AN HP A-PRIORI ERROR ANALYSIS OF THE DG TIME-STEPPING METHOD FOR INITIAL VALUE PROBLEMS

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Abstract

The Discontinuous Galerkin (DG) time-stepping method for the numerical solution of initial value ODEs is analyzed in the context of the *hp*-version of the Galerkin method. New a-priori error bounds explicit in the time steps and in the approximation orders are derived and it is proved that the DG method gives spectral and exponential accuracy for problems with smooth and analytic time dependence, respectively. It is further shown that temporal singularities can be resolved at exponential rates of convergence if geometrically refined time steps are employed.

1 Introduction

The origins of Galerkin methods for the numerical solution of initial value ODEs can be traced back to the seventies. In 1972, Hulme [8, 9] introduced and analyzed such methods with continuous approximations. The first analysis of a Galerkin method for ODEs with discontinuous approximations, the so-called Discontinuous Galerkin (DG) method, seems to be contained in Lesaint and Raviart [11]. Generalizations of this method has been proposed by Delfour, Hager and Trochu [2]. The DG method is an implicit single-step scheme which allows for arbitrary variation in the time step Δt as well as in the approximation order r. Despite the underlying Galerkin approach, the DG methods of order r correspond to the first subdiagonal Padé approximations of the function $\exp(\lambda t)$ and, hence, are equivalent to certain implicit schemes of Runge-Kutta (RK) type [11].
For recent developments of the DG time-stepping method we mention here the papers of Böttcher and Rannacher [1], Estep [3] and Johnson [10] where issues such as optimal order error estimates, a posteriori error analysis and adaptivity have been addressed. DG methods have also been applied successfully to partial differential equations and, especially, in the context of parabolic problems

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a series of important papers has been written by Eriksson, Johnson, Thomée and their coworkers (we refer here only to the recent monograph [18] and the references there).

The known error analyses, however, are mainly concerned with the so-called "h-version" of the DG method where convergence is achieved by decreasing the time step Δt at a fixed approximation order r. For solutions depending smoothly on t, this h-version approach yields asymptotic error estimates of the form $C\Delta t^{r+1}$. Error bounds that are explicit in the order r as well, i.e., where the dependence of $C = C(r)$ on r is given explicitly, do not seem to be available in the literature. Such bounds are not known either for the corresponding RK schemes.

Typically, exact solutions of initial value problems become very smooth (in many cases even analytic) after a possibly non-smooth initial phase. This behavior suggests that concepts from the p- and hp-version of the Galerkin method can also be applied in the time integration of ODEs (see [17] for a survey on p- and hp-methods). In this context, the "p-version" approach is to increase the approximation order r on fixed time time steps, whereas the "hp-approach" combines judiciously h- and p-refinement techniques. In [15], such hp-concepts have recently been applied in the DG time discretization of linear parabolic problems.

In the present work we introduce and analyze the hp-version of the DG method for a class of non-linear initial value ordinary differential equations. We derive a-priori error bounds that are completely explicit in the time step Δt, in the approximation order r, and in the regularity of the exact solution. While these estimates allow us to recover the optimal convergence rates in Δt, they also show that the DG method converges if the order $r \to \infty$ and the time steps are kept fixed. We are able to prove that this p-version DG approach gives spectral accuracy for solutions with smooth time dependence, i.e., the convergence rates are of arbitrarily high algebraic order. Moreover, for analytic solutions, the convergence is even exponential. In conjunction with geometric time steps and linearly increasing approximation orders, we show that the hp-version of the DG method can approximate piecewise analytic solutions exhibiting start-up singularities at exponential rates of convergence.

The outline of this paper is as follows: In Section 2, we describe the DG method and prove the existence of discrete DG solutions. Section 3 is the main part of this paper. It is devoted to the hp a-priori error analysis of the DG method. In Section 4 we perform some numerical experiments that verify our theoretical results and in Section 5 we end with concluding remarks.

Throughout, standard notations and conventions are followed: We denote by $(\cdot,\cdot)$ (respectively by $\|\cdot\|$) the Euclidean inner product (respectively the Euclidean norm) in $\mathbb{R}^d$. We write $L^p(I;\mathbb{R}^d)$, $1 \leq p \leq \infty$, for the Lebesgue spaces of functions $I \to \mathbb{R}^d$. The norm in $L^\infty(I;\mathbb{R}^d)$ is denoted $\|\cdot\|_I$. $W^{k,p}(I;\mathbb{R}^d)$ are the Sobolev spaces of order $k \in \mathbb{N}_0$ equipped with the usual norms $\|\cdot\|_{W^{k,p}(I;\mathbb{R}^d)}$ and seminorms $|\cdot|_{W^{k,p}(I;\mathbb{R}^d)}$. The fractional order spaces $W^{s,p}(I;\mathbb{R}^d)$, $s \geq 0$, are defined via the K-method of interpolation. We set $H^s(I;\mathbb{R}^d) = W^{s,2}(I;\mathbb{R}^d)$. $\mathcal{P}^r(I;\mathbb{R}^d)$ is the set of all polynomials of degree $\leq r$ with coefficients in $\mathbb{R}^d$. We denote by $c, d, C, D, C_1, C_2, \ldots$ generic constants not necessarily identical at different places, but always independent of the parameters of interest (such as time steps and approximation orders).

2 The Discontinuous Galerkin Method

In this section we introduce the DG method and prove the existence of DG solutions provided that a certain "CFL condition" is obeyed. It turns out, however, that this condition is completely independent of the approximation order.

2.1 DG Discretization of Initial Value Problems

Let $J = (0, T)$ for some $T > 0$. For a given continuous function $f : \overline{J} \times \mathbb{R}^d \to \mathbb{R}^d$ and a vector $u_0 \in \mathbb{R}^d$ we consider the initial value problem (IVP)

$$u'(t) = f(t, u(t)),\ t \in \overline{J}, \qquad u(0) = u_0. \tag{2.1}$$

Let the function $f(t, u)$ be uniformly Lipschitz continuous with respect to u, i.e.

$$\|f(t, u) - f(t, v)\| \le L\|u - v\|, \qquad u, v \in \mathbb{R}^d,\ t \in \overline{J} \tag{2.2}$$

for some Lipschitz constant $L > 0$. Under assumption (2.2) there exists a unique solution $u : \overline{J} \to \mathbb{R}^d$ of (2.1) which is continuously differentiable.

Let now $\mathcal{M}$ be a partition of J into N time intervals $\{I_n\}_{n=1}^N$ given by $I_n = (t_{n-1}, t_n)$ with nodes $0 =: t_0 < t_1 < \ldots < t_{N-1} < t_N := T$. k_n is the length of I_n, i.e. $k_n := t_n - t_{n-1}$. We set further $k := \max_{n=1}^N k_n$. At the nodes $\{t_n\}_{n=0}^N$ the left- and right-sided limits of piecewise continuous functions $\varphi : J \to \mathbb{R}^d$ will be important. They are defined as follows:

$$\varphi_n^+ = \lim_{s\to 0,\ s>0} \varphi(t_n + s),\ 0 \le n \le N-1, \qquad \varphi_n^- = \lim_{s\to 0,\ s>0} \varphi(t_n - s),\ 1 \le n \le N.$$

The jumps across the nodes are also of interest and are given by $[\varphi]_n = \varphi_n^+ - \varphi_n^-$.

On $\mathcal{M}$ we introduce the space

$$C(\mathcal{M}; \mathbb{R}^d) = \{\varphi \in L^2(J; \mathbb{R}^d) :\ \varphi|_{I_n} \text{ is continuous and bounded}\}. \tag{2.3}$$

Then the solution u of (2.1) satisfies

$$\sum_{n=1}^N \int_{I_n} (u'(t) - f(t, u(t)), \varphi(t))dt + \sum_{n=2}^N ([u]_{n-1}, \varphi_{n-1}^+) + (u_0^+, \varphi_0^+) = (u_0, \varphi_0^+) \tag{2.4}$$

for all $\varphi \in C(\mathcal{M}; \mathbb{R}^d)$.

We assign to each time interval I_n an approximation order $r_n \ge 0$ and store these elemental orders in the vector $\underline{r} := \{r_n\}_{n=1}^N$. We set $|\underline{r}| := \max_{n=1}^N r_n$. The tupel $(\mathcal{M}, \underline{r})$ is called an hp-discretization of (2.1).

In the DG method the continuous space in (2.3) is replaced by the discrete space

$$\mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d) = \{\varphi \in L^2(J; \mathbb{R}^d) :\ \varphi|_{I_n} \in \mathcal{P}^{r_n}(I_n; \mathbb{R}^d),\ 1 \le n \le N\}. \tag{2.5}$$

If $r_n = r$ in each time step I_n, we simply write $\mathcal{V}(\mathcal{M}, r; \mathbb{R}^d)$.

