omega_0,\omega) \approx 1 - \cos(\omega_0-\omega) \le |\omega_0-\omega|^2.$$

By Taylor's expansion

$$g(\omega_0,\omega) = g(\omega,\omega) + G_1\cdot(\omega-\omega_0) + (\omega-\omega_0)^T G_2(\omega-\omega_0)$$

where $G_1, G_2$ are $C^\infty$ functions. Since $g(\omega,\omega)=0$ and

$$|g(\omega_0,\omega)| \le C|\omega_0-\omega|^2,$$

$G_1$ must also vanish, so that

$$g(\omega_0,\omega) = (\omega_0-\omega)^T G_2(\omega_0-\omega).$$

Using also (8.4) we see that

$$g(\omega_0,\omega) = (\epsilon d)^2 B_1(r,\omega,\epsilon) \tag{8.7}$$

where $B_1(r,\omega,\epsilon)$ is $C^\infty$.

The function $1+\epsilon f(\omega_0) - r$ may be viewed as a $C^\infty$ function in $(d,\omega,\epsilon)$ and, by (8.5), it vanishes linearly in $d$. By Taylor's theorem we then have

$$1+\epsilon f(\omega_0) - r = dC(r,\omega,\epsilon)$$

where $C(r,\omega,\epsilon)$ is $C^\infty$ in $(r,\omega,\epsilon)$. Using this and (8.7) in (8.6), we find that

$$d = (1+\epsilon f(\omega_0) - r) + B_2(r,\omega,\epsilon)\epsilon^2 d$$

where $B(r,\omega,\epsilon)$ is $C^\infty$ in $(r,\omega,\epsilon)$. By (8.4) we also have

$$\epsilon f(\omega_0) - \epsilon f(\omega) = B_3(r,\omega,\epsilon)\epsilon^2 d.$$

Hence

$$d = 1 + \epsilon f(\omega) - r + B(r,\omega,\epsilon)\epsilon^2 d$$

where $B(r,\omega,\epsilon)$ is $C^\infty$ in $(r,\omega,\epsilon)$. Writing

$$d(r,\omega,\epsilon) = \frac{1+\epsilon f(\omega) - r}{1 - B(r,\omega,\epsilon)\epsilon^2} \tag{8.8}$$

we find that

$$\begin{aligned}\Delta d &= \left(\frac{\partial^2}{\partial r^2} + \frac{(\nu-1)}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega\right)(1+\epsilon f(\omega) - r) + O(\epsilon^2)\\ &= -\frac{\nu-1}{r} + \frac{\epsilon}{r^2}\Delta_\omega f + O(\epsilon^2).\end{aligned}$$

Using the relation $r = 1 - d + \epsilon f + O(\epsilon^2 d)$ we conclude that

$$\Delta d = -\frac{\nu-1}{1-d} + \epsilon\left(\frac{(\nu-1)}{(1-d)^2} f + \Delta_\omega f\right) + O(\epsilon d) + O(\epsilon^2),$$

and letting $d \to 0$ we obtain

$$\Delta d \,|_{d=0} = (\nu-1)[-1 + \epsilon(f + \frac{1}{\nu-1}\Delta_\omega f)] + O(\epsilon^2). \tag{8.9}$$

On the other hand, we have the well known formula (see, for instance, [12; Lemma 14.7])

$$\Delta d + \sum_{i=1}^{\nu-1} \frac{\kappa_i}{1-\kappa_i d} = 0 \quad (d = d(r,\omega,\epsilon))$$

where $\kappa_i = \kappa_i(\omega_0)$ are the principal curvatures of $S_\epsilon$ at $(1+\epsilon f(\omega_0), \omega_0)$. Letting $d \to 0$ we obtain

$$\Delta d \,|_{d=0} + (\nu-1)\kappa = 0 \quad (\kappa = \frac{1}{\nu-1}\sum \kappa_i(\omega))$$

where $\kappa = \kappa(\omega)$ is the mean curvature at $(1+\epsilon f(\omega), \omega)$. Comparing this formula with (8.9), the assertion (8.2) follows. //

If we proceed as in §2, then Theorem 8.1 will allow us to rewrite (2.10) in the form

$$p + \gamma(\lambda - 1 + \frac{1}{\nu-1}\Delta_\omega \lambda) = F^3. \tag{8.10}$$

We note that the surface $\partial\Omega(t)$ to which we apply Theorem 8.1 is in $C^{2+\beta}$ (a posteriori); this is a consequence of the estimate on $\Lambda$ in (8.23) below. If $\partial\Omega(t)$ is in $C^{2+\beta}$ then, at each point $(r,\omega)$ on $\partial\Omega(t)$, the direction of the normal differs from the direction of the ray $\{s\omega, s>0\}$ by $O(\epsilon)$ and therefore the boundary condition (1.2) yields

$$\frac{\partial\lambda}{\partial t} + \frac{\partial p}{\partial r} = O(\epsilon). \tag{8.11}$$

where $O(\epsilon)$ involves first order derivatives of $p$ and of the free boundary. It follows that the boundary condition (2.9) does not change for $\nu \geq 2$.

We next consider the boundary condition (2.5). We write it, for general $\nu \geq 2$, as

$$\frac{\partial p}{\partial r} = T(p) \tag{8.12}$$

where $T$ is the Dirichlet-to-Neumann mapping in $B_\delta = \{r < \delta\}$. If we expand

$$p = \sum r^n A_n(\omega)$$

then

$$\Delta_\omega A_n + n(n + \nu - 2)A_n = 0$$

and we can write

$$A_n = \sum_j a_{nj} Y_{nj}(\omega), \; j = (j_1, j_2, \ldots, j_{\nu-1})$$

where the $Y_{nj}$ are the $\nu$-dimensional spherical harmonics. We thus have the representation

$$T(p) = \frac{1}{\delta} \sum_n n\delta^n A_n = \frac{1}{\delta} \sum_n \sum_j n\delta^n a_{nj} Y_{nj}(\omega). \tag{8.13}$$

Before proceeding to extend Lemma 4.1 to $\nu \geq 2$ we need to establish two lemmas.

