Completeness of multiseparable superintegrability in $E_{2,C}$

E. G. Kalnins
Department of Mathematics and Statistics,
University of Waikato,
Hamilton, New Zealand.
W. Miller, Jr.
School of Mathematics, University of Minnesota,
Minneapolis, Minnesota, 55455, U.S.A.
and G. S. Pogosyan*
Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research,
Dubna, Moscow Region 141980, Russia

December 28, 1999

Abstract
The possibility that Schrödinger's equation with a given potential can separate in more than one coordinate system is intimately connected with the notion of superintegrability. Examples of this type of system are well known. In this article we demonstrate how to establish a complete list of such potentials using essentially algebraic means. Our approach is to classify all nondegenerate potentials that admit a pair of second order constants of the motion. Here "nondegenerate means that the potentials depend on four in-

*Work supported in part by the Russian Foundation for Basic Research under grant 98-01-00330

dependent parameters. This is carried out for two dimensional complex Euclidean space, though the method generalizes to other spaces and dimensions. We show that all these superintegrable systems correspond to quadratic algebras, and we work out the detailed structure relations and their quantum analogs.

1 Introduction

It has long been known that Schrödinger's equation with certain special potentials can admit (multiplicative) separation of variables in more than one coordinate system. This is intimately related to the notion of superintegrability, [1, 2]. This subject has been studied by a number of authors, based on the use of the corresponding differential equations that that are implied by the requirement of simultaneous separability, [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Specifically, superintegrability means that for a Schrödinger equation in dimension N there exist 2N-1 functionally independent quantum mechanical observables (i.e., self-adjoint operators that commute with the Hamiltonian). There is an analogous concept of superintegrability for classical mechical systems. This relates to the corresponding additive separation of variables of the Hamilton-Jacobi equation. A first step in studying separability in the classical case is to realise that the direct formulation of the simultaneous separability requirement is not obviously tractable. An additional observation is that if we do have simultaneous separability then the resulting constants of the motion close quadratically under repeated application of the Poisson bracket, [12]. We also know that for spaces of constant curvature separable coordinate systems of the free motion are described by quadratic elements of the corresponding first order symmetries, [17, 18, 19].

Although concrete examples of superintegrable systems are easily at hand, a complete classification of all such systems has presented major difficulties. How can one be sure that all systems for free motion have been found? (For example, Rañada's classification [16] omits our system (5a) below.) Once these are determined, how can one be sure that the most general additive potential term has been calculated?

Here we take a new approach to the problem and apply it for the case of two dimensional complex Euclidean space. In §2 we classify all nondegenerate potentials that admit a pair of second order constants of the motion. Here "nondegenerate" means that the potentials depend on four independent parameters. The requirement that a potential admit two constants of the motion leads to two second order partial differential equations obeyed by the potential, and the integrability conditions for these two simultaneous equations permit us to classify all possibilities. The classification is greatly simplified by the equivalence of two potentials that are related by an action of the complex Euclidean motion group. We then prove that each nondegenerate potential is associated with a pair of constants of the motion in the classical case, and a pair of symmetry operators in the quantum case, that generate a quadratic algebra. Furthermore, we verify that there is a one-to-one correspondence between superintegrable systems and free-field symmetry operators that generate quadratic algebras. Finally, we demonstrate explicitly that superintegrability implies multiseparability, i.e., separability in more than one coordinate system.

This systematic classification approach introduces a "fine structure" into our problem. it is easy to show that potentials admitting two constants of the motion cannot depend on more than four parameters. However, potentials that depend on fewer parameters, i.e., that cannot be embedded in a four parameter family, are not associated with a quadratic algebra.

2 Completeness in two dimensional Euclidean space

Due to the close connection between separation of variables and constants of the motion, [20], a common approach to the classification and study of superintegrable systems is to search for potentials that permit varable separation in more than one coordinate system. Let us consider what our problem is in these circumstances. The Hamilton-Jacobi equation is

$$H = p_x^2 + p_y^2 + V(x,y) = (\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2 + V(x,y) = E. \tag{1}$$

The additive separation ansatz implies a solution

$$S = U(u, E, \alpha) + V(v, E, \alpha) \tag{2}$$

in a suitable coordinate system $x = x(u,v), y = y(u,v)$. Here α is the separation constant. In the complex Euclidean plane there are six different separable coordinate systems, listed in the Appendix, in what we take to be a standard form. One approach to our problem relates to the following question. If the Hamilton-Jacobi equation with a potential $V(x,y)$ admits additive separation in a given coordinate system

$$x = x(u,v), y = y(u,v),$$

when does it also admit a separation in a coordinate system

$$x = x(p,q)\cos\beta + y(p,q)\sin\beta + a, \quad y = -x(p,q)\sin\beta + y(p,q)\cos\beta + b?$$

Here coordinates u,v and p,q are two different systems taken from the list in the Appendix, and we have allowed for the possibility that the second coordinate system can be subjected to a Euclidean motion consisting of a rotation through the angle β and a translation by the vector (a,b). To attempt to solve this problem one could relate the two different coordinate systems to each other and apply the additive separability conditions in one coordinate system to the equation when written in terms of the other. This is not a revealing approach. Instead we adopt a different method, one that does not directly relate to variable separation.

Let us assume that in addition to the classical Hamiltonian we have two quadratic constants of the motion

$$L_h = \sum_{k,j=1}^{2} a^{kj}_{(h)}(x,y)p_k p_j + W_{(h)}(x,y) \equiv \ell_h + W_{(h)}, \quad h = 1,2 \tag{3}$$

which must satisfy

$$\{H, L_h\} = 0$$

with $\{\}$ the usual Poisson bracket. We require that the set $\{dH, dL_1, dL_2\}$ is linearly independent, so that H, L_1, L_2 is a maximal set of functionally independent constants of the motion. It is clear that $R = \{L_1, L_2\}$ is a constant of the motion, so it and R^2 must be expressible as an analytic function of H, L_1, L_2:

$$R^2 = F(L_0, L_1, L_2), \qquad H \equiv L_0. \tag{4}$$

Note that R has the form

$$R = \sum_{k,l,m=1}^{2} c^{klm} p_k p_l p_m + \sum_{k=1}^{2} d^k p_k, \tag{5}$$

but that it doesn't follow that R^2 is necessarily a polynomial as a function of L_0, L_1, L_2. We will find conditions that guarantee that F is a third-order polynomial in its arguments.

Using the identity

$$\{K, G\} = \sum_{h=0}^{2} \{K, L_h\} \frac{\partial G}{\partial L_h} \tag{6}$$

for a continuously differentiable function $G(L_h)$, we find the relations

$$\{L_1, R\} = \frac{1}{2}\frac{\partial F}{\partial L_2}, \quad \{L_2, R\} = -\frac{1}{2}\frac{\partial F}{\partial L_1}. \tag{7}$$

Thus, the constants of the motion $\{L_1, R\}, \{L_2, R\}$ are easily computed once F is known. Further, if F is a polynomial in the invariants, then so are $\{L_1, R\}$, and $\{L_2, R\}$.