We set also $N = \mathrm{NRDOF}(\mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d)) := \sum_{n=1}^{N}(r_n+1)$ for the number of degrees of freedom of the time discretization. Since the DG method amounts in each step I_n to the solution of a stationary problem of size $r_n + 1$ (see (2.6) ahead), the number N can be viewed as a crude measure for the cost of the discretization.

Definition 2.1 *The Discontinuous Galerkin (DG) Method for the IVP (2.1) is:*
Find $U \in \mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d)$ *such that*

$$\sum_{n=1}^{N} \int_{I_n} (U'(t) - f(t, U(t)), \varphi(t))dt + \sum_{n=2}^{N}([U]_{n-1}, \varphi_{n-1}^{+}) + (U_0^{+}, \varphi_0^{+}) = (u_0, \varphi_0^{+})$$

for all $\varphi \in \mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d)$.

Remark 2.2 The DG method can be interpreted as a time stepping scheme: If U is given on the time intervals I_k, $1 \le k \le n-1$, we find $U|_{I_n} \in \mathcal{P}^{r_n}(I_n; \mathbb{R}^d)$ by solving

$$\int_{I_n} (U'(t) - f(t, U(t)), \varphi(t))dt + (U_{n-1}^{+}, \varphi_{n-1}^{+}) = (U_{n-1}^{-}, \varphi_{n-1}^{+}), \qquad \varphi \in \mathcal{P}^{r_n}(I_n; \mathbb{R}^d). \tag{2.6}$$

Here, we set $U_0^{-} = u_0$.

Remark 2.3 In practice, the integrals in (2.6) have to be evaluated by numerical quadrature which introduces additional quadrature errors. In order to concentrate on the discretization of the differential equation only, the impact of numerical quadrature is not considered in this paper and has to be addressed in future work. We note that, by the use of interpolatory quadrature rules of Gauss-Radau type, (2.6) coincides with certain RK methods [11]. In this context we mention also [12] and the references there where fast integration techniques are presented for *hp*-methods.

2.2 Existence of DG Solutions

The DG method amounts in each time step to the solution of a nonlinear system of the form (2.6). It is well known that this system is uniquely solvable provided that the "CFL" condition

$$k < c \cdot L^{-1} \tag{2.7}$$

is satisfied for some sufficiently small constant $c > 0$ (see, e.g., [1, 3]). An analogous condition is encountered for implicit RK methods (see [5, 6]). In the *hp*-context, however, the dependence of the constant c on the approximation orders $\underline{r}$ has to be clarified as well. This will be done in the present section. We start with the subsequent lemma:

Lemma 2.4 *Let* $I = (a, b)$ *and* $k = b - a > 0$. *There holds*

$$\int_a^b \|\varphi(t)\|^2 dt \le \frac{1}{k} \sum_{i=1}^{d} (\int_a^b \varphi_i(t)dt)^2 + \frac{1}{2} \int_a^b (b-t)(t-a)\|\varphi'(t)\|^2 dt \tag{2.8}$$

for all $\varphi(t) = (\varphi_1(t), \ldots, \varphi_d(t)) \in \mathcal{P}^r((a, b); \mathbb{R}^d)$, $r \in \mathbb{N}_0$. *(This estimate is in particular independent of* r.*)*

Remark 2.5 Estimate (2.8) is a kind of a weighted Poincaré inequality and does even hold for more general classes of functions (e.g., for $(\varphi_1, \dots, \varphi_d) \in H^1((a,b);\mathbb{R}^d)$). However, we need this inequality only for polynomial functions and the extension to other function spaces is straight forward.

Proof: Up to summation it is enough to show (2.8) componentwise and therefore we may assume $d = 1$. We consider first the case $I = (-1,1)$. We denote by L_i, $i \geq 0$, the usual Legendre polynomial of degree i on I and develop φ, φ' into the series

$$\varphi(t) = \sum_{i=0}^{r} a_i L_i(t), \qquad \varphi'(t) = \sum_{i=1}^{r} a_i L_i'(t).$$

There holds $2a_0 = \int_{-1}^{1} \varphi(t)dt$. Since

$$\int_{-1}^{1} L_i'(t)L_j'(t)(1-t^2)dt = j(j+1)\int_{-1}^{1} L_i(t)L_j(t)dt = \frac{2j(j+1)}{2j+1}\delta_{ij}$$

for $i,j \geq 1$, the $\{L_i'\}$ are orthogonal with respect to the measure $(1-t^2)dt$. We get

$$\begin{aligned}\int_{-1}^{1} \varphi'(t)^2(1-t^2)dt + (\int_{-1}^{1}\varphi(t)dt)^2 &= \sum_{i=1}^{r} a_i^2 \int_{-1}^{1} L_i'(t)^2(1-t^2)dt + 4a_0^2 \\ = \sum_{i=1}^{r} a_i^2 i(i+1)\int_{-1}^{1} L_i(t)^2 dt + 4a_0^2 &\geq 2\sum_{i=0}^{r} a_i^2 \int_{-1}^{1} L_i(t)^2 dt = 2\int_{-1}^{1}\varphi(t)^2 dt.\end{aligned}$$

This proves (2.8) in the case where $I = (-1,1)$. If $I = (a,b)$, the assertion is obtained by scaling (a,b) back on $(-1,1)$ via the transformation $Q : (-1,1) \to (a,b)$, $\tau \mapsto t = a + \frac{k}{2}(\tau+1)$. □

We have:

Theorem 2.6 *Let $(\mathcal{M}, \underline{r})$ be an hp-discretization of (2.1) with $kL < 1$, i.e. satisfying (2.7) with $c = 1$. Then there exists a unique DG solution $U \in \mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d)$. (In particular, this existence criterion is independent of $\underline{r}$ and the dimension d.)*

Remark 2.7 The result in Theorem 2.6 shows that the constant c in (2.7) can be chosen completely independent of the approximation orders $\underline{r}$. In this sense, it extends the h-version results of e.g. [1, 3] as in the approaches there c is growing for increasing approximation orders.

Remark 2.8 If $r_n \geq 1$ on time step I_n, the local problem in (2.6) is even solvable for $k_n L < \sqrt{\frac{6}{5}}$, as can be inferred from the proof of Theorem 2.6 below.

Proof: Given the initial value U_{n-1}^- (with $U_0^- = u_0$) it is sufficient to show that the problem (2.6) on time step I_n has a unique solution $U|_{I_n} \in \mathcal{P}^{r_n}(I_n;\mathbb{R}^d)$. To do so, we define for $\tilde{U} \in \mathcal{P}^{r_n}(I_n;\mathbb{R}^d)$ the polynomial $U = T\tilde{U} \in \mathcal{P}^{r_n}(I_n;\mathbb{R}^d)$ as the solution of

$$\int_{I_n} (U'(t), \varphi(t))dt + (U_{n-1}^+, \varphi_{n-1}^+) = (U_{n-1}^-, \varphi_{n-1}^+) + \int_{I_n} (f(t,\tilde{U}(t)), \varphi(t))dt \tag{2.9}$$

for all $\varphi \in \mathcal{P}^{r_n}(I_n, \mathbb{R}^d)$. (2.9) is a linear system of $r_n + 1$ equations in $\mathbb{R}^d$ which is uniquely solvable and hence $T\tilde{U}$ is well defined. A fixed point $\tilde{U}$ of T (i.e. $\tilde{U} = T\tilde{U}$) is a solution of (2.6). We show that the operator T is a contraction on $\mathcal{P}^{r_n}(I_n; \mathbb{R}^d)$ for k sufficiently small and the assertion then follows from Banach's fixed point theorem.
To prove the contraction property, fix $\tilde{U}$, $\tilde{V}$ in $\mathcal{P}^{r_n}(I_n; \mathbb{R}^d)$ and set $U = T\tilde{U}$, $V = T\tilde{V}$, $W = U - V$ and $\tilde{W} = \tilde{U} - \tilde{V}$. We have by definition of T in (2.9)

$$\int_{I_n} (W', \varphi) dt + (W_{n-1}^+, \varphi_{n-1}^+) = \int_{I_n} (f(t, \tilde{U}) - f(t, \tilde{V}), \varphi) dt, \qquad \varphi \in \mathcal{P}^{r_n}(I_n; \mathbb{R}^d). \tag{2.10}$$

Remark that by integration by parts (2.10) is equivalent to

$$-\int_{I_n} (W, \varphi') dt + (W_n^-, \varphi_n^-) = \int_{I_n} (f(t, \tilde{U}) - f(t, \tilde{V}), \varphi) dt, \qquad \varphi \in \mathcal{P}^{r_n}(I_n; \mathbb{R}^d). \tag{2.11}$$

