Some results on the evolution of thin viscous jets

Marco A. Fontelos
Institute for Mathematics and its Applications, University of Minnesota,
Minneapolis, Minnesota 55455.

August 1, 1999

Abstract

In this paper we present some new resuls concerning the evolution of thin fluid jets. Well posedness up to break-up time of the one-dimensional system discribing this evolution is proved and a lower bound on the break-up time is given. We prove the existence of solutions which are infinitesimal perturbations of the solutions of the nonviscous jets in the limit of very low viscosities and the existence of solutions which develop very long and thin threads previous to break-up in the limit of very large viscosities.

1 Introduction

This paper contains some results on the evolution and break-up of thin fluid jets with a free boundary. This evolution is described by the so-called one-dimensional system which consists on the following equations:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = \frac{1}{\delta}\frac{1}{h^2}\frac{\partial h}{\partial z} + \mu \frac{1}{h^2}\frac{\partial}{\partial z}\left(h^2\frac{\partial v}{\partial z}\right) , \tag{1.1}$$

$$\frac{\partial h}{\partial t} + v\frac{\partial h}{\partial z} = \frac{h}{2}\frac{\partial v}{\partial z} \tag{1.2}$$

where δ and μ are adimensional parameters. The parameter μ depends linearly on the viscosity, and we will occasionally refer to it as "viscosity". The z coordinate is considered along the axis of the jet and runs from $-\infty$ to $+\infty$. The function $h(z,t)$ represents the distance of a point of the free boundary to the axis while the function $v(z,t)$ represents the velocity of the fluid along the axis.

The system (1.1), (1.2) has to be complemented with appropriate initial and boundary conditions. As initial conditions, we will take the initial geometry and velocity distribution along the jet:

$$h(z,0) = h_0(z) ,$$
$$v(z,0) = v_0(z) .$$

Both $h_0(z)$ and $v_0(z)$ will be assumed to be smooth enough. If we model the evolution of a free jet, it is natural to impose $h_0(z) > 0$, and

$$h(z,t) \to R > 0 \text{ as } |z| \to \infty ,$$
$$v(z,t) \to 0 \text{ as } |z| \to \infty .$$

These conditions imply that the jet is stationary at infinity. We will assume without loss of generality that $R = 1$.

The system (1.1), (1.2) has been known for a long time (cf. [2] for a general reference on the subject). It is deduced as an asymptotic limit of the Navier-Stokes system corresponding to longitudinal perturbations with a characteristic wave length much larger than the diameter of the jet.

There are several phenomena that are thought to be accurately described by the one dimensional system. One of them is the break-up locally near the break-up point and time (cf. [3] and [4]). Another one is the formation of iterated structures consisting of long and thin threads previous to break-up in the very viscous regime (cf. [10]).

Break-up analysis has been almost exclusively done by searching for the self-similar solutions of (1.1), (1.2). In [3] and [4] a self-similar solution was numerically found. In the non-viscous regime, there is a two parameter family of self-similar solutions (cf. [11], [5]) and even a much larger family of solutions which develop other kind of singularities including some of fractal type (see [5], [6]).

In Section 2 we introduce a reformulation of (1.1), (1.2) in terms of lagrangian coordinates which is more suitable for analysis. Section 3 is devoted to the analysis of the well-posedness of the system. Sections 4 and 5 contain the analysis of the asymptotic limits corresponding (formally) to $\mu \to \infty$ and $\delta \to \infty$ respectively. In Section 6 we deduce a lower bound to the break-up time and introduce a new reformulation of the system. Sections 7 and 8 are devoted to the analysis of certain solutions of (1.1), (1.2) in the limits of very high and very low viscosities.

2 Reduction of the 1-dimensional system to a single PDE

In this Section we shall describe a new representation of the one-dimensional system for the evolution of the free boundary of a thin fluid tube, in terms of a single Partial Differential Equation. This formulation will be useful in the analysis performed in the rest of the paper. We define the lagrangian evolution of the position of a point $z(\zeta, t)$ by means of the equation:

$$\frac{Dz(\zeta,t)}{Dt} = v(z(\zeta,t),t) \, , \tag{2.1}$$

$$z(\zeta,0) = \zeta \, , \tag{2.2}$$

where from now on $\frac{D}{Dt}$ stands by the lagrangian derivative (i.e. for fixed ζ). Notice that for a given function $f(z,t)$ there holds:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z}, \tag{2.3}$$

where $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial t}$ are the partial derivatives for fixed t and z respectively.

Differentiating (2.1), (2.2) with respect to ζ, we readily derive the equations:

$$\frac{D}{Dt}\left(\frac{\partial z}{\partial \zeta}\right) = \frac{\partial v(z(\zeta,t),t)}{\partial z}\frac{\partial z}{\partial \zeta}\,, \tag{2.4}$$

$$\frac{\partial z}{\partial \zeta}(\zeta,0) = 1\,. \tag{2.5}$$

Using (2.3) we can rewrite (2.4) as:

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \zeta}\right) + v\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial \zeta}\right) = \frac{\partial v}{\partial z}\frac{\partial z}{\partial \zeta}\,. \tag{2.6}$$

We then define $G(\zeta,t) = h^2(z(\zeta,t),t)\frac{\partial z}{\partial \zeta}(\zeta,t)$. Taking into account (1.2) and (2.6) we deduce that:

$$\frac{DG}{Dt} = 0\,. \tag{2.7}$$

Then:

$$h^2(z(\zeta,t),t) = \frac{h^2(\zeta,0)}{\frac{\partial z}{\partial \zeta}(\zeta,t)}\,. \tag{2.8}$$

Using (2.1), (2.3) into (1.1) it follows that:

$$\frac{Dv}{Dt} = \left(\delta h^2\frac{\partial z}{\partial \zeta}\right)^{-1}\frac{\partial h}{\partial \zeta} + \mu\left(h^2\frac{\partial z}{\partial \zeta}\right)^{-1}\frac{\partial}{\partial \zeta}\left(h^2\left(\frac{\partial z}{\partial \zeta}\right)^{-1}\frac{\partial v}{\partial \zeta}\right)\,. \tag{2.9}$$

Taking into account (2.8) we can rewrite (2.9) as:

$$\frac{Dv}{Dt} = \frac{1}{\delta h^2(\zeta,0)}\frac{\partial}{\partial \zeta}\left(\left(\frac{\partial z}{\partial \zeta}\right)^{-\frac{1}{2}}h(\zeta,0)\right) + \mu\frac{1}{h^2(\zeta,0)}\frac{\partial}{\partial \zeta}\left(h^2(\zeta,0)\left(\frac{\partial z}{\partial \zeta}\right)^{-2}\frac{\partial v}{\partial \zeta}\right)\,. \tag{2.10}$$

