Continuous dependence on the nonlinearity of viscosity solutions of parabolic equations

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This paper establishes an upper bound for $u - v$ where u is a subsolution of

$$u_t + F(u, D_x u, D_x^2 u) = 0,$$

and v is a supersolution of

$$v_t + G(v, D_x v, D_x^2 v) = 0,$$

in $(0, \infty) \times \Omega$ with Neumann bondary conditions and where Ω is an open convex set.

1. INTRODUCTION

In this paper we study the difference between a viscosity subsolution of the parabolic equation

$$u_t + F(u, D_x u, D_x^2 u) = 0, \tag{1}$$

and a viscosity supersolution of the equation

$$v_t + G(v, D_x v, D_x^2 v) = 0, \tag{2}$$

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in $(0,T)\times\Omega$ when $u(0,\underline{x})$ and $v(0,\underline{x})$ are given. We take Ω to be an open and convex subset of $\boldsymbol{R}^d$, and Neumann boundary conditions are imposed at the boundary points, if there are any.

The fundamental monotonicity condition is that F and G are nondecreasing in their first and nonincreasing in their third variable. This will, for example, be the case for the equations

$$u_t - \phi(D_x u)\Delta u = 0, \tag{3}$$

and

$$v_t - \gamma(D_x v)\Delta v = 0, \tag{4}$$

provided both ϕ and γ are nonnegative functions. But it turns out that for equations of this form we can obtain better results than those one obtains by considering them to be special cases of (1) and (2). For this reason we shall actually state our result for the equations

$$u_t + f(u, D_x u, D_x^2 u) - \phi(D_x u)\Delta u = 0, \tag{5}$$

and

$$v_t + g(v, D_x v, D_x^2 v) - \gamma(D_x v)\Delta v = 0. \tag{6}$$

Obviously, equations (5) and (6) include, respectively, (1), (3) and (2), (4) as particular cases.

The upper bound for $u-v$ that we get involves the difference between the intial values, a function depending on the moduli of continuity of the initial values and a parameter α, and the supremum of $g - f + 3\alpha d(\sqrt{\phi} - \sqrt{\gamma})^2$ taken over a set depending on the parameter α. Here α is strictly positive but otherwise unrestricted. This allows us, in each particular case but not in the general case, to choose α so as to obtain optimal results.

In its crudest form the result we obtain says that if $f = g$ and, for example, the initial value $u(0,\underline{x})$ satisfies the Hölder condition $|u(0,x) - u(0,y)| \le C|x-y|^\lambda$ for all x, $y \in \overline{\Omega}$, then

$$u(t,x) - v(t,x) \le \sup_{x\in\overline{\Omega}}\big(u(0,x) - v(0,x)\big)^+ + KT^{\frac{\lambda}{2}} \sup_{\boldsymbol{q}\in\boldsymbol{R}^d}\left|\sqrt{\phi(\boldsymbol{q})} - \sqrt{\gamma(\boldsymbol{q})}\right|^\lambda,$$

for all $x \in \overline{\Omega}$ and $t \in [0,T)$, where K is a constant that depends on C, λ, and d (and can be calculated explicitly). Thus we get for example [4, Thm. 5.1], a case not covered by the comparison result given in [3, p. 50].

The proof uses the parabolic version of the Theorem on Sums [3, Thm. 8.3] combined with a technical approach employed earlier in [1]. Another source of inspiration for this paper is [2] where the difference between the solutions of the equations $u_t - \Delta(\phi(u)) = 0$ and $v_t - \Delta(\gamma(u)) = 0$ is estimated. These estimates are given in terms of L^1-norms instead of sup-norms, however.