**Lemma 8.2**. *Set $B = \{\delta < r < 1\}$, $\partial B = \{r = 1\}$, let $s$ be a nonnegative integer, and let*

$$F(r, \omega) = \sum_{n \geq 0, m} F_{nm}(r) Y_{nm}(\omega),$$
$$f(\omega) = \sum_{n \geq 0, m} f_{nm} Y_{nm}(\omega).$$

*Then, there exists positive constants $c_1, c_2$ independent of $F, f$ such that*

$$c_1 \|F\|^2_{H^s(B)} \leq \sum_{j=0}^{s} \sum_{n \geq 0} (1 + n^{2(s-j)}) \sum_m \int_\delta^1 |D_r^j F_{nm}|^2 r^{\nu-1} dr \leq c_2 \|F\|^2_{H^s(B)}, \tag{8.14}$$

$$c_1 \|f\|^2_{H^{s+1/2}(\partial B)} \leq \sum_{n \geq 0} (1 + n^{2s+1}) \sum_m |f_{nm}|^2 \leq c_2 \|f\|^2_{H^{s+1/2}(\partial B)}. \tag{8.15}$$

**Proof**. To prove (8.14) we first note that, by effecting an appropriate radial extension of $F \in H^s(B)$ we may assume without loss of generality that $F$ is compactly supported in $B$. Then, from classical elliptic estimates, for each integer $k$ there exists a constant $K_1$ such that

$$\|\Delta^k F\|_{L^2(B)} \leq \|F\|_{H^{2k}(B)} \leq K_1 \|\Delta^k F\|_{L^2(B)},$$

and from Poincare's inequality (and elliptic estimates)

$$\|\nabla\Delta^k F\|_{L^2(B)} \le \|F\|_{H^{2k+1}(B)} \le K_1\|\nabla\Delta^k F\|_{L^2(B)}.$$

Thus, it suffices to show that (8.14) holds with $\|F\|_{H^s(B)}$ replaced by $\|\Delta^k F\|_{L^2(B)}$ and $\|\nabla\Delta^k F\|_{L^2(B)}$ when $s = 2k$ and $s = 2k+1$, respectively. On the other hand, since [14]

$$\Delta \sum_{m,n} F_{nm}(r)Y_{nm}(\omega) = \sum_{n,m}\left(\frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r F_{nm}) - n(n+\nu-2)F_{nm}\right)Y_{nm},$$

we have, using the othogonality properties of $Y_{mn}$,

$$\|\Delta^k F\|_{L^2(B)} = \sum_{m,n}\int_\delta^1 r^{\nu-1}|[\frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r) - n(n+\nu-2)]^k F_{nm}(r)|^2 dr \tag{8.16}$$

and

$$\begin{aligned}\|\nabla\Delta^k F\|_{L^2(B)} &= \int_B \nabla\Delta^k F\cdot\nabla\Delta^k F = -\int_B(\Delta^{k+1}F)(\Delta^k F)\\ &= -\sum_{n,m}\int_\delta^1 r^{\nu-1}(\frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r) - n(n+\nu-2))^{k+1}F_{nm}\\ &\quad\times(\frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r) - n(n+\nu-2))^k F_{nm}\\ &= \sum_{n,m}\int_\delta^1 r^{\nu-1}\{[\partial_r(\frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r) - n(n+\nu-2))^k F_{nm}]^2\\ &\quad + n(n+\nu-2)[(\frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r) - n(n+\nu-2))^k F_{nm}]^2\}\end{aligned} \tag{8.17}$$

and therefore we need only establish that the norms in the right hand sides of (8.16), (8.17) are equivalent to

$$\sum_{j=0}^{s}\sum_n(1+n^{2(s-j)})\sum_m\int_\delta^1 |D_r^j F_{nm}|^2 r^{\nu-1}dr \tag{8.18}$$

when $s = 2k$ and $s = 2k+1$, respectively. Since the norms in (8.16), (8.17) are clearly majorized by a multiple of (8.18), it is enough to show that this latter quantity can be dominated by constant multiples of (8.16), (8.17) for $s = 2k$ and $s = 2k+1$, respectively.

To see this we shall first show that if we set

$$\varphi = \frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r)F - n(n+\nu-2)F$$

then for each integer $L$ we have

$$\begin{aligned}&\sum_{j=0}^{L}\int_\delta^1(|D_r^j\partial_r^2F|^2 + n^2|D_r^j\partial_r F|^2 + n^4|D_r^j F|^2)r^{\nu-1}dr\\ &\qquad\le K_0\sum_{j=0}^{L}\int_\delta^1 |D_r^j\varphi|^2 r^{\nu-1}dr.\end{aligned} \tag{8.19}$$

Note that once (8.19) has been established, an easy inductive procedure will show that (8.18) can be majorized by a multiple of (8.16) (or (8.17)). Indeed, this is obviously true if $s = 2k = 0$, and if we assume that

$$\sum_{j=0}^{2k}\sum_n (1+n^{2(s-j)})\sum_m \int_\delta^1 |D_r^j F_{nm}|^2 r^{\nu-1} dr$$
$$\leq K_2 \sum_{n,m}\int_\delta^1 r^{\nu-1}|[\frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r) - n(n+\nu-2)]^k F_{nm}|^2 dr$$

we then have

$$K_2 \sum_{m,n}\int_\delta^1 r^{\nu-1}|[\frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r) - n(n+\nu-2)]^{k+1} F_{nm}|^2 dr$$
$$\geq \sum_{j=0}^{2k}\sum_n (1+n^{2(2k-j)})\sum_m \int_\delta^1 r^{\nu-1}|D_r^j[\frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r F_{nm} - n(n+\nu-2)F_{nm}]|^2$$
$$\geq \frac{C(k)}{K_0}\sum_{j=0}^{2k}\sum_n (1+n^{2(2k-j)})\sum_m \int_\delta^1 r^{\nu-1}[|D_r^j\partial_r^2 F_{nm}|^2 + n^2|D_j^r\partial_r F_{nm}|^2 + n^4|F_{nm}|^2],$$

by(8.19), which proves the assertion for $s = 2k+1$. A similar argument then also shows that (8.18) with $s = 2k+2$ is dominated by a constant multiple of (8.19).

