# Non-Lipschitz minimizers of smooth strongly convex functionals

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## 1 Introduction

We consider variational integrals of the form

$$I(u) = \int_\Omega f(Du(x))dx, \tag{1.1}$$

where $\Omega$ is a bounded open set with smooth boundary in $\mathbf{R}^n$, $u\colon \Omega \to \mathbf{R}^m$, $Du$ is the gradient matrix of $u$ and $f\colon M^{m\times n} \to \mathbf{R}$ is a smooth strongly convex function with uniformly bounded second derivatives. Here $M^{m\times n}$ denotes the set of real $m \times n$ matrices. (Recall that we say $f$ is strongly convex if there exists a constant $\nu > 0$, such that for all $\xi \in M^{m\times n}, X \in M^{m\times n}$, the inequality $f_{p^i_\alpha p^j_\beta}(X)\xi^i_\alpha \xi^j_\beta \geq \nu|\xi|^2$ holds.)

We shall consider the problem of regularity of minimizers of $I$ belonging to $W^{1,2}(\Omega, \mathbf{R}^m)$. By a minimizer we mean a mapping $u \in W^{1,2}(\Omega, \mathbf{R}^m)$ such that for any smooth mapping $\phi\colon \Omega \to \mathbf{R}^m$ compactly supported in $\Omega$ the inequality $I(u + \phi) \geq I(u)$ holds. When $f$ is strongly convex with uniformly bounded second derivatives, it is not difficult to see that $u$ is a minimizer of $I$ if and only if $u$ is a weak solution of the $Euler - Lagrange$ equation of $I$, i.e., $u$ is a weak solution of

$$\partial_\alpha f_{p^i_\alpha}(Du(x)) = 0, \qquad i = 1, \cdots, m. \tag{1.2}$$

(Here and in what follows we use the summation convention.)

A classical result of C.B. Morrey [Mo] shows that when $n = 2, m \geq 1$ and $f$ is a smooth strongly convex function with uniformly bounded second derivatives every weak solution of (1.2) is smooth, this is also the case when $n \geq 2, m = 1$ and $f$ satisfies the same condition by fundamental work of De Giorgi [De1] and Nash [Na]. The method used in the proof of De Giorgi and Nash cannot be extended to the case $m \geq 2$ as shown by a counterexample of De Giorgi [De2]. The first example of a nonsmooth minimizer for a smooth strongly convex functional of type (1.1) was constructed by Nečas in high dimensions (see [Ne]). He considered $u\colon \mathbf{R}^n \to \mathbf{R}^{n^2}$ defined by

$$u_{ij} = \frac{x_i x_j}{|x|}, \tag{1.3}$$

and for large $n$ constructed a smooth strongly convex function $f$ with bounded second derivatives defined on $M^{n\times n^2}$ for which $u$ is a minimizer of the corresponding functional $I$. Later Nečas, Hao and Leonardi [HLN] were able to modify this construction and make it work for $n \geq 5$. They modified the original $u$ in the following way:

$$u_{ij} = \frac{x_i x_j}{|x|} - \frac{|x|}{n}\delta_{ij}. \tag{1.4}$$

Recently the authors (see [SY]) constructed a nonsmooth minimizer of a smooth strongly convex functional of type (1.1) in the case $n = 3, m = 5$ by considering the same function $u$ defined by (1.4).

The main idea of our construction is the following. Let $K = \{\nabla u(x), x \in \Omega\}$ be the set of gradients of $u$. We find a null Lagrangian $L$ (See definition 2.1 below) such that

$$\nabla L(X) = \nabla f(X), \qquad \forall X \in K, \tag{1.5}$$

for a smooth strongly convex function $f$ with bounded second derivatives. Then $u$ will satisfy the $Euler - Lagrange$ equation of $I$ automatically.

All the counterexamples of nonsmooth minimizers above are Lipschitz continuous. In fact, it was an open problem whether minimizers with unbounded gradients exist. Partial results in this direction can be found in [ME] , where local Lipschitz continuity of minimizers for a special class of functionals was obtained.

In this paper we use the null-Lagrangian approach to construct counterexamples showing, among other things, that in general for $n \geq 3$ we cannot expect Lipschitz continuity of the minimizer of a smooth strongly convex functional. Moreover, for $n = 5$ we find a locally unbounded solution to (1.2). We recall that $n = 5$ is the first possible dimension where such an example is possible. (When $n \leq 4$ each minimizer must be Hölder continuous, since it belongs to $W^{2,2+\delta}$ for some $\delta > 0$, see [Gi].) We also construct in Section 4 a completely new example for $n = 4$, $m = 3$. The important new feature in this example is the low dimension of the target space. The construction also gives a non-Lipschitz minimizer in this case. The mapping used in that example is derived from the Hopf fibration $S^3 \to S^2$, which can be thought of as a complex version of (1.4). In addition, as a byproduct of our methods, we found an example (with $n = m = 3$) of non-uniqueness of weak solutions of (1.2) in the spaces $W^{1,p}$ with $1 < p < 2$. This is briefly explained in Section 5.

