

Head-Media Interaction in Magnetic Recording

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Abstract

The tape-head interaction in magnetic recording is modelled by a coupled system of a second order differential equation for the pressure and a fourth-order differential equation for the tape deflection. There is also the constraint that the spacing between the head and tape remains positive. In this paper, we study the stationary one-dimensional case and establish the existence of a smooth solution.

1 The Model

Figure 1 shows the magnetic tape given by $\hat{y} = \hat{u}(\hat{x})$, $0 < \hat{x} < \hat{L}$, and the magnetic head profile $\hat{y} = \hat{\delta}(\hat{x})$, $\hat{L}_1 \leq \hat{x} \leq \hat{L}_2$. The spacing between the head and the tape is denoted by $\hat{h}(\hat{x})$, i.e.,

$$\hat{h}(\hat{x}) = \hat{u}(\hat{x}) - \hat{\delta}(\hat{x}), \quad \hat{h}(\hat{x}) > 0, \quad \hat{L}_1 \leq \hat{x} \leq \hat{L}_2.$$

The tape is driven with velocity V , and its motion entrains air in the space between the head and the tape, with pressure $\hat{p}(\hat{x})$, $\hat{L}_1 \leq \hat{x} \leq \hat{L}_2$. At the endpoints $\hat{x} = \hat{L}_1$, $\hat{x} = \hat{L}_2$ the pressure is equal to the atmospheric pressure.

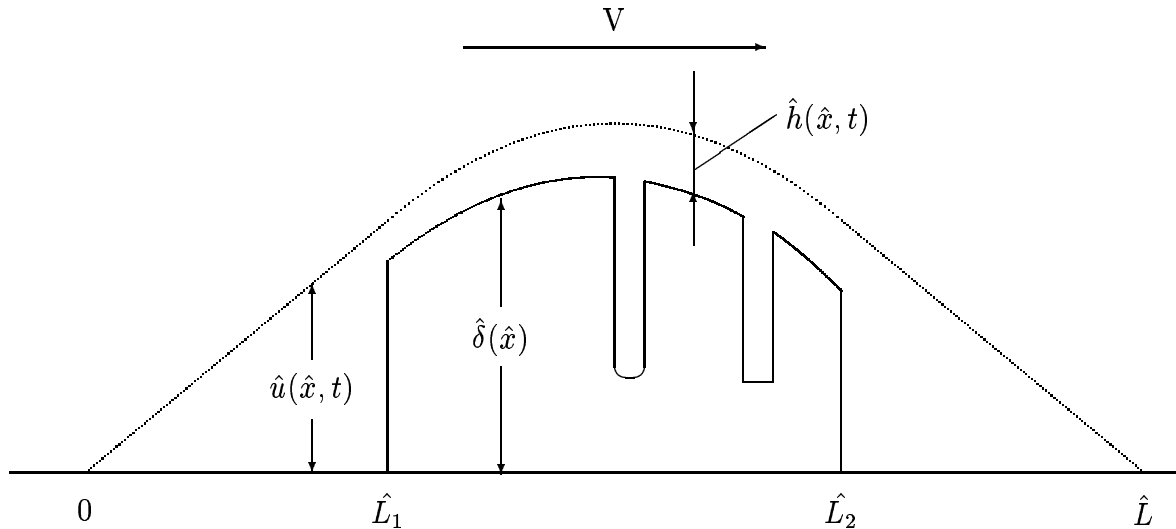


Figure 1

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After nondimensionalization one obtains the following system for the tape $y = u(x)$ and the pressure $p(x)$ [1][3;Chap.6]:

$$(1.1) \quad \frac{\partial(ph)}{\partial x} - \epsilon \frac{\partial}{\partial x} (\alpha h^2 \frac{\partial p}{\partial x} + \beta h^3 p \frac{\partial p}{\partial x}) = 0, \quad L_1 < x < L_2,$$

$$(1.2) \quad -\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^4 u}{\partial x^4} = K(p-1)\chi_{[L_1, L_2]}, \quad 0 < x < L,$$

$$(1.3) \quad u(x) = h(x) + \delta(x), \quad h(x) > 0 \text{ if } L_1 \leq x \leq L_2$$

where $0 < L_1 < L_2 < L$, $\chi_{[L_1, L_2]}$ is the characteristic function of the interval $[L_1, L_2]$, and, typically,

$$\alpha \sim \frac{1}{10}, \quad \beta \sim 1, \quad L_2 - L_1 \sim 1, \quad L \sim 10,$$

$$K \sim 10^4, \quad \epsilon \sim 10^{-2}, \quad \mu \sim 10^{-3};$$

L_1 and L_2 lie near of the middle of the interval $(0, L)$.