We now eliminate v_0 from (2.10) using (2.1). We then obtain the equation:

$$\frac{\partial^2 z}{\partial t^2} = \frac{1}{\delta h^2(\zeta,0)}\frac{\partial}{\partial \zeta}\left(\left(\frac{\partial z}{\partial \zeta}\right)^{-\frac{1}{2}}h(\zeta,0)\right) + \mu\frac{1}{h^2(\zeta,0)}\frac{\partial}{\partial \zeta}\left(h^2(\zeta,0)\left(\frac{\partial z}{\partial \zeta}\right)^{-2}\frac{\partial^2 z}{\partial \zeta\partial t}\right)\,. \tag{2.11}$$

Finally, equation (2.11) can be further simplified by introducing a new spatial variable:

$$ds = h^2(\zeta,0)d\zeta\,, \tag{2.12}$$

with the initial condition $s = 0$ at $\zeta = 0$.

Notice that (2.8) and (2.12) imply that:

$$\frac{\partial z}{\partial s} = \frac{1}{h^2(\zeta,0)}\frac{\partial z}{\partial \zeta} = \frac{1}{h^2(\zeta,t)} \equiv u\,. \tag{2.13}$$

Thus, the function u solves the equation:

$$\frac{\partial^2 u}{\partial t^2} - \delta^{-1}\frac{\partial^2}{\partial s^2}\left(\frac{1}{\sqrt{u}}\right) = -\mu\frac{\partial^3}{\partial s^2\partial t}\left(\frac{1}{u}\right)\,, \tag{2.14}$$

that using the relation between u and h given in (2.13) can be written in the equivalent way

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{h^2}\right) = \delta^{-1}\frac{\partial^2}{\partial s^2}(h) - \mu\frac{\partial^3}{\partial s^2 \partial t}\left(h^2\right) . \tag{2.15}$$

Let us remark that the three terms in (2.14) have physical meaning. The term u_{tt} comes from the inertial term $v_t + vv_z$ in (1.1). The term $\delta^{-1}\left(\frac{1}{\sqrt{u}}\right)_{ss}$ is associated to the surface tension (first term at the right-hand side of (1.1)), and, finally, the term $\mu\left(\frac{1}{u}\right)_{sst}$ is associated to the viscous effects (second term at the right-hand side of (1.1)).

Among the equivalent formulations (2.14), (2.15) we will use the equation (2.14) for u because, although the function h has a more clear physical meaning, the function u will be more convenient for our analysis. We need to complement (2.14) with suitable initial data. Taking into account (2.8) it follows that

$$u(s,0) = u_0(s) = \frac{1}{h^2(s,0)}, \quad -\infty < s < \infty . \tag{2.16}$$

On the other hand, differentiating (2.1) with respect to ζ and multiplying by $\frac{1}{h^2(\zeta,0)}$ we deduce

$$\frac{D}{Dt}\left(\frac{1}{h^2(\zeta,0)}\frac{\partial z}{\partial \zeta}\right) = \frac{\partial v}{\partial z}\frac{1}{h^2(\zeta,0)}\frac{\partial z}{\partial \zeta} ,$$

whence, using (2.14) we obtain:

$$\frac{Du}{Dt} = \frac{\partial v}{\partial s}(s,0) . \tag{2.17}$$

We then obtain that $\frac{\partial u}{\partial t}(s,0)$ can be computed as $\frac{\partial v}{\partial s}$. Summarizing, we need to add to (2.14), (2.16) the condition

$$\frac{\partial u}{\partial t}(s,0) = \frac{\partial v}{\partial s}(s,0) = w_0(s), \quad -\infty < s < \infty . \tag{2.18}$$

We have then reduced the problem (1.1), (1.2) for $v(z,t)$ and $h(z,t)$ with initial conditions $h(z,0)$, $v(z,0)$ to the partial differential equation (2.14), (2.16), (2.18). Notice that once the initial data $h(z,0)$ is prescribed, (2.12) as well as the normalization condition $s=0$ at $\zeta=0$ provide the dependence $s=s(\zeta)$ and $\zeta=\zeta(s)$. From now on we then freely consider as equivalent to give a function depending on the variables (ζ,t) or (s,t).

Let us describe now the method that allows us to solve (1.1), (1.2) after a solution of the problem (2.14), (2.16), (2.18) $u=u(s,t)$ has been found. By (2.13) we already have $h=h(s,t)$, and by (2.17) it follows that we can obtain $\frac{\partial v_0}{\partial s}(s,t)$. In order to obtain $v(s,t)$ we need to prescribe the value of $v(\cdot,t)$ at a particular point. If we restrict our attention to symmetric solutions we have $v(0,t)$. Alternatively it is possible to prescribe $v(s,t)$ as $s\to+\infty$ or $s\to-\infty$. After $v(s,t)$ (or equivalently $v(\zeta,t)$) we can use (2.1), (2.2) to obtain the relation between z and ζ (or s).

Alternatively, if we restrict our attention to symmetric solutions, (2.1), (2.2) imply that $z = 0$ at $\zeta = 0$, and (2.8) implies that for each fixed value of t $dz = \frac{h^2(\zeta,0)}{h^2(\zeta,t)} d\zeta$, whence we can compute also the $z = z(\zeta,t)$ and $\zeta = \zeta(z,t)$.

If we formally integrate equation (2.14) in s and use the conditions $u_s, u_t, u_{st} \to 0$ as $|s| \to \infty$, (that follow from the fact $u \to 1$ as $|s| \to \infty$ and the structure of the equation) we obtain

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} ds - \delta^{-1} \frac{\partial}{\partial s}\left(\frac{1}{\sqrt{u}}\right)\Big|_{-\infty}^{\infty} - \mu \frac{\partial}{\partial s}\left(\frac{u_t}{u^2}\right)\Big|_{-\infty}^{\infty} = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} ds = \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} u ds = 0 \,,$$

from which it holds

$$\int_{-\infty}^{\infty} u ds = A + Bt = \int_{-\infty}^{\infty} u_0 ds + \left(\int_{-\infty}^{\infty} w_0 ds\right) t \,. \tag{2.19}$$