Consider first the case where $r_n = 0$: Selecting $\varphi = W$ in (2.10) yields

$$\|W\|_{I_n}^2 = \|W_{n-1}^+\|^2 \le k_n L \|\tilde{W}\|_{I_n} \|W\|_{I_n}$$

or

$$\|T\tilde{U} - T\tilde{V}\|_{I_n} \le k_n L \|\tilde{U} - \tilde{V}\|_{I_n}. \tag{2.12}$$

Hence, T is a contraction for $k_n L < 1$.
Assume now that $r_n \ge 1$: Selecting $\varphi(t) = W'(t)(t - t_{n-1})$ in (2.10) yields (since $\varphi_{n-1}^+ = (0, \dots, 0)$)

$$\begin{aligned} \int_{I_n} \|W'\|^2 (t - t_{n-1}) dt &= \int_{I_n} (f(t, \tilde{U}) - f(t, \tilde{V}), W'(t)(t - t_{n-1})) dt \\ &\le L \int_{I_n} \|\tilde{W}\| \|W'\| (t - t_{n-1}) dt \\ &\le L \Big(\int_{I_n} (t - t_{n-1}) \|\tilde{W}\|^2 dt\Big)^{\frac{1}{2}} \Big(\int_{I_n} (t - t_{n-1}) \|W'\|^2 dt\Big)^{\frac{1}{2}} \end{aligned}$$

and therefore

$$\int_{I_n} \|W'\|^2 (t - t_{n-1}) dt \le L^2 k_n \int_{I_n} \|\tilde{W}\|^2 dt. \tag{2.13}$$

Fix a component $i \in \{1, \dots, d\}$ and choose in (2.11) $\varphi = (0, \dots, 0, t_n - t, 0, \dots, 0) \in \mathcal{P}^1(I_n; \mathbb{R}^d)$ with non-vanishing i-th entry. This results in (since $\varphi_n^- = (0, \dots, 0)$)

$$\int_{I_n} W_i(t) dt = \int_{I_n} f_i(t)(t_n - t) dt \le \Big(\int_{I_n} f_i^2 dt\Big)^{\frac{1}{2}} \Big(\int_{I_n} (t_n - t)^2 dt\Big)^{\frac{1}{2}}$$

where W_i is the i-th component of W and $f_i(t) = [f(t, \tilde{U}) - f(t, \tilde{V})]_i$. We get

$$\Big(\int_{I_n} W_i dt\Big)^2 \le \frac{k_n^3}{3} \int_{I_n} f_i^2 dt, \qquad i = 1, \dots, d.$$

Summing over all components gives

$$\begin{aligned}\sum_{i=1}^{d}\Big(\int_{I_n} W_i dt\Big)^2 &\le \frac{k_n^3}{3}\int_{I_n}\Big(\sum_{i=1}^{d} f_i^2\Big)dt \\ &= \frac{k_n^3}{3}\int_{I_n}\|f(t,\tilde U)-f(t,\tilde V)\|^2 dt \le \frac{k_n^3}{3}L^2\int_{I_n}\|\tilde W\|^2 dt. \end{aligned} \tag{2.14}$$

Applying Lemma 2.4, (2.13) and (2.14) yields

$$\begin{aligned}\int_{I_n}\|W\|^2 dt &\le \frac{1}{k_n}\sum_{i=1}^{d}\Big(\int_{I_n} W_i(t)dt\Big)^2 + \frac{1}{2}\int_{I_n}(t_n-t)(t-t_{n-1})\|W'(t)\|^2 dt \\ &\le \frac{1}{k_n}\sum_{i=1}^{d}\Big(\int_{I_n} W_i(t)dt\Big)^2 + \frac{1}{2}k_n\int_{I_n}(t-t_{n-1})\|W'(t)\|^2 dt \\ &\le \Big(\frac{k_n^2L^2}{3}+\frac{k_n^2L^2}{2}\Big)\int_{I_n}\|\tilde W\|^2 dt = \frac{5k_n^2L^2}{6}\int_{I_n}\|\tilde W\|^2 dt,\end{aligned}$$

which is

$$\|T\tilde U - T\tilde V\|_{L^2(I_n;\mathbb{R}^d)} \le \sqrt{\frac{5}{6}}k_n L\|\tilde U-\tilde V\|_{L^2(I_n,\mathbb{R}^d)}. \tag{2.15}$$

Thus, the operator T is a contraction for $k_n < \sqrt{\frac{6}{5}}L^{-1}$.
Referring to (2.12) and (2.15) finishes the proof. □

3 Error Analysis

In this section we derive our a-priori estimates. They are completely explicit in the time steps k_n, the approximation orders r_n, and in the regularity of the exact solution. From an approximation theory point of view, the bounds are optimal in terms of the step size k and slightly suboptimal in terms of the approximation order r. However, the estimates give spectral accuracy for smooth solutions and exponential accuracy for solutions which are analytic on $\overline{J}$. They enable us also to prove exponential rates of convergence in the approximation of start-up singularities on geometrically refined time partitions.

3.1 Preliminaries

We need the following inverse inequality:

Lemma 3.1 *For $I=(a,b)$ and any $r\in\mathbb{N}_0$, there holds*

$$\|\varphi\|_I^2 \le C\log(r+1)\int_a^b\|\varphi'(t)\|^2(t-a)dt + C\|\varphi(b)\|^2$$

for all $\varphi(t) = (\varphi_1(t), \dots, \varphi_d(t)) \in \mathcal{P}^r((a,b);\mathbb{R}^d)$*. The constant* C *is independent of* r*,* a*,* b *and the dimension* d*.*
This estimate can not be improved asymptotically as $r \to \infty$*, i.e., there is a constant* C *such that for each* $r > 2$ *there exists* $\varphi_r \in \mathcal{P}^r((a,b);\mathbb{R}^d)$ *with* $\varphi_r(b) = (0,\dots,0)$*,* $\int_a^b \|\varphi_r'(t)\|^2(t-a)dt \le C$ *and* $\max_{t\in(a,b)} \|\varphi_r(t)\|^2 \ge C\log(r)$*.*

Proof: Since $\max_{t\in(a,b)} \|\varphi(t)\|^2 \le \sum_{l=1}^d \max_{t\in(a,b)} \varphi_l^2(t)$, it is sufficient to prove this estimate componentwise and we may thus assume that $d=1$. Further, we assume $r \ge 1$ (the case $r=0$ being trivial).
We consider first $I = (-1,1)$. Denote by $P_i^{(\alpha,\beta)}$ the (α,β)-Jacobi polynomial of order i on I. The polynomials $\{P_i^{(0,1)}\}_{i\ge 0}$ are orthogonal with respect to the measure $(1+t)dt$, i.e.

$$\int_I P_i^{(0,1)}(t)P_j^{(0,1)}(t)(1+t)dt = \frac{2}{i+1}\delta_{ij}, \qquad i,j \ge 0. \tag{3.1}$$

We will also make use of the following relations (cf. [4, formulae 8.96])

$$\begin{aligned} (2i+2)P_i^{(0,1)}(t) &= (i+2)P_i^{(1,1)}(t) - (i+1)P_{i-1}^{(1,1)}(t), \\ P_{i-1}^{(1,1)}(t) &= \frac{2}{i+1}\frac{d}{dt}P_i^{(0,0)}(t), \end{aligned}$$

which yields

$$P_i^{(0,1)}(t) = \frac{1}{i+1}(L_{i+1}'(t) - L_i'(t)), \tag{3.2}$$

where $\{L_i\}_{i\ge 0}$ are the Legendre polynomials on $(-1,1)$ (note that $L_i(t) = P_i^{(0,0)}(t)$). If now $\varphi \in \mathcal{P}^r(I;\mathbb{R})$, we write $\varphi(t) = -\int_t^1 \varphi'(s)ds + \varphi(1)$ and develop $\varphi'(t)$ into the series

$$\varphi'(t) = \sum_{i=0}^{r-1} a_i P_i^{(0,1)}(t). \tag{3.3}$$

Because of (3.1),

$$\int_I \varphi'(t)^2(1+t)dt = 2\sum_{i=0}^{r-1} a_i^2 \frac{1}{i+1}. \tag{3.4}$$

Integrating (3.3) and using (3.2) gives then

$$\begin{aligned} \varphi(t) &= -\sum_{i=0}^{r-1} a_i \int_t^1 P_i^{(0,1)}(t)dt + \varphi(1) \\ &= \sum_{i=0}^{r-1} \frac{a_i}{i+1}(L_{i+1}(t) - L_i(t)) + \varphi(1) \end{aligned} \tag{3.5}$$

and

$$\varphi(t)^2 \leq 2(\sum_{i=0}^{r-1} \frac{2|a_i|}{i+1})^2 + 2\varphi(1)^2.$$

The Cauchy-Schwarz inequality yields with (3.4)

$$\begin{aligned}\varphi(t)^2 &\leq 2(\sum_{i=0}^{r-1} \frac{2a_i^2}{i+1})(\sum_{i=0}^{r-1} \frac{2}{i+1}) + 2\varphi(1)^2 \\ &\leq C(1+\log(r)) \int_a^b \varphi'(t)^2(t-a)dt + C\varphi(1)^2.\end{aligned}$$

This is the desired estimate on $(-1,1)$, the general case follows by a scaling argument.
To prove the optimality of this estimate, let $\varphi_r'(t)$ be given again on $I=(-1,1)$ by (3.3) with $a_i = (-1)^{i+1}/\log(r)^{1/2}$ and set $\varphi_r(t) = -\int_t^1 \varphi_r'(s)ds$. With the use of (3.4) we have $\int_I \varphi_r'(t)^2(1+t)dt \leq C$. From (3.5), we get $\varphi_r(-1) = 2/\log(r)^{1/2}\sum_{i=0}^{r-1}\frac{1}{i+1} \geq C\log(r)^{1/2}$. Therefore, $\max_{t\in I}\varphi_r(t)^2 \geq \varphi_r(-1)^2 \geq C\log(r)$. This finishes the proof. □

Further, our error analysis is based on the discrete form of the Gronwall inequality which we review here for the convenience of the reader. The proof can be found, e.g., in [19].