The boundary conditions are

$$(1.4) \quad p(L_1) = p(L_2) = 1,$$

$$(1.5) \quad u = \frac{\partial u}{\partial x} = 0 \text{ at } x = L_1 \text{ and } x = L_2.$$

We assume throughout the paper that

$$(1.6) \quad \delta \text{ is piecewise continuous with jump discontinuous at } \xi_1, \xi_2, \dots, \xi_s \text{ where } \xi_0 = L_1 < \xi_1 < \dots < \xi_s < L_2 = \xi_{s+1}, \text{ and } \delta \in C^1[\xi_i, \xi_{i+1}] \text{ for } 0 \leq i \leq s,$$

and

$$(1.7) \quad \delta(L_1) < \delta'(L_1)L_1, \quad \delta(L_2) < (L_2 - L)\delta'(L_2);$$

the case where $\delta(x)$ has no discontinuities may be considered as a special case of (1.6) (with $s=0$).

Note that the inequality $\delta(L_1) < \delta'(L_1)L_1$ means that the tangent to the head at $x = L_1$ intersects the x -axis in the interval $(0, L_1)$. Similarly, the second inequality in (1.7) means that the tangent to the head at $x = L_2$ intersects the x -axis in the interval (L_2, L) .

Note also that ϵ and μ are small numbers. In this paper we prove:

Theorem 1.1 *Assume that (1.6) and (1.7) are satisfied. Then there exist positive constants ϵ_* , μ_* such that if $0 < \epsilon < \epsilon_*$, $0 < \mu < \mu_*$, then the system (1.1)-(1.5) has a solution with p in $W^{1,\infty}[L_1, L_2]$, $u \in W^{4,\infty}[0, L]$ and $p > 0$, $h > 0$ in $[L_1, L_2]$.*

Theorem 1.1 was proved in [4] under the assumption that

$$(1.8) \quad \delta \in C^2 \text{ and } \delta''(x) < 0, \quad L_1 \leq x \leq L_2.$$

This assumption is very restrictive, not only mathematically, but also physically: Magnetic heads do not generally satisfy the concavity condition (1.8). Indeed, in order to reduce the effect of air entrainment (which causes boundary layer for the pressure p near $x = L_2$), trenches are dug into the head (see [2]) and, of course, $\delta(x)$ is discontinuous at the edges of the trench. But even if a trench is smoothed near the edges so that δ is a smooth function in a neighbourhood of trench, $\delta''(x)$ will change sign across the trench.

For clarity we shall first prove Theorem 1.1 replacing (1.6) by the stronger assumption:

$$(1.9) \quad \delta \in C^2[L_1, L_2];$$

the proof for this special case is given in Sections 2-4. In Section 2 we establish the existence of a solution in the case $\epsilon = \mu = 0$ and in Section 3 we prove that the problem for $\epsilon = \mu = 0$ can be written as a variational inequality. The approach we use to establish these results is entirely different from the approach in [4]; instead of the shooting method used in [4] we use here a method based in sub- and super-solutions. In Section 4 we prove Theorem 1.1 (under the stronger assumption (1.9)) by combining the method used in [4] with the results of Section 2 and 3. The proof of Theorem 1.1 in the general case is given in Section 5.

2 The case $\epsilon = 0, \mu = 0$

In the special case $\epsilon = \mu = 0$, the system (1.1)-(1.3) reduces to

$$(2.1) \quad \frac{\partial(ph)}{\partial x} = 0, \quad L_1 < x < L_2,$$

$$(2.2) \quad -\frac{\partial^2 u}{\partial x^2} = K(p-1)\chi_{[L_1, L_2]}, \quad 0 < x < L.$$

Some of the boundary conditions in (1.4), (1.5) need to be dropped, and we take

$$(2.3) \quad p(L_1) = 1,$$

$$(2.4) \quad u(0) = u(L) = 0.$$

From (2.1) we see that $ph = \text{constant} = C$ and, since $p(L_1) = 1$, $C = h(L_1) = u(L_1) - \delta(L_1)$, so that

$$p(x) = \frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)}.$$

Hence (2.2) becomes

$$(2.5) \quad -\frac{\partial^2 u}{\partial x^2} = K\left(\frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1\right)\chi_{[L_1, L_2]}, \quad 0 < x < L.$$

Theorem 2.1 *There exists a solution $u(x)$ of (2.5),(2.4).*

The proof requires several Lemmas. Let

$$A = \|\delta'\|_{L^\infty(L_1, L_2)} \quad (A > 0), \quad B = \sup_{L_1 \leq x \leq L_2} [-\delta''(x)] \quad (B > 0).$$

For any

$$\lambda \in I \equiv (\delta(L_1), AL_1],$$

consider the problem

$$(2.6) \quad \begin{aligned} -\frac{\partial^2 u}{\partial x^2} &= K \left(\frac{\lambda - \delta(L_1)}{u(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]}, \quad 0 < x < L, \\ u(0) &= u(L) = 0, \\ u(x) - \delta(x) &> 0 \quad \text{if } L_1 \leq x \leq L_2. \end{aligned}$$

Lemma 2.1 *For any $\lambda \in I$ there exists a unique solution $u(x, \lambda)$ of (2.6).*