For counterexamples to regularity of solutions of elliptic systems which are not of the form (1.2) we refer the reader to [GM] and [NJS]. A comprehensive treatment of regularity questions can be found in [Gi]. Interesting sufficient conditions for regularity are discussed in [Ko].

# 2 Preliminaries

First we introduce some basic facts about null Lagrangians.

**Definition 2.1** *(see [Ba1]) $L\colon M^{m\times n} \to \mathbf{R}$ is a null Lagrangian if for each smooth $u\colon \mathbf{R}^n \to \mathbf{R}^m$,*

$$\operatorname{div} \nabla L(\nabla u(x)) = 0. \tag{2.1}$$

We recall the following classical theorem about null Lagrangian (see [Da] or [BCO]).

**Proposition 1** *Let $L\colon M^{m\times n} \to \mathbf{R}$, the following conditions are equivalent:*

*i) $L$ is a null Lagrangian.*

*ii) $L$ is a linear combination of subdeterminants.*

*iii) $L$ is rank-one affine, i.e. $t \to L(A + tB)$ is affine for each $A \in M^{m\times n}$ and each $B \in M^{m\times n}$ with rank $(A - B) = 1$.*

*Moreover, if $L$ is quadratic, then any of the above conditions is satisfied if and only if $L(B) = 0$ for each $B \in M^{m\times n}$ with rank(B)=1.*

# 3 The case $n \geq 3, m = n(n+1)/2 - 1$.

Let $\Omega$ be the unit ball in $\mathbf{R}^n$. Consider $u^\epsilon(x) = (u^\epsilon_{ij}(x))$ given by

$$\begin{aligned} u_{ij}(x) &= \frac{x_i x_j}{|x|} - \frac{|x|}{n}\delta_{ij}, \\ u^\epsilon_{ij}(x) &= |x|^{-\epsilon} u_{ij}(x), \qquad \epsilon > 0, \qquad i,j = 1, ..., n. \end{aligned} \tag{3.1}$$

Then for each $x \in \Omega, u^\epsilon(x) \in \{A \in M^{n\times n}, A = A^t, \mathrm{Tr}\ A = 0\} \cong \mathbf{R}^{\frac{n(n+1)}{2}-1}$. For each $R \in SO(n)$ we have

$$u^\epsilon(Rx) = Ru^\epsilon(x)R^t.$$

Denote $K^\epsilon = \{\nabla u^\epsilon(x), x \in \Omega\}, K_1^\epsilon = \{\nabla u^\epsilon(x), x \in S^{n-1}\}$. Following [SY], we identify $M^{m\times n}$ with $T = \{a_{ijk} \in (\mathbf{R}^n)^{\otimes 3} | a_{ijk} = a_{jik}, a_{iik} = 0\}$ in the obvious way. We recall that $m = \frac{n(n+1)}{2} - 1$. Then we use a classical procedure to decompose $T$ into irreducible subspaces ( see [We1]). We first decompose $T$ into the trace-free part $T'$ and its orthogonal supplement $T_3$, i.e., $T = T' \oplus T_3$. An easy calculation shows that the projection on $T_3$ is given by $a_{ijk} \to -\frac{2}{(n+2)(n-1)}\delta_{ij}\eta_k + \frac{n}{(n+2)(n-1)}\delta_{ki}\eta_j + \frac{n}{(n+2)(n-1)}\delta_{jk}\eta_i$ with $\eta_k = a_{kii}, k = 1, \cdots, n$. Then we decompose $T'$ by using symmetrizations. We have $T' = T_1 \oplus T_2$, where the projection on $T_1$ is given by symmetrizations, i.e., $a_{ijk} \to \frac{1}{3}(a_{ijk} + a_{jki} + a_{kij})$; the projection on $T_2$ is given by $a_{ijk} \to \frac{1}{3}(a_{ijk} + a_{jik} - a_{kji} - a_{kij})$, which corresponds to the following Young tableau:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

We remark that the antisymmetric part of any tensor in $T$ is 0.