We first determine the conditions that the function

$$L = \sum_{j,k=1}^{2} a^{jk}(x, y) p_k p_j + W(x, y), \quad a^{jk} = a^{kj}, \tag{8}$$

must satisfy to be a constant of the motion. The requirement is $\{H, L\} = 0$ where

$$\{f, g\} = \sum_{j=1}^{2} \left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j}\right), \quad (x_1, x_2) = (x, y), \tag{9}$$

and

$$H = p_1^2 + p_2^2 + V(x, y). \tag{10}$$

The conditions are thus

$$\begin{aligned} \frac{\partial a^{11}}{\partial x} = 0 \qquad 2\frac{\partial a^{12}}{\partial x} + \frac{\partial a^{11}}{\partial y} &= 0 \\ \frac{\partial a^{22}}{\partial y} = 0 \qquad \frac{\partial a^{22}}{\partial x} + 2\frac{\partial a^{12}}{\partial y} &= 0, \end{aligned} \tag{11}$$

and

$$\frac{\partial W}{\partial x} - a^{11}\frac{\partial V}{\partial x} - a^{12}\frac{\partial V}{\partial y} = 0, \quad \frac{\partial W}{\partial y} - a^{12}\frac{\partial V}{\partial x} - a^{22}\frac{\partial V}{\partial y} = 0. \tag{12}$$

The solution for the terms quadratic in the p_j is

$$\begin{aligned} a^{11} &= \alpha_1 y^2 + \alpha_2 y + \alpha_3' &(13)\\ a^{12} &= -\alpha_1 xy - \frac{1}{2}\alpha_2 x - \frac{1}{2}\alpha_4 y + \frac{1}{2}\alpha_5 &(14)\\ a^{22} &= \alpha_1 x^2 + \alpha_4 x + \alpha_3'', &(15)\end{aligned}$$

where the α_k are constants. The requirement that $\partial_x W_y = \partial_y W_x$ leads from (12) to the second order partial differential equation for the potential

$$\begin{aligned} \tfrac{1}{2} \quad & (2\alpha_1 xy + \alpha_2 x + \alpha_4 y - \alpha_5)(V_{xx} - V_{yy}) + (\alpha_1[y^2 - x^2] + \alpha_2 y - \alpha_4 x + \alpha_3)V_{xy} \\ = \quad & (-3\alpha_1 y - \frac{3}{2}\alpha_2)V_x + (3\alpha_1 x + \frac{3}{2}\alpha_4)V_y, \end{aligned} \tag{16}$$

where $\alpha_3 = \alpha_3' - \alpha_3''$. We denote the solution space of this equation by

$$[\alpha_1, \cdots, \alpha_5]. \tag{17}$$

Let us now return to our assumption that the Hamilton-Jacobi equation admits two constants of the motion:

$$L_h = \sum_{j,k=1}^{2} a_{(h)}^{jk} p_k p_j + W_{(h)}, \qquad h = 1, 2.$$

These two operators together with H are assumed functionally independent. The constant of the motion L_1 leads to the condition (17) on the potential V; whereas L_2 leads to the second condition

$$[\beta_1, \cdots, \beta_5]. \tag{18}$$

The potential must lie in the intersection of the solution spaces (17,18) for these two conditions. It follows that the equations

$$V_{xx} - V_{yy} = AV_x + BV_y, \qquad V_{xy} = CV_x + DV_y \tag{19}$$

must hold, where

$$
\begin{aligned}
A\mathcal{E} &= \frac{3}{2}H_{12}(x^2+y^2) - 3H_{14}xy + 3H_{13}y - \frac{3}{2}H_{24}x + \frac{3}{2}H_{23} \\
B\mathcal{E} &= \frac{3}{2}H_{14}(x^2+y^2) - 3H_{12}xy - 3H_{13}x + \frac{3}{2}H_{24}y + \frac{3}{2}H_{34} \\
2C\mathcal{E} &= -3H_{14}y^2 + (-\frac{3}{2}H_{24} + 3H_{15})y + \frac{3}{2}H_{25} \qquad (20) \\
2D\mathcal{E} &= 3H_{12}x^2 + (-\frac{3}{2}H_{24} - 3H_{15})x - \frac{3}{2}H_{45} \\
2\mathcal{E} &= -H_{12}xy^2 + H_{14}x^2y - H_{12}x^3 + H_{14}y^3 - 2H_{13}xy + H_{24}(x^2+y^2) \\
&+ H_{15}(x^2-y^2) + (H_{34} - H_{25})y + (H_{45} - H_{23})x - H_{35},
\end{aligned}
$$

and $H_{k\ell} = -H_{\ell k} = \alpha_k\beta_\ell - \alpha_\ell\beta_k$.

From the fundamental equations (19) we can compute all of the third partial derivatives of V. Indeed

$$
\begin{aligned}
V_{xxx} &= (A_x + BC + C_y + C^2 + A^2)V_x + (B_x + DB + D_y + CD + AB)V_y \\
&+ (A+D)V_{yy} \\
V_{xxy} &= (C_x + DC + AC)V_x + (D_x + D^2 + BC)V_y + CV_{yy} \\
V_{xyy} &= (C_y + C^2)V_x + (D_y + CD)V_y + DV_{yy} \qquad (21) \\
V_{yyy} &= (-A_y + C_x + DC)V_x + (-B_y - AD + D_x + D^2 + BC)V_y \\
&+ (C-B)V_{yy}.
\end{aligned}
$$

Thus if the potential V is subject to the two conditions (17,18), then V can depend on at most 3 parameters, in addition to a trivial additive constant. We can choose these parameters to be $V_x(x_0,y_0), V_y(x_0,y_0), V_{yy}(x_0,y_0)$ for any fixed regular point (x_0,y_0). Then $V_{xx}(x_0,y_0)$ and all higher derivatives can be computed by successive differentiation of relations (21). We require that our potential be *nondegenerate*, i.e., that it depend on 3 arbitrary parameters.