PROOF. Set

$$F(u, x) = K \left(\frac{\lambda - \delta(L_1)}{u - \delta(x)} - 1 \right) \chi_{[L_1, L_2]}, \quad u > \delta(x)$$

and note that

$$(2.7) \quad F_u < 0.$$

Consider the function

$$\underline{u}(x) = \bar{\delta}(x) + \frac{\lambda - \delta(L_1)}{1 + c(\lambda)}$$

where

$$\bar{\delta}(x) = \begin{cases} \delta(L_1) + \delta'(L_1)(x - L_1), & 0 \leq x < L_1 \\ \delta(x), & L_1 \leq x \leq L_2 \\ \delta(L_2) + \delta'(L_2)(x - L_2), & L_2 < x \leq L, \end{cases}$$

$$c(\lambda) = \max \left\{ \frac{B}{K}, \frac{\lambda - \delta(L_1)}{-\bar{\delta}(0)}, \frac{\lambda - \delta(L_1)}{-\bar{\delta}(L)} \right\}.$$

Then

$$\begin{aligned} -\underline{u}_{xx} - F(\underline{u}(x), x) &\leq \left\{ B - K \left(\frac{\lambda - \delta(L_1)}{\left(\delta(x) + \frac{\lambda - \delta(L_1)}{1 + c(\lambda)} \right) - \delta(x)} - 1 \right) \right\} \chi_{[L_1, L_2]} = \\ &= [B - Kc(\lambda)] \chi_{[L_1, L_2]} \end{aligned}$$

so that, by the choice of $c(\lambda)$,

$$-\underline{u}_{xx}(x) \leq F(\underline{u}(x), x).$$

Furthermore,

$$\underline{u}(0) = \delta(L_1) - \delta'(L_1)L_1 + \frac{\lambda - \delta(L_1)}{1 + c(\lambda)} < 0,$$

and, similarly, $\underline{u}(L) < 0$. Thus \underline{u} is a subsolution for the problem (2,6).

The function $\bar{u}(x) = Ax$ is a supersolution. Indeed, since $(\bar{u} - \delta)' = A - \delta' \geq 0$,

$$\bar{u}(x) - \delta(x) \geq \bar{u}(L_1) - \delta(L_1) = AL_1 - \delta(L_1) \geq \lambda - \delta(L_1)$$

if $L_1 \leq x \leq L_2$, so that

$$-\bar{u}_{xx}(x) - K\left(\frac{\lambda - \delta(L_1)}{\bar{u}(x) - \delta(x)} - 1\right)\chi_{[L_1, L_2]} \geq -\bar{u}_{xx}(x) = 0.$$

Furthermore, $\bar{u}(0) = 0$ and $\bar{u}(L) \geq 0$.

Introduce the convex set of functions

$$G = \{\tilde{u} \in C^0[0, L], \underline{u}(x) \leq \tilde{u} \leq \bar{u}\}.$$

For any $\tilde{u} \in G$ we consider the problem

$$-u_{xx} = F(\tilde{u}, x), \quad 0 < x < L,$$

$$u(0) = u(L) = 0.$$

Using (2.7) and the maximum principle (or comparison) we deduce that the solution u satisfies

$$\underline{u} \leq u \leq \bar{u}, \quad 0 \leq x \leq L.$$

If we define a mapping T by $T(\tilde{u}) = u$, then T maps G into itself. It is easily seen that T is continuous and $T(G)$ lies in a compact subset of G . Appealing to the Schauder fixed point theorem we conclude that T has a fixed point, which is clearly the solution to (2.6). Finally, if u_1 is another solution, then the function $w = u - u_1$ satisfies

$$w_{xx} + F_u w = 0$$

where F_u is evaluated at some intermediate point. Since $F_u < 0$, $w \equiv 0$ and thus $u_1 = u$. \square

We denote the solution of (2.6) by $u(x, \lambda)$ and introduce the function

$$(2.8) \quad f(\lambda) = u(L_1, \lambda), \quad \lambda \in I.$$

Lemma 2.2 *The function f is continuous.*

PROOF. If $\lambda_n \rightarrow \lambda_0 \in I$ then any subsequence of λ_n has a sub-subsequence $\lambda_{n'}$ for which $u(x, \lambda_{n'})$ is uniformly convergent to a function $u_0(x)$, and $u_0(x)$ is the solution of (2.6) for $\lambda = \lambda_0$ (by uniqueness). It follows that

$$f(\lambda_n) = u(L_1, \lambda_n) \rightarrow u(L_1, \lambda_0) = f(\lambda_0). \quad \square$$

PROOF OF THEOREM 2.1. We need to show that the mapping $\lambda \rightarrow f(\lambda)$ has a fixed point. If $f(\lambda_1) \geq \lambda_1$ for some $\lambda_1 \in I$, then, since $f(AL_1) \leq AL_1$ and $f(\lambda)$ is continuous, f will have a fixed point in the interval $[\lambda_1, AL_1]$. Thus it remains to prove that the inequality

$$(2.9) \quad f(\lambda) < \lambda \text{ for all } \lambda \in I$$

cannot hold. We shall assume that (2.9) holds and proceed to derive the contradiction. To do that we define

$$\lambda_{n+1} = f(\lambda_n), \quad n > 1$$

for some $\lambda_0 \in I$. Then, in view of (2.9), $\lambda_n \downarrow \lambda_*$, and $\lambda_* = \delta(L_1)$ since, otherwise, $f(\lambda_*) = \lambda_*$ by Lemma 2.2. Thus

$$(2.10) \quad \lambda_n \downarrow \delta(L_1) \text{ as } n \rightarrow \infty.$$