2. STATEMENT OF RESULTS

We let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\boldsymbol{R}^d$, $\mathcal{S}(d)$ is the space of symmetric real $d \times d$ matrices with order defined by $A \geq B$ if $\langle \boldsymbol{p}, A\boldsymbol{p} \rangle \geq \langle \boldsymbol{p}, B\boldsymbol{p} \rangle$ for every $\boldsymbol{p} \in \boldsymbol{R}^d$, and $(a)^+ \stackrel{\text{def}}{=} \max\{a, 0\}$. Furthermore, $\mathcal{USC}$, $\mathcal{LSC}$, and $\mathcal{C}$ stand for uppper semi-continuous, lower semi-continuous, and continuous, respectively. If Ω is a convex set in $\boldsymbol{R}^d$ with closure $\overline{\Omega}$ and boundary $\partial\Omega$, the set of outward normals is $N_\Omega(x) \stackrel{\text{def}}{=} \{\, \boldsymbol{n} \in \boldsymbol{R}^d \mid |\boldsymbol{n}| = 1, \quad \langle \boldsymbol{n}, x - y \rangle \geq 0, \quad y \in \overline{\Omega}\,\}$ for $x \in \partial\Omega$.

THEOREM 2.1. *Assume that $d \geq 1$, and that*

(i) *$\Omega \subset \boldsymbol{R}^d$ is an open and convex set;*

(ii) *$f \in \mathcal{LSC}(\boldsymbol{R} \times \boldsymbol{R}^d \times \mathcal{S}(d); \boldsymbol{R})$ and $g \in \mathcal{USC}(\boldsymbol{R} \times \boldsymbol{R}^d \times \mathcal{S}(d); \boldsymbol{R})$, and both functions are nondecreasing in their first and nonincreasing in their third argument;*

(iii) *ϕ and $\gamma \in \mathcal{C}(\boldsymbol{R}^d; \boldsymbol{R})$ are nonnegative;*

(iv) *$u \in \mathcal{USC}(\boldsymbol{R}^+ \times \overline{\Omega}; \boldsymbol{R})$, $\lceil u \rceil\!]_T^\infty \stackrel{\text{def}}{=} \sup_{t \in [0,T), x \in \overline{\Omega}} u(t,x) < \infty$ for all $T > 0$, and u is a viscosity subsolution of (5) on $(0,\infty) \times \boldsymbol{R}^d$ with boundary condition*

$$\inf_{\boldsymbol{n} \in N_\Omega(x)} \langle \boldsymbol{n}, D_x u \rangle = 0,$$

on $(0,\infty) \times \partial\Omega$;

(v) *$v \in \mathcal{LSC}(\boldsymbol{R}^+ \times \overline{\Omega}; \boldsymbol{R})$, $\lfloor v \rfloor\!]_T^\infty \stackrel{\text{def}}{=} \inf_{t \in [0,T), x \in \overline{\Omega}} v(t,x) > -\infty$ for all $T > 0$ and v is a viscosity supersolution of (6) on $(0,\infty) \times \boldsymbol{R}^d$ with boundary condition*

$$\sup_{\boldsymbol{n} \in N_\Omega(x)} \langle \boldsymbol{n}, D_x v \rangle = 0,$$

on $(0,\infty) \times \partial\Omega$;

Then, for each $T > 0$,

$$
\begin{aligned}
&u(t,x) - v(t,x) \le \sup_{x\in\overline{\Omega}}\big(u(0,x)-v(0,x)\big)^+ \\
&+ T\left(\sup_{(r,\boldsymbol{q},X)\in D_{\alpha,T}}\Big(g(r,\boldsymbol{q},X) - f(r,\boldsymbol{q},X) + 3\alpha d\left(\sqrt{\phi(\boldsymbol{q})}-\sqrt{\gamma(\boldsymbol{q})}\right)^2\Big)\right)^+ \\
&\quad + \sup_{x,y\in\overline{\Omega}}\Big(\min\big\{|u(0,x)-u(0,y)|, |v(0,x)-v(0,y)|\big\} - \tfrac{\alpha}{2}|x-y|^2\Big), \qquad (7)
\end{aligned}
$$

for all $t \in [0,T)$, $x \in \overline{\Omega}$, and $\alpha > 0$, where

$$
\begin{aligned}
D_{\alpha,T} \overset{\text{def}}{=} \Big[\llfloor v \rrbracket_T^\infty, \llceil u \rrbracket_T^\infty\Big] \times \Big\{\, \boldsymbol{q}\in \boldsymbol{R}^d \;\Big|\; |\boldsymbol{q}| \le \sqrt{2(\llceil u\rrbracket_T^\infty - \llfloor v\rrbracket_T^\infty)\alpha}\,\Big\} \\
\times \{\, X \in \mathcal{S}(d) \mid \|X\| \le 3\alpha \,\}.
\end{aligned}
$$