Lemma 3.2 *Let $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ be two sequences of non-negative real numbers with $b_1 \leq b_2 \leq \ldots \leq b_N$. Assume that for $C \geq 0$ and weights $(k_1,\ldots,k_{N-1}) \in \mathbb{R}_+^{N-1}$ there holds*

$$a_1 \leq b_1, \qquad a_n \leq b_n + C\sum_{i=1}^{n-1} k_i a_i, \quad n=2,\ldots,N.$$

Then we have $a_n \leq b_n \exp(C\sum_{i=1}^{n-1}k_i)$ for $n=1,\ldots,N$.

3.2 An Abstract Error Bound

We introduce the projector Π^r which is also used in h-version approaches (see [1, 18]).

Definition 3.3 *Let $I=(-1,1)$. For $u \in L^\infty(I;\mathbb{R}^d)$ let the polynomial $\Pi^r u \in \mathcal{P}^r(I;\mathbb{R}^d)$ be defined by*

$$\Pi^r u(+1) = u(+1), \tag{3.6}$$

$$\int_I (\Pi^r u, q)dt = \int_I (u,q)dt, \qquad \forall q \in \mathcal{P}^{r-1}(I;\mathbb{R}^d) \qquad (\textit{if } r>0). \tag{3.7}$$

Lemma 3.4 *Π^r in Definition 3.3 is well defined. If $u = \sum_{i=0}^\infty u_i L_i$ is the expansion of u into Legendre polynomials $\{L_i\}$, we have the unique representations*

$$\Pi^r u = \sum_{i=0}^{r-1} u_i L_i + (\sum_{i=r}^\infty u_i)L_r, \qquad u - \Pi^r u = \sum_{i=r+1}^\infty u_i L_i - (\sum_{i=r+1}^\infty u_i)L_r. \tag{3.8}$$

Proof: The proof can be found in [15]. □

On an arbitrary interval $I = (a,b)$ of length $k = b - a$ we define the projector Π_I^r as $\Pi_I^r u = [\Pi^r(u \circ Q)] \circ Q^{-1}$ where $Q : (-1,1) \to (a,b)$ is the linear transformation $\tau \mapsto t = 1/2(a + b + \tau k)$. For the exact solution u of (2.1) the interpolant $\mathcal{I}u \in \mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d)$ is defined intervalwise as

$$\mathcal{I}u|_{I_n} = \Pi_{I_n}^{r_n} u, \qquad n = 1, \dots, N. \tag{3.9}$$

If U is the DG solution in $\mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d)$, it can be seen that the difference $\eta := \mathcal{I}u - U$ satisfies

$$\int_{I_n} (\eta', \varphi) dt + (\eta_{n-1}^+, \varphi_{n-1}^+) = \int_{I_n} (f(t,u) - f(t,U), \varphi) dt + (\eta_{n-1}^-, \varphi_{n-1}^+) \tag{3.10}$$

for all $\varphi \in \mathcal{P}^{r_n}(I_n; \mathbb{R}^d)$. Equivalently,

$$-\int_{I_n} (\eta, \varphi') dt + (\eta_n^-, \varphi_n^-) = \int_{I_n} (f(t,u) - f(t,U), \varphi) dt + (\eta_{n-1}^-, \varphi_{n-1}^+) \tag{3.11}$$

for all $\varphi \in \mathcal{P}^{r_n}(I_n; \mathbb{R}^d)$. In (3.10) and (3.11), we set $\eta_0^- = (0, \dots, 0)$.

Lemma 3.5 *We have*

$$\|\eta_n^-\|^2 \leq 2L \int_{I_n} \|\xi\|^2 dt + 3L \int_{I_n} \|\eta\|^2 dt + \|\eta_{n-1}^-\|^2, \tag{3.12}$$

$$\int_{I_n} \|\eta'\|^2 (t - t_{n-1}) dt \leq 2L^2 k_n \int_{I_n} \|\xi\|^2 dt + 2L^2 k_n \int_{I_n} \|\eta\|^2 dt, \tag{3.13}$$

$$\sum_{i=1}^{d} \left(\int_{I_n} \eta_i(t) dt\right)^2 \leq \frac{4L^2 k_n^3}{3} \int_{I_n} \|\xi\|^2 dt + \frac{4L^2 k_n^3}{3} \int_{I_n} \|\eta\|^2 dt + 2k_n^2 \|\eta_n^-\|^2. \tag{3.14}$$

Proof: To prove (3.12), we take $\varphi = \eta$ in (3.10) and get

$$\frac{1}{2}\|\eta_n^-\|^2 + \frac{1}{2}\|\eta_{n-1}^+\|^2 \leq L \int_{I_n} \|e\| \|\eta\| dt + \frac{1}{2}\|\eta_{n-1}^-\|^2 + \frac{1}{2}\|\eta_{n-1}^+\|^2,$$

which yields

$$\|\eta_n^-\|^2 \leq 2L \int_{I_n} \|e\| \|\eta\| dt + \|\eta_{n-1}^-\|^2. \tag{3.15}$$

Since $\|e\| \|\eta\| \leq \|\xi\|^2 + \frac{3}{2}\|\eta\|^2$, the assertion (3.12) follows.
To establish (3.13), select now $\varphi = \eta'(t)(t - t_{n-1})$ in (3.10). As in the proof of (2.13) in Theorem 2.6 we derive with the Cauchy-Schwarz inequality that

$$\int_{I_n} \|\eta'\|^2 (t - t_{n-1}) dt \leq L^2 k_n \int_{I_n} \|e\|^2 dt. \tag{3.16}$$

The estimate (3.13) follows then with $\|e\|^2 \le 2\|\xi\|^2 + 2\|\eta\|^2$.
Finally, to show (3.14), we can assume that $r_n \ge 1$ (for $r_n = 0$ the estimate is trivial). Fix $i \in \{1, \dots, d\}$ and choose $\varphi = (0, \dots, 0, t_{n-1} - t, 0, \dots, 0) \in \mathcal{P}^{r_n}(I_n; \mathbb{R}^d)$ with a non-vanishing i-th component in (3.10). We get

$$\int_{I_n} \eta_i(t)dt - k_n(\eta_i)_n^- = \int_{I_n} f_i(t_{n-1} - t)dt.$$

Hence, with Cauchy-Schwarz

$$\left(\int_{I_n} \eta_i(t)dt\right)^2 \le 2k_n^2[(\eta_i)_n^-]^2 + 2\int_{I_n} f_i^2 dt \cdot \int_{I_n} (t_{n-1} - t)^2 dt.$$

Here, $f_i = [f(t,u) - f(t,U)]_i$ is the i-th component of $f(t,u) - f(t,U)$. Since $\int_{I_n}(t_{n-1} - t)^2 dt = \frac{k_n^3}{3}$, summing up yields

$$\sum_{i=1}^{d} (\int_{I_n} \eta_i(t)dt)^2 \le 2k_n^2\|\eta_n^-\|^2 + \frac{2k_n^3 L^2}{3} \int_{I_n} \|e\|^2 dt.$$

Observing again that $\|e\|^2 \le 2\|\xi\|^2 + 2\|\eta\|^2$, finishes the proof of (3.14). □

We can now state and prove the following error bound for the DG method:

Theorem 3.6 *Let $(\mathcal{M}, \underline{r})$ be an hp-discretization of (2.1) which satisfies*

$$k \cdot L \le c \tag{3.17}$$

for a sufficiently small constant $c > 0$ (independent of $\underline{r}$ and d). Let U be the discrete DG solution in $\mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d)$ and let $\mathcal{I}u \in \mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d)$ be the interpolant of u defined in (3.9). Then we have the error bound

$$\|u - U\|_J \le K(L, T, \underline{r})\|u - \mathcal{I}u\|_J \tag{3.18}$$

with

$$K(L, T, \underline{r})^2 \le C_1 \log(\max(|\underline{r}|, 2))(1 + LT\exp(C_2 LT)). \tag{3.19}$$

Remark 3.7 Up to the logarithmic factor in (3.19), the estimate (3.18) is independent of $\underline{r}$ and the convergence rates of the DG method are completely determined by the *hp* approximation properties of $\mathcal{I}u$ which are investigated in Section 3.3 ahead. Theorem 3.6 generalizes the h-version bounds as e.g. in [1, 3].