Recall that $u_{xx}(x, \lambda_n) = 0$ if $0 < x < L_1$, so that

$$u_x(L_1, \lambda_n) = \frac{u(L_1, \lambda_n)}{L_1} = \frac{\lambda_{n+1}}{L_1} \rightarrow \frac{\delta(L_1)}{L_1}.$$

Since, by (1.7),

$$\delta'(L_1) > \frac{\delta(L_1)}{L_1} + 3\epsilon_0$$

for some $\epsilon_0 > 0$, it follows that

$$(2.11) \quad u_x(L_1, \lambda_n) < \delta'(L_1) - 2\epsilon_0$$

if n is sufficiently large. We also have

$$(2.12) \quad u_{xx}(L_1 + 0, \lambda_n) = K \left(1 - \frac{\lambda_n - \delta(L_1)}{\lambda_{n+1} - \delta(L_1)}\right) < 0$$

since $\lambda_{n+1} = f(\lambda_n) < \lambda_n$.

Let

$$(2.13) \quad \bar{x} = \max\{x \in (L_1, L_2]; \delta'(L_1) - \delta'(x') \leq \epsilon_0, \text{ for all } L_1 \leq x' \leq x\}.$$

We claim that

$$(2.14) \quad u_{xx}(x, \lambda_n) < 0 \text{ if } L_1 \leq x < \bar{x}.$$

Indeed, if this is not true then, setting for simplicity $u(x) = u(x, \lambda_n)$, we have, by (2.12),

$$(2.15) \quad \begin{aligned} u_{xx}(x) &< 0 \text{ if } L_1 \leq x < x^*, \\ u_{xx}(x^*) &= 0 \end{aligned}$$

for some $x^* = x_n^* \in (L_1, \bar{x})$, and then also

$$u_{xxx}(x^*) \geq 0.$$

From the differential equation in (2.6) we get

$$(2.16) \quad u_{xxx}(x) = K \frac{\lambda - \delta(L_1)}{(u(x) - \delta(x))^2} (u_x(x) - \delta'(x)) \text{ in } (L_1, L_2),$$

so that, by (2.11),

$$u_{xxx}(L_1 + 0) < 0.$$

It follows that there exists a point $x^{**} = x_n^{**}$ in $(L_1, x^*]$ such that

$$u_{xxx}(x) < 0 \text{ if } L_1 < x < x^{**},$$

$$u_{xxx}(x^{**}) = 0.$$

Appealing again to (2.16) we deduce that

$$u_x(x^{**}) - \delta'(x^{**}) = 0$$

and then by (2.15) and (2.11)

$$\delta'(x^{**}) = u_x(x^{**}) < u_x(L_1) < \delta'(L_1) - 2\epsilon_0$$

which is a contradiction to (2.13) since $x^{**} < \bar{x}$.

From (2.14) and (2.11),(2.13) we deduce that

$$u_x(x, \lambda_n) < u_x(L_1, \lambda_n) < \delta'(L_1) - 2\epsilon_0 \leq \delta'(x) - \epsilon_0 \text{ if } L_1 < x \leq \bar{x}.$$

Hence

$$u(\bar{x}, \lambda_n) - \delta(\bar{x}) < \lambda_{n+1} - \delta(L_1) - \epsilon_0(\bar{x} - L_1).$$

Recalling (2.10) and the fact that \bar{x} is independent of n , we get

$$u(\bar{x}, \lambda_n) - \delta(\bar{x}) < 0$$

if n is sufficiently large, which contradicts the inequality $u(x, \lambda_n) > \delta(x)$ in $[L_1, L_2]$. \square

3 Variational formulation for (2.5), (2.4)

Theorem 2.1 can be extended to the system

$$(3.1) \quad \begin{aligned} -\frac{\partial^2 g}{\partial x^2} &= K \left(\frac{g(L_1) - \delta(L_1)}{g(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]}, \quad 0 < x < L, \\ u(0) &= u(L) = -\gamma, \\ g(x) - \delta(x) &> 0 && \text{if } L_1 \leq x \leq L_2, \end{aligned}$$

provided

$$(3.2) \quad \max\{\delta(L_1) - \delta'(L_1)L_1, \delta(L_2) - \delta'(L_2)(L_2 - L)\} < -\gamma < 0.$$

Note that the construction of the solution to

$$(3.3) \quad \begin{aligned} -\frac{\partial^2 g}{\partial x^2} &= K\left(\frac{\lambda - \delta(L_1)}{g(x) - \delta(x)} - 1\right)\chi_{[L_1, L_2]}, \quad 0 < x < L, \\ u(0) &= u(L) = -\gamma, \\ g(x) - \delta(x) &> 0 \quad \text{if } L_1 \leq x \leq L_2, \end{aligned}$$

we use the same sub- and super-solutions $\underline{u}(x)$ and $\bar{u}(x)$, respectively, as before. We shall denote the solution of (3.3) by $g(x, \lambda, \gamma)$.