The boundary conditions are taken in the viscosity sense as defined in [3, Def. 7.4] and the infimum and supremum are needed for the boundary conditions to be lower and upper semi-continuous, respectively, at corner points where there is not a unique outward normal. Note also that if the functions f and g do not depend on the value of the solution, then the term $\sup_{x\in\overline{\Omega}}\big(u(0,x)-v(0,x)\big)^+$ can be replaced by $\sup_{x\in\overline{\Omega}}\big(u(0,x)-v(0,x)\big)$.

The term $\sup_{x,y\in\overline{\Omega}}\Big(\min\big\{|u(0,x)-u(0,y)|, |v(0,x)-v(0,y)|\big\} - \frac{\alpha}{2}|x-y|^2\Big)$ can in most cases be simplified since we have the following:

Remark 2.1. Suppose that w is Hölder continuous in $\overline{\Omega}$, that is, $|w(x) - w(y)| \le C|x-y|^\lambda$ for x, $y \in \overline{\Omega}$ where C is some constant and $\lambda \in (0,1]$. Then

$$
\sup_{x,y\in\overline{\Omega}}\Big(|w(x)-w(y)| - \tfrac{\alpha}{2}|x-y|^2\Big) \le \tfrac{1}{2}C^{\frac{2}{2-\lambda}}\alpha^{-\frac{\lambda}{2-\lambda}}, \quad \alpha > 0.
$$

3. PROOF OF THEOREM 2.1

We use the notation $F(\underline{r},\underline{\boldsymbol{q}},\underline{X}) = f(\underline{r},\underline{\boldsymbol{q}},\underline{X}) - \phi(\underline{\boldsymbol{q}})\mathrm{tr}(\underline{X})$ and $G(\underline{r},\underline{\boldsymbol{q}},\underline{X}) = g(\underline{r},\underline{\boldsymbol{q}},\underline{X}) - \gamma(\underline{\boldsymbol{q}})\mathrm{tr}(\underline{X})$ where $\mathrm{tr}(X)$ denotes the trace of X. Since the equations are translation invariant with respect to the space variable, we may without loss of generality assume that 0 is an interior point of Ω.

Define

$$
E_0 \overset{\text{def}}{=} \sup_{x\in\overline{\Omega}}\big(u(0,x)-v(0,x)\big)^+ \quad \text{and} \quad M_T \overset{\text{def}}{=} \llceil u\rrbracket_T^\infty - \llfloor v\rrbracket_T^\infty.
$$

Let $T > 0$ and $\epsilon > 0$ be arbitrary and let

$$\sigma \stackrel{\text{def}}{=} \sup_{t\in[0,T)x\in\overline{\Omega}} \Big(u(t,x) - v(t,x) - \frac{\epsilon}{T-t} - \epsilon|x|^2 \Big) - E_0$$

Suppose for the moment that $\sigma > 0$ (so that $M_T > 0$), let $\delta \in (0,1)$ and $\alpha > 5\epsilon$ be arbitrary and define

$$\begin{aligned}\psi(t,x,y) \stackrel{\text{def}}{=} u(t,x) - v(t,y) - \frac{\epsilon}{T-t} - \frac{(1-\delta)\sigma}{T}t \\ - \tfrac{1}{2}\alpha|x-y|^2 - \tfrac{3}{2}\epsilon|x|^2 + \tfrac{1}{2}\epsilon|y|^2, \quad t \in [0,T), \quad x,y \in \overline{\Omega}.\end{aligned}$$