By the above formula, a rank one matrix $a_{ijk} = C_{ij}\xi_k$ in $M^{m\times n}$ with $C = C^t$, $\mathrm{Tr}\ C = 0$ can be decomposed as

$$a_{ijk} = a_{ijk}^1 + a_{ijk}^2 + a_{ijk}^3,$$

with

$$\begin{aligned}
|a_{ijk}^1|^2 &= \frac{1}{3}|C|^2|\xi|^2 + \frac{2n}{3(n+2)}|C\xi|^2, \\
|a_{ijk}^2|^2 &= \frac{2}{3}|C|^2|\xi|^2 - \frac{2n}{3(n-1)}|C\xi|^2, \\
|a_{ijk}^3|^2 &= \frac{2n}{(n+2)(n-1)}|C\xi|^2.
\end{aligned}$$

For $X = X_1 + X_2 + X_3, X_i \in T_i$, we let $L(X) = -2|X_1|^2 + |X_2|^2 + n|X_3|^2$. From the above formula, we see that $L$ vanishes on all rank one matrices in $M^{m\times n}$, hence $L$ is a quadratic null Lagrangian on $M^{m\times n}$. Moreover, we have the following lemma:

**Lemma 3.1** *We have* $L(X) \equiv l_\epsilon = \frac{2(n-1)}{n+2}[(n+1-\epsilon)^2 - (1+\epsilon)^2]$ *on* $K_1^\epsilon$ *and for* $0 < \epsilon < \frac{n+1-\sqrt{\frac{3(n+1)}{n-1}}}{\sqrt{\frac{3(n+1)}{n-1}}+1}$, *there exist constant* $\delta_0(\epsilon) > 0$, *such that for any* $X = \nabla u^\epsilon(x), Y = \nabla u^\epsilon(y) \in K_1^\epsilon$, *we have*

$$\nabla L(X) \cdot (Y - X) \leq -\delta_0(\epsilon)|Y - X|^2. \tag{3.2}$$

*Proof:* First we note that on $K_1^\epsilon$ we can decompose $\nabla u^\epsilon(x) = \{u_{ij,k}^\epsilon\}$ as follows:

$$u_{ij,k}^\epsilon(x) = u_{ij,k}^{\epsilon,1}(x) + u_{ij,k}^{\epsilon,2}(x) + u_{ij,k}^{\epsilon,3}, \qquad x \in S^{n-1},$$

where $u^{\epsilon,i} \in T_i$ with

$$
\begin{aligned}
u^{\epsilon,1}_{ij,k} &= (1+\epsilon)[-x_i x_j x_k + \frac{1}{n+2}(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij})], \\
u^{\epsilon,2}_{ij,k} &= 0 \\
u^{\epsilon,3}_{ij,k} &= \frac{n+1-\epsilon}{n+2}(x_i\delta_{jk} + x_j\delta_{ik} - \frac{2}{n}\delta_{ij}x_k).
\end{aligned} \tag{3.3}
$$

and

$$
|u^{\epsilon,1}_{ij,k}|^2 = \frac{(1+\epsilon)^2(n-1)}{n+2}, \quad |u^{\epsilon,3}_{ij,k}|^2 = \frac{2(n+1-\epsilon)^2(n-1)}{n(n+2)}.
$$

Hence $\forall x \in S^{n-1}$,

$$
\begin{aligned}
L(\nabla u^\epsilon(x)) &\equiv \frac{2(n-1)}{n+2}[(n+1-\epsilon)^2 - (1+\epsilon)^2] \\
&= l_\epsilon, \\
|\nabla u^\epsilon(x)|^2 &\equiv \frac{n-1}{n+2}[(1+\epsilon)^2 + \frac{2(n+1-\epsilon)^2}{n}] \\
&= N_\epsilon^2, \\
|\nabla L(\nabla u^\epsilon(x))|^2 &\equiv 8\frac{n-1}{n+2}[2(1+\epsilon)^2 + n(n+1-\epsilon)^2] \\
&= m_\epsilon^2.
\end{aligned}
$$

Since $L$ is quadratic, we have

$$
L(\nabla u^\epsilon(x) - \nabla u^\epsilon(y)) = 2L(\nabla u^\epsilon(x)) - 2L(\nabla u^\epsilon(x), \nabla u^\epsilon(y)),
$$

where we also use $L$ for the symmetric bilinear form corresponding to the quadratic form $L$.

$$
\begin{aligned}
& L(\nabla u^\epsilon(x), \nabla u^\epsilon(y)) \\
= \; & -2u^{\epsilon,1}_{ij,k}(x) \cdot u^{\epsilon,1}_{ij,k}(y) + n u^{\epsilon,3}_{ij,k}(x) \cdot u^{\epsilon,3}_{ij,k}(y) \\
= \; & -2(1+\epsilon)^2 \left(-x_i x_j x_k + \frac{1}{n+2}(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij})\right) \cdot \left(-y_i y_j y_k + \frac{1}{n+2}(y_i\delta_{jk} + y_j\delta_{ik} + y_k\delta_{ij})\right) \\
& + n\left(\frac{n+1-\epsilon}{n+2}\right)^2 (x_i\delta_{jk} + x_j\delta_{ki} - \frac{2}{n}x_k\delta_{ij}) \cdot (y_i\delta_{jk} + y_j\delta_{ik} - \frac{2}{n}y_k\delta_{ij}) \\
= \; & -2(1+\epsilon)^2\langle x,y\rangle^3 + \left(\frac{6}{n+2}(1+\epsilon)^2 + \frac{2(n+1-\epsilon)^2(n-1)}{n+2}\right)\langle x,y\rangle.
\end{aligned}
$$