Set

$$F(u, x, \lambda) = K\left(\frac{\lambda - \delta(L_1)}{u - \delta(x)} - 1\right)\chi_{[L_1, L_2]}, \quad u > \delta(x).$$

Clearly

$$F_u < 0, \quad F_\lambda > 0.$$

By the strong maximum principle we then get

$$(3.4) \quad g(x, \lambda_1, \gamma_1) > g(x, \lambda_2, \gamma_2) \text{ in } (0, L) \text{ if } \lambda_1 \geq \lambda_2, \quad 0 \leq \gamma_1 < \gamma_2.$$

Lemma 3.1 *Let $g(x)$ the any solution of (3.1). Then there exists a solution $u(x)$ of (2.5), (2.4) such that*

$$(3.5) \quad u(x) > g(x) \text{ if } 0 < x < L$$

and, consequently,

$$(3.6) \quad u(x) \geq g(x) + \epsilon_0, \quad \text{if } L_1 \leq x \leq L_2.$$

for some $\epsilon_0 > 0$.

PROOF. By (3.4) and the maximum principle,

$$(3.7) \quad u(x, \lambda) > g(x, \lambda, \gamma) \text{ if } 0 < x < L.$$

Take $\lambda = \lambda_0$ such that $g(x, \lambda_0, \gamma)$ is the solution $g(x)$. Then

$$u(L_1, \lambda_0) > g(L_1, \lambda_0, \gamma) = \lambda_0,$$

so that $f(\lambda_0) > \lambda_0$. It follows that there is a fixed point λ_* of the mapping $\lambda \rightarrow f(\lambda)$ in the interval $(\lambda_0, AL_1]$. The function $u(x) = u(x, \lambda_*)$ is the solution of (2.5), (2.4) and by (3.4), (3.7)

$$u(x) = u(x, \lambda_*) \geq u(x, \lambda_0) > g(x) \text{ in } (0, L). \quad \square$$

Introduce the function

$$\psi(s) = \begin{cases} s & s < 1 + AL_1 \\ 1 + AL_1 & s \geq 1 + AL_1. \end{cases}$$

Take any solution $g(x)$ of (3.1) and consider the variational inequality

$$(3.8) \quad \begin{aligned} -\frac{\partial^2 u}{\partial x^2} &\geq K\left(\frac{\psi(u(L_1)) - \delta(L_1)}{u(x) - \delta(x)} - 1\right)\chi_{[L_1, L_2]}, \\ u(x) &\geq g(x), \\ (u - g)\left[\frac{\partial^2 u}{\partial x^2} + K\left(\frac{\psi(u(L_1)) - \delta(L_1)}{u(x) - \delta(x)} - 1\right)\chi_{[L_1, L_2]}\right] &= 0 \quad \text{in } [0, L], \\ u(0) = u(L) &= 0. \end{aligned}$$

The truncation ψ is introduced for a technical reason, so that we can carry out the proof of Lemma 4.1 in Section 4; see also [4].

Note that the solution u of (2.5), (2.4) established in Lemma 3.1 satisfies the variational inequality (3.8) since $\psi(u(L_1)) = u(L_1)$ (as $u(L_1) \leq AL_1$). We now prove the converse:

Theorem 3.1 *Any solutions $u(x)$ of the variational inequality (3.8) is a solution of (2.5), (2.4), and it satisfies the inequalities (3.5), (3.6).*

PROOF. Since $u(L_1) \geq g(L_1)$, $\psi(s)$ is monotone increasing in s , and $g(L_1) \leq AL_1$,

$$\psi(u(L_1)) - \delta(L_1) \geq \psi(g(L_1)) - \delta(L_1) = g(L_1) - \delta(L_1).$$

It follows that

$$-\frac{\partial^2 u}{\partial x^2} \geq K\left(\frac{g(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1\right)\chi_{[L_1, L_2]}.$$

Since, furthermore,

$$u(0) = u(L) > -\gamma = g(0) = g(L),$$

the strong maximum principle yields the inequalities (3.5), (3.6). It follows that

$$(3.9) \quad -\frac{\partial^2 u}{\partial x^2} = K\left(\frac{\psi(u(L_1)) - \delta(L_1)}{u(x) - \delta(x)} - 1\right)\chi_{[L_1, L_2]}.$$

From the inequality $\psi(s) \leq s$ we then have that

$$-\frac{\partial^2 u}{\partial x^2} \leq K\left(\frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1\right)\chi_{[L_1, L_2]},$$

and, since $\bar{u}(x) = Ax$ is a supersolution, we get, by comparison, $u(x) \leq Ax$. This implies that $\psi(u(L_1)) = u(L_1)$, so that (3.9) reduces to (2.5). Thus $u(x)$ is a solution of (2.5), (2.4) satisfying (3.5), (3.6). \square