Proof: We split the error $e = u - U = \xi + \eta$ into $\xi := u - \mathcal{I}u$ and $\eta := \mathcal{I}u - U$ where $\mathcal{I}u \in \mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d)$ is the interpolant in (3.9). Assuming that (quasioptimal) estimates for ξ are available we must control η. We proceed in several steps:

Step 1: We claim that for c in (3.17) small enough

$$\sum_{i=1}^{d}(\int_{I_n}\eta_i(t)dt)^2 \le Ck_n^2\|\eta_n^-\|^2 + Ck_n^3L^2\int_{I_n}\|\xi\|^2dt + Ck_n^4L^2\int_{I_n}\|\eta'(t)\|^2(t-t_{n-1})dt. \tag{3.20}$$

To see (3.20), we combine (3.14) and Lemma 2.4 into

$$\begin{aligned}\sum_{i=1}^{d}(\int_{I_n}\eta_i(t)dt)^2 \quad &\le \quad Ck_n^2\|\eta_n^-\|^2 + Ck_n^3L^2\int_{I_n}\|\xi\|^2dt \\ &\quad +Ck_n^2L^2\sum_{i=1}^{d}(\int_{I_n}\eta_i(t)dt)^2 + Ck_n^4L^2\int_{I_n}\|\eta'(t)\|^2(t-t_{n-1})dt.\end{aligned}$$

For kL small enough the third term on the right-hand side can be hidden on the left-hand side. This proves (3.20).

Step 2: We have

$$\begin{aligned}\int_{I_n}\|\eta'(t)\|^2(t-t_{n-1})dt + \|\eta_n^-\|^2 \quad &\le \quad Ck_nL(\int_{I_n}\|\eta'(t)\|^2(t-t_{n-1})dt + \|\eta_n^-\|^2) \\ &\quad +CL\int_{I_n}\|\xi\|^2dt + \|\eta_{n-1}^-\|^2.\end{aligned} \tag{3.21}$$

To prove (3.21), we combine (3.12), (3.13) and Lemma 2.4 to obtain

$$\begin{aligned}&\int_{I_n}\|\eta'(t)\|^2(t-t_{n-1})dt + \|\eta_n^-\|^2 \\ &\le CL\int_{I_n}\|\xi\|^2dt + CL\int_{I_n}\|\eta\|^2dt + \|\eta_{n-1}^-\|^2 \\ &\le CL\int_{I_n}\|\xi\|^2dt + \frac{CL}{k_n}\sum_{i=1}^{d}(\int_{I_n}\eta_i dt)^2 + CLk_n\int_{I_n}\|\eta'\|^2(t-t_{n-1})dt + \|\eta_{n-1}^-\|^2.\end{aligned}$$

Applying now (3.20) in Step 1 proves the assertion.

Step 3: Iterating the estimate (3.21) in Step 2 yields

$$\begin{aligned}&\int_{I_n}\|\eta'(t)\|^2(t-t_{n-1})dt + \|\eta_n^-\|^2 \\ &\le CL\sum_{i=1}^{n}k_i\|\xi\|_{I_i}^2 + CL\sum_{i=1}^{n}k_i(\int_{I_i}\|\eta'\|^2(t-t_{n-1})dt + \|\eta_n^-\|^2).\end{aligned}$$

For kL small enough Gronwall's Lemma (Lemma 3.2) can be applied and gives

$$\int_{I_n}\|\eta'(t)\|^2(t-t_{n-1})dt + \|\eta_n^-\|^2 \le CLT\|\xi\|_J^2\exp(CLT). \tag{3.22}$$

Referring to Lemma 3.1 results in

$$\|\eta\|_{I_n}^2 \leq \log(\max(r_n, 2))CLT\|\xi\|_J^2 \exp(CLT).$$

The triangle inequality finishes the proof. $\square$

3.3 Approximation Properties of Π^r

We analyze the approximation properties of the projector Π^r in Definition 3.3.

Lemma 3.8 *Let $I = (-1, 1)$, $u \in H^1(I; \mathbb{R}^d)$ and $r \in \mathbb{N}_0$. Then we have*

$$\|u - \Pi^r u\|_I \leq C(r+1)\|u - q\|_{L^2(I;\mathbb{R}^d)} + C\|u' - q'\|_{L^2(I;\mathbb{R}^d)}$$

for any $q \in \mathcal{P}^r(I; \mathbb{R}^d)$.

Proof: Let $u = \sum_{i=0}^{\infty} u_i L_i$ be the Legendre series of u and denote by P^r the $L^2(I; \mathbb{R}^d)$-projection onto $\mathcal{P}^r(I; \mathbb{R}^d)$. Then we have $u - P^r u = \sum_{i=r+1}^{\infty} u_i L_i$ and obtain with Lemma 3.4 (since $|L_i(t)| \leq 1$)

$$\|u - \Pi^r u\|_I^2 \leq 2\|u - P^r u\|_I^2 + 2\|\sum_{i=r+1}^{\infty} u_i\|^2.$$

In [15] it is proved that

$$\|\sum_{i=r+1}^{\infty} u_i\|^2 \leq \frac{1}{2r+1}\|u'\|_{L^2(I;\mathbb{R}^d)}^2.$$

Thus, inserting $u - q$ for a polynomial $q \in \mathcal{P}^r(I; \mathbb{R}^d)$ into the above bounds and observing that $\Pi^r q = q$ yields

$$\|u - \Pi^r u\|_I^2 \leq 4\|u - q\|_I^2 + 4\|P^r(u - q)\|_I^2 + \frac{2}{2r+1}\|u' - q'\|_{L^2(I;\mathbb{R}^d)}^2. \tag{3.23}$$

Due to the imbedding $H^1(I; \mathbb{R}^d) \hookrightarrow L^\infty(I; \mathbb{R}^d)$, the first term on the right hand side of (3.23) can be estimated as

$$\|u - q\|_I^2 \leq C\|u - q\|_{L^2(I;\mathbb{R}^d)}^2 + C\|u' - q'\|_{L^2(I;\mathbb{R}^d)}^2.$$

For the second term on the right hand side of (3.23) we use an L^∞-stability result for the L^2-projection from [7], i.e.,

$$\|P^r w\|_I^2 \leq (r+1)^2\|w\|_{L^2(I;\mathbb{R}^d)}^2, \qquad w \in L^2(I; \mathbb{R}^d).$$

Hence, we obtain

$$\|u - \Pi^r u\|_I^2 \leq C(r+1)^2\|u - q\|_{L^2(I;\mathbb{R}^d)}^2 + C\|u' - q'\|_{L^2(I;\mathbb{R}^d)}^2 \tag{3.24}$$

for any $q \in \mathcal{P}^r(I; \mathbb{R}^d)$. $\square$

Theorem 3.9 *Let $I = (-1, 1)$, $u \in H^{s_0+1}(I; \mathbb{R}^d)$ and $r \in \mathbb{N}_0$. Then we have*

$$\|u - \Pi^r u\|_I^2 \leq C \frac{\Gamma(r+1-s)}{\Gamma(r+1+s)} |u|^2_{H^{s+1}(I;\mathbb{R}^d)}$$

for any integer $0 \leq s \leq \min(s_0, r)$.

Proof: In Schwab [17] it is proved that there exists $q \in \mathcal{P}^r(I; \mathbb{R}^d)$ with

$$\begin{aligned}
\|u' - q'\|^2_{L^2(I;\mathbb{R}^d)} &\leq C \frac{\Gamma(r+1-s)}{\Gamma(r+1+s)} |u|^2_{H^{s+1}(I;\mathbb{R}^d)}, \\
\|u - q\|^2_{L^2(I;\mathbb{R}^d)} &\leq C \frac{1}{(r+1)^2} \frac{\Gamma(r+1-s)}{\Gamma(r+1+s)} |u|^2_{H^{s+1}(I;\mathbb{R}^d)}
\end{aligned}$$

for any $0 \leq s \leq \min(s_0, r)$. Combining these *hp*-approximation results with Lemma 3.8 finishes the proof of the assertion. $\square$

Interpolating between integral Sobolev spaces and scaling to an arbitrary time interval I yields immediately:

Corollary 3.10 *Let $I = (a, b)$, $k = b - a$, $r \in \mathbb{N}_0$ and $u \in H^{s_0+1}(I; \mathbb{R}^d)$ for $s_0 \geq 0$. Then we have*

$$\|u - \Pi_I^r u\|_I^2 \leq C (\frac{k}{2})^{2s+1} \frac{\Gamma(r+1-s)}{\Gamma(r+1+s)} \|u\|^2_{H^{s+1}(I;\mathbb{R}^d)}$$

for any real $0 \leq s \leq \min(r, s_0)$. Further, if $u \in W^{s_0+1,\infty}(I; \mathbb{R}^d)$, we have also

$$\|u - \Pi_I^r u\|_I^2 \leq C (\frac{k}{2})^{2s+2} \frac{\Gamma(r+1-s)}{\Gamma(r+1+s)} \|u\|^2_{W^{s+1,\infty}(I;\mathbb{R}^d)}$$

for any real $0 \leq s \leq \min(r, s_0)$.