Let $t = \langle x,y\rangle$. Then $-1 \le t \le 1$, and we have

$$
\begin{aligned}
& L(\nabla u^\epsilon(x) - \nabla u^\epsilon(y)) \\
= \; & 2L(\nabla u^\epsilon(x)) - 2L(\nabla u^\epsilon(x), \nabla u^\epsilon(y)) \\
= \; & 2(1-t)\left(-2(1+\epsilon)^2(1+t+t^2) + \frac{6}{n+2}(1+\epsilon)^2 + \frac{2(n+1-\epsilon)^2(n-1)}{n+2}\right) \\
\ge \; & 2(1-t)\left(-6(1+\epsilon)^2 + \frac{6}{n+2}(1+\epsilon)^2 + \frac{2(n+1-\epsilon)^2(n-1)}{n+2}\right) \\
= \; & |x-y|^2\left(-\frac{6(n+1)}{n+2}(1+\epsilon)^2 + \frac{2(n+1-\epsilon)^2(n-1)}{n+2}\right).
\end{aligned}
$$

Since $L(X) = l_\epsilon$ on $K_1^\epsilon$, therefore

$$\begin{aligned} & \nabla L(\nabla u^\epsilon(x)) \cdot (\nabla u^\epsilon(y) - \nabla u^\epsilon(x)) \\ = \; & -L(\nabla u^\epsilon(x) - \nabla u^\epsilon(y)) + L(\nabla u^\epsilon(x)) + L(\nabla u^\epsilon(y)) - 2L(\nabla u^\epsilon(x)) \\ = \; & -L(\nabla u^\epsilon(x) - \nabla u^\epsilon(y)) \\ \leq \; & -\left( -\frac{6(n+1)}{n+2}(1+\epsilon)^2 + \frac{2(n+1-\epsilon)^2(n-1)}{n+2} \right) |x-y|^2. \end{aligned}$$

Note there exist constants $c_1(\epsilon), c_2(\epsilon) > 0$, such that for $X = \nabla u^\epsilon(x), Y = \nabla u^\epsilon(y) \in K_1^\epsilon$,

$$c_1(\epsilon)|x-y|^2 \leq |X-Y|^2 \leq c_2(\epsilon)|x-y|^2.$$

It is clear that when $\epsilon < \frac{n+1-\sqrt{\frac{3(n+1)}{n-1}}}{\sqrt{\frac{3(n+1)}{n-1}}+1}$, we can always find $\delta_0(\epsilon) > 0$, such that (3.2) is satisfied.

We proved in [SY] that (3.2) together with the fact that $L$ is constant on $K_1^\epsilon$ is also sufficient for the existence of a smooth strongly convex function with bounded second derivatives satisfying (1.5) on $K_1^\epsilon$. We explain the main idea here for the convenience of the reader. A natural attempt to make such an extension would be to take the convex hull of $K_1^\epsilon$ and consider a modification of the corresponding Minkowski function then use the homogeneity of $L$ and the Minkowski function. However, since the convex hull of $K_1^\epsilon$ may not be smooth at $K_1^\epsilon$, we need to slightly modify this construction.

We fix $\mu > 0$ (the exact value will be specified later) and for each $X \in K_1^\epsilon$, consider the ball in $T_1 \oplus T_3$ of radius $r_\mu = \mu|\nabla L(X)| = \mu m_\epsilon$ passing through $X$ centered at $X' = X - \nabla L(X)\mu$. We will denote the ball as $B_{X',r_\mu}$.