Thus, it remains to prove (8.19), which again can be established inductively in $L$. When $L = 0$, (8.19) follows from straightforward energy estimates. Indeed

$$\int_\delta^1 \varphi\partial_r(r^{\nu-1}\partial_r F) = \int_\delta^1 \frac{1}{r^{\nu-1}}(\partial_r(r^{\nu-1}\partial_r F))^2 - n(n+\nu-2)\int_\delta^1 F\partial_r(r^{\nu-1}\partial_r F)$$
$$= \int_\delta^1 r^{\nu-1}\{(\partial_r^2 F + \frac{(\nu-1)}{r}\partial_r F)^2 + n(n+\nu-2)|\partial_r F|^2\},$$

from which one can deduce that

(8.20) $$\int_\delta^1 r^{\nu-1}\{(\partial_r^2 F)^2 + n^2|\partial_r F|^2\} \leq K_3\int_\delta^1 \varphi^2.$$

Similarly

$$\int \varphi r^{\nu-1} n(n+\nu-2)F = \int \partial_r(r^{\nu-1}\partial_r F)n(n+\nu-2)F - \int (n(n+\nu-2))^2|F|^2 r^{\nu-1}$$
$$= -n(n+\nu-2)\int r^{\nu-1}|\partial_r F|^2 - (n(n+\nu-2))^2\int |F|^2 r^{\nu-1}$$

implies that

$$n^2\int_\delta^1 r^{\nu-1}|\partial_r F|^2 + n^4\int_\delta^1 |F|^2 r^{\nu-1} \leq K_4\int_\delta^1 \varphi^2,$$

which together with (8.20) establishes (8.19) for $L = 0$.

Now assume that (8.19) if valid for $L = M$. Then, since

$$\partial_r \varphi + \frac{(\nu-1)}{r^2}\partial_r F = \frac{1}{r^{\nu-1}}\partial_r(r^{\nu-1}\partial_r)\partial_r F - n(n+\nu-2)\partial_r F,$$

we have, from the inductive assumption,

$$\begin{aligned}\sum_{j=0}^{M}\int_\delta^1 (|D_r^j\partial_r^2\partial_r F|^2 + n^2|D_r^j\partial_r\partial_r F|^2 + n^4|D_r^j\partial_r F|^2)r^{\nu-1}dr \\ \le K_0\sum_{j=0}^{M}\int_\delta^1 |D_r^j(\partial_r\varphi + \frac{(\nu-1)}{r^2}\partial_r F)|^2 dr \\ \le \tilde{K}_0\sum_{j=0}^{M+1}\int_\delta^1 |D_r^j\varphi|^2\end{aligned}$$

which proves (8.19) for $L = M+1$.

Finally to establish (8.15) we consider the harmonic extension $F(r,\omega)$ of $f(\omega)$,

$$F(r,\omega) = \sum_{n\ge 0,m} f_{nm} r^n Y_{nm}(\omega)$$

Then, from the trace theorem [1],

$$\|f\|_{H^{s+1/2}(\partial B)} \le C\|F\|_{H^{s+1}(B)}$$

and from classical elliptic estimates

$$\|F\|_{H^{s+1}(B)} \le C\|f\|_{H^{s+1/2}(\partial B)}.$$

Thus, since

$$\int_\delta^1 |D_r^j r^n|^2 r^{\nu-1}dr = (\frac{n!}{(n-j)!})^2(\frac{1-\delta^{2n-2}}{2n-2j+\nu}) = O(n^{2j-1}),$$

the inequalities in (8.15) immediately follow. //

**Lemma 8.3**. *For any harmonic function $p$ in $\overline{B_\delta}$ and for any derivative $D_\omega^\alpha = D_{\omega_1}^{\alpha_1}\ldots D_{\omega_{\nu-1}}^{\alpha_{\nu-1}}$ there holds:*

$$\|T(D_\omega^\alpha p)\|_{L^2(\partial B_\delta)} = \|D_\omega^\alpha T(p)\|_{L^2(\partial B_\delta)} \le C\|p\|_{H^{|\alpha|+1}(\partial B_\delta)} \tag{8.21}$$

*where $|\alpha| = \alpha_1 + \ldots + \alpha_{\nu-1}$ and $C$ is a constant independent of $p$.*

**Proof**. We can write

$$p(\delta,\omega) = \sum a_{nj}\delta^n Y_{nj}.$$

Then, by (8.13) and the orthonormality of the $Y_{nj}$,

$$\|T(p)\|^2_{L^2(\partial B_\delta)} = \sum_{n,j} n^2 |a_{nj}|^2 \delta^{2n-2}.$$

On the other hand, since by Lemma 8.2,

$$\|p(\delta,\cdot)\|^2_{H^1(\partial B_\delta)} \approx \sum_{n,j} (1+|n|^2)|a_{nj}|^2\delta^{2n},$$

(8.21) follows for $|\alpha| = 0$. The proof for $|\alpha| > 0$ is similar. //

We can now proceed to extend Lemma 4.1. Assuming that

$$\Lambda^0 \in H^{2+3(\nu+1)}(\partial B), \tag{8.22}$$

we need to replace in the statement of the lemma, (4.16) by

$$\begin{aligned} &\|P - P_\infty\|_{2+3(\nu+1)} + \|\Lambda - \Lambda_\infty\|_{3(\nu+1)+3\frac{1}{2},\partial B} + \|\Lambda_t\|_{3(\nu+1)+\frac{1}{2},\partial B} \\ &\quad \le C\{\|F^1\|_{3(\nu+1),B} + \|F^2\|_{3(\nu+1)+\frac{1}{2},\partial B} + \|F^3 - F^3_\infty\|_{3(\nu+1)+\frac{3}{2},\partial B} \\ &\quad + \|F^4\|_{3(\nu+1)+\frac{1}{2},\partial B} + \|\Lambda^0\|_{H^{2+3(\nu+1)}(\partial B)}\}. \end{aligned} \tag{8.23}$$

In the expansion of $P$ we replace (4.12) by

$$P(r,\omega,t) = \sum_{n,m}\sum P_{nm}(r,t)Y_{nm}(\omega)$$

where $m = (m_1,\ldots,m_{\nu-1})$, and similarly make the corresponding changes in (4.9)-(4.11),(4.13).