Remark 3.11 The error bounds in Corollary 3.10 are optimal in terms of the length of the interval, k. They are, however, not optimal in terms of the asymptotic rate of convergence as $r \to \infty$. Using Stirling's formula, we find that, as $r \to \infty$ at fixed s,

$$\|u - \Pi_I^r u\|_I \leq C k^{s+1/2} r^{-s} \|u\|_{H^{s+1}(I)}, \qquad \|u - \Pi_I^r u\|_I \leq C k^{s+1} r^{-s} \|u\|_{W^{s+1,\infty}(I)}.$$

Hence, in the L^∞-setting the estimate falls a power of r short of being optimal. However, the L^∞-bound for the L^2-projection employed in the proof of Lemma 3.8 is optimal in terms of the regularity $u \in H^{s+1}(I; \mathbb{R}^d)$, as can be inferred from the estimates in Melenk and Schwab [13].

3.4 Convergence Rates

We combine now the abstract error estimate in Theorem 3.6 and the approximation properties of Π^r explored in Section 3.3. We assume always that the discretization $(\mathcal{M}, \underline{r})$ of (2.1) satisfies (3.17).

Theorem 3.12 *Let U be the DG solution in $\mathcal{V}(\mathcal{M}, \underline{r}; \mathbb{R}^d)$. Let the exact solution u satisfy*

$$u|_{I_n} \in W^{s_{0,n}+1,\infty}(I_n; \mathbb{R}^d), \qquad s_{0,n} \geq 0, \; n = 1, \ldots, N.$$

Then we have

$$\|u - U\|_J^2 \leq CK(L, T, \underline{r})^2 \max_{n=1}^{N} \left\{ (\frac{k_n}{2})^{2s_n+2} \frac{\Gamma(r_n + 1 - s_n)}{\Gamma(r_n + 1 + s_n)} \|u\|^2_{W^{s_n+1,\infty}(I_n;\mathbb{R}^d)} \right\},$$

for any real $0 \leq s_n \leq \min(s_{0,n}, r_n)$. Here, $K(L, T, \underline{r})$ is the bound in (3.19).

Remark 3.13 The estimates in Theorem 3.12 are explicit in the time steps k_n, in the approximation order r_n and in the regularity of the exact solution s_n.

Remark 3.14 The result in Theorem 3.12 holds also when $\mathbb{R}^d$ is replaced by Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$.

From the general error bound in Theorem 3.12 the following convergence rates can be deduced for the h- and p-version DG method:

Corollary 3.15 *Let $r_n = r$, $k = \max\{k_n\}$ and $U \in \mathcal{V}(\mathcal{M}, r; \mathbb{R}^d)$. For $u \in W^{s_0+1,\infty}(J; \mathbb{R}^d)$ there holds*

$$\|u - U\|_J \leq CK(L, T, \underline{r}) \frac{k^{\min(s_0,r)+1}}{r^{s_0}} \|u\|_{W^{s_0+1,\infty}(J;\mathbb{R}^d)}.$$

Proof: The assertion follows from Theorem 3.12 and Stirling's formula. □

Remark 3.16 The estimates in Corollary 3.15 are uniform in r and k. They show the DG method converges either as the time steps are refined ($k \to 0$) or as r is increased ($r \to \infty$). In k they are optimal and have already been obtained e.g. in [1, 3], whereas in the approximation degree r they are slightly suboptimal (see Remark 3.11). However, this seems to be the first error bound of the DG method where the dependence on the approximation order r is estimated explicitly.

Remark 3.17 In the h-version DG, the error at the endpoints of the time intervals I_n is of order k^{2r+1} for smooth solutions, due to superconvergence properties of the method (see [2, 3]). For linear (parabolic) problems, analogous results hold true in the p-version as well (see [16]).

In terms of $N = \mathrm{NRDOF}(\mathcal{V}(\mathcal{M}, r; \mathbb{R}^d))$, we have in the "$h$-version" of the DG method where convergence is achieved by decreasing the time steps at a fixed approximation order r

$$\|u - U\|_J \leq CN^{-\min(s_0,r)-1}. \tag{3.25}$$

For the "p-version" of the DG method where convergence is obtained by increasing the approximation order on a fixed time partition $\mathcal{M}$ we get

$$\|u-U\|_J \leq CN^{-s_0}. \tag{3.26}$$

Hence, it can be seen that for smooth solutions for which s_0 is large it is more advantageous to increase r rather than to reduce k at fixed, low r. Indeed, arbitrarily high algebraic convergence rates are possible if the approximation order r is raised. This is referred to as spectral convergence. It turns out, however, that the p-version of the DG method converges in fact exponentially if the solution u is analytic in $\overline{J}$. This result can be derived immediately from Lemma 3.10, Theorem 3.6 and standard approximation theory for analytic functions.

Theorem 3.18 *Let the exact solution u be analytic in $\overline{J}$. Let the $r_n = r$ and let U be the DG solution in $\mathcal{V}(\mathcal{M}, r; \mathbb{R}^d)$ on a fixed partition $\mathcal{M}$. Then there holds*

$$\|u-U\|_J \leq C\exp(-br)$$

with constants $C, b > 0$ which are independent of r.

3.5 Exponential Convergence for Singular Solutions

Typically, solutions of initial value problems become very smooth after a non-smooth initial phase due to start-up singularities. The numerical resolution of such singular solution components usually requires locally refined time steps. We show in this section that the hp-version of the DG method where geometrically refined time steps are combined with linearly increasing approximation orders leads to exponential rates of convergence for piecewise analytic solutions exhibiting singularities. Consider $J = (0,1)$ for simplicity. Assume that the exact solution u is analytic in $\overline{J} \setminus \{0\}$ and has a radical t^θ-singularity at $t = 0$, i.e., there are constants $C_u, d_u > 0$ such that

$$\|u^{(s)}(t)\| \leq C_u d_u^s \Gamma(s+1) t^{\theta - s}, \qquad s \in \mathbb{N}_0,\ t \in J,\ \theta \geq 1. \tag{3.27}$$

The exponent θ is a measure for the strength of the singularity at $t = 0$. Note that $\theta \geq 1$ ensures the continuity of u' on $\overline{J}$, as required in (2.1).

Definition 3.19 *A geometric partition $\mathcal{M}_{M,\sigma} = \{I_n\}_{n=1}^{M+1}$ of $J = (0,1)$ with grading factor $\sigma \in (0,1)$ and $N := M+1$ time intervals I_n is given by the nodes*

$$t_0 = 0, \qquad t_n = \sigma^{M-n+1}, \qquad 1 \leq n \leq M+1.$$

For $2 \leq n \leq M+1$ the time steps $k_n = t_n - t_{n-1}$ satisfy $k_n = \lambda t_{n-1}$ with $\lambda = \frac{1-\sigma}{\sigma}$.

Lemma 3.20 *Assume (3.27). Let $\mathcal{M}_{M,\sigma} = \{I_n\}_{n=1}^{M+1}$ be a geometric partition of J. Then we have*

$$\begin{aligned}
\|u\|^2_{W^{1,\infty}(I_1;\mathbb{R}^d)} &\leq C, \\
\|u\|^2_{W^{s_n+1,\infty}(I_n;\mathbb{R}^d)} &\leq Cd^{2s_n}\Gamma(2s_n+1)\sigma^{2(M-n+2)(\theta-s-1)}, \qquad 2 \leq n \leq M+1,\ s_n \geq 0
\end{aligned}$$

with constants C, d independent of n, M and s_n.

Proof: This is a simple consequence of (3.27), Definition 3.19 and of properties of the Gamma function. □

Definition 3.21 *An approximation degree vector* $\underline{r} = \{r_n\}_{n=1}^{M+1}$ *is called linear with slope* $\mu > 0$ *on the geometric partition* $\mathcal{M}_{M,\sigma}$ *if* $r_n = \lfloor \mu n \rfloor$ *for* $1 \leq n \leq M+1$.