**Lemma 3.2** *When $\mu$ is sufficiently small we have*

$$\nabla L(X) \cdot (\tilde{Y} - X) \leq -\frac{\delta_0(\epsilon)}{2}|\tilde{Y} - X|^2, \tag{3.4}$$

*for each $X \in K_1^\epsilon$ and each $\tilde{Y} \in B_{Y',r_\mu}$, where $B_{Y',r_\mu}$ is defined above, with $Y$ being an arbitrary point of $K_1^\epsilon$.*

**Proof**: The inequality

$$|\tilde{Y} - Y'|^2 \leq \mu^2 |\nabla L(Y)|^2$$

gives

$$\nabla L(Y) \cdot (\tilde{Y} - Y) \leq -\frac{1}{2\mu}|\tilde{Y} - Y|^2.$$

Hence

$$\begin{aligned} \nabla L(X) \cdot (\tilde{Y} - X) \;\; = \;\; & (\nabla L(X) - \nabla L(Y)) \cdot (\tilde{Y} - Y) + \nabla L(Y) \cdot (\tilde{Y} - Y) \\ & + \nabla L(X) \cdot (Y - X) \\ \leq \;\; & (4+2n)|Y-X||\tilde{Y} - Y| - \frac{1}{2\mu}|\tilde{Y} - Y|^2 - \delta_0(\epsilon)|Y-X|^2, \end{aligned}$$

and the statement follows easily.

Let $S^\epsilon = \cup_{X \in K_1^\epsilon} B_{X',r_\mu}$. When $\mu$ is small, the boundary of $S^\epsilon$ is smooth by elementary results about tubular neighborhoods (see [Hi] or [We2]). Lemma 3.2 implies that (for sufficiently small $\mu$) all the eigenvalues of the second fundamental form of $\partial S^\epsilon$ is negative and bounded above uniformly on $K_1^\epsilon$ by a negative constant $\gamma_\epsilon$ (i.e. the principle curvatures $k_i(X) \leq \gamma_\epsilon < 0, \forall i$ and $\forall X \in K_1^\epsilon$.)

Since $\partial S^\epsilon$ is smooth, we conclude that $\partial S^\epsilon$ is locally strongly convex at any point of $U^\epsilon \cap \partial S^\epsilon$, where $U^\epsilon$ is a small neighborhood of $K_1^\epsilon$ in $T_1 \oplus T_3$.

Now take $G^\epsilon$ to be the convex hull of $S^\epsilon$ in $T_1 \oplus T_3$. Using Lemma 3.2 and the fact that $\partial S^\epsilon$ is smooth and locally strongly convex in $U^\epsilon \cap \partial S^\epsilon$, we infer that $U^\epsilon \cap \partial G^\epsilon = U^\epsilon \cap \partial S^\epsilon$ when the neighborhood $U^\epsilon$ of $K_1^\epsilon$ is chosen to be sufficiently small. Let

$$F_1^\epsilon(X) = \min\{t \geq 0, X \in tG^\epsilon\}, \quad F^\epsilon(X) = l_\epsilon(F_1^\epsilon(X))^2.$$

Then $F^\epsilon$ is two homogeneous, smooth in $U^\epsilon$ and strongly convex in $U^\epsilon$ (see [Ro]), and $\nabla L(X) = \nabla F^\epsilon(X)$ for each $X \in K_1^\epsilon$.

Let $\phi$ be a nonnegative $C^\infty$ function with support in $[\frac{1}{2}, 1]$, such that

$$\int_{\mathbf{R}^{nm}} \phi(|Z|)dZ = 1.$$

Define

$$\tilde{F}_\delta^\epsilon(X) = \left(\int_{\mathbf{R}^{nm}} F_1^\epsilon(X + |X|Z)\phi_\delta(|Z|)dZ\right)^2,$$

where $\phi_\delta(X) = \delta^{-n}\phi(\frac{X}{\delta})$. Then $\tilde{F}_\delta^\epsilon$ is convex and 2-homogeneous (see [Sc]). Let

$$H_{\delta,\tau}^\epsilon(X) = \tilde{F}_\delta^\epsilon + \tau|X|^2, X \in T_1 \oplus T_3.$$

Let $\tilde{\eta}^\epsilon(X)$ be a smooth cut off function defined on $T_1 \oplus T_3$ which vanishes outside $U^\epsilon$ and is 1 in a smaller neighborhood $V^\epsilon$ of $K_1^\epsilon$.

Consider

$$\eta^\epsilon(X) = \tilde{\eta}^\epsilon(N_\epsilon X/|X|) \qquad \text{when } |X| \neq 0.$$

Define

$$G_{\delta,\tau}^\epsilon(X) = \begin{cases} (1 - \eta^\epsilon(X))H_{\delta,\tau}^\epsilon(X) + \eta^\epsilon(X)F^\epsilon(X) & |X| \neq 0, \\ 0 & X = 0. \end{cases}$$

A direct calculation shows that when $\delta, \tau$ are small enough, $G_{\delta,\tau}^\epsilon$ is 2-homogeneous, smooth away from 0 and strongly convex away from 0 with

$$\nabla G_{\delta,\tau}^\epsilon(X) = \nabla L(X), \quad X \in K_1^\epsilon. \tag{3.5}$$

Moreover by the fact that

$$G_{\delta,\tau}^\epsilon(\lambda X) = \lambda^2 G_{\delta,\tau}^\epsilon(X) \qquad \forall \lambda \geq 0. \tag{3.6}$$

we see (3.5) holds on $K^\epsilon$.