We need to make just a few changes in the proof of Lemma 4.1. In (4.20) (with $P_n$ replaced by $P_{nm}$) we replace $n^2/r^2$ by $n(n+\nu-2)/r^2$ and $1/r$ by $(\nu-1)/r$, in (4.22) we again replace $P_n$ by $P_{nm}$ and $n^2$ by $n(n+\nu-2)$, and (4.23) is replaced by using the explicit expression for $T(p)$, as described above. Then no changes are made in (4.25), but in (4.29),(4.31) and (4.33) we replace $n^2$ by $n(n+\nu-2)/(\nu-1)$. We also take, in (4.33),

$$0 < \alpha < 2\gamma(\frac{2\nu}{\nu-1} - 1). \tag{8.24}$$

We next proceed as in §6, but, for clarity, consider first the case $\nu = 3$. We need to move the origin into a new center

$$(\rho_0, \omega_0) \text{ where } \omega_0 = (\theta_0, \varphi_0)$$

where

$$\rho_0 = \sum_{m\ge1} \rho_{0m}\epsilon^m,$$

$$\omega_0 = \omega_{00} + \sum_{m\ge1}\omega_{0m}\epsilon^m.$$

Take any point $(r, \omega)$ and denote by $\rho$ its distance to $(\rho_0, \omega_0)$. Then

$$\rho^2 = r^2 + \rho_0^2 - 2\rho_0 r \cos\sigma.$$

On the other hand, using spherical coordinates of $(r, \omega)$ and $(\rho_0, \omega_0)$ we can write

$$\begin{aligned}\rho^2 = (r\cos\theta - \rho_0\cos\theta_0)^2 &+ (r\sin\theta\cos\varphi - \rho_0\sin\theta_0\cos\varphi_0)^2 \\ &+ (r\sin\theta\sin\varphi - \rho_0\sin\theta_0\sin\varphi)^2\end{aligned}$$

so that, by comparison,

$$\begin{aligned}\cos\sigma = \cos\theta\cos\theta_0 &+ \sin\theta\cos\varphi\sin\theta_0\cos\varphi_0 \\ &+ \sin\theta\sin\varphi\sin\theta_0\sin\varphi_0.\end{aligned} \tag{8.25}$$

The functions $\cos\theta$, $\sin\theta\cos\varphi$ and $\sin\theta\sin\varphi$ are linear combinations of the spherical harmonics $Y_{1,-1}, Y_{1,0}$ and $Y_{1,1}$.

The relation (6.4) holds with $\theta - \theta_0$ replaced by $\sigma$ so that, by (6.24) and (8.25), we find that in order to eliminate the first three modes for $N^0$ (analogously to (7.9)) we need to choose $\rho_{01}, \theta_{00}, \varphi_{00}$ such that

$$\begin{aligned}\rho_{01}\cos\theta_{00} + \lambda_{01} &= 0, \\ \rho_{01}\sin\theta_{00}\cos\varphi_{00} + \lambda_{02} &= 0, \\ \rho_{01}\sin\theta_{00}\sin\varphi_{00} + \lambda_{03} &= 0\end{aligned} \tag{8.26}$$

where $\lambda_{01}, \lambda_{02}, \lambda_{03}$ are linear combinations of the first three modes $\lambda^0_{1,-1}, \lambda^0_{1,0}$ and $\lambda^0_{1,1}$ of $\lambda^0$. If

$$\lambda_{01}^2 + \lambda_{02}^2 + \lambda_{03}^2 > 0 \tag{8.27}$$

then there exists a unique solution $(\rho_{01}, \theta_{00}, \varphi_{00})$ of (8.26) and we can proceed as in the proof of Lemma 7.1; here use Appendix B and Theorems A.5, A.6. Although the forms of the $F^j$ are more complicated than in (6.20*b*)-(6.23*b*), the structure is the same, and the estimates are similar although lengthier.

If (8.27) is not satisfied, then $\rho_{01} = 0$ and we proceed to consider $\epsilon^2$ terms, exactly as we did in §7.

In case $\nu > 3$ we can proceed in the same manner, using the more complicated expressions for $\cos\gamma$ that result from the representation of $\rho$ by means of spherical coordinates.

We summarize the final result that is obtained from the above considerations:

**Theorem 8.2** *If $\lambda^0 \in H^{2+3(\nu+1)}(\partial B)$ then there exists a unique solution of (1.1)-(1.3), (1.6) with free boundary*

$$r = 1 + \lambda(\omega, t) \tag{8.28}$$

*having the following properties:*

$$\lambda(\omega, t) = \sum_{n\geq 1} \lambda_n(\omega, t)\epsilon^n \tag{8.29}$$

*where the series is uniformly convergent for* $|\epsilon| \le \epsilon_0$*, for some* $\epsilon_0 > 0$*;*

*(ii) there exist convergent series*

$$\rho_0 = \sum_{m \ge 1} \rho_{0m} \epsilon^m,$$
$$\omega = \omega_{00} + \sum_{m \ge 1} \omega_{0m} \epsilon^m$$
$$N_\infty = 1 + \sum_{m \ge 1} N_{m,\infty} \epsilon^m$$

*for* $|\epsilon| \le \epsilon_0$*;*

*(iii) if the free boundary* $r = 1 + \lambda(\omega, t)$ $(\lambda(\omega, t)$ *as in (8.29)) is written as* $\rho = \hat{N}(\omega, t, \epsilon)$ *where* $\rho$ *is the distance from* $(r, \omega)$ *to* $(\rho_0, \omega_0)$*, then*

$$\|\hat{N} - N_\infty\|_{3(\nu+1)+3\frac{1}{2}, \partial B} < \infty,$$

$$\|\hat{N}_t\|_{3(\nu+1)+\frac{1}{2}, \partial B} < \infty.$$

**Remark 8.2**. Remark 7.1 extends to the present case. In particular (i) if $\lambda^0(\omega) \in C^\infty$ then the free boundary is $C^\infty$ in $\omega$; (ii) if $\lambda^0(\omega)$ is analytic then the free boundary is analytic jointly in $(\omega, \epsilon)$, and then $p(x, t)$ is analytic in $(x, \epsilon)$ for $|x| \le 1 + \epsilon_0$, $|\epsilon| \le \epsilon_0$.