Then, the conditions $\partial_x V_{xxy} = \partial_y V_{xxx}, \partial_y V_{xxy} = \partial_x V_{xyy}, \partial_y V_{xyy} = \partial_x V_{yyy}$ for the fourth partial derivatives lead to the integrability conditions

$$
\begin{aligned}
\partial_x(2C - B) &= \partial_y(2D + A) \quad (\mathit{satisfied\ identically}) \qquad (22) \\
C_{xx} - C_{yy} - A_{xy} &= 2CC_y - DA_y - 2CD_x + AA_y - AC_x \\
&+ CB_y + BC_y \qquad (23) \\
D_{xx} - D_{yy} - B_{xy} &= -2DD_x - CB_x + 2DC_y - BB_x \\
&- BD_y + DA_x + AD_x. \qquad (24)
\end{aligned}
$$

Note that if we have another constant of the motion L_3 associated with a nondegenerate potential, then L_3 must be a linear combination of H, L_1, L_2. Indeed, if L_3 is not a linear combination of the basis functions, then the potential V must satisfy an equation (16) that is linearly independent of the equations associated with L_1, L_2. This means an additional constraint on the solution space and that V can depend on at most two parameters, which is a contradiction.

We will use the conditions (23,24) to classify the possible potentials V and the corresponding constants of the motion L_1, L_2. For this we note that it is only the three-dimensional subspace spanned by H, L_1, L_2 that matters; we can choose any basis for this subspace. Hence we can replace the conditions (17, 18) by linear combinations of themselves without changing the potential. Moreover, to simplify the results we note that we can always subject the coordinates (x, y), and L_1, L_2 to a simultaneous Euclidean motion, i.e., we regard all translated and rotated potentials as members of the same equivalence class.

Multiplying both sides of (23) and (24) by $\mathcal{E}^3$ we obtain polynomial identities in x and y. Equating the coefficients of the various powers $x^n y^m$ we obtain conditions on the parameters H_{jk}. The simplest non-trivial condition, which is associated with the coefficient of a fifth order power in either of the equations, is

$$2H_{15}(H_{14}^2 - H_{12}^2) + H_{24}(H_{14}^2 + H_{12}^2) - 4H_{14}H_{12}H_{13} = 0. \tag{25}$$

We exploit these and the remaining conditions, and Euclidean motions to classify the possibilities for the L_j. The full conditions (23) and (24), expressed in terms of the parameter H_{ij}, take several pages to list and are complicated to solve directly. (Indeed a symbol manipulation program was an important aid to our computations.) However, by dividing the problem up into special cases and using Euclidean motions, we can simplify the conditions and obtain a full solution. In the listing that follows we use the fact that the constants of the motion can each be expressed as a quadratic element in the enveloping algebra of the Euclidean group in the plane with basis elements

$$p_x, \ p_y, \ M = xp_y - yp_x,$$

to which a potential term $W(x, y)$ is added. (Strictly speaking, conditions (23) and (24) are only necessary conditions for existence of nondegenerate potentials. However, in our case-by-case study we have found that they are also sufficient: all solutions of these equations lead to nondegenerate potentials.)

Suppose $I \equiv H_{12}^2 + H_{14}^2 \neq 0$. Via an appropriate coordinate rotation through complex angle θ we obtain a new set of equations (19) in the rotated coordinates where the new parameters H'_{12}, H'_{14} are related to the original ones by

$$H'_{12} = H_{12}\cos\theta + H_{14}\sin\theta, \quad H'_{14} = H_{14}\cos\theta - H_{12}\sin\theta, \tag{26}$$

and for which $I = I'$. Thus, by an appropriate choice of θ, we can assume $H_{12} = 0$. Then, by an appropriate Euclidean translation that leaves H_{12}, H_{14} unchanged, it follows from (20) that we can assume $H_{13} = H_{24} = 0$. Then (25) implies $H_{15} = 0$, so $H_{35} = H_{23} = 0$. Further, the fourth order integrability conditions give $H_{34}H_{45} = 0$, and under appropriate translations, we can take $H_{34} = H_{45} = 0$.

Case (1) $H_{12}^2 + H_{14}^2 \neq 0$

$$[1, 0, 0, 0, 0], \quad [0, 0, 0, 1, 0] \tag{27}$$

Here,

$$L_1 = 4M^2 + W^{(1)}, \qquad L_2 = -2Mp_y + W^{(2)} \tag{28}$$

$$V(x) = \frac{\alpha}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{x^2+y^2}}\left[\frac{\beta}{\sqrt{x^2+y^2+x}} + \frac{\gamma}{\sqrt{x^2+y^2-x}}\right]. \tag{29}$$

This potential allows separation in parabolic or polar coordinates.

If, on the other hand, $H_{12} = \pm iH_{14} \neq 0$, then, via translation, we can assume $H_{24} = 0$. In this case (25) implies $H_{15} = iH_{13}$.

Case (2) $H_{12} = \pm iH_{14} \neq 0$

$$[1, 0, \alpha_3, 0, -i\alpha_3 - i\beta_3^2], \quad [0, -1, \beta_3, i, i\beta_3] \tag{30}$$

Here,

$$L_1 = M^2 - \frac{\beta_3^2}{4}p_+^2 + \left(\frac{\alpha_3}{2} + \frac{\beta_3^2}{4}\right)p_-^2 + W^{(1)}, \qquad L_2 = Mp_+ + \frac{\beta_3}{2}p_+^2 + W^{(2)}, \tag{31}$$

where $p_\pm = p_x \pm ip_y$, $z = x + iy$, and $\bar{z} = x - iy$. There are two subcases to consider. If $\mu = 2\alpha_3 + \beta_3^2 \neq 0$, then via a rotation about the origin we can achieve

Case (2a)

$$[1, 0, 2c^2, 0, 0], \quad [0, -1, ic, i, -c], \tag{32}$$

where,

$$\begin{aligned} L_1 &= M^2 + c^2p_x^2 + W^{(1)}, \qquad L_2 = Mp_+ + \frac{ic}{2}p_+^2 + W^{(2)} \\ V(x) &= \frac{\alpha z}{(c^2 - z^2)^{\frac{1}{2}}} + \frac{\beta}{\sqrt{(c-z)(c+\bar{z})}} + \frac{\gamma}{\sqrt{(c+z)(c+\bar{z})}}. \end{aligned} \tag{33}$$

The corresponding Hamilton-Jacobi and Schrödinger equations for this system separates in elliptical coordinates, see the Appendix, as well as shifted elliptical coordinates. In terms of the cartesian coordinates x_E, y_E associated with the elliptic coordinates, the shifted coordinates are $X = x_E - c$, $Y = y_E - ic$. The corresponding operator in this case is $(M + ic(p_x + ip_y))^2 + c^2p_x^2$.

If $\mu = 0$ we have

Case (2b)

$$[1, 0, 2, 0, 2i], \quad [0, -1, 2i, i, -2], \tag{34}$$

where,

$$\begin{aligned} L_1 &= M^2 + p_+^2 + W^{(1)}, \qquad L_2 = (M + 2ip_+)^2 + p_+^2 + W^{(2)} \\ V(x) &= \frac{\alpha}{z^2} + \frac{\beta}{\sqrt{z^3(\bar{z}+2)}} + \frac{\gamma}{\sqrt{z(\bar{z}+2)}}. \end{aligned} \tag{35}$$

This system separates in terms of hyperbolic coordinates, see the Appendix, and displaced hyperbolic coordinates. In terms of the cartesian coordinates x_H, y_H associated with the hyperbolic coordinates, the displaced coordinates are $X = x_H - 2, Y = y_H - 2i$.