3 Local existence

In this Section, we notice that equation (2.14) (or (2.15)) is very convenient in order to prove well-posedness for the problem (1.1), (1.2) in the set $(s,t) \in \mathbb{R} \times [0,T)$. Formally, integrating (2.14) once in the variable t we deduce the equation

$$\frac{\partial u}{\partial t} + \mu \frac{\partial^2}{\partial s^2}\left(\frac{1}{u}\right) = \delta^{-1} \int_0^t \frac{\partial^2}{\partial s^2}\left(\frac{1}{\sqrt{u}}\right) dt' + f(s) \,, \tag{3.1}$$

where

$$f(s) = w_0(s) + \mu \frac{\partial^2}{\partial s^2}\left(\frac{1}{u_0}\right) \,,$$

can be computed by means of the initial data. The equation (3.1) must be complemented with an initial data:

$$u(s,0) = u_0(s) \,. \tag{3.2}$$

We will prove the existence and uniqueness result in the classical Hölder spaces (cf. [7] and [9]). Let $l = n + \alpha$ with $n \in \mathbb{N}$ and $\alpha \in (0,1)$. Remember that $C^{n+\alpha}(\mathbb{R})$ is the set of functions f such that

$$\|f\|_{C^{n+\alpha}(\mathbb{R})} = \|f\|_{C^n(\mathbb{R})} + \sum_{i=0}^{n} \sup_{s,s' \in \mathbb{R}} \frac{\left|f^{(i)}(s) - f^{(i)}(s')\right|}{|s - s'|^\alpha} < \infty$$

and if we denote $Q_T = \mathbb{R} \times [0,T]$ then the space $C^{n,\frac{n}{2}}(Q_T)$ is defined as the set of functions f such that

$$\|f\|_{C^{n,\frac{n}{2}}(Q_T)} = \sum_{i+2j \le n} \sup_{(s,t) \in Q_T} \left|D_s^i D_t^j f(s,t)\right| < \infty$$

and the space $C^{n+\alpha,\frac{n+\alpha}{2}}(Q_T)$ is the set of functions f such that

$$\|f\|_{C^{n+\alpha,\frac{n+\alpha}{2}}(Q_T)} = \|f\|_{C^{n,\frac{n}{2}}(Q_T)} + \sum_{i+2j \le n} \sup_{(s,t),(s',t') \in Q_T} \frac{\left|D_s^i D_t^j f(s,t) - D_s^i D_t^j f(s',t')\right|}{|s - s'|^\alpha + |t - t'|^{\frac{\alpha}{2}}} < \infty \,.$$

Theorem 3.1 *Let $0 < \alpha < 1$ and $u_0(s) \in C^{2+\alpha}(\mathbb{R})$ such that*

$$0 < m < u_0(s) < M < +\infty \ , \tag{3.3}$$

$v_0(s) \in C^{2+\alpha}(\mathbb{R})$ and $T > 0$ small enough. Then there exists a unique solution $u(s,t)$ of problem (3.1) in the space $C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)$.

Proof. Let $w = u - u_0 - v_0 t$. We can then write (3.1) in the form

$$w_t - \mu\left(\frac{w}{u_0^2}\right)_{ss} = \mu\left(\frac{v_0}{u_0^2}\right)_{ss} t + \left(\frac{1}{\sqrt{u_0}}\right)_{ss} t + \mu\frac{\partial^2}{\partial s^2}\left(\frac{1}{u_0}\sum_{n=2}^{\infty}(-1)^{n+1}\left(\frac{w+v_0 t}{u_0}\right)^n\right) +$$

$$+\int_0^t \frac{\partial^2}{\partial s^2}\left(\frac{1}{\sqrt{u_0}}\sum_{n=1}^{\infty}\frac{(-1)^n\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\Gamma(\frac{1}{2})}\left(\frac{w+v_0 t}{u_0}\right)^n\right) . \tag{3.4}$$

The equation (3.4) may then be written in the following fashion:

$$w_t - \mu\left(\frac{w}{u_0^2}\right)_{ss} = f(s,t) \ , \tag{3.5}$$

with

$$f(s,t) = f_1(s,t) + f_2(s,t) + f_3(s,t)$$

and

$$\begin{aligned} f_1(s,t) &= \mu\left(\frac{v_0}{u_0^2}\right)_{ss} t + \left(\frac{1}{\sqrt{u_0}}\right)_{ss} t \\ f_2(s,t) &= \mu\frac{\partial^2}{\partial s^2}\left(\frac{1}{u_0}\sum_{n=2}^{\infty}(-1)^{n+1}\left(\frac{w+v_0 t}{u_0}\right)^n\right) \\ f_3(s,t) &= \int_0^t \frac{\partial^2}{\partial s^2}\left(\frac{1}{\sqrt{u_0}}\sum_{n=1}^{\infty}\frac{(-1)^n\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\Gamma(\frac{1}{2})}\left(\frac{w+v_0 t}{u_0}\right)^n\right) . \end{aligned} \tag{3.6}$$

We look for solutions of (3.5) such that $w(s,0) = 0$. Classical estimates for parabolic equations (see [7] and [9]) establish then that:

$$\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} \le C\,\|f\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}$$

with C depending on $\left\|u_0^{-2}\right\|_{C^{2+\alpha}(\mathbb{R})}$.

Next we estimate $\|f_i\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}$, $i = 1,2,3$. The estimate for f_1 is immediate given the definition of Hölder spaces:

$$\|f_1\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le CT\left(\|v_0\|_{C^{2+\alpha}(\mathbb{R})} + \left\|u_0^{-1/2}\right\|_{C^{2+\alpha}(\mathbb{R})}\right) . \tag{3.7}$$

f_2 may be estimated in the following fashion:

$$\|f_2\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le \sum_{n=2}^{\infty} n^2 C^n\left(\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} + \|v_0\|_{C^{2+\alpha}(\mathbb{R})}\,T\right)^n \tag{3.8}$$

where the factor n^2 comes from the fact that we have taken two derivatives on s in (3.6) We have also used the following inequality for functions w_1 and w_2 such that $w_1(s,0) = w_2(s,0) = 0$:

$$\sup \frac{|w_1(s,t)w_2(s,t) - w_1(s',t')w_2(s',t')|}{|s-s'|^\alpha + |t-t'|^{\frac{\alpha}{2}}} \leq$$

$$\leq \sup |w_2(s,t)| \sup \frac{|w_1(s,t) - w_1(s',t')|}{|s-s'|^\alpha + |t-t'|^{\frac{\alpha}{2}}} + \sup |w_1(s',t')| \sup \frac{|w_2(s,t) - w_2(s',t')|}{|s-s'|^\alpha + |t-t'|^{\frac{\alpha}{2}}} =$$