**Remark 8.3**. We have not considered in this paper higher regularity in $t$. But it is clear that we can use norms with higher derivatives in $t$ than just the two derivatives we have used. In particular, if $\lambda^0(\omega)$ is $C^\infty$ then we can deduce that the free boundary is $C^\infty$ in $(\omega, \epsilon, t)$.

## Appendix A: Interpolation inequalities

**Lemma A.1**. *Let $B = \{|x| < 1\} \subset \mathbf{R}^2$ and let $F(x,t) = 0$ if $t \geq 2$, and*

$$\|F\|_{s,B} < \infty \text{ where } s \geq 6.$$

*Then*

$$\|D_x^j F\|_{L^r} \leq C\|F\|_{s,B} \quad \text{where } r = \begin{cases} \infty & \text{if } 0 \leq j \leq s-5 \\ \frac{18}{9-2(s-j)} & \text{if } s-4 \leq j \leq s, \end{cases} \tag{A.1}$$

and

$$\|D_t D_x^j F\|_{L^r} \leq C\|F\|_{s,B} \quad \text{where } r = \begin{cases} 6 & \text{if } 0 \leq j \leq s-6 \\ \frac{18}{3-2(s-j)} & \text{if } s-5 \leq j \leq s-3. \end{cases} \tag{A.2}$$

**Proof**. We extend $F(x,t)$ to all $\mathbf{R}^3$ in such a way that $F(x,t) = 0$ if $t \leq -1$ or $|x| \geq 2$ and at the same time the norm $\|F\|_{s,B_2}$ (defined as in (4.6) but with $B$ replaced by $\{|x| \leq 2\}$ and the t-integration starting at $t = -1$) is bounded by $C\|F\|_{s,B}$. Denoting by $\tilde{F}(\xi,\tau)$ the Fourier transform of $F$, we then have:

$$\int_{\mathbf{R}^3} [(1+|\xi|)^{2s} + (1+|\xi|)^{2(s-3)}(1+|\tau|)^2 + (1+|\xi|)^{2(s-6)}(1+|\tau|)^4]|F(\xi,\tau)|^2 \leq C\|F\|_{6,B}. \tag{A.3}$$

We want to find the largest $\beta > 0$ such that

$$\|D_x^j F\|_{H^\beta} \leq C\|F\|_{s,B}; \tag{A.4}$$

then, by Sobolev's imbedding [1] we shall obtain the estimate (A.1) with

$$r = \begin{cases} (\frac{1}{2} - \frac{\beta}{3})^{-1} & \text{if } \beta < \frac{3}{2} \\ \infty & \text{if } \beta > \frac{3}{2}. \end{cases} \tag{A.5}$$

Taking the Fourier transform equivalent of (A.4) and recalling (A.3), the inequality (A.4) will follow from the inequality

$$(|\xi|^{2\beta} + |\tau|^{2\beta})|\xi|^{2j} \leq C(|\xi|^{2s} + |\xi|^{2(s-3)}|\tau|^2 + |\xi|^{2(s-6)}|\tau|^4) \tag{A.6}$$

for $|\xi| \geq 1,\ |\tau| \geq 1$.

Clearly we must then have

$$0 < \beta \leq 2,\ \beta \leq s-j \tag{A.7}$$

and it remains to show that $|\tau|^{2\beta}|\xi|^{2j}$ is bounded by the right-hand side of (A.6). This is evidently the case if

$$\beta = 2,\ 0 \le j \le s-6, \tag{A.8}$$

which completes the proof of (A.1) in case $j \le s-6$.

If $j > s-6$ then we can write

$$\begin{aligned} |\tau|^{2\beta}|\xi|^{2j} &= |\tau|^{2\beta}|\xi|^{(s-6)\beta} \times |\xi|^{2j-(s-6)\beta} \\ &\le [|\tau|^{2\beta}|\xi|^{(s-6)\beta}]^p + [|\xi|^{2j-(s-6)\beta}]^q, \quad \frac{1}{p}+\frac{1}{q} = 1, \end{aligned}$$

and to show that this is bounded by the right-hand side of (A.6) we choose

$$2\beta p = 4,\ (2j-(s-6)\beta)q = 2s$$

(note that $2j-(s-6)\beta > 0$ since $j > s-6$). This yields

$$\beta = \frac{1}{3}(s-j), \tag{A.9}$$

a choice which satisfies the earlier restrictions in (A.7), and substituting this into (A.5), the proof of (A.1) for $j \le s$ follows.

To prove (A.2) we proceed in the same way, but now we have to show that

$$(|\xi|^{2\beta}+|\tau|^{2\beta})|\tau|^2|\xi|^{2j} \le C(|\xi|^{2s}+|\xi|^{2(s-3)}|\tau|^2+|\xi|^{2(s-6)}|\tau|^4) \tag{A.10}$$

for $|\xi| \ge 1$, $|\tau| \ge 1$; from this inequality it follows that (A.2) holds with the same $r$ as in (A.5).

For (A.10) to hold it suffices to show that

$$|\tau|^2|\xi|^{2\beta+2j} \le C|\tau|^2|\xi|^{2(s-3)} \tag{A.11}$$

and

$$|\tau|^{2+2\beta}|\xi|^{2j} \le C(|\xi|^{2s}+|\xi|^{2(s-6)}|\tau|^4). \tag{A.12}$$

We take

$$\beta + j \le s-3. \tag{A.13}$$

Then (A.11) is satisfied. If

$$\beta = 1,\ j \le s-6$$

then (A.12) is clearly satisfied and then (A.2) holds with $r = 6$. Thus it remains to consider the case

$$s-6 < j \le s-3. \tag{A.14}$$

As before, if $\beta < 1$, we estimate the left-hand of (A.12) by writing

$$|\tau|^{2+2\beta}|\xi|^{2j} \le (|\tau|^{2+2\beta}|\xi|^{(1+\beta)(s-6)})^p + (|\xi|^{2j-(1+\beta)(s-6)})^q$$

where

$$\frac{1}{p} + \frac{1}{q} = 1, \ (2+2\beta)p = 4, \ [2j-(1+\beta)(s-6)]q = 2s.$$