Now suppose $H_{12} = H_{14} = 0$.

Case (3) $H_{12} = H_{14} = 0$, $\alpha_1 \neq 0$

$$[1, 0, 0, 0, \alpha_5], \quad [0, 0, \beta_3, 0, \beta_5]. \tag{36}$$

This breaks up into three subcases. A rotation through complex angle θ has the effect

$$H'_{15} = H_{15}\cos 2\theta + H_{13}\sin 2\theta, \quad H'_{13} = H_{13}\cos 2\theta - H_{15}\sin 2\theta, \quad H'_{24} = H_{24}. \tag{37}$$

Thus, if $H_{15}^2 + H_{13}^2 \neq 0$ we can achieve $H_{15} = 0$. Then, we can assume via translation that $\alpha_2 = \alpha_4 = 0$, so $H_{24} = 0$. An integrability condition gives $H_{35} = 0$. Thus,

Case (3a) $H_{15}^2 + H_{13}^2 \neq 0$

$$[1,0,0,0,0], \quad [0,0,1,0,0] \tag{38}$$

Here,

$$\begin{aligned} L_1 &= M^2 + W^{(1)}, \qquad L_2 = p_x^2 + W^{(2)} && (39) \\ V(x) &= \alpha(x^2+y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}. && (40) \end{aligned}$$

This potential permits separation in polar, elliptic and cartesian coordinates.

If, however, $H_{15} = \pm i H_{13} \neq 0$, we can again translate to get $H_{24} = 0$, and find two possibilities, depending on whether $\alpha_5 = 0$:

Case (3b) $H_{15}^2 + H_{13}^2 = 0$

$$[1,0,0,0,0], \quad [0,0,2,0,\pm 2i] \tag{41}$$

Here,

$$\begin{aligned} L_1 &= M^2 + W^{(1)}, \qquad L_2 = p_+^2 + W^{(2)} \\ V(x) &= \alpha\frac{x^2+y^2}{(x+iy)^4} + \frac{\beta}{(x+iy)^2} + \gamma(x^2+y^2). \end{aligned} \tag{42}$$

(There is a similar solution where the term p_+^2 in L_2 is replaced by p_-^2.) The potential permits separation in hyperbolic and polar coordinates.

Case (3c) $H_{15}^2 + H_{13}^2 = 0$

$$[1,0,c^2,0,0], \quad [0,0,2,0,\pm 2i] \tag{43}$$

Here,

$$\begin{aligned} L_1 &= M^2 + c^2 p_x^2 + W^{(1)}, \qquad L_2 = p_+^2 + W^{(2)} \\ V(x) &= \frac{\alpha z}{\sqrt{z^2-c^2}} + \frac{\beta \bar{z}}{\sqrt{z^2-c^2}(z+\sqrt{z^2-c^2})^2} + \gamma z \bar{z} \end{aligned} \tag{44}$$

The potential permits separation in hyperbolic and elliptic coordinates. Indeed, instead of the basis L_1, L_2 let us consider the basis $M^2 + c^2 p_x^2$ and $M^2 + \frac{1}{4}c^2(p_x - ip_y)^2$. Corresponding to these operators are (1) the normal choice of elliptic coordinates and (2) the choice of hyperbolic coordinates $x = \frac{1}{2}cx_H$ and $y = -\frac{i}{2}cy_H$, (see the Appendix).
Case (4) $H_{12} = H_{13} = H_{14} = H_{15} = 0$, $\alpha_2 \neq 0$, $H_{24} \neq 0$

$$[0,1,0,0,0], \quad [0,0,0,1,0] \tag{45}$$

Here,

$$L_1 = -2Mp_x + W^{(1)}, \qquad L_2 = -2Mp_y + W^{(2)} \tag{46}$$

$$V(x) = \frac{\alpha}{\sqrt{x^2+y^2}} + \beta\frac{(\sqrt{x^2+y^2}+x)^{\frac{1}{2}}}{\sqrt{x^2+y^2}} + \gamma\frac{(\sqrt{x^2+y^2}-x)^{\frac{1}{2}}}{\sqrt{x^2+y^2}}. \tag{47}$$

Separation of variables is possible in two types of parabolic coordinates, the usual parabolic coordinates and the interchanged parabolic coordinates $x = \mu\nu, y = \frac{1}{2}(\mu^2 - \nu^2)$.
Case (5) $H_{12} = H_{13} = H_{14} = H_{15} = 0$, $\alpha_2 \neq 0$, $H_{24} = 0$

$$[0,1,\alpha_3,\alpha_4,\alpha_5], \quad [0,0,\beta_3,0,\beta_5] \tag{48}$$

If $(H_{34}-H_{25})^2 + (H_{45}-H_{23})^2 \neq 0$ we can make a complex rotation to achieve $H_{45} = H_{23}$. Then we can take $H_{25} = 1$ and the constant term integrability conditions yield $H_{34} = -H_{23}^2 = 1$.
Case (5a) $(H_{34}-H_{25})^2 + (H_{45}-H_{23})^2 \neq 0$

$$[0,1,\alpha_3,\pm i,0], \quad [0,0,\pm i,0,1] \tag{49}$$

Here we choose the typical case

$$\begin{aligned} L_1 &= 4iMp_- + p_+^2 + W^{(1)}, \quad L_2 = p_-^2 + W^{(2)} \qquad (50) \\ V(x) &= \alpha(x-iy) + \beta(x+iy-\frac{3}{2}(x-iy)^2) + \gamma(x^2+y^2-\frac{1}{2}(x-iy)^3). \qquad (51) \end{aligned}$$

The possible separable coordinates are semihyperbolic coordinates corresponding to operator $Mp_- + p_+^2$ and shifted semihyperbolic coordinates with operator $Mp_- + \delta p_-^2 + p_+^2$. This corresponds to the standard coordinates shifted via the transformation $x \to x+\delta, y \to y+i\delta$.

Case (5b) $(H_{34}-H_{25})^2 + (H_{45}-H_{23})^2 = 0$

Here $(H_{34}-H_{25}) = \pm i(H_{45}-H_{23}) \neq 0$. The constant term integrability conditions, and a translation in y, yield the solutions

$$[0,1,0,\alpha_4,0], \quad [0,0,\beta_3,0,1], \ \beta_3 = \pm i, \alpha_4 \neq \beta_3. \qquad (52)$$

Then an appropriate rotation about the origin takes this to

$$[0,0,0,1,0], \quad [0,0,1,0,\pm i].$$

This system, for which now $\alpha_2 = 0$, corresponds to case (6b) below.