Fix $\delta, \tau$ such that $h^\epsilon(X) = G_{\delta,\tau}^\epsilon(X)$ is smooth and strongly convex away from 0. Let $\psi(x)$ be a smooth mollifier defined on $\mathbf{R}^{mn}$ with support in $B_1$, let $\lambda^\epsilon(X)$ be a smooth cut off function defined on $\mathbf{R}^{mn}$ such that $\lambda^\epsilon(X) = 1$ in $B_{\frac{N_\epsilon}{4}}$ and vanishes outside $B_{\frac{N_\epsilon}{2}}$. Define

$$f_{\alpha,\beta}^\epsilon(X) = \lambda^\epsilon(X)(h_\alpha^\epsilon(X) + \beta|X|^2) + (1 - \lambda^\epsilon(X))h^\epsilon(X),$$

where $h_\alpha^\epsilon(X) = h^\epsilon * \psi_\alpha(X)$, $\psi_\alpha(X) = \alpha^{-n}\psi(\frac{X}{\alpha})$. It is not difficult to see that when $\alpha, \beta$ are small enough, $f_{\alpha,\beta}$ is smooth, strongly convex and satisfies (3.5) on $K^\epsilon$.

In the last step, define

$$f^\epsilon(A) = f_{\alpha,\beta}^\epsilon(X + Y) + |Z|^2,$$

where $A = X + Y + Z, X \in T_1, Z \in T_2, Y \in T_3$. Then $f^\epsilon$ is a smooth strongly convex function with bounded second derivatives which satisfies (1.5) on $K^\epsilon$.

We remark that we can take $\epsilon > 1$ when $n \geq 5$, and thus get an unbounded minimizer of a functional of the type (1.1).

# 4 The case $n = 4, m = 3$

In this section, let $\Omega$ be the unit ball in $\mathbf{R}^4$. For $\epsilon \geq 0$, consider mapping $v^\epsilon : \mathbf{R}^4 \to \mathbf{R}^3$ given by

$$v^\epsilon(z,w) = \left( \frac{\Re(\bar{z}w)}{r^{\epsilon+1}}, \frac{\Im(\bar{z}w)}{r^{\epsilon+1}}, \frac{|w|^2 - |z|^2}{2r^{\epsilon+1}} \right), \tag{4.1}$$

where $(z,w) \in \mathbf{C}^2 \cong \mathbf{R}^4$, $r^2 = |z|^2 + |w|^2$ and $\Re f, \Im f$ denote respectively the real and imaginary part of $f$.

For $R = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$, $|a|^2 + |b|^2 = 1$, $a, b \in \mathbf{C}$, denote by $\rho(R)$ the real 4 dimensional representation of $SU(2)$ given by

$$\rho(R)x = \begin{pmatrix} a_0 & -a_1 & b_0 & -b_1 \\ a_1 & a_0 & b_1 & b_0 \\ -b_0 & -b_1 & a_0 & a_1 \\ b_1 & -b_0 & -a_1 & a_0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad x \in \mathbf{R}^4, \quad z = x_1 + ix_2, \quad w = x_3 + ix_4.$$

Where

$$a = a_0 + ia_1, \quad b = b_0 + ib_1.$$

For $R \in SU(2)$, we have

$$v^\epsilon(\rho(R)x) = \tilde{\rho_3}(R) v^\epsilon(x), \quad x \in \mathbf{R}^4,$$

where $\tilde{\rho_3}(R)$ is the real representation of $SU(2)$ on $\wedge^2 \mathbf{R}^3$ induced by $\rho_3(R)$, the three dimensional irreducible representation of $SU(2)$. We remark that $\rho_3$ is of the real type. We have

$$\nabla v^\epsilon(\rho(R)x) = \tilde{\rho_3}(R) \nabla v^\epsilon(x) \rho(R)^t, \qquad x \in \mathbf{R}^4, \tag{4.2}$$

where $\nabla f$ denotes $(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4})$.

In another way, we can write (4.2) as

$$\nabla v^\epsilon(Rx) = \tilde{\rho_3}(R) \otimes \rho(R) \nabla v^\epsilon(x) \tag{4.3}$$

for $x \in \mathbf{R}^4$. We then have the following lemma.