This yields

$$1+\beta = \frac{1}{3}(s-j).$$

For this choice of $\beta$, $\beta < 1$ and (A.13) holds (since $s-6 < j \le s-3$), so that both (A.11) and (A.12) hold. Substituting this $\beta$ into (A.5), the proof of (A.2) is complete. //

**Theorem A.2**. *Let $B = \{|x| < 1\} \subset \mathbf{R}^2$. If $s \ge 6$ then there is a constant $M_0$ such that*

$$\|FG\|_{s,B} \le M_0 \|F\|_{s,B} \|G\|_{s,B} \tag{A.14}$$

*for any two functions $F, G$ with finite $\| \ \|_{s,B}$ norm.*

**Proof**. We introduce a partition of unity $\{\chi_m\}$, $m \ge 0$, of $\{-1 \le t < \infty\}$ where $\chi_m(t)$ is supported in the interval $m-1 \le t \le m+1$, and set

$$F_m = F\chi_m, \quad G_m = G\chi_m.$$

Then

$$F = \sum F_m, \quad G = \sum G_m,$$

and

$$D^k F \cdot D^\ell G = \sum_m D^k F_m \cdot D^\ell G_m + \sum_m D^k F_m \cdot D^\ell G_{m\pm 1} \tag{A.15}$$

for any derivatives $D^k, D^\ell$.

To prove (A.14) we need to show that

$$\int_0^\infty \int_B e^{2\alpha s}|A|^2 \le C\|F\|^2_{s,B}\|G\|^2_{s,R} \tag{A.16}$$

where $A$ is any one of the functions

$$D_x^j F \cdot D_x^{s-j} G, \ 0 \le j \le s, \tag{A.17}$$

$$D_t D_x^j F \cdot D_x^{s-j} G, \quad D_x^j F \cdot D_t D_x^{s-j} G, \ 0 \le j \le s-3, \tag{A.18}$$

and

$$D_t^2 D_x^j F \cdot D_x^{s-j} G, \ D_t D_x^j F \cdot D_t D_x^{s-j} G, \ D_x^j F \cdot D_t^2 D_x^{s-j} G, \ 0 \le j \le s-6. \tag{A.19}$$

Consider first the case where $j \geq s-4,\ j \leq 4$. By Lemma A.1, if $F_0 = G_0 = 0$ for $t \geq 2$ or $t \leq -1$, then

$$\begin{aligned} &\|D_x^j F_0\|_{L^r} \leq C\|F_0\|_{s,B}, \\ &\|D_x^{s-j} G_0\|_{L^{r'}} \leq C\|G_0\|_{s,B}, \\ &\text{where } 2(\frac{1}{r} + \frac{1}{r'}) \leq 1. \end{aligned} \tag{A.20}$$

We shall prove that (A.16) holds with $A = D_x^j F \cdot D_x^{s-j} G$.

Consider first the series

$$I = \sum_m e^{2\alpha m} \int_m^{m+1} \int_B |D_x^j F_m \cdot D_x^{s-j} G_m|^2.$$

By (A.20), if $r = 2p,\ r' = 2q$.

$$\begin{aligned} I &\leq \sum_m e^{2\alpha m} (\int_m^{m+1} \int_B |D_x^j F_m|^{2p})^{\frac{1}{p}} (\int_m^{m+1} \int_B |D_x^{s-j} G_m|^{2q})^{\frac{1}{q}} \\ &\leq C\|F\|_{s,B}^2 \sup_m (\int_m^{m+1} \int_B |D_x^{s-j} G_m|^{2q})^{\frac{1}{q}} \\ &\leq C\|F\|_{s,B}^2 \|G\|_{s,B}^2. \end{aligned}$$

Similarly one can estimate the other two series on the right-hand side of (A.15). We conclude that (A.16) holds for $A = D_x^j F \cdot D_x^{s-j} G$ provided $j \geq s-4,\ j \geq 5$.

If $j \leq s-5$ then, by Lemma A.1,

$$\|D_x^j F_0\|_{L^\infty} \leq C\|F_0\|_{6,B} \quad (F_0 = 0 \text{ if } t \geq 2 \text{ or } t \leq -1)$$

and we can again prove (A.16) , in fact much more simply. The same applies to the case $j \geq 5$.

We can now examine all the functions in (A.17) and check that each of them is estimated by the above procedure. The same arguments apply to the functions in (A.18) and (A.19), and thus the proof of the theorem is complete. //

**Lemma A.3**. *Let $B = \{|x| < 1\} \subset \mathbf{R}^2,\ \partial B = \{|x| = 1\}$ and let $f(\theta, t) = 0$ if $t \geq 2$ and*

$$\|f\|_{s,\partial B} < \infty \text{ where } s \geq 6.$$

*Then*

$$\begin{aligned} &\|D_\theta^j f\|_{L^r(\partial B)} \leq C\|f\|_{s,\partial B} \\ &\text{where } r = \begin{cases} \infty & \text{if } j \leq s-4 \\ \text{arbitrarily large} & \text{if } j = s-3 \\ 6/(3-(s-j)) & \text{if } s-2 \leq j \leq s \end{cases} \end{aligned} \tag{A.21}$$

and

$$\|D_t D_\theta^j f\|_{L^r} \le C\|f\|_{s,B} \tag{A.22}$$
$$\text{where } r = \begin{cases} \text{arbitrarily large} & \text{if } j \le s-6 \\ \frac{6}{6-(s-j)} & \text{if } s-5 \le j \le s-3. \end{cases}$$

**Proof**. As before, to prove (A.21) we want to find the largest number $\beta > 0$ such that

$$(|\xi|^{2\beta} + |\tau|^{2\beta})|\xi|^{2j} \le C(|\xi|^{2s} + |\xi|^{2(s-3)}|\tau|^2 + |\xi|^{2(s-6)}|\tau|^4) \tag{A.23}$$

where $|\xi| \ge 1,\ |\tau| \ge 1$; here $\tilde{f}(\xi,\tau)$ is the Fourier transform of $f(\theta,t)$, so that $\xi$ is actually a discrete variable (i.e., $\xi$ varies over the integers). Once (A.23) is proven, it follows that (A.21) holds with

$$\begin{aligned} &\frac{1}{r} = \frac{1}{2} - \frac{\beta}{2} \text{ if } \beta < 1, \\ &r = \infty \text{ if } \beta > 1, \\ &r \text{ any positive number if } \beta = 1. \end{aligned} \tag{A.24}$$

As before, if $0 \le j \le s-6$, we can take $\beta = 2$ so that $r = \infty$. If $j > s-6$, we can choose $\beta$ as in the proof of Lemma A.1, and then (A.24) gives $r = \infty$ if $j \le s-4$, $r$ arbitrarily large if $j = s-3$, and $r = 6/(3-(s-j))$ if $s-2 \le j \le s$.