$$= T^{\frac{\alpha}{2}} \sup \frac{|w_1(s,t) - w_1(s,0)|}{t^{\frac{\alpha}{2}}} \sup \frac{|w_2(s,t) - w_2(s',t')|}{|s-s'|^\alpha + |t-t'|^{\frac{\alpha}{2}}} +$$

$$+ T^{\frac{\alpha}{2}} \sup \frac{|w_2(s,t) - w_2(s,0)|}{t^{\frac{\alpha}{2}}} \sup \frac{|w_1(s,t) - w_1(s',t')|}{|s-s'|^\alpha + |t-t'|^{\frac{\alpha}{2}}}$$

$$\leq 2T^{\frac{\alpha}{2}} \|w_1\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \|w_2\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}$$

and estimated the terms of order higher than linear on w. We have also used the following estimate for functions of the form $g(s)w(s,t)$

$$\sup \frac{|g(s)w(s,t) - g(s')w(s',t')|}{|s-s'|^\alpha + |t-t'|^{\frac{\alpha}{2}}} \leq$$

$$\leq \sup |g(s)| \sup \frac{|w(s,t) - w(s',t')|}{|s-s'|^\alpha + |t-t'|^{\frac{\alpha}{2}}} + \sup |g(s')| \sup \frac{|w(s,t) - w(s',t')|}{|s-s'|^\alpha + |t-t'|^{\frac{\alpha}{2}}} =$$

$$= 2 \|g\|_{C^0(I\!R)} \|w\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} . \tag{3.9}$$

The constant C in (3.8) depends on $\left\|u_0^{-1}\right\|_{C^{2+\alpha}(I\!R)}$ or, equivalently, on $\left\|u_0^{-1/2}\right\|_{C^{2+\alpha}(I\!R)}$ as

$$\left\|u_0^{-1}\right\|_{C^{2+\alpha}(I\!R)} \leq K \left\|u_0^{-1/2}\right\|_{C^{2+\alpha}(I\!R)} \left\|u_0^{-1/2}\right\|_{C^{2}(I\!R)} \tag{3.10}$$

by (3.9).

Finally, we estimate f_3:

$$\|f_3\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \leq \sum_{n=1}^{\infty} n^{\frac{5}{2}} T C^n \left(\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} + \|v_0\|_{C^{2+\alpha}(I\!R)} T \right)^n \tag{3.11}$$

where we have used the following estimate for a function g such that $g(s,0) = 0$

$$\sup \frac{\left| \int_0^t g(s,\tau) d\tau - \int_0^{t'} g(s',\tau) d\tau \right|}{|s-s'|^\alpha + |t-t'|^{\frac{\alpha}{2}}} =$$

$$= \sup \frac{\left| \int_0^t \left(g(s,\tau) - g(s',\tau)\right) d\tau - \int_t^{t'} \left(g(s',\tau) - g(s',0)\right) d\tau \right|}{|s-s'|^\alpha + |t-t'|^{\frac{\alpha}{2}}} \leq CT \|g\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}$$

and estimates for the nonlinear terms identical to those used in order to estimate f_2. The constant C in (3.11) depends on $\left\|u_0^{-1/2}\right\|_{C^{2+\alpha}(I\!R)}$.

From the previous estimates, we deduce that for T small enough the series in (3.8) and (3.11) converge and in fact

$$\|f\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \leq KT + \|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} T + \|w\|^2_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} \tag{3.12}$$

when $\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} \leq R$. The constant K depends on μ, $\|v_0\|_{C^{2+\alpha}(I\!R)}$, $\left\|u_0^{-1/2}\right\|_{C^{2+\alpha}(I\!R)}$ and T by (3.7) and (3.10).

In order to solve the complete nonlinear problem, we use a Banach fixed point theorem. We consider the application $\mathcal{L}$ which assigns to a function $\overline{w} \in W$ with

$$W = \left\{w \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T) \text{ such that } \|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} \leq R \quad \text{and} \quad w(s,0) = 0\right\}$$

the solution of (3.5) satisfying $w(s,0) = 0$ for $f(s,t)$ based on $\overline{w}(s,t)$. The estimate (3.12) shows that $\mathcal{L}$ applies W on itself for T small enough.

The application $\mathcal{L}$ is contractive. In order to prove it, we take $\overline{w_1}$, $\overline{w_2} \in W$ and the corresponding solutions w_1, w_2 of (3.5) with $f(s,t)$ based on $\overline{w}_1(s,t)$ and $\overline{w}_2(s,t)$ respectively. The function $w = w_2 - w_1$ satisfies the equation (3.5) with

$$f(s,t) = F_1(s,t) + F_2(s,t)\ ;$$

$$F_1(s,t) = \sum_{n=2}^{\infty} \mu \frac{\partial^2}{\partial s^2}\left(\frac{1}{u_0}(-1)^{n+1}\left(\left(\frac{\overline{w_2}+v_0 t}{u_0}\right)^n - \left(\frac{\overline{w_1}+v_0 t}{u_0}\right)^n\right)\right), \tag{3.13}$$

$$F_2(s,t) = \sum_{n=1}^{\infty}\int_0^t \frac{\partial^2}{\partial s^2}\left(\frac{1}{\sqrt{u_0}}\frac{(-1)^n\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\Gamma(\frac{1}{2})}\left(\left(\frac{\overline{w_2}+v_0 t}{u_0}\right)^n - \left(\frac{\overline{w_1}+v_0 t}{u_0}\right)^n\right)\right). \tag{3.14}$$

We perform the derivatives on s and estimate the terms appearing in the series (3.13) and (3.14) in the following fashion

$$\|g_2^n - g_1^n\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \leq \left\|(g_2-g_1)\left(\sum_{i=0}^{n-1} g_2^{n-1-i} g_1^i\right)\right\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \leq$$

$$\leq C^n \|g_2 - g_1\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}$$

with C depending on R, T, $\|v_0\|_{C^{2+\alpha}(I\!R)}$, $\left\|u_0^{-1/2}\right\|_{C^{2+\alpha}(I\!R)}$ and $\lim_{R,T\to 0} C = 0$. We have then that for T and R small enough

$$\|w_2 - w_1\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} \leq L\,\|\overline{w_2} - \overline{w_1}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)}$$

with $L < 1$ and therefore $\mathcal{L}$ is a contraction. Under these conditions the Banach fixed point theorem guaranties the existence of a unique solution in W.