Case (6) $H_{12} = H_{13} = H_{14} = H_{15} = 0$, $\alpha_2 = 0$, $\alpha_3 \neq 0$,

The constant term integrabilty condition is $H_{45}(H_{45}^2 + H_{34}^2) = 0$. There are two cases.

Case (6a) $H_{45} = 0$

$$[0,0,1,0,0], \quad [0,0,0,1,0] \qquad (53)$$

Here,

$$\begin{aligned} L_1 &= p_x^2 + W^{(1)}, \qquad L_2 = -2Mp_y + W^{(2)} \qquad (54) \\ V(x) &= \alpha(4x^2+y^2) + \beta x + \frac{\gamma}{y^2}. \qquad (55) \end{aligned}$$

The possible separable coordinates are cartesian and parabolic.

Case (6b) $H_{45} \neq 0$

$$[0,0,1,0,\pm i], \quad [0,0,0,1,0] \qquad (56)$$

Here we take

$$L_1 = 2p_yp_+ + W^{(1)}, \qquad L_2 = Mp_y + W^{(2)} \tag{57}$$

$$V(x) = \frac{\alpha}{\sqrt{x+iy}} + \beta x + \gamma\frac{2x+iy}{\sqrt{x+iy}}. \tag{58}$$

There is the possibility of separability in parabolic coordinates $\{Mp_y\}$ or displaced parabolic coordinates $\{(M+\delta(p_x \pm ip_y))p_y\}$ for suitable δ.

Now we demonstrate that there is a quadratic algebra associated with each nondegenerate potential. Because we are working in two dimensions there can only be three functionally independant constants at most. Consequently all Poisson brackets must be functionally dependent on $H = L_0, L_1$ and L_2. We want to show that in fact $R^2 = \{L_1, L_2\}^2 = F(L_0, L_1, L_2)$ is a polynomial in these variables.

Note that for arbitrary L_1, L_2, the F is in general *not* a polynomial. Consider the example:

$$L_0 = p_x^2 + p_y^2, \quad L_1 = M^2 + p_xp_y, \quad L_2 = p_x^2.$$

Then we have $R = \{L_1, L_2\} = 4Mp_xp_y$ and

$$R^2 = F(L_0, L_1, L_2) = 16L_1L_2(L_0 - L_2) - 16L_2^{\frac{3}{2}}(L_0 - L_2)^{\frac{3}{2}}.$$

Here, although F is defined and bounded at the point $(L_0, L_1, L_2) = (0,0,0)$, it is not analytic at this point. Thus it has no power series expansion about the origin. We conjecture that this is an illustration of the general problem: if F is not a polynomial, then there are branch points or cuts at $(0,0,0)$.

We will show, however, that for nondegenerate potentials the associated F *is* a polynomial. First, we can verify that this is true when the potential is turned off, i.e., if we consider only the functions

$$\ell_h = \sum_{j,k=1}^{2} a^{jk}_{(h)} p_kp_j, \ i = h, 2 \quad \ell_0 = p_x^2 + p_y^2,$$

where $L_h = \ell_h + W^{(h)}$. Let $\mathcal{R} = \{\ell_1, \ell_2\}$. Then for each of the cases listed above it is straightforward to check that $\mathcal{R}^2 = \mathcal{P}_3(\ell_0, \ell_1, \ell_2)$ where $\mathcal{P}_3$ is a homogeneous third order polynomial in its arguments. [1] It follows that

$$R^2 = F(L_0, L_1, L_2) = \mathcal{P}_3(L_0, L_1, L_2) + F_4(\mathbf{s}, L_0, L_1, L_2), \tag{59}$$

where F_4 is a fourth, second and zeroth order polynomial in the momenta p_x, p_y, and $F_4(\mathbf{0}, L_0, L_1, L_2) = 0$. Here, the parameters in the potential are denoted by $\mathbf{s} = (V_x^0, V_y^0, V_{yy}^0)$, evaluated at some fixed point (x_0, y_0) and F_4 is a polynomial function of these parameters.

From (7) we have

$$\{\ell_1, \mathcal{R}\} = \frac{1}{2}\frac{\partial \mathcal{P}_3}{\partial \ell_2}(\ell_0, \ell_1, \ell_2),$$

$$\{\ell_2, \mathcal{R}\} = -\frac{1}{2}\frac{\partial \mathcal{P}_3}{\partial \ell_1}(\ell_0, \ell_1, \ell_2),$$

hence

$$\{L_1, R\} = \frac{1}{2}\frac{\partial \mathcal{P}_3}{\partial L_2}(L_0, L_1, L_2) + \frac{1}{2}\frac{\partial F_4}{\partial L_2}(\mathbf{s}),$$
$$\{L_2, R\} = -\frac{1}{2}\frac{\partial \mathcal{P}_3}{\partial L_1}(L_0, L_1, L_2) - \frac{1}{2}\frac{\partial F_4}{\partial L_1}(\mathbf{s}),$$

where the $\partial F_4/\partial L_h(\mathbf{s})$ have only terms of orders two and zero in the momenta. It follows that the $\partial F_4/\partial L_h(\mathbf{s})$ must be expressible as linear combinations of the L_h. This shows that the commutators $\{L_h, R\}$ can be expressed as polynomials in L_0, L_1, L_2. It is then a simple matter to verify that F itself is a polynomial in L_0, L_1, L_2.

We now list the quadratic algebra relations for each of the cases studied above. In view of relations (7) it is sufficient to give the relation $R^2 = F(L_0, L_1, L_2)$ for each case.

Case (1) $[1,0,0,0,0]$, $[0,0,0,1,0]$

$$R^2 = 16L_1^2H - 16L_2^2L_1 - 32(\beta+\gamma)L_2^2 + 64\alpha(\beta-\gamma)L_2 + 16\alpha^2L_1 - 256\beta\gamma H - 32\alpha^2(\beta+\gamma).$$

[1] Moreover, it is straightforward to verify that the cases corresponding to nondegenerate potentials are the *only* cases where $\mathcal{P}_3$ is a homogeneous third order polynomial in its arguments. Thus the possible quadratic algebras generated by second order elements in the Euclidean Lie algebra correspond one-to-one with nondegenerate potentials.