First we observe, by considering the possibility that $x = y$, that we must have

$$\sup_{t\in[0,T)x,y\in\overline{\Omega}} \psi(t,x,y) \geq \sigma + E_0 - (1-\delta)\sigma = \delta\sigma + E_0. \tag{8}$$

Next, note that because

$$\tfrac{1}{2}\alpha|x-y|^2 + \tfrac{3}{2}\epsilon|x|^2 - \tfrac{1}{2}\epsilon|y|^2 = \tfrac{1}{2}(\alpha - 2\epsilon)|x-y|^2 + \tfrac{1}{2}\epsilon|x|^2 + \tfrac{1}{2}\epsilon|2x-y|^2, \tag{9}$$

and because u is bounded from above and v is bounded from below there cannot be a sequence $\{(t_n, x_n, y_n)\}_{n=1}^{\infty}$ in $[0,T) \times \overline{\Omega} \times \overline{\Omega}$ such that we would have $\lim_{n\to\infty} \psi(t_n, x_n, y_n) = \sup_{t\in[0,T),x,y\in\overline{\Omega}} \psi(t,x,y)$ and $\sup_{n\geq 1}(|x_n| + |y_n| + 1/(T - t_n)) = \infty$. It follows that there is a point $(t_0, x_0, y_0) \in [0,T) \times \overline{\Omega} \times \overline{\Omega}$ such that

$$\psi(t_0, x_0, y_0) \geq \psi(t,x,y), \quad (t,x,y) \in [0,T) \times \overline{\Omega} \times \overline{\Omega}. \tag{10}$$

On the other hand, we have by (8), (9), and because $E_0 \geq 0$ and $\sigma > 0$ that

$$0 \leq \psi(t_0, x_0, y_0) \leq M_T - \tfrac{1}{2}(\alpha - 2\epsilon)|x_0 - y_0|^2 - \tfrac{1}{2}\epsilon|x_0|^2,$$

and since $\alpha > 3\epsilon$ it follows that

$$|x_0 - y_0| \leq \sqrt{\frac{2M_T}{\alpha - 2\epsilon}}, \quad |x_0| \leq \sqrt{\frac{2M_T}{\epsilon}}, \quad |y_0| \leq 2\sqrt{\frac{2M_T}{\epsilon}}. \tag{11}$$

Suppose that $t_0 > 0$. Then we can apply [3, Thm. 8.3] to conclude (cf. the proof of [3, Thm. 8.2]) that there are numbers a and b and symmetric matrices X and Y such that

$$\begin{aligned}&(a, \alpha(x_0 - y_0) + 3\epsilon x_0, X) \in \overline{P}_{\Omega}^{2,+} u(t_0, x_0) \quad \text{and} \\ &(b, \alpha(x_0 - y_0) + \epsilon y_0, Y) \in \overline{P}_{\Omega}^{2,-} v(t_0, y_0),\end{aligned}$$

such that

$$a - b = \frac{(1-\delta)\sigma}{T} + \frac{\epsilon}{(T - t_0)^2},$$

and

$$\begin{aligned}-3(\alpha+\epsilon)\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y\end{pmatrix} \le (3\alpha + 2\epsilon)\begin{pmatrix} I & -I \\ -I & I\end{pmatrix} \\ + \begin{pmatrix} (\frac{9\epsilon^2}{\alpha} + 4\epsilon)I & 0 \\ 0 & (\frac{\epsilon^2}{\alpha} - 4\epsilon)I\end{pmatrix}. \quad (12)\end{aligned}$$

By the definition of a subsolution we know that if $(a, \boldsymbol{q}, X) \in \overline{P}_{\Omega}^{2,+} u(t_0, x_0)$, then

$$\begin{aligned}a + F(u(t_0, x_0), \boldsymbol{q}, X) \le 0, &\quad \text{if} \quad x_0 \in \Omega, \\ \min\left\{\inf_{\boldsymbol{n} \in N_\Omega(x)} \langle \boldsymbol{n}, \boldsymbol{q}\rangle, a + F(u(t_0, x_0), \boldsymbol{q}, X)\right\} \le 0, &\quad \text{if} \quad x_0 \in \partial\Omega.\end{aligned}$$