We finally remark that the conditions imposed on $u_0(s)$ guarantee the boundedness of $\left\|u_0^{-1/2}\right\|_{C^{2+\alpha}(I\!R)}$ and $\|v_0\|_{C^{2+\alpha}(I\!R)}$ on which all the constants that have appeared along the proof do depend.□

Remark 3.1 *The only requirements on $u_0(s)$ are regularity and boundedness above and below. This means that we can extend the existence results up to any time previous to the moment in which $u = 0$ or $u \to \infty$ at some point s_0.*

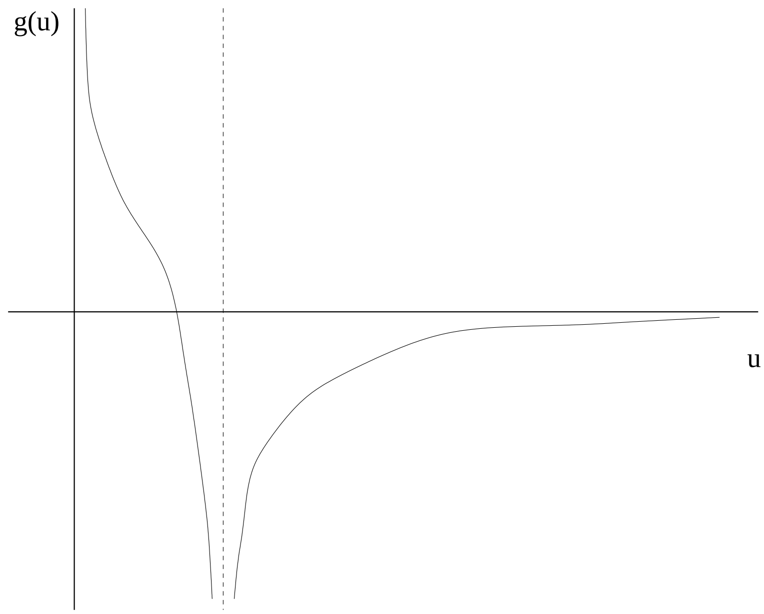

Figure 1: The function $g(u)$

4 The solution for Stokes fluids

In the case of a Stokes (inertialess) flow, the equivalent to equation (2.14) is:

$$\left(\frac{1}{\sqrt{u}}\right)_{ss} = \mu \left(\frac{1}{u}\right)_{sst} , \tag{4.1}$$

(we have made $\delta = 1$ for simplicity) with initial data

$$u(s, t = 0) = u_0(s) , \tag{4.2}$$

which implies, integrating twice on s

$$\frac{1}{\sqrt{u}} = \mu \left(\frac{1}{u}\right)_t + c_1(t)s + c_2(t) . \tag{4.3}$$

Imposing the boundary condition $u(s, t) \to 1$ as $|s| \to \infty$ (free boundary tends to a cylinder as $|z| \to \infty$), we have $c_1(t) = 0$, $c_2(t) = 1$. We arrive the to the following equation:

$$\mu u_t = u^2 - u^{\frac{3}{2}} ,$$

which can be integrated easily and give

$$g(u) = \frac{1}{\sqrt{u}} + \log\left|1 - \frac{1}{\sqrt{u}}\right| = \frac{t}{2\mu} + f(s)$$

with $f(s) = g(u_0(s))$. The graphical representation of g is given in Figure 1:

For a given initial data $u_0(s)$ with an absolute maximum at $s = 0$, it is easy to conclude that u blow-up at $s = 0$ when $t = -2\mu \left(\frac{1}{\sqrt{u_0(0)}} + \log\left|1 - \frac{1}{\sqrt{u_0(0)}}\right|\right) =$

t_0. We are now interested in the profile near that singularity. If we consider that generically $u_0(s) \sim u_0(0) - as^2$ near $s = 0$, we can deduce that, locally,

$$\frac{1}{\sqrt{u}} + \log\left|1 - \frac{1}{\sqrt{u}}\right| \sim \frac{1}{2u} \sim \frac{t_0 - t}{2\mu} + \gamma s^2 \ ,$$

i.e., the singularity is of the form

$$u = \frac{1}{(t_0 - t)} f\left(\frac{s}{(t_0 - t)^{\frac{1}{2}}}\right) \ .$$

If we remember now the relation between $u(s,t)$ and the function $z(s,t)$:

$$u(s,t) = z_s(s,t) = \frac{1}{h^2(z(s,t),t)} \ ,$$

we conclude that the function $h(z,t)$ behaves, near the singularity, in the form

$$h(z,t) = (t_0 - t)^{\frac{1}{2}} \, \phi\left((t_0 - t)^{\frac{1}{2}} \, z\right)$$

which represents a closing cylinder. This cylinder has a characteristic length

$$L \sim (t_0 - t)^{-\frac{1}{2}}$$

and a characteristic cross-section

$$S \sim (t_0 - t)^{\frac{1}{2}} \ .$$

We have just proved the following theorem with respect to system (4.1), (4.2):

Theorem 4.1 *Given a regular initial data $h_0(z)$ such that*

$$h_0 \underset{|z| \to \infty}{\to} 1$$

and such that, locally around its absolute minimum (that we suppose located at z_0),

$$h_0 \underset{z \to z_0}{\sim} a + b(z - z_0)^2$$

with $b < 0$; then, in regions

$$z - z_0 = O((t_0 - t)^{-\frac{1}{2}}) \ ,$$

we have

$$h(z,t) \sim (t_0 - t)^{\frac{1}{2}} \, \phi\left((t_0 - t)^{\frac{1}{2}} \, (z - z_0)\right) \ .$$

Let us remark two facts. The first one is that this explicit solution originates very long and thin threads previous to break-up, a fact which is seen experimentally (see [2]). The second remark is about the global validity of this solution. It is important to notice that the solution of the one-dimensional limit for Stokes system does not satisfy the relation given by (2.19), as at the break-up time and close to the break-up point s_0 we have

$$u \underset{s \to s_0}{\sim} (s - s_0)^{-2} ,$$

which implies that u is not an integrable function. Nevertheless, we shall prove in section 7 that the qualitative fact, i.e. the formation of a long and thin thread, is present in the complete one-dimensional system when the parameter μ is very large.

This kind of qualitative behavior has been the subject of several analysis. Most of them (cf. [12], [2]) have focused in the existence of self-similar solution for the Stokes system. Nevertheless, as we will see, the mechanism described in this Section is the right one when considering infinite jets.