To prove (A.22) we need to establish the inequality

$$(|\xi|^{2\beta} + |\tau|^{2\beta})|\tau|^2|\xi|^{2j} \le C(|\xi|^{2s} + |\xi|^{2(s-3)}|\tau|^2 + |\xi|^{2(s-6)}|\tau|^4) \tag{A.25}$$

for $|\xi| \ge 1,\ |\tau| \ge 1$. This inequality holds if

$$\beta = 1,\ j \le s-6$$

so that, by (A.24), we obtain the estimate (A.22) if $j \le s-6$.

If $j > s-6$, we can proceed as in Lemma A.1 and find that (A.25) holds if

$$1 + \beta = \frac{1}{3}(s-j)$$

and, together with (A.24), the assertion (A.22) follows. //

**Theorem A.4**. *Let $B = \{|x| < 1\} \subset \mathbf{R}^2,\ \partial B = \{|x| = 1\}$. If $s \ge 6$ then there is a constant $M_0$ such that*

$$\|fg\|_{s,\partial B} \le M_0\|f\|_{s,\partial B}\|g\|_{s,\partial B} \tag{A.26}$$

*for any two functions $f, g$ with finite $\|\ \|_{s,\partial B}$ norm.*

The proof is the same as for Theorem A.2.

We need to extend the previous results to the case where

$$B = \{|x| < 1\} \subset R^\nu, \ \nu \geq 2. \tag{A.27}$$

Following the proof of Lemma A.1 we find that, for $B$ as in (A.27), (A.1) holds with

$$r = \frac{6(\nu+1)}{3(\nu+1) - 2(s-j)}$$

if the right-hand side is positive, and (A.2) holds with

$$r = \frac{6(\nu+1)}{3(\nu+1) + 6 - 2(s-j)}$$

if the right-hand side is positive. Similarly Lemma A.3 extends to the case where $B$ is given by (A.27) and $f = f(\omega, t)$, with

$$r = \frac{6\nu}{3\nu - 2(s-j)}$$

in (A.21) if the right-hand side is positive and

$$r = \frac{6\nu}{3\nu + 6 - 2(s-j)}$$

in (A.22) if the right-hand side is positive.

**Theorem A.5**. *If $B$ is as in (A.27) then the inequality (A.14) holds provided $s \geq 3(\nu+1)$.*

**Proof**. Indeed, as in the proof of Theorem A.2, we need to show that

$$\|D^j F \cdot D^{s-j} G\|_{L^2(B)} \leq C \|F\|_{s,B} \|G\|_{s,B}.$$

By Hölder's inequality we deduce that this holds if

$$\frac{3(\nu+1) - 2(s-j)}{6(\nu+1)} + \frac{3(\nu+1-2j)}{6(\nu+1)} \leq 2,$$

which is true when $s \geq 3(\nu+1)$. One can easily check that also the other products $D_t D^{j-3} F \cdot D^{s-j} G$, $D_t^2 D^{j-6} F \cdot D^{s-j} G$ are bounded in the $L^2$ norm by $C\|F\|_{s,B}\|G\|_{s,B}$. //

Similarly one can prove:

**Theorem A.6**. *If $B$ is as in (A.27) then the inequality (A.26) holds provided $s \geq 3(\nu+1)$.*

## Appendix B: Estimating derivatives of composite functions

Let $D$ be a domain in $\mathbf{R}^d$ and let $\| \ \|$ be a norm defined for functions $f(x),\ x \in D$, such that

$$\|fg\| \le M_0 \|f\| \|g\|, \quad M_0 \ge 1.$$

**Lemma B.1**. *Let $\mu(x,\epsilon)$ be a function of the form*

$$(B.1) \qquad \mu(x,\epsilon) = \sum_{n=1}^{\infty} \mu_n(x)\epsilon^n, \quad x \in D$$

*such that*

$$(B.2) \qquad \|\mu_n - \mu_n^{\infty}\| \le C_0 \frac{H^{n-1}}{n^2},\ |\mu_n^{\infty}| \le C_0 \frac{H^{n-1}}{n^2}$$

*for some constants $\mu_n^{\infty}$. Let*

$$(B.3) \qquad G(s) = s^N,\ N \text{ positive integer},$$

*and set*

$$(B.4) \qquad G(\mu(x,\epsilon)) \equiv (\sum_{n\ge 1} \mu_n(x)\epsilon^n)^N \equiv \sum_{n \ge N} \Phi_n^N(x)\epsilon^n,$$

$$(B.5) \qquad (\sum_{n\ge 1} \mu_n^{\infty}\epsilon^n)^N = \sum_{n\ge N} \Phi_n^{N,\infty}\epsilon^n.$$

*Then*

$$(B.6) \qquad |\Phi_n^{N,\infty}| \le C_0(C_0A_0)^{N-1}\frac{H^{n-N}}{n^2},\ n \ge N$$

*and*

$$(B.7) \qquad \|\Phi_n^N - \Phi_n^{N,\infty}\| \le C_0(C_0A_0M_0)^{N-1}\frac{H^{n-N}}{n^2},\ n \ge N$$

*where $A_0$ is a universal constant.*

**Proof**. Consider first the case $N = 2$. Then

$$|\Phi_n^{2,\infty}| = |\sum_k \mu_{n-k}^{\infty}\mu_k^{\infty}| \le \sum_{k=1}^{n-1} C_0^2 \frac{H^{n-k-1}}{(n-k)^2}\frac{H^{k-1}}{k^2}$$
$$\le C_0^2 A_0 \frac{H^{n-2}}{n^2} \quad (n \ge 2),$$

provided

$$(B.8)\qquad A_0 \geq \sup_n \sum_{k=1}^{n-1} \frac{n^2}{(n-k)^2 k^2} \equiv \tilde{A}_0.$$