Case (2a) $[1, 0, 2c^2, 0, 0]$, $[0, -1, ic, i, -c]$

$$R^2 = \frac{1}{2}c^4H^3 - 4icL_2^3 + 2c^2L_2^2H - 4L_2^2L_1 - c^2H^2L_1 - ic^3H^2L_2 + \frac{i}{2}c^4\alpha H^2 + 2i\alpha c^2L_2^2 - c^2\alpha^2L_1 + ic(\beta^2 + \gamma^2 - c^2\alpha^2)L_2 + \frac{1}{2}(-c^2\beta^2 + c^4\alpha^2 + c^2\gamma^2)H + \frac{1}{2}(2\beta\gamma + ic^2\alpha^2)c^2\alpha.$$

Case (2b) $[1, 0, 2, 0, 2i]$, $[0, -1, 2i, i, -2]$

$$R^2 = -2L_1^3 - 2L_2^3 + 2L_2^2L_1 + 2L_1^2L_2 + 32\alpha HL_1 + 32\alpha HL_2 - 8\beta\gamma L_1 + 8\beta\gamma L_2 + 16\beta^2H + 16\alpha\gamma^2.$$

Case (3a) $[1, 0, 0, 0, 0]$, $[0, 0, 1, 0, 0]$

$$R^2 = -16L_2^2L_1 + 16L_2L_1H - 16\beta H^2 - 16(\beta + \gamma)L_2^2 - 16\alpha L_1^2 + 32HL_2 + 64\alpha\beta\gamma.$$

Case (3b) $[1, 0, 0, 0, 0]$, $[0, 0, 2, 0, \pm 2i]$

$$R^2 = -16L_1^3 + 32L_1^2L_2 - 16L_2^2L_1 + 16\alpha H^2 + 16\beta HL_2 - 16\beta HL_1 - 64\alpha\gamma L_1 - 16\beta^2\gamma.$$

Case (3c) $[1, 0, c^2, 0, 0]$, $[0, 0, 2, 0, \pm 2i]$

$$\begin{aligned} R^2 &= 4c^2L_2^3 + 8c^2L_2^2H - 16L_2^2L_1 + 4c^2H^2L_2 + (4i\beta - 2c^4\gamma)H^2 + 8ic^2\alpha L_2^2 \\ &+ (4c^4\gamma^2 - 16i\beta\gamma)L_1 + (4\alpha^2c^2 - c^6\gamma^2 + 4ic^2\gamma\beta)L_2 \\ &+ (-2c^6\gamma^2 + 8ic^2\gamma\beta + 8\beta\alpha)H + 2c^4\alpha^2\gamma - 2ic^6\gamma^2\alpha - 8c^2\alpha\beta\gamma - 4i\alpha^2\beta. \end{aligned}$$

Case (4) $[0, 1, 0, 0, 0]$, $[0, 0, 0, 1, 0]$

$$R^2 = 4HL_1^2 + 4HL_2^2 + 4(\beta^2 - \gamma^2)L_2 - 8\beta\gamma L_1 - 4\alpha^2H - 4\alpha(\beta^2 + \gamma^2).$$

Case (5a) $[0, -4i, 2, 1, 2i]$, $[0, 0, 2, 0, -2i]$

$$R^2 = 64L_2^3 - 64\gamma H^2 - 128\alpha L_2^2 + 128\beta HL_2 + 64\gamma L_2L_1 + 64\alpha^2L_2 + 64\beta^2L_1 - 128\beta\alpha H.$$

Case (6a) $[0, 0, 1, 0, 0]$, $[0, 0, 0, 1, 0]$

$$R^2 = 16L_1^3 - 32L_1^2H + 16H^2L_1 - 16\alpha L_2^2 - 8\beta HL_2 + 8\beta L_2L_1 - 64\alpha\gamma L_1 - 4\beta^2\gamma.$$

Case (6b) $[0, 0, -2i, 0, 2]$, $[0, 0, 0, 1, 0]$

$$R^2 = 2iL_1^3 + L_1^2H - \beta HL_2 - 2i\beta L_2L_1 - \gamma^2L_2 - i\alpha\gamma L_1 + \frac{1}{4}\beta\alpha^2.$$

3 Quantum Superintegrability in two dimensional Euclidean space

Here we give the analogous quantum algebras for superintegrable systems arising from the potentials we have already computed. The only difference is that the Poisson bracket is now replaced by the commutator bracket $[A, B] = AB - BA$ and the operators H, L_1 and L_2 are the obvious (formally self-adjoint) symmetry partial differential operators.

$$H = \partial_x^2 + \partial_y^2 + V(x, y), \quad L_h = \sum_{k,j=1}^{2} \partial_k (a_{(h)}^{kj}) \partial_j + W_{(h)}(x, y), \quad h = 1, 2. \tag{60}$$

Just as for the Hamilton-Jacobi case, if we have another constant of the motion L associated with a maximal potential, then L must be a linear combination of H, L_1, L_2. Indeed, if L is in self-adjoint form, then the conditions that $[H, L] = 0$ are identical with (11), (12). Thus, if L is not a linear combination of the basis functions, then the potential V must satisfy an equation (16) that is linearly independent of the equations associated with L_1, L_2. This means an additional constraint on the solution space and that V can depend on at most two parameters, which is a contradiction.

Furthermore the proof of the existence of quadratic algebra relations at the end of §2 goes through almost unchanged for the operator case: $[L_1, L_2]^2 = R^2$ and $[L_1, R], [L_2, R]$ can be expressed as (symmetric) polynomials in the operators H, L_1, L_2. To make the prior construction go through, one need only note that since R^2 is a formally self-adjoint 6th order differential symmetry operator, the 5th order terms are fixed linear functions of the 6th order terms. The expressions $\{A, B\} = AB + BA$ and $\{A, B, C\} = ABC + CAB + BCA$ are operator symmetrizers. The explicit relations are:

Case (1)

$$[L_2, R] = 8L_2^2 + 8HL_2 + 8\alpha^2,$$

$$[L_1, R] = 8\{L_2, L_1\} + 16(1 + 2\beta + 2\gamma)L_2 + 32\alpha(\gamma - \beta),$$

$$\begin{aligned} R^2 &= 16L_1^2 H - \frac{8}{3}\{L_2, L_2, L_1\} - 16(2\beta + 2\gamma + \frac{11}{3})L_2^2 - \frac{176}{3}HL_1 \\ &+ 64\alpha(\beta - \gamma)L_2 + 16\alpha^2 L_1 + (-\frac{32}{3} + 96\gamma + 96\beta + 256\beta\gamma)H \\ &- \frac{32}{3}\alpha^2(3\beta + 3\gamma - 1). \end{aligned}$$