5 The solution in the limit $\delta \to \infty$

The one-dimensional system for a fluid without surface tension is equivalent to the following PDE

$$u_{tt} + \mu \left(\frac{1}{u}\right)_{sst} = 0 ,$$

with initial data

$$\begin{aligned} u(s,0) &= u_0(s) , \\ u_t(s,0) &= u_1(s) , \end{aligned}$$

or, equivalently, integrating in t

$$u_t + \mu \left(\frac{1}{u}\right)_{ss} = f(s) , \tag{5.1}$$

with initial data

$$u(s,0) = u_0(s) . \tag{5.2}$$

The equation (5.1) is a nonlinear and no homogeneous equation. We can write it in the form

$$u_t = \mu \left(\frac{u_{ss}}{u^2} - \frac{2u_s^2}{u^3} \right) + f(s) . \tag{5.3}$$

We shall consider that functions f and u_0 are both continuous and bounded.

Theorem 5.1 *The solution of equation (5.3) with initial data (5.2) does not blow-up at finite time.*

Proof. If we denote by $s(t)$ a curve starting at a maximum (in space) of $u(s,t)$ and running, as time progresses, along the maxima of the solution, we will have that along the curve,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{ds}{dt}\frac{\partial u}{\partial s} = \mu \left(\frac{u_{ss}(s(t),t)}{(u(s(t),t))^2} - \frac{2\,(u_s(s(t),t))^2}{(u(s(t),t))^3} \right) + f(s(t)) \leq f(s(t)) ,$$

and therefore

$$\sup_s u \leq \sup_s u_0 + \left(\sup_s f\right) t \ ,$$

i.e., there is no finite time blow-up.□

This result supports the idea that surface tension is responsible for the break-up of jets, as such a break-up doesn't take place (in the one-dimensional approximation and at finite time) if it is absent.

6 A lower bound for the break-up time

In this section we will introduce a new formulation of problem (1.1), (1.2) that will be very useful in order to get a lower estimate of the break-up time. In next section we will use this new formulation in order to prove the existence of solutions of (1.1), (1.2) that develop very long and thin fluid threads previous to break-up.

The equation

$$u_{tt} - \left(\frac{1}{\sqrt{u}}\right)_{ss} = -\mu \left(\frac{1}{u}\right)_{sst}$$

may be written in the form

$$\begin{aligned} u_t &= v_s \ , & (6.1)\\ v_t &= \left(\frac{1}{\sqrt{u}} - \mu\left(\frac{1}{u}\right)_t\right)_s \ . & (6.2) \end{aligned}$$

Let us define now

$$w = \frac{1}{\sqrt{u}} - \mu\left(\frac{1}{u}\right)_t = \frac{1}{\sqrt{u}} + \mu\frac{u_t}{u^2} = \frac{1}{\sqrt{u}} + \mu\frac{v_s}{u^2} \ . \qquad (6.3)$$

We have then by (6.6) and (6.1)

$$v_s = \frac{1}{\mu}\left(w - \frac{1}{\sqrt{u}}\right) u^2 \ . \qquad (6.4)$$

By (6.2) and (6.4)

$$\begin{aligned} w_{ss} = v_{st} = \frac{\partial}{\partial t}\left[\frac{1}{\mu}\left(w - \frac{1}{\sqrt{u}}\right)u^2\right] &= \frac{1}{\mu}\left(\left(w_t + \frac{1}{2u^{\frac{3}{2}}}u_t\right)u^2 + 2\left(w - \frac{1}{\sqrt{u}}\right)uu_t\right) \\ &= \frac{1}{\mu}\left(u^2 w_t + \frac{1}{2}u^{\frac{1}{2}}u_t + 2wuu_t - 2\sqrt{u}u_t\right) \\ &= \frac{1}{\mu}\left(u^2 w_t + \left(\frac{1}{2}u^{\frac{1}{2}} + 2wu - 2\sqrt{u}\right)\frac{1}{\mu}\left(w - \frac{1}{\sqrt{u}}\right)u^2\right) \ , \end{aligned}$$

and we can then write

$$w_t - \mu u^{-2} w_{ss} = -\frac{2}{\mu}u\left(w - \frac{3}{4\sqrt{u}}\right)\left(w - \frac{1}{\sqrt{u}}\right) \ .$$

If we make the rescaling

$$t \to \frac{t}{\mu} \,, \tag{6.5}$$

we have the system

$$\left\{ \begin{array}{c} w_t - \mu^2 u^{-2} w_{ss} = -2u \left(w - \frac{3}{4\sqrt{u}} \right) \left(w - \frac{1}{\sqrt{u}} \right) \\ u_t = u^2 \left(w - \frac{1}{\sqrt{u}} \right) \end{array} \right. \,, \tag{6.6}$$

which subject to the initial conditions

$$\begin{aligned} u(s,0) &= k_1(s) \,, \\ w(s,0) &= k_2(s) \,, \end{aligned} \tag{6.7}$$

is equivalent to problem (1.1), (1.2).

Theorem 6.1 *There exists a unique regular solution of system (6.6) with initial conditions (6.7) up to a time*

$$t_0 = 32 \left[\sqrt{\sup |k_2(s)|^2 + \frac{1}{16 \sup k_1(s)}} - \sup |k_2(s)| \right] \,. \tag{6.8}$$

Proof. Let us call $F(w,u)$ the right hand side of the first equation in (6.6), i.e.,

$$F(w,u) = -2u \left(w - \frac{3}{4\sqrt{u}} \right) \left(w - \frac{1}{\sqrt{u}} \right) \,.$$

The function $F(w,u)$ has a maximum in $w_{\max}$ and this maximum is achieved when

$$\frac{\partial}{\partial w} F(w,u) = -2u \left(w - \frac{1}{\sqrt{u}} \right) - 2u \left(w - \frac{3}{4\sqrt{u}} \right) = -\frac{1}{2}\sqrt{u}\,(8w\sqrt{u} - 7) = 0 \,,$$

that is,:

$$w_{\max} = \frac{7}{8} \frac{1}{\sqrt{u}} \,.$$

At $w = w_{\max}$ we have

$$F(w_{\max}, u) = -\frac{2}{\mu} u \left(\frac{7}{8} \frac{1}{\sqrt{u}} - \frac{3}{4\sqrt{u}} \right) \left(\frac{7}{8} \frac{1}{\sqrt{u}} - \frac{1}{\sqrt{u}} \right) \,.$$