4 Proof of Theorem 1.1 (special case)

Consider the system

$$(4.1) \quad \frac{\partial(ph)}{\partial x} - \epsilon \frac{\partial}{\partial x} (\alpha h^2 \frac{\partial p}{\partial x} + \beta h^3 p \frac{\partial p}{\partial x}) = 0, \quad L_1 < x < L_2,$$

$$(4.2) \quad -\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^4 u}{\partial x^4} \geq K(p-1)\chi_{[L_1, L_2]}, \quad u(x) \geq g(x),$$

$$(u-g)[-\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^4 u}{\partial x^4} - K(p-1)\chi_{[L_1, L_2]}] = 0 \quad \text{in } [0, L],$$

$$(4.3) \quad h(x) = u(x) - \delta(x), \quad L_1 \leq x \leq L_2$$

with the boundary conditions

$$(4.4) \quad p(L_1) = p(L_2) = \frac{\psi(h(L_1))}{h(L_1)},$$

$$(4.5) \quad u = u_x = 0 \text{ at } x = 0 \text{ and } x = L.$$

Let

$$(4.6) \quad G = \{p \in C^0[L_1, L_2], 0 \leq p \leq 1 + \frac{1 + AL_1}{l}\}$$

where

$$l = \inf_{L_1 \leq x \leq L_2} [g(x) - \delta(x)] > 0.$$

As in [4] one shows that for any $p \in G$ there exists a unique solution u of (4.2),(4.5) and

$$|u_x| \leq C$$

where C is a constant independent of p

Set $h = u - \delta$ and denote by $\hat{p}(x)$ the solution of (4.1),(4.4). It is obtained by solving the initial value problem

$$(4.7) \quad \hat{p}h - \epsilon(\alpha h^2 \frac{\partial \hat{p}}{\partial x} + \beta h^3 \hat{p} \frac{\partial \hat{p}}{\partial x}) = \eta, \quad L_1 < x < L_2,$$

$$\hat{p}(L_1) = \frac{\psi(h(L_1))}{h(L_1)}$$

for any constant $\eta \geq 0$, and then choosing η in the unique way such that

$$(4.8) \quad \hat{p}(L_2) = \frac{\psi(h(L_1))}{h(L_1)};$$

for details see [4].

We define the mapping S by

$$S(p) = \hat{p}$$

and wish to show that S has a fixed point in G .

Lemma 4.1 *If ϵ is small enough then S maps G into itself and the parameter η determined by (4.7),(4.8) satisfies:*

$$(4.9) \quad |\eta - \psi(h(L_1))| \leq C\epsilon$$

where C is a constant independent of ϵ .

PROOF. We introduce the operator

$$\mathcal{L}(w) \equiv \frac{\partial(wh)}{\partial x} - \epsilon \frac{\partial}{\partial x} (\alpha h^2 \frac{\partial w}{\partial x} + \beta h^3 w \frac{\partial w}{\partial x}), \quad L_1 < x < L_2$$

and consider the function

$$v(x) = \frac{\psi(h(L_1))}{h(L_1)} - \frac{C}{l}(x - L_1) - \nu, \quad L_1 \leq x \leq L_2$$

for C positive and large and ν positive and arbitrarily small. Denote by \hat{x} the point where $v(\hat{x}) = 0$. Then, as in [4],

$$\mathcal{L}(v) < 0 \text{ in } [L_1, \hat{x}]$$

if ϵ is small enough. A comparison argument used in [4] then shows that

$$\hat{p}(x) \geq v(x) \text{ in } [L_1, \hat{x}].$$

Taking $\nu \downarrow 0$ we deduce that

$$p_x(L_1) > -\frac{C}{l}$$

and, then, by (4.7),

$$\eta - \psi(h(L_1)) \leq C\epsilon.$$

The complementary inequality

$$\eta - \psi(h(L_1)) \geq -C\epsilon.$$

is derived in the same way by working with the function

$$\frac{\psi(h(L_1))}{h(L_1)} + \frac{C}{l}(x - L_1) + \nu. \quad \square$$

Since

$$\psi(h(L_1)) \leq 1 + AL_1,$$

it follows from (4.9) that

$$\eta \leq 1 + AL_1 + C\epsilon.$$

(This is where we need the truncation ψ , since we do not have a uniform bound on $h(L_1)$.) But, then, from (4.7) we conclude that

$$\hat{p} \leq 1 + \frac{1 + AL_1}{l},$$

if ϵ is sufficient small.

From (4.7) we also deduce that \hat{p} cannot take negative minimum, and thus $\hat{p} \in G$ and S maps G into itself. \square

PROOF OF THEOREM 1.1. By Lemma 4.1, S maps G into itself. It is also easy to show that S is continuous and $S(G)$ is contained in a compact subset of G . Hence, by Schauder's fixed point theorem, S has a fixed point in G , which is solution (u, p) of the system (4.1)-(4.5). If we can prove that

$$(4.10) \quad u(x) > g(x) \text{ in } (0, L),$$

$$(4.11) \quad u(L_1) < 1 + AL_1$$

for $0 < \epsilon < \epsilon_*$ and $0 < \mu < \mu_*$, then (u, p) and h form a solution of (1.1)-(1.5) as asserted in Theorem 1.1.