Case (2a)

$$[L_1,R]=-\frac{1}{2}ic^3H^2-6icL_2^2+2c^2HL_2-2\{L_1,L_2\}+(2i\alpha c^2-1)L_2-\frac{1}{2}ic^3\alpha^2+\frac{1}{2}ic\beta^2+\frac{1}{2}ic\gamma^2,$$

$$[L_2,R]=\frac{1}{2}c^2H^2+2L_2^2+\frac{1}{2}c^2\alpha^2,$$

$$\begin{aligned}R^2 &= \frac{1}{2}c^4H^3-4icL_2^3+2c^2L_2^2H-\frac{2}{3}\{L_1,L_1,L_2\}-c^2H^2L_1-ic^3H^2L_2\\ &+ (\frac{i}{2}c^2\alpha+\frac{1}{12})H^2+(2i\alpha c^2-\frac{11}{3})L_2^2-c^2\alpha^2L_1+ic(\beta^2+\gamma^2-c^2\alpha^2)L_2\\ &+ \frac{1}{2}(-c^2\beta^2+c^4\alpha^2+c^2\gamma^2)+\frac{1}{2}(2\beta\gamma+ic^2\alpha^2+\frac{1}{12}\alpha)c^2\alpha.\end{aligned}$$

Case (2b)

$$[L_1,R]=L_1^2-3L_2^2+\{L_1,L_2\}+16\alpha H+L_1-L_2+4\beta,\gamma$$

$$[L_2,R]=3L_1^2-L_2^2-\{L_1,L_2\}-16\alpha H+L_1-L_2+4\beta,\gamma$$

$$\begin{aligned}R^2 &= -2L_1^3-2L_2^3+\frac{1}{3}\{L_2,L_2,L_1\}+\frac{1}{3}\{L_1,L_1,L_2\}-\frac{11}{3}L_1^2-\frac{11}{3}L_2^2\\ &+ 32\alpha HL_1+32\alpha HL_2+\frac{11}{3}\{L_1,L_2\}-8\gamma\beta L_1+8\gamma\beta L_2\\ &+ (-\frac{16}{3}\alpha+16\beta^2)H+16\alpha\gamma^2.\end{aligned}$$

Case (3a)

$$[L_2,R]=-8L_2^2+8HL_2-16\alpha L_1+8\alpha,$$

$$[L_1,R]=-8HL_1+8\{L_2,L_1\}-8(1+2\beta)H+16(1+\beta+\gamma)L_2,$$

$$\begin{aligned}R^2 &= -\frac{8}{3}\{L_2,L_2,L_1\}+8H\{L_2,L_1\}-4(3+4\beta)H^2-16(\beta+\gamma-\frac{11}{3})L_2^2-16\alpha L_1^2\\ &+ 16(2\beta+\frac{11}{3})HL_2+\frac{176}{3}\alpha L_2+16\alpha(3\beta+3\gamma+4\beta\gamma+\frac{2\alpha}{3}).\end{aligned}$$

Case (3b)

$$[L_2,R]=-8L_2^2-32\alpha\gamma,$$

$$[L_1,R]=16\{L_2,L_1\}-8\beta H+16L_2,$$

$$R^2=-\frac{8}{3}\{L_2,L_2,L_1\}+16\alpha H^2-\frac{176}{3}L_2^2+16\beta HL_2-64\alpha\gamma L_1+(\frac{64}{3}\alpha\gamma-16\beta^2\gamma).$$

Case (3c)

$$[L_2,R]=L_2^2+8i\beta\gamma-2c^4\gamma^2,$$

$$[L_1,R]=2c^2H^2+6c^2L_2^2+8c^2HL_2-8\{L_1,L_2\}+(-16+8ic^2\alpha)L_2-\frac{1}{2}c^6\gamma^2+2\alpha^2c^2+2ic^2\beta\gamma,$$

$$R^2=4c^2L_2^3+8c^2L_2^2H-\frac{8}{3}\{L_2,L_2,L_1\}+4c^2H^2L_2+(4i\beta-2c^4\gamma)H^2+(8ic^2\alpha-\frac{176}{3})L_2^2$$
$$+(4c^4\gamma^2-16i\beta\gamma)L_1+(4\alpha^2c^2-c^6\gamma^2+4ic^2\gamma\beta)L_2+$$
$$(-2c^6\gamma^2+8ic^2\gamma\beta+8\beta\alpha)H+2c^4\alpha^2\gamma-2ic^6\gamma^2\alpha-8c^2\alpha\beta\gamma-4i\alpha^2\beta-\frac{4}{3}\gamma^2c^4+\frac{16i}{3}\beta\gamma.$$

Case (4)

$$[L_2,R]=4HL_1-4\beta\gamma,$$

$$[L_1,R]=-4HL_2+2(\gamma^2-\beta^2),$$

$$R^2=4L_1^2H+4L_2^2H+4H^2+4(\beta^2-\gamma^2)L_2-8\beta\gamma L_1-4\alpha^2H-4\alpha(\gamma^2+\beta^2).$$

Case (5a)

$$[L_2,R]=32\gamma L_2+32\beta^2,$$

$$[L_1,R]=-96L_2^2-64\beta H+128\alpha L_2-32\gamma L_1-32\alpha^2,$$

$$\begin{aligned}R^2 &= 64L_2^3-64\gamma H^2-128\alpha L_2^2+128\beta HL_2+32\gamma\{L_2,L_1\}+64\alpha^2L_2\\ &+ 64\beta^2L_1-128\beta\alpha H-256\gamma^2.\end{aligned}$$

Case (6a)

$$[L_2,R]=8H^2+24L_1^2-32L_1H+4\beta L_2-\alpha(24+32\gamma),$$

$$[L_1,R]=4\beta H+16\alpha L_2-4\beta L_1,$$

$$\begin{aligned}R^2 &= 16L_1^3-32HL_1^2+16H^2L_1-16\alpha L_2^2-8\beta HL_2+4\beta\{L_2,L_1\}\\ &- (64\alpha\gamma+176\alpha)L_1+128\alpha H-(4\gamma\beta^2+3\beta^2).\end{aligned}$$

Case (6b)

$$[L_2,R]=3iL_1^2+HL_1-i\beta L_2-\frac{i}{2}\alpha\gamma,$$

$$[L_1, R] = \frac{1}{2}\beta H + i\beta L_1 + \frac{1}{2}\gamma^2,$$

$$R^2 = 2iL_1^3 + L_1^2 H - \beta H L_2 - i\beta\{L_2, L_1\} - \gamma^2 L_2 - i\alpha\gamma L_1 + \frac{1}{4}(\beta^2 + \alpha^2\beta).$$

We note that the quadratic relations in the quantum case provide useful information relating the special functions that occur as (separable) eigenfunctions for each superintegrable case [15].

4 Conclusions

In this paper we have used the concept of a "nondegenerate potential" to add structure to the study of superintegrable classical and quantum mechanical systems in $E(2, C)$. We have shown how to classify all such systems in a straightforward manner, so that gaps can be avoided. Furthermore, we have shown the following:

1. Each system is associated with a pair of constants of the motion in the classical case, and a pair of symmetry operators in the quantum case, that generate a quadratic algebra.