Theorem 3.22 *Assume (3.27). Let* U *be the DG solution in* $\mathcal{V}(\mathcal{M}_{M,\sigma}, \underline{r}; \mathbb{R}^d)$ *for a geometric partition* $\mathcal{M}_{M,\sigma}$ *and a linearly increasing degree vector* $\underline{r}$. *Then there exists* $\mu_0 > 0$ *such that for all linear polynomial degree vectors* $\underline{r} = \{r_n\}_{n=1}^{M+1}$ *with slope* $\mu \geq \mu_0$ *we have the error estimate*

$$\|u - U\|_J^2 \leq C \exp(-bN^{\frac{1}{2}})$$

with constants C *and* b *independent of* $N = \mathrm{NRDOF}(\mathcal{V}(\mathcal{M}_{M,\sigma}, \underline{r}; \mathbb{R}^d))$.

Proof: From Theorem 3.12 we have

$$\|u - U\|_J^2 \leq C(L,T) \log(\max(r_{M+1}, 2))^2 \max_{n=1}^{M+1} e_n$$

with

$$e_n = (\frac{k_n}{2})^{2s_n+2} \frac{\Gamma(r_n + 1 - s_n)}{\Gamma(r_n + 1 + s_n)} \|u\|^2_{W^{s_n+1,\infty}(I_n;\mathbb{R}^d)}, \qquad 0 \leq s_n \leq \min(s_{0,n}, r_n).$$

Due to (3.27), $s_{0,n}$ can be chosen arbitrarily large on the elements away from $t = 0$.
On the first element I_1 near $t = 0$, we select $s_1 = 0$ and have from Lemma 3.20

$$e_1 \leq Ck_1^2 = C\sigma^{2M}.$$

Now, fix an element I_n away from $t = 0$, i.e., with $2 \leq n \leq M+1$. From Lemma 3.20, we get

$$\begin{aligned} e_n &\leq C(\frac{\lambda \sigma^{M-n+2}}{2})^{2s_n+2} \frac{\Gamma(r_n + 1 - s_n)}{\Gamma(r_n + 1 + s_n)} (\sigma^{M-n+2})^{2(\theta - s_n - 1)} d^{2s_n} \Gamma(2s_n + 1) \\ &= C\sigma^{(M-n+2)2\theta} [(\lambda d)^{2s_n} \frac{\Gamma(r_n + 1 - s_n)}{\Gamma(r_n + 1 + s_n)} \Gamma(2s_n + 1)]. \end{aligned}$$

Setting now $s_n = \alpha_n r_n$ with $\alpha_n \in (0,1)$, we obtain with Stirling's formula

$$e_n \leq C\sigma^{(M-n+2)2\theta} r_n^{1/2} [(\lambda d)^{2\alpha_n} [\frac{(1-\alpha_n)^{1-\alpha_n}}{(1+\alpha_n)^{1+\alpha_n}}]]^{r_n}.$$

The function $f_{\lambda,d}(\alpha) = (\lambda d)^{2\alpha} \frac{(1-\alpha)^{1-\alpha}}{(1+\alpha)^{1+\alpha}}$ satisfies

$$0 < \inf_{0<\alpha<1} f_{\lambda,d}(\alpha) =: f_{\lambda,d}(\alpha_{min}) < 1 \qquad \text{with } \alpha_{min} = \frac{1}{\sqrt{1 + \lambda^2 d^2}}.$$

Set $f_{min} = f_{min}(\lambda, d) =: f_{\lambda,d}(\alpha_{min})$ and select $\alpha_m = \alpha_{min}$ for $2 \leq n \leq M+1$. Hence, for $r_n = \lfloor \mu n \rfloor$ we have

$$e_n \leq C\sigma^{(M-n+2)2\theta} r_n^{\frac{1}{2}} f_{min}^{r_n} \leq C\sigma^{(M-n+2)2\theta} (\mu n)^{\frac{1}{2}} f_{min}^{\mu n} \leq C\sigma^{M2\theta} (\mu(M+1))^{\frac{1}{2}} (\sigma^{(-n+2)2\theta} f_{min}^{\mu n}).$$

Let

$$\mu \geq \frac{2\theta \ln(\sigma)}{\ln(f_{min})}. \tag{3.28}$$

Then, $f_{min}^{\mu n} \leq \sigma^{n 2\theta}$ and, consequently,

$$e_n \leq C\sigma^{M2\theta}(\mu(M+1))^{\frac{1}{2}}(\sigma^{4\theta}) \leq C\sigma^{M2\theta}(\mu(M+1))^{\frac{1}{2}}.$$

Combining the above estimates yields

$$\|u-U\|_J^2 \leq C \log[\max(\mu(M+1), 2)] \max[\sigma^{2M}, \sigma^{M2\theta}(\mu(M+1))^{\frac{1}{2}}].$$

Observing that $N = \mathrm{NRDOF}(\mathcal{V}(\mathcal{M}_{M,\sigma}, \underline{r}; \mathbb{R}^d)) \leq CM^2$ completes the proof. □

Remark 3.23 From a practical point of view it may be more convenient to use a fixed approximation order r on a geometric partition $\mathcal{M}_{M,\sigma}$. In this case, exponential convergence results for all $\sigma \in (0,1)$ provided that r is proportional to the number of layers, i.e., $r = \lfloor \mu(M+1) \rfloor$. Indeed, we see from the proof of Theorem 3.22, that

$$\|u-U\|_J \leq C \max(\sigma^{2M}, r^{\frac{1}{2}} f_{min}^r) \leq C \exp(-br) \leq C \exp(-bN^{1/2}).$$

Note that condition (3.28) on the slope $\mu > 0$ is not necessary in this case.

4 Numerical Experiments

In this section we verify the theoretical results of Section 3 in a series of numerical experiments. The non-linear problems of the form (2.6) are solved very accurately with Newton's iteration method so that the overall error is governed by the error of the DG discretization.

4.1 Smooth Solution

We consider the DG performance for

$$u'(t) = \cos(\frac{t+u(t)}{5}),\ t \in J = (0,10), \qquad u(0) = 0 \tag{4.1}$$

with exact solution $u(t) = -t + 10\arctan(\frac{t}{5})$. Obviously, this solution is analytic in $\overline{J}$. In Figure 1 we present the h- and p-version of the DG method for this problem and plot the relative $L^\infty(J)$-errors against $\mathrm{NRDOF}(\mathcal{V}(\mathcal{M}, r; \mathbb{R}^d))$.

For the h-version DG method, $\mathcal{M}$ is chosen to be an equidistant time mesh consisting of 2^i time steps, $i \geq 0$. The approximation r order is kept fixed and results for $r = 0, \ldots, 4$ are shown. We can clearly see the slopes $-(r+1)$ predicted by Corollary 3.15 and (3.25). These slopes correspond to algebraic convergence rates of order $r+1$.

In the p-version DG method, we increase the order r on fixed partitions consisting of 1, 5, 10 and 20 time steps, respectively. For the analytic solution of (4.1), the p-version results in exponential rates of convergence, in agreement with Theorem 3.18. Note that we achieve a relative error of 10^{-14} with less than 100 degrees of freedom, whereas an order of magnitude more degrees of freedom has to be employed in the h-version with $r = 4$ to obtain the same accuracy.

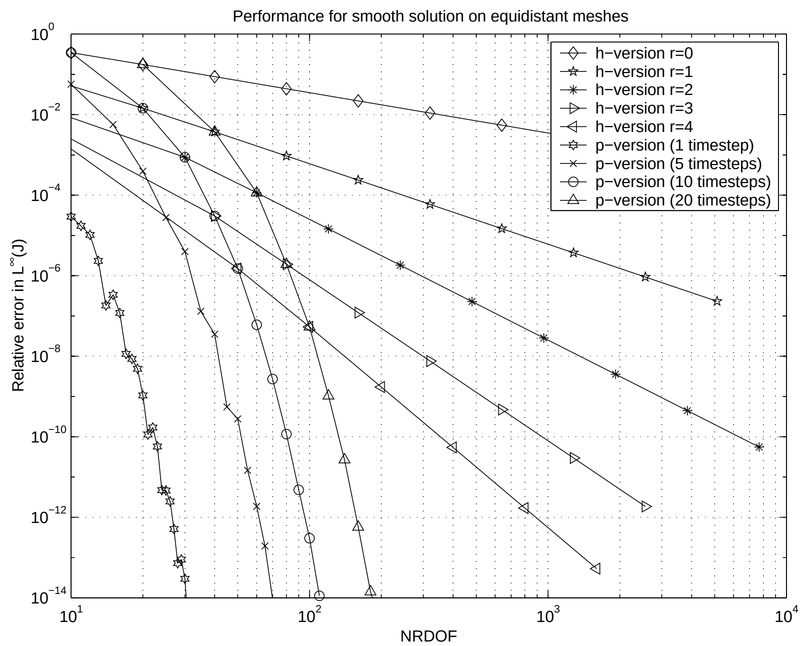

Figure 1: Results for smooth solution on equidistant time steps.