Proceeding inductively from $N-1$ to $N$,

$$\begin{aligned}
|\Phi_n^{N,\infty}| = |\sum_k \Phi_{n-k}^{N-1,\infty} \mu_k^\infty| &\leq C_0^N A_0^{N-2} \sum_k \frac{H^{n-(N-1)-k}}{(n-k)^2} \frac{H^{k-1}}{k^2} \\
&\leq C_0^N A_0^{N-1} \frac{H^{n-N}}{n^2} \quad (n \geq N).
\end{aligned}$$

Having proved (B.6), we next prove (B.7) for $N=2$. Writing

$$\begin{aligned}
\Phi_n^2 - \Phi_n^{2,\infty} &= \sum_{k=1}^{n-1} \mu_k \mu_{n-k} - \sum_{k=1}^{n-1} \mu_{n-k}^\infty \mu_k^\infty \\
&= \sum_{k=1}^{n-1} [(\mu_k - \mu_k^\infty)(\mu_{n-k} - \mu_{n-k}^\infty) + \mu_k^\infty(\mu_{n-k} - \mu_{n-k}^\infty) + \mu_{n-k}^\infty(\mu_k - \mu_k^\infty)],
\end{aligned}$$

we get

$$\|\Phi_n^2 - \Phi_n^{2,\infty}\| \leq 3 \sum_{k=1}^{n-1} C_0^2 M_0 \frac{H^{n-k-1}}{(n-k)^2} \frac{H^{k-1}}{k^2} \leq C_0^2 M_0 A_0 \frac{H^{n-2}}{n^2},\ n \geq 2$$

provided $A_0 \geq 3\tilde{A}_0$, $\tilde{A}_0$ as in (B.8).

Proceeding by induction from $N-1$ to $N$, we write

$$\begin{aligned}
\Phi_n^N - \Phi_n^{\infty,N} &= \sum_{k=N-1}^{n-1} (\Phi_k^{N-1} \mu_{n-k} - \Phi_k^{N-1,\infty} \mu_{n-k}^\infty) \\
&= \sum_{k=N-1}^{n-1} [(\Phi_k^{N-1} - \Phi_k^{N-1,\infty})(\mu_{n-k} - \mu_{n-k}^\infty) \\
&\quad + \mu_{n-k}^\infty(\Phi_k^{N-1} - \Phi_k^{N-1,\infty}) + \Phi_k^{N-1,\infty}(\mu_{n-k} - \mu_{n-k}^\infty)]
\end{aligned}$$

so that

$$\begin{aligned}
\|\Phi_n^N - \Phi_n^{N,\infty}\| &\leq 3 \sum_{k=N-1}^{n-1} C_0^2 (C_0 A_0 M_0)^{N-2} M_0 \frac{H^{k-N+1}}{k^2} \frac{H^{n-k-1}}{(n-k)^2} \\
&\leq C_0 (C_0 A_0 M_0)^{N-1} \frac{H^{n-N}}{n^2},\ n \geq N.
\end{aligned}$$

//

**Lemma B.2**. *Let $\mu(x,\epsilon)$ be as in Lemma B.1 and let*

$$G(s)=\sum_{n=1}^{\infty}\alpha_n s^n,\ \ s\in\mathbf{R} \tag{$B.9$}$$

*where*

$$|\alpha_n|\le C^n,\ \ C>0.$$

Set

$$G(\mu(x,\epsilon))=\sum_{n=1}^{\infty}G_n(x)\epsilon^n, \tag{$B.10$}$$

$$G(\sum_{n=1}^{\infty}\mu_n^{\infty}\epsilon^n)=\sum_{n=1}^{\infty}G_n^{\infty}\epsilon^n. \tag{$B.11$}$$

Then

$$|G_n^{\infty}|\le 2CC_0\frac{H^{n-1}}{n^2}, \tag{$B.12$}$$

$$\|G_n-G_n^{\infty}\|\le 2CC_0\frac{H^{n-1}}{n^2}, \tag{$B.13$}$$

provided $H\ge 2CC_0A_0M_0$.

**Proof**. Writing

$$\sum_{n\ge1}G_n^{\infty}\epsilon^n=\sum_{m\ge1}\alpha_m(\sum_{n\ge m}\Phi_n^{m,\infty}\epsilon^n)$$

we get

$$\begin{aligned}|G_n^{\infty}|&=|\sum_{m\le n}\alpha_m\Phi_n^{m,\infty}|\\ &\le\sum_{1\le m\le n}C^mC_0(C_0A_0)^{m-1}\frac{H^{n-m}}{n^2}\quad\text{(by Lemma B.1)}\\ &\le 2CC_0\frac{H^{n-1}}{n^2}\end{aligned}$$

if $H\ge 2CC_0A_0$, thus proving (B.12). To prove (B.13) we note that

$$\begin{aligned}G(\mu(x,\epsilon))-G(\sum_{n=1}^{\infty}\mu_n^{\infty}\epsilon^n)&=\sum_m\alpha_m(\sum_{n\ge m}\Phi_n^m\epsilon^n)-\sum_m\alpha_m(\sum_{n\ge m}\Phi_n^{m,\infty}\epsilon^n)\\ &=\sum_{n\ge1}\epsilon^n\sum_{m=1}^{n}\alpha_m(\Phi_n^m-\Phi_n^{m,\infty})\equiv\sum_{n\ge1}\epsilon^n(G_n-G_n^{\infty}),\end{aligned}$$

so that

$$\|G_n - G_n^\infty\| \le \sum_{m=1}^{n} |\alpha_m| C_0 (C_0 A_0 M_0)^{m-1} \frac{H^{n-m}}{n^2} \quad \text{(by Lemma B.1)}$$

$$\le C C_0 \sum_{m=1}^{n} \frac{(C C_0 A_0 M_0)^{m-1}}{H^{m-1}} \frac{H^{n-1}}{n^2} \le 2 C C_0 \frac{H^{n-1}}{n^2}.$$

//

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