2. There is a one-to-one correspondence between superintegrable systems and free-field symmetry operators that generate quadratic algebras.

3. Superintegrability implies multiseparability, i.e., separability in more than one coordinate system.

In a forthcoming paper we will prove the analogous results for superintegrable systems on the complex 2-sphere.

5 Appendix

As is well known [3, 17, 19] there are essentially six coordinate systems on the complex Euclidean plane in which the free particle Hamilton-Jacobi equation separates: Cartesian, polar, parabolic, elliptic, hyperbolic and semi-hyperbolic. We describe these coordinate systems and their corresponding free particle constants of the motion L. (We adopt the basis $p_x, p_y, M =$

$xp_y - yp_x$ for the Lie algebra $e(2,C)$ and define $p_\pm = p_x \pm ip_y$.) There is one orbit of constants of the motion, with representative Mp_+, that is not associated with variable separation [20]. The systems are:

Cartesian coordinates

$$x, y, \qquad L = p_x^2 \tag{61}$$

Polar Coordinates

$$\begin{array}{ll} x = r\cos\theta \ , & \\ y = r\sin\theta \ , & \qquad L = M^2 \end{array} \tag{62}$$

Parabolic Coordinates.

$$x_P = \frac{1}{2}(\xi^2 - \eta^2) \ , \qquad y_P = \xi\eta \ , \qquad L = Mp_y \tag{63}$$

Elliptic Coordinates (in algebraic form)

$$\begin{aligned} x_E^2 &= c^2(u-1)(v-1), \quad y_E^2 = -c^2uv, \\ L &= M^2 + c^2p_x^2 \end{aligned} \tag{64}$$

Hyperbolic Coordinates

$$\begin{aligned} x_H &= \frac{r^2 + r^2s^2 + s^2}{2rs}, \quad y_H = i\frac{r^2 - r^2s^2 + s^2}{2rs}, \\ L &= M^2 + p_+^2 \end{aligned} \tag{65}$$

Semi-Hyperbolic Coordinates

$$\begin{aligned} x_{SH} &= -\frac{1}{4}(w-u)^2 + \frac{1}{2}(w+u), \quad iy_{SH} = -\frac{1}{4}(w-u)^2 - \frac{1}{2}(w+u), \\ L &= 2Mp_+ + p_-^2 \end{aligned} \tag{66}$$

References

[1] N.W.Evans. Superintegrability in Classical Mechanics; *Phys.Rev.* **A 41** (1990) 5666; Group Theory of the Smorodinsky-Winternitz System; *J.Math.Phys.* **32**, 3369 (1991).

[2] N.W.Evans. Superintegrability of the Calogero-Moser System. *Phys. Lett.* **A 95**, 279 (1983). Super-Integrability of the Winternitz System; *Phys.Lett.* **A 147**, 483 (1990).

[3] L.P.Eisenhart. Enumeration of Potentials for Which One-Particle Schrödinger Equations Are Separable; *Phys.Rev.* **74**, 87 (1948).

[4] J.Friš, V.Mandrosov, Ya.A.Smorodinsky, M.Uhlir and P.Winternitz. On Higher Symmetries in Quantum Mechanics; *Phys.Lett.* **16**, 354 (1965).

[5] J.Friš, Ya.A.Smorodinskii, M.Uhlír and P.Winternitz. Symmetry Groups in Classical and Quantum Mechanics; *Sov.J.Nucl.Phys.* **4**, 444 (1967).

[6] A.A.Makarov, Ya.A.Smorodinsky, Kh.Valiev and P.Winternitz. A Systematic Search for Nonrelativistic Systems with Dynamical Symmetries; *Nuovo Cimento* **A 52**, 1061 (1967).

[7] D.Bonatos, C.Daskaloyannis and K.Kokkotas. Deformed Oscillator Algebras for Two-Dimensional Quantum Superintegrable Systems; *Phys. Rev.* **A 50**, 3700 (1994).

[8] F.Calogero. Solution of a Three-Body Problem in One Dimension; *J.Math.Phys.* **10**, 2191 (1969).

[9] A.Cisneros and H.V.McIntosh. Symmetry of the Two-Dimensional Hydrogen Atom; *J.Math.Phys.* **10**, 277 (1969).

[10] L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian and V.M.Ter-Antonyan. Elliptic Basis for a Circular Oscillator. *Nuovo Cimento*, **B 88**, 43 (1985), Two-Dimensional Hydrogen Atom: I. Elliptic Bases; *Theor.Math.Phys.* **61**, 1021 (1984); Hidden symmetry, Separation of Variables and Interbasis Expansions in the Two-Dimensional Hydrogen Atom. *J.Phys.*, **A 18**, 455 (1985).

[11] B.Zaslow and M.E.Zandler. Two-Dimensional Analog of the Hydrogen Atom; *Amer. J.Phys.* **35**, 1118 (1967).

[12] Ya.A.Granovsky, A.S.Zhedanov and I.M.Lutzenko. Quadratic Algebra as a 'Hidden' Symmetry of the Hartmann Potential; *J.Phys.* **A 24**, 3887 (1991).

[13] C.Grosche, G.S.Pogosyan, A.N.Sissakian. Path Integral Discussion for Smorodinsky - Winternitz Potentials:I. Two - and Three Dimensional Euclidean Space. *Fortschritte der Physik*, **43**, 453 (1995).

[14] P.Letourneau and L.Vinet. Superintegrable systems: Polynomial Algebras and Quasi-Exactly Solvable Hamiltonians. *Ann. Phys.* **243**, 144-168, (1995).

[15] E.G.Kalnins, W.Miller Jr. and G.S.Pogosyan. Superintegrability and associated polynomial solutions. Euclidean space and the sphere in two dimensions. *J.Math.Phys.* **37**, 6439, (1996).

[16] M.F. Rañada. Superintegrable n=2 systems, quadratic constants of motion, and potentials of Drach. *J.Math.Phys.* **38**, 4165, (1997).

[17] W.Miller, Jr. Symmetry and Separation of Variables. *Addison-Wesley Publishing Company*, Providence, Rhode Island, 1977.

[18] E.G. Kalnins and W. Miller, Jr. Separation of variables on n-dimensional Riemannian manifolds 1. The n-sphere S_n and Euclidean n-space R_n *J. Math. Phys.* **27**, 1721, (1986).

[19] E.G. Kalnins. *Separation of Variables for Riemannian Spaces of Constant Curvature*, Pitman, Monographs and Surveys in Pure and Applied Mathematics 28, Longman, Essex, England, 1986.

[20] W. Miller Jr. Mechanisms for variable separation in partial differential equations and their relationship to group theory. In *Symmetries and Non-linear Phenomena* pp. 188. World Scientific, 1988.