4.2 Singular Solution

The second example we consider is

$$u'(t) = \sqrt{t}\sin(u(t)),\ t \in J = (0,1), \qquad u(0) = 2\arctan(\exp(0)), \tag{4.2}$$

with exact solution $u(t) = 2\arctan(\exp(\frac{2x^{\frac{3}{2}}}{3}))$. This solution is analytic in $\overline{J} \setminus \{0\}$ and has a singularity of the form (3.27) near $t = 0$. Globally, we only have $u \in W^{1,\infty}(J)$ and $u \notin W^{2,\infty}(J)$. Due to this lack of global smoothness the performance on equidistant time meshes is rather poor. In Figure 2 we present results for the h- and p-version. In the h-version on uniform temporal meshes, we see an optimal convergence rate of -1 for $r = 0$, whereas for higher orders r the optimal rates are not achieved anymore. We obtain a slope of about -1.5, which is justified theoretically by the $N^{-\min(r,s_0)-1}$ bound in (3.25) (see also Corollary (3.15)). The p-version for (4.2) results also in algebraic rates of convergence as predicted in Section 3. Corollary 3.15 gives an algebraic rate of at least -0.5. Since this estimate is slightly suboptimal with respect to r (see Remarks 3.11 and 3.16), a rate of -1.5 can be expected. In Figure 2, however, we see an algebraic rate of -3 for the p-version. This doubling of the convergence rate is a well known phenomenon in the p-version (see, e.g., [17]) and can be explained if the regularity of the exact solution is measured in certain weaker, weighted Sobolev spaces. We refer also to [14] for results in this direction for the DG discretization of parabolic partial differential equations. However, for non-smooth solutions the use of high-order approximations is not particularly advantageous without any local h-refinement towards $t = 0$.

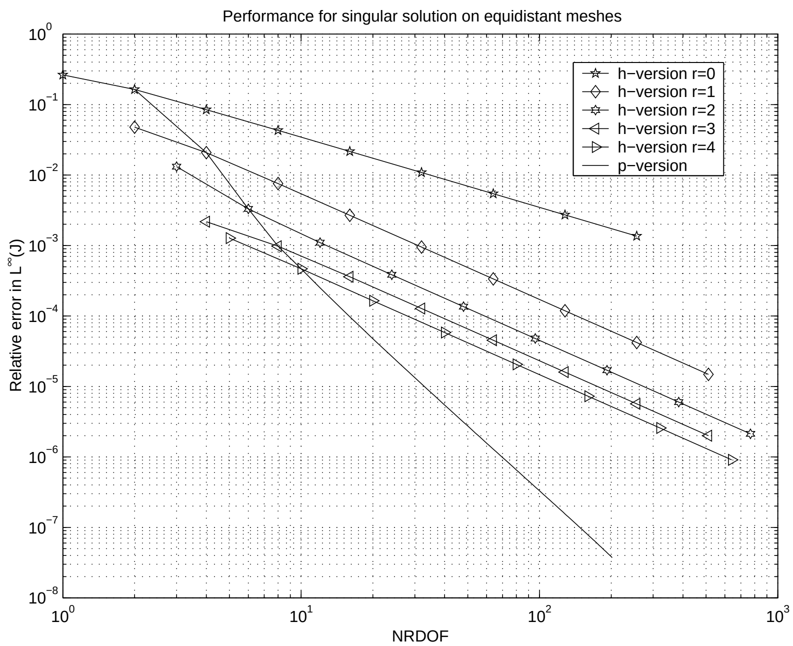

Figure 2: Results for singular solution on equidistant time steps.

In the hp-version DG method we employ geometrically refined partitions $\mathcal{M}_{M,\sigma}$ and we select an approximation order r that is proportional to the number of the layers, $r = \text{int}(\mu(M+1))$. In Figure 3 the performance of the hp-version is considered for various values of μ and for $\sigma = 0.2$. All the curves show exponential rates of convergence and confirm the result in Theorem 3.22. From the proof of Theorem 3.22, we can see that the "optimal" value of μ depends on the strength of the singularity. However, this dependence on μ seems not to be very sensitive. A relative error of 10^{-15} can be obtained with already around $N = 256$ degrees of freedom. It is also known that the error in the hp-version of the DG method can be orders of magnitude smaller when the grading factor σ is optimally chosen. This question is addressed numerically in Figure 4 where the performance for $\mu = 2.25$ and various grading factors is depicted. The best results are obtained for σ in the range of 0.2 and 0.25.

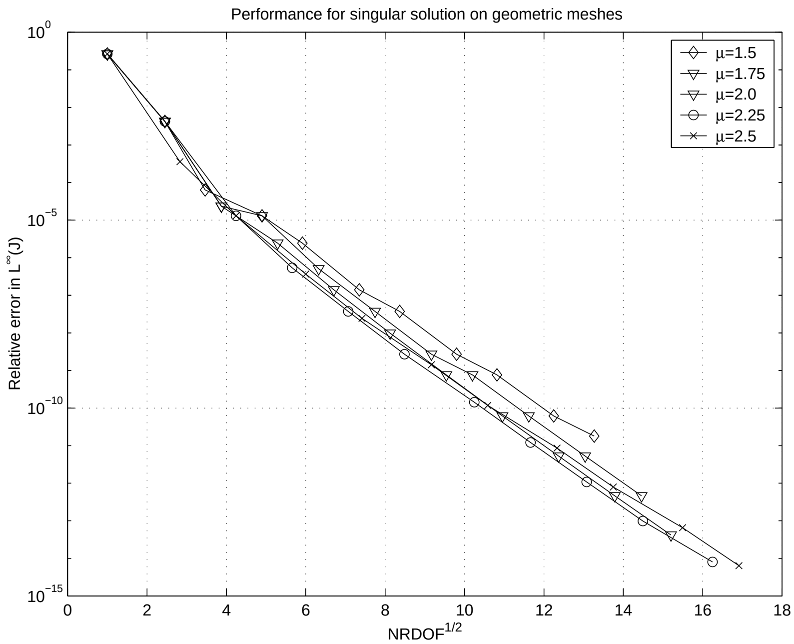

Figure 3: Results for singular solution on geometric time steps.

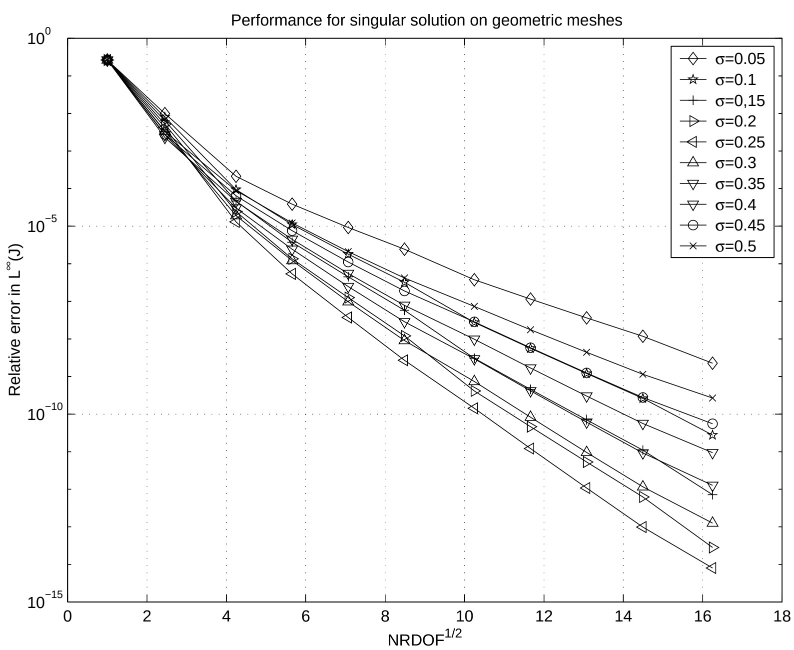

Figure 4: Results for singular solution on geometric time steps.

5 Conclusions

In this paper we have derived a-priori error estimates for the DG method that are explicit in the time steps, the approximation orders, and in the regularity of the exact solution. To our knowledge, these are the first estimates that rigorously prove that p-version and spectral convergence is possible in the integration of initial value problems. Our results and numerical experiments indicate in fact that the use of high-order approximations is very advantageous in those parts of the time interval where the solution is very smooth or analytic, whereas in the presence of solution singularities an appropriate h-refinement is necessary. For the hp-version of the DG method, it has been shown that a geometric refinement gives exponential rates of convergence for piecewise analytic solutions with temporal singularities.

The adaptive selection of the temporal meshwidth k and the approximation order r will be dealt with in future work.

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