**Lemma 4.1** *There exists a unique (up to a multiplication by a real scalar) quadratic null Lagrangian on $M^{4\times 3}$ which is invariant on the above action of $SU(2)$.*

*Proof:*

We identify the quadratic null Lagrangian on $M^{4\times 3}$ with $\wedge^2 \mathbf{R}^4 \otimes \wedge^2 \mathbf{R}^3 \cong \mathrm{Hom}(\wedge^2 \mathbf{R}^4, \wedge^2 \mathbf{R}^3)$ and consider the representation $\tau$ of $SU(2)$ on $\mathrm{Hom}(\wedge^2 \mathbf{R}^4, \wedge^2 \mathbf{R}^3)$ induced by $\tilde{\rho_3} \otimes \rho$. Using elementary group representation theory, we can easily determine that over the field of complex numbers, $\tau$ decomposes into irreducible representations as

$$\tau = \rho_5 \oplus 4\rho_3 \oplus \rho_1.$$

Where $\rho_i$ denotes the unique $i$-dimensional irreducible representation of $SU(2)$ over $\mathbf{C}$. Since all $\rho_i$ appearing in this decomposition are of the real type, we can have the same decomposition over the real numbers. The statement follows easily.

A straightforward calculation along standard lines gives the following expressions for the null Lagrangian from lemma 4.1:

$$L(X) = -X_{1212} + X_{1234} + X_{1313} + X_{1324} - X_{2323} + X_{2314}. \tag{4.4}$$

We recall that the invariant null Lagrangian is defined only up to a multiplication by a real scalar. In what follows we will use the normalization given by the formula (4.4).

Now we follow the same method used in Section 3 to construct the convex function $f^\epsilon$. First we have the following lemma.

**Lemma 4.2** *For $0 \le \epsilon < \sqrt{7}-2$, $x, y \in S^3$, $L(\nabla v^\epsilon(x)) \equiv 2-\epsilon$, and there exists constant $c_0(\epsilon) > 0$, such that*

$$\nabla L(\nabla v^\epsilon(x)) \cdot (\nabla v^\epsilon(y) - \nabla v^\epsilon(x)) \le -c_0(\epsilon)|\nabla v^\epsilon(x) - \nabla v^\epsilon(y)|^2. \tag{4.5}$$

*Proof:* When $r = 1$,

$$\begin{aligned}
X_{1212} &= -(x_3^2+x_4^2) + (1+\epsilon)(x_1^2+x_2^2)(x_3^2+x_4^2),\\
X_{1234} &= x_1^2+x_2^2 - (1+\epsilon)(x_1^2+x_2^2)(x_3^2+x_4^2),\\
X_{1313} &= x_3^2+x_1^2 - (x_3^2-x_1^2)\frac{(x_3^2+x_4^2-x_1^2-x_2^2)(1+\epsilon)}{2} - 2(1+\epsilon)x_3x_1(x_1x_3+x_2x_4),\\
X_{1324} &= x_4^2+x_2^2 - (x_4^2-x_2^2)\frac{(x_3^2+x_4^2-x_1^2-x_2^2)(1+\epsilon)}{2} - 2(1+\epsilon)x_2x_4(x_4x_2+x_1x_3),\\
X_{2323} &= -x_2^2-x_3^2 + (x_3^2-x_2^2)\frac{(x_3^2+x_4^2-x_1^2-x_2^2)(1+\epsilon)}{2} - 2(1+\epsilon)x_2x_3(x_1x_4-x_3x_2),\\
X_{2314} &= x_4^2+x_1^2 - (x_4^2-x_1^2)\frac{(x_3^2+x_4^2-x_1^2-x_2^2)(1+\epsilon)}{2} - 2(1+\epsilon)x_1x_4(x_4x_1-x_2x_3).
\end{aligned}$$

Then

$$L(X) \equiv 2-\epsilon \qquad \forall X \in K_1^\epsilon. \tag{4.6}$$

Here we use the same notation for the gradient set of $v^\epsilon$ as in section 3.

When $x, y \in S^3$, a straightforward calculation yields

$$\begin{aligned}
L(\nabla v^\epsilon(x) - \nabla v^\epsilon(y)) &= (4-2\epsilon)(1-<x,y>) - (\epsilon+1)^2\frac{<x,y>(1-<x,y>^2)}{2}\\
&\quad +(\epsilon+1)^2\frac{<x,y>}{2}(x_2y_1-x_1y_2-x_3y_4+x_4y_3)^2.
\end{aligned}$$

Write $<x,y>= t$, when $t \ge 0$, we have

$$\begin{aligned}
& L(\nabla v^\epsilon(x) - \nabla v^\epsilon(y))\\
\ge\ & (4-2\epsilon - \frac{(\epsilon+1)^2}{2}<x,y>(1+<x,y>))(1-<x,y>)\\
\ge\ & (3-4\epsilon-\epsilon^2)(1-<x,y>);
\end{aligned}$$