Therefore, by the first equation in (6.6),

$$w_t - \mu^2 u^{-2} w_{ss} \leq \frac{1}{32} \,.$$

If we write now $w = \frac{1}{32} t + \widetilde{w}$, we have the following equation for $\widetilde{w}$

$$\widetilde{w}_t - \mu^2 u^{-2} \widetilde{w}_{ss} \leq 0 \,, \tag{6.9}$$

with

$$\widetilde{w}(s,0) = k_2(s) \ . \tag{6.10}$$

By application of the maximum principle to (6.9), (6.10) we get

$$\widetilde{w}(s,t) \leq \sup |k_2(s)| \ ,$$

that is,

$$w < \sup |k_2(s)| + \frac{1}{32} t \ .$$

Considering now the second equation in (6.6), we have

$$u_t < u^2 \left(w - \frac{1}{\sqrt{u}} \right) \leq u^2 \left(\sup |k_2(s)| + \frac{1}{32} t \right)$$

and therefore

$$- \left(\frac{1}{u} \right)_t < \sup |k_2(s)| + \frac{1}{32} t$$

or integrating

$$\frac{1}{u} - \frac{1}{k_1(s)} > - \sup |k_2(s)| \, t - \frac{1}{64} t^2 \ ,$$

that is,

$$u < \frac{1}{\frac{1}{k_1(s)} - \sup |k_2(s)| \, t - \frac{1}{64} t^2} \ . \tag{6.11}$$

The function u is finite for every t such the denominator in the right hand side of (6.11) is positive. Thus, for every $t < t_0$ where

$$t_0 = 32 \left[\sqrt{\sup |k_2(s)|^2 + \frac{1}{16 \sup k_1(s)}} - \sup |k_2(s)| \right] \ ,$$

and the proof of the theorem is complete.□

If we write now the equation (6.8) in terms of (h_0, v_0) which are related to $(k_1(s), k_2(s))$ and undo the change of variables (6.5), we obtain that there exists a solution of system (2.1), (2.2) with initial conditions (h_0, v_0) for every $t < t^*$, where

$$t^* = 32\mu \left(\sqrt{\sup \left| h_0^2 v_{0,z} + h_0 \right|^2 + \frac{1}{16} \inf h_0^2} - \sup \left| h_0^2 v_{0,z} + h_0 \right| \right) \ .$$

This result implies that for bounded (above and below) initial data (h_0, v_0) the solution of (2.1), (2.2) with initial data (h_0, v_0) may be extended up to times that grow at least linearly in μ. Thus viscosity retards the break-up time of a viscous jet, a fact that is commonly observed in experiments (cf. [2] and the references included there).

7 Discussion of the solutions for very large viscosities

In this section we will construct solutions of the system (6.6) that exhibit, at very high (but finite) viscosities, a behavior similar to the solutions of the system corresponding to infinite viscosity. More precisely, we will show the appearance of filaments previous to breakup. In the first subsection we prove the main result, while the second subsection is devoted to proving a technical result used in the first one.

7.1 Filament formation

Consider again the system

$$u^2 w_t - \mu^2 w_{ss} = -u^3 (w - \frac{3}{4} u^{-\frac{1}{2}})(w - u^{-\frac{1}{2}}) \,, \tag{7.1}$$

$$u_t = u^2 (w - u^{-\frac{1}{2}}) \,. \tag{7.2}$$

Let us write

$$u = u_0 + \varepsilon \widetilde{u} \,, \tag{7.3}$$

$$w = 1 + \varepsilon \widetilde{w} \,, \tag{7.4}$$

where u_0 satisfies the equation

$$u_{0,t} = u_0^2 (1 - u_0^{-\frac{1}{2}}) \,, \tag{7.5}$$

with initial condition

$$u_0(s, 0) = U(s) \,. \tag{7.6}$$

We will assume later that ε is small enough.

We have then

$$\begin{aligned} u_0^2 \widetilde{w}_t - \mu^2 \widetilde{w}_{ss} + f_1(u_0) \widetilde{w} &= \frac{1}{\varepsilon} f_2(u_0) + f_3(u_0) \widetilde{u} \,, \\ &\quad -\varepsilon (2 u_0 \widetilde{u} + \varepsilon \widetilde{u}^2) \widetilde{w}_t + \varepsilon O(\varepsilon; u_0, \widetilde{u}, \widetilde{w}) \qquad (7.7) \\ \widetilde{u}_t &= f_4(u_0) \widetilde{u} + f_5(u_0) \widetilde{w} + \varepsilon O(\varepsilon; u_0, \widetilde{u}, \widetilde{w}) \,, \qquad (7.8) \end{aligned}$$

where

$$\begin{aligned} f_1(u_0) &= 2 u_0^3 \left(1 - \frac{7}{8} u_0^{-\frac{1}{2}}\right) > 0 \,, \\ f_2(u_0) &= -u_0^3 (1 - \frac{3}{4} u_0^{-\frac{1}{2}})(1 - u_0^{-\frac{1}{2}}) < 0 \,, \\ f_3(u_0) &= u_0^2 \left(\frac{7}{8} u_0^{-\frac{1}{2}} + \frac{3}{4} u_0^{-2}\right) > 0 \,, \\ f_4(u_0) &= u_0^2 \left(1 - \frac{3}{2} u_0^{\frac{1}{2}}\right) < 0 \,, \\ f_5(u_0) &= u_0^4 > 0 \,, \end{aligned} \tag{7.9}$$

and $O(\varepsilon; u_0, \widetilde{u}, \widetilde{w})$ denotes a series of products of powers of ε, $\widetilde{u}, u_0$ and $\widetilde{w}$ which converges for bounded $\widetilde{u}, u_0$ and $\widetilde{w}$ and ε small enough.

The form of the system (7.7)-(7.8) suggests to write $\widetilde{w} = w_1 + \widehat{w}$, where

$$u_0^2 w_{1,t} - \mu^2 w_{1,ss} + f_1(u_0) w_1 = \frac{1}{\varepsilon} f_2(u_0) \;, \tag{7.10}$$

with initial condition

$$w_1(s, 0) = 0 \;, \tag{7.11}$$

and

$$\begin{aligned} & u^2 \widehat{w}_t - \mu^2 \widehat{w}_{ss} + f_1(u_0) \widehat{w} \\ = \; & f_3(u_0)\widetilde{u} - \varepsilon(2u_0\widetilde{u} + \varepsilon\widetilde{u}^2) w_{1,t} + \varepsilon O(\varepsilon; u_0, \widetilde{u}, \widetilde{w}) \;, \quad (7.12) \\ \widetilde{u}_t \; = \; & f_4(u_0)\widetilde{u} + f_5(u_0)\widehat{w} + f_5(u_0) w_1 + \varepsilon O(\varepsilon; u_0, \widetilde{u}, \widetilde{w}) \;, \quad (7.13) \end{aligned}$$

with initial conditions

$$\widehat{w}(s, 0) = 0 \;, \;\; \widetilde{u}(s, 0) = 0 \;. \tag{7.14}$$

The solution of (7.10)-(7.11) will give us the leading order asymptotics for $\widetilde{w}$. Our goal is to show some solution exhibiting some particular behavior. Let us choose for the shake of simplicity $U(s)$ in the following way

$$U(s) = \begin{cases} 1 & \text{if } |s| > 1 \\ 1 + \delta & \text{if } |s| \le 1 \end{cases} \;. \tag{7.15}$$