Suppose that $x_0 \in \partial\Omega$ and that

$$\inf_{\boldsymbol{n} \in N_\Omega(x_0)} \langle \boldsymbol{n}, \alpha(x_0 - y_0) + 3\epsilon x_0\rangle \le 0.$$

But this is a contradiction by the definition of $N_\Omega(x_0)$ and the assumption that 0 is an interior point of Ω so we must have $a + F(u(t_0, x_0), \alpha(x_0 - y_0) + 3\epsilon x_0, X) \le 0$. A similar argument can be applied to the second equation and we conclude that

$$\begin{aligned}\frac{(1-\delta)\sigma}{T} + \frac{\epsilon}{(T - t_0)^2} + F(u(t_0, x_0), \boldsymbol{p} + 3\epsilon x_0, X) \\ - G(v(t_0, y_0), \boldsymbol{p} + \epsilon y_0, Y) \le 0, \quad (13)\end{aligned}$$

where $\boldsymbol{p} \stackrel{\text{def}}{=} \alpha(x_0 - y_0)$.

Let $\boldsymbol{p}_F \overset{\text{def}}{=} \boldsymbol{p} + 3\epsilon x_0$ and $\boldsymbol{p}_G = \boldsymbol{p} + \epsilon y_0$. By (12) we have for an arbitrary $\boldsymbol{e} \in \boldsymbol{R}^d$,

$$\begin{aligned}
&\phi(\boldsymbol{p}_F)\langle \boldsymbol{e}, X\boldsymbol{e}\rangle - \gamma(\boldsymbol{p}_G)\langle \boldsymbol{e}, Y\boldsymbol{e}\rangle \\
&\qquad = \left\langle \begin{pmatrix} \sqrt{\phi(\boldsymbol{p}_F)}\boldsymbol{e} \\ \sqrt{\gamma(\boldsymbol{p}_G)}\boldsymbol{e} \end{pmatrix}, \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \begin{pmatrix} \sqrt{\phi(\boldsymbol{p}_F)}\boldsymbol{e} \\ \sqrt{\gamma(\boldsymbol{p}_G)}\boldsymbol{e} \end{pmatrix} \right\rangle \\
&\qquad \le (3\alpha + 2\epsilon)\left(\sqrt{\phi(\boldsymbol{p}_F)} - \sqrt{\gamma(\boldsymbol{p}_G)}\right)^2 \langle \boldsymbol{e}, \boldsymbol{e}\rangle + 6\epsilon\phi(\boldsymbol{p}_F)\langle \boldsymbol{e}, \boldsymbol{e}\rangle,
\end{aligned}$$

where we used $\alpha \ge 5\epsilon$. By choosing $\boldsymbol{e}$ to be the vectors in an orthonormal basis, and then adding, one gets

$$\begin{aligned}
&\phi(\boldsymbol{p}_F)\operatorname{tr}(X) - \gamma(\boldsymbol{p}_G)\operatorname{tr}(Y) \\
&\qquad \le (3\alpha + 2\epsilon)d\left(\sqrt{\phi(\boldsymbol{p}_F)} - \sqrt{\gamma(\boldsymbol{p}_G)}\right)^2 + 6d\epsilon\phi(\boldsymbol{p}_F). \qquad (14)
\end{aligned}$$

From (12) it also follows that

$$X \le Y + 2\epsilon I \quad \text{and} \quad \|X\| \le 3\alpha + 8\epsilon. \qquad (15)$$