When $t < 0$,

$$\begin{aligned}
& L(\nabla v^\epsilon(x) - \nabla v^\epsilon(y))\\
\ge\ & (4-2\epsilon)(1-<x,y>) + \frac{<x,y>}{2}(\epsilon+1)^2(x_2y_1-x_1y_2-x_3y_4+x_4y_3)^2\\
\ge\ & (4-2\epsilon)(1-<x,y>) - \frac{(\epsilon+1)^2}{2}|x-y|^2\\
\ge\ & (3-4\epsilon-\epsilon^2)(1-<x,y>).
\end{aligned}$$

Hence if we take $0 \le \epsilon < \sqrt{7}-2$, the conclusion follows by the following fact:

$$\begin{aligned}
& -\nabla L(\nabla v^\epsilon(x)) \cdot (\nabla v^\epsilon(y) - \nabla v^\epsilon(x)) = L(\nabla v^\epsilon(x) - \nabla v^\epsilon(y)),\\
& d_1(\epsilon)|y-x|^2 \le |\nabla v^\epsilon(x) - \nabla v^\epsilon(y)|^2 \le d_2(\epsilon)|y-x|^2.
\end{aligned}$$

We then can follow the procedure in Section 3 to finish the construction of $f^\epsilon$.

## 5 An example of nonuniqueness of weak solutions in $W^{1,2-\delta}$

Let $\Omega$ be the unit ball in $\mathbf{R}^3$, we consider $w^\epsilon : \Omega \to \mathbf{R}^3$ given by:

$$w^\epsilon(x) = \frac{x}{|x|^\epsilon}, \qquad \frac{3}{2} < \epsilon < 3. \tag{5.1}$$

Direct calculation shows that for $L(X) = -\mathrm{Tr}\ \mathrm{cof}\,(X)$, we have

$$\begin{aligned} L(X) &= 2\epsilon - 3, \qquad X \in K_1^\epsilon; \\ L(X-Y) &= \frac{\epsilon^2}{2}|X-Y|^2, \qquad X, Y \in K_1^\epsilon. \end{aligned}$$

Where $K_1^\epsilon = \{\nabla w^\epsilon(x), x \in S^2\}$. Then we can follow the same procedure to construct smooth strongly convex $f^\epsilon$, such that $w^\epsilon$ satisfies

$$\mathrm{div}\, \nabla f^\epsilon(\nabla w^\epsilon) = 0. \tag{5.2}$$

in the sense of distributions. On the other hand, we know $u \equiv x$ is the unique $W^{1,2}$ weak solution of (5.2) from general theory. Note that for our choice of $\epsilon$, $w^\epsilon$ is in $W^{1,p}(\Omega, \mathbf{R}^3)$ for $1 < p < 3/\epsilon$, but not in $W^{1,2}(\Omega, \mathbf{R}^3)$, thus gives a counterexample to uniqueness of equations of type (1.2) in $W^{1,p}$ space.

We summarize what we have proved in the following theorem:

**Theorem 1**

*i) Let $u^\epsilon : \mathbf{R}^n \to \mathbf{R}^m$ be given by (3.1), where $m = \frac{n(n+1)}{2} - 1$. Then for $0 \le \epsilon < \frac{n+1-\sqrt{\frac{3(n+1)}{n-1}}}{\sqrt{\frac{3(n+1)}{n-1}}+1}$, there exists a smooth strongly convex function $f^\epsilon : M^{m\times n} \to \mathbf{R}$ such that $|D^2 f^\epsilon| \le c$ in $M^{m\times n}$ and*

$$\mathrm{div}\, \nabla f^\epsilon(\nabla u^\epsilon) = 0 \qquad \text{in} \quad \mathbf{R}^n.$$

*ii) Let $v^\epsilon : \mathbf{R}^4 \to \mathbf{R}^3$ be given by (4.1). For $0 \le \epsilon < \sqrt{7} - 2$, there exists a smooth strongly convex function $f^\epsilon : M^{3\times 4} \to \mathbf{R}$ such that $|D^2 f^\epsilon| \le c$ in $M^{3\times 4}$ and*

$$\mathrm{div}\, \nabla f^\epsilon(\nabla v^\epsilon) = 0 \qquad \text{in} \quad \mathbf{R}^4.$$

*iii) Let $w^\epsilon : \mathbf{R}^3 \to \mathbf{R}^3$ be given by (5.1). For $\frac{3}{2} < \epsilon < 3$, there exists a smooth strongly convex function $f^\epsilon : M^{3\times 3} \to \mathbf{R}$ such that $|D^2 f^\epsilon| \le c$ in $M^{3\times 3}$ and*

$$\mathrm{div}\, \nabla f^\epsilon(\nabla w^\epsilon) = 0 \qquad \text{in} \quad \mathbf{R}^3.$$

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