But if either (4.10) or (4.11) is not satisfied, then, by taking sequences $\epsilon_j \downarrow 0, \mu_j \downarrow 0$ as in [4], we obtain limit functions (u_0, p_0, h_0) with

$$p_0(x)h_0 = \frac{\psi(h_0(L_1))}{h_0(L_1)} \text{ (using (4.9))}$$

and $u_0(x)$ satisfying the variational inequality (3.8). By Theorem 3.1,

$$u_0(x) \geq g(x) + \epsilon_0 \text{ for } L_1 \leq x \leq L_2$$

and clearly also

$$u_0(x) \leq AL_1.$$

But then, for the solution of (4.1)-(4.5), with $\epsilon = \epsilon_j, \mu = \mu_j$ small,

$$(4.12) \quad u(x) \geq g(x) + \frac{1}{2}\epsilon_0 \text{ for } L_1 \leq x \leq L_2,$$

and $u(L_1) < 1 + AL_1$. Since (4.12) implies (4.10), we get a contradiction to both inequalities (4.10) and (4.11). \square

5 Proof of Theorem 1.1

We introduce the function

$$(5.1) \quad \hat{\delta}(x) = \delta(L_1) + ((\xi_1 - L_1)\|\delta'\|_{L^\infty[L_1, \xi_1]} + \|\delta\|_{L^\infty[\xi_1, L_2]}) \frac{x - L_1}{\xi_1 - L_1}$$

for $0 \leq x \leq L$ and note that

$$(5.2) \quad \delta(x) \leq \hat{\delta}(x) \text{ if } L_1 \leq x \leq L_2.$$

We first prove Theorem 2.1 (under the assumption (1.6)). Since $-\delta''$ is not bounded from above (if $s \geq 1$ in (1.6)), we cannot construct a subsolution as before. We therefore first approximate $\delta(x)$ by functions δ_n in $C^2[L_1, L_2]$ such that

$$(5.3) \quad \int_{L_1}^{L_2} |\delta'_n(x) - \delta'(x)|^p dx \longrightarrow 0 \text{ for any } 1 < p < \infty,$$

$$(5.4) \quad \delta_n(L_1) = \delta(L_1),$$

$$(5.5) \quad \sup_{\substack{L_1 \leq x \leq L_1 + \eta_0 \\ L_2 - \eta_0 \leq x \leq L_2}} |\delta'_n(x) - \delta'(x)| \longrightarrow 0$$

for some small $\eta_0 > 0$,

$$(5.6) \quad \delta'_n(x) \longrightarrow \delta'(x) \text{ for all } x \neq \xi_i$$

as $n \longrightarrow \infty$, and

$$(5.7) \quad \delta_n(x) \leq \hat{\delta}(x).$$

Consider the problem

$$(5.8) \quad \begin{aligned} -v'' &= K \left(\frac{\lambda - \delta(L_1)}{v(x) - \delta_n(x)} - 1 \right) \chi_{[L_1, L_2]}, \quad 0 < x < L, \\ v(0) &= v(L) = 0. \end{aligned}$$

Lemma 5.1 *The function $\bar{u}(x) = \hat{\delta}(x) - \hat{\delta}(0)$ is a supersolution of (5.8) provided $\lambda \leq \bar{u}(L_1)$.*

PROOF. Clearly

$$\bar{u}(x) - \delta_n(x) = \hat{\delta}(x) - \delta_n(x) - \hat{\delta}(0) \geq -\hat{\delta}(0), \text{ by (5.7).}$$

Also

$$\hat{\delta}(L_1) - \delta(0) = \bar{u}(L_1) \geq \lambda$$

so that

$$-\hat{\delta}(0) \geq \lambda - \hat{\delta}(L_1) = \lambda - \delta(L_1).$$

Hence

$$\bar{u}(x) - \delta_n(x) \geq \lambda - \delta(L_1)$$

and, consequently,

$$-\bar{u}_{xx}(x) - K \left(\frac{\lambda - \delta(L_1)}{\bar{u}(x) - \delta_n(x)} - 1 \right) \chi_{[L_1, L_2]} \geq -\bar{u}_{xx}(x) = 0.$$

Since also $\bar{u}(0) = 0$ and $\bar{u}(L) > 0$, the lemma follows. \square

It seems difficult to construct a subsolution to the u_n which is independent of n . But we can nevertheless apply Theorem 2.1 (for the case where (1.9) is satisfied) to deduce that there exists a solution $u_n(x)$ of

$$(5.9) \quad \begin{aligned} -u_n''(x) &= K \left(\frac{u_n(L_1) - \delta(L_1)}{u_n(x) - \delta_n(x)} - 1 \right) \chi_{[L_1, L_2]}, \quad 0 < x < L, \\ u_n(0) &= u_n(L) = 0, \end{aligned}$$

and

$$(5.10) \quad \delta_n(x) < u_n(x) \leq \bar{u}(x), \quad 0 < x < L.$$

We may assume that

$$(5.11) \quad u_n(L_1) \longrightarrow \lambda_*$$

for some $\delta(L_1) \leq \lambda_* \leq \bar{u}(L_1)$.