When we use (11), (14), and (15) together with the fact that $u(t_0, x_0) \ge v(t_0, y_0)$ because $\sigma > 0$, we get from (13) and from the monotonicity properties of f and g that

$$\begin{aligned}
\sigma \le \frac{T}{1-\delta} &\sup_{|\boldsymbol{q}_G|, |\boldsymbol{q}_F| \le 8\sqrt{\epsilon M_T}} \ \sup_{(r, \boldsymbol{q}, X) \in D_{\alpha, T}} \Big(g(r, \boldsymbol{q} + \boldsymbol{q}_G, X - 2\epsilon I) \\
&- f(r, \boldsymbol{q} + \boldsymbol{q}_F, X) + (3\alpha + 2\epsilon)d\left(\sqrt{\phi(\boldsymbol{q} + \boldsymbol{q}_F)} - \sqrt{\gamma(\boldsymbol{q} + \boldsymbol{q}_G)}\right)^2 \\
&\qquad + 6d\epsilon\phi(\boldsymbol{q} + \boldsymbol{q}_F)\Big). \qquad (16)
\end{aligned}$$

Suppose next that $t_0 = 0$. By (8) and (9) we must have

$$\begin{aligned}
\delta\sigma + E_0 &\le u(0, x_0) - v(0, y_0) - \tfrac{\alpha - 2\epsilon}{2}|x_0 - y_0|^2 \\
&\qquad \le |u(0, x_0) - u(0, y_0)| + E_0 - \tfrac{\alpha - 2\epsilon}{2}|x_0 - y_0|^2,
\end{aligned}$$

and since we can get the same inequality with u replaced by v we have

$$\begin{aligned}
\sigma &\le \frac{1}{\delta}\Big(\min\big\{|u(0, x_0) - u(0, y_0)|, |v(0, x_0) - v(0, y_0)|\big\} - \tfrac{\alpha - 2\epsilon}{2}|x_0 - y_0|^2\Big) \\
&\le \frac{1}{\delta} \sup_{x, y \in \overline{\Omega}} \Big(\min\big\{|u(0, x) - u(0, y)|, |v(0, x) - v(0, y)|\big\} - \tfrac{\alpha - 2\epsilon}{2}|x - y|^2\Big).
\end{aligned}$$

Now we have two upper bounds for σ, i.e., (16) and the inequality above, depending on whether t_0 is positive or not. Thus we get an upper bound of the form $\max\{\frac{\mu}{1-\delta}, \frac{\eta}{\delta}\}$ and by choosing $\delta = \frac{\eta}{\mu+\eta}$ it becomes $\mu + \eta$. Hence we have (trivially in the case $\sigma \le 0$ as well)

$$\begin{aligned}\sigma \le \sup_{x,y\in\overline{\Omega}} & \Big(\min\big\{|u(0,x)-u(0,y)|, |v(0,x)-v(0,y)|\big\} - \tfrac{\alpha-2\epsilon}{2}|x-y|^2\Big) \\ &+ T \sup_{|\boldsymbol{q}_G|,|\boldsymbol{q}_F|\le 8\sqrt{\epsilon M_T}} \Bigg(\sup_{(r,\boldsymbol{q},X)\in D_{\alpha,T}} \Big(g(r,\boldsymbol{q}+\boldsymbol{q}_G, X-2\epsilon I) \\ &- f(r,\boldsymbol{q}+\boldsymbol{q}_F, X) + (3\alpha+2\epsilon)d\left(\sqrt{\phi(\boldsymbol{q}+\boldsymbol{q}_F)} - \sqrt{\gamma(\boldsymbol{q}+\boldsymbol{q}_G)}\right)^2 \\ &+ 6d\epsilon\phi(\boldsymbol{q}+\boldsymbol{q}_F)\Big)\Bigg)^+. \end{aligned} \tag{17}$$

If now $x \in \overline{\Omega}$ and $t \in [0,T)$ are arbitrary, then we have by the definition of σ

$$u(t,x) - v(t,x) \le E_0 + \sigma + \frac{\epsilon}{T-t} + \epsilon|x|^2.$$

If we now use (16) in this inequality and then let $\epsilon \downarrow 0$, then we get (7) since f is lower semicontinuous, g is upper semicontinuous and the functions ϕ and γ are continuous. ▌

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