Lemma 5.2 *There holds:*

$$(5.12) \quad \lambda_* > \delta(L_1).$$

PROOF. Suppose $\lambda_* = \delta(L_1)$. Then, for any small $\epsilon_0 > 0$,

$$u_n(L_1) < \delta(L_1) + L_1 \epsilon_0 \quad \text{if } n \geq n_0(\epsilon_0),$$

and

$$(5.13) \quad u_n'(L_1) = \frac{u_n(L_1)}{L_1} < \frac{\delta(L_1)}{L_1} + \epsilon_0 < \delta'(L_1) - 2\epsilon_0 < \delta_n'(L_1) - \epsilon_0$$

if ϵ_0 is small enough, by (1.7), (5.5).

We can now argue as in the proof of (2.14) (with $u = u_n$, $\delta = \delta_n$ and $\lambda_n = u_n(L_1)$) to deduce that

$$u_n''(x) < 0 \quad \text{if } L_1 \leq x \leq \bar{x}$$

where \bar{x} is such that $\bar{x} \leq \eta_0$ and $\delta_n'(L_1) - \delta_n'(x) < \frac{1}{2}\epsilon_0$ for all $L_1 \leq x \leq \bar{x}$, $n \geq n_0(\epsilon_0)$. Using also (5.13), we deduce that

$$u_n'(x) < \delta_n'(x) - \frac{\epsilon_0}{2} \quad \text{if } L_1 \leq x \leq \bar{x}.$$

It follows that

$$u_n(\bar{x}) - \delta_n(\bar{x}) < (u_n(L_1) - \delta_n(L_1)) - \frac{\epsilon_0}{2}(\bar{x} - L_1).$$

Hence $u_n(\bar{x}) - \delta_n(\bar{x}) < 0$ if n is sufficiently large, a contradiction. \square

Since $u_n''(x) = 0$ and $0 < u_n(x) \leq \bar{u}(x)$ for $0 < x < L_1$ and $L_2 < x < L$, we have

$$(5.14) \quad 0 \leq u_n'(0) \leq \bar{u}'(0), \quad \text{and } 0 > u_n'(L) = \frac{u_n(L_2)}{L_2 - L} \geq \frac{\bar{u}(L_2)}{L_2 - L}.$$

Integrating the differential equation in (5.9) and using (5.14) we obtain the inequality

$$\int_{L_1}^{L_2} \frac{u_n(L_1) - \delta(L_1)}{u_n(x) - \delta_n(x)} dx \leq C.$$

In view of (5.10) and (5.11), (5.12), the last inequality implies that

$$(5.15) \quad \int_{L_1}^{L_2} \frac{dx}{u_n(x) - \delta_n(x)} \leq C.$$

Integrating (5.9) and using (5.15), (5.14) we see that

$$|u_n'(x)| \leq C, \quad 0 < x < L.$$

We may then assume that

$$(5.16) \quad u_n(x) \longrightarrow u(x) \text{ uniformly in } x \in [0, L],$$

$$(5.17) \quad |u'(x)| \leq C \text{ a.e.}$$

We next deduce, by Fatou's Lemma and (5.15), that

$$(5.18) \quad \int_{L_1}^{L_2} \frac{dx}{u(x) - \delta(x)} \leq C.$$

Lemma 5.3 *There holds:*

$$(5.19) \quad u(x) - \delta(x) > 0 \text{ if } L_1 \leq x \leq L_2.$$

PROOF. Suppose $u(x_0) - \delta(x_0) = 0$ for some $x_0 \in [L_1, L_2]$. Then, by the Mean Value Theorem,

$$|u(x) - \delta(x)| \leq C|x - x_0|$$

in an interval with endpoint at x_0 , and this contradicts the estimate (5.18). \square

Having proved Lemmas 5.2, 5.3, we can now pass to the limit in (5.9) and conclude that

$$-u'' = K \left(\frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]}$$

for $0 < x < L$, $x \neq \xi_i$, and this completes the proof of Theorem 2.1. Note that $u(x)$ is continuously differentiable in $0 \leq x \leq L$ and $u''(x)$ is piecewise continuous with jump discontinuities at $\xi_1, \xi_2, \dots, \xi_s, \xi_{s+1}$.

In the same way we can proceed to construct a solution g to (3.1).

We next claim that Lemma 3.1 extends to the present case. Indeed, since both $u(x)$ and $g(x)$ are larger than $\delta(x)$ and $u''(x)$ and $g''(x)$ are piecewise continuous, the maximum principle can be applied to deduce (3.7).

The proofs of Theorem 3.1 and of Theorem 1.1 can now proceed exactly as before.

REMARK. As in [4], the solutions asserted in Theorem 1.1 exhibit a boundary layer behavior at $x = L_2$.

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References

1. B.BHUSHAN, *Tribology and Mechanics of Magnetic Storage Devices*, Springer-Verlag, New York, 1990.
2. B.BHUSHAN , *Mechanics and Reliability of Flexible Magnetic Media*, Springer-Verlag, New York, 1992.
3. A.FRIEDMAN, *Mathematics in Industrial Problems*, Part 7, IMA Volume 67, Springer-Verlag, New York, 1994.
4. A.FRIEDMAN & B. HU, *Head-Media interaction in magnetic recording*, Archive Rational Mech. Anal., **140** (1997), 79-101.