The corresponding solution of (7.5) is then simply

$$u_0(s, t) = \begin{cases} 1 & \text{if } |s| > 1 \\ g(t) & \text{if } |s| \le 1 \end{cases} \;,$$

where

$$\begin{aligned} \int_{1+\delta}^{g(t)} \frac{dy}{(y^2 - y^{\frac{3}{2}})} &= t \;, \\ g(0) &= 1 + \delta \;. \end{aligned}$$

As we saw in Section 4, $g(t)$ blows-up at some time T^*. We will show that given any $T < T^*$, it is possible to find a solution (u, w) of (7.1), (7.2) in the form (7.3), (7.4) with $|\varepsilon\widetilde{u}| << |u_0|$ and $|\varepsilon\widetilde{w}| << 1$ in $\mathbb{R} \times [0, T]$.

Observe that at the right hand side of (7.10) there is a factor $\frac{1}{\varepsilon}$. We will show in the next lemma that despite this fact, w_1 is not $O(\varepsilon^{-1})$ but $O(1)$ if μ is large enough.

Lemma 7.1 *The solution of (7.10) with initial condition (7.11) satisfies the estimates*

$$\sup_{\mathbb{R}\times[0,T]} |w_1| \le \frac{C}{\mu\varepsilon} \sup_{\mathbb{R}\times[0,T]} |u_0|^3 \;,$$

$$\sup_{\mathbb{R}\times[0,T]} |w_{1,t}| \le \frac{C}{\varepsilon} \sup_{\mathbb{R}\times[0,T]} |u_0|^3 \;,$$

where C is independent of ε.

Proof. The equation (7.10) is parabolic with non-constant coefficients. As $f_2(u_0) < 0$, then by the maximum principle $w_1(s,t) < 0$. We can therefore write

$$u_0^2 \left|w_1\right|_t - \mu^2 \left|w_1\right|_{ss} + f_1(u_0) \left|w_1\right| = \frac{1}{\varepsilon} \left|f_2(u_0)\right| . \tag{7.16}$$

In fact, as $f_1(u_0) \geq 0$, we have that $|w_1(s,t)| \leq v(s,t)$ where $v(s,t)$ satisfies the equation

$$u_0^2 v_t - \mu^2 v_{ss} = \frac{1}{\varepsilon} \left|f_2(u_0)\right| \equiv g(s,t) , \tag{7.17}$$

with initial condition

$$v(s,0) = 0 . \tag{7.18}$$

Notice that

$$f_2(u_0) \leq C \left(\sup |u_0|^3\right) e^{-s^2} .$$

where sup stands here and in the rest of the section for $\sup_{\mathbb{R}\times[0,T]}$.

We can write

$$v(s,t) = \int_{-\infty}^{\infty} \int_0^t \widetilde{K}(s,t;\sigma,\tau) \frac{1}{\varepsilon} \left|f_2(u_0)\right| d\sigma d\tau ,$$

where $K(s,t;\sigma,\tau)$ is the fundamental solution of the parabolic operator $u_0^2 v_t - \mu^2 v_{ss}$, and, using the Corollary to Lemma 7.3 (see next subsection), we have

$$|v(s,t)| \leq \frac{1}{\varepsilon} \sup |u_0|^3 \int_0^t \frac{C_1}{\mu\sqrt{t-\tau}} \int_{-\infty}^{\infty} e^{-\frac{(s-\xi)^2}{C_2\mu^2(t-\tau)}} e^{-\xi^2} d\xi d\tau .$$

On the other hand,

$$\int_{-\infty}^{\infty} e^{-\frac{(s-\xi)^2}{a}} e^{-\xi^2} d\xi = \frac{\pi}{\sqrt{1+\frac{1}{a}}} e^{-\frac{s^2}{1+a}} \leq \frac{\pi}{\sqrt{1+\frac{1}{a}}} ,$$

and therefore

$$|v(s,t)| \leq C_1 \frac{\pi}{\varepsilon\mu} \sup |u_0|^3 \int_0^t \frac{1}{\sqrt{t-\tau}\sqrt{1+\frac{1}{\mu^2(t-\tau)}}} d\tau \leq C \frac{1}{\varepsilon\mu} \sup |u_0|^3 .$$

In order to estimate $w_{1,t}$, we take one derivative on t of equation (7.10) and obtain then the equation

$$u_0^2 w_{1,tt} - \mu^2 w_{1,tss} + f_1'(u_0) u_{0,t} w_1 + f_1(u_0) w_{1,t} + 2u_0 u_{0,t} w_{1,t} = \frac{1}{\varepsilon} f_2'(u_0) u_{0,t} .$$

Using formula (7.5) we can deduce the following equivalent parabolic equation for $c \equiv w_{1,t}$:

$$u_0^2 c_t - \mu^2 c_{ss} + \widetilde{f_1}(u_0) c = \frac{1}{\varepsilon} \widetilde{f_2}(u_0) , \tag{7.19}$$

where

$$\begin{aligned} \widetilde{f_1}(u_0) &= f_1(u_0) + 2u_0^3 (1 - u_0^{-\frac{1}{2}}) , \\ \widetilde{f_2}(u_0) &= \left(-f_1'(u_0) w_1 + \frac{1}{\varepsilon} f_2'(u_0)\right) u_0^2 (1 - u_0^{-\frac{1}{2}}) . \end{aligned}$$

Notice that, by equation (7.10),

$$c(s,0)=\frac{1}{\varepsilon}f_2(u_0)\ , \tag{7.20}$$

and, as $f_2'(u_0)\le 0$,

$$-f_1'(u_0)w_1+\frac{1}{\varepsilon}f_2'(u_0)=f_1'(u_0)\,|w_1|+\frac{1}{\varepsilon}f_2'(u_0)\le C\frac{1}{\varepsilon\mu}\sup|u_0|^2-\frac{1}{\varepsilon}\sup|f_2'(u_0)|\le 0\ ,$$

provided μ is taken large enough. The negativity of the terms at the right hand side of (7.19) and (7.20) implies $c(s,t)\le 0$. Moreover, given the fact that $(1-u_0^{-\frac{1}{2}})=0$ for $|s|>1$, we can estimate

$$\left|\widetilde{f_2}(u_0)\right|\le\frac{C}{\varepsilon}\left(\sup|u_0|^3\right)e^{-s^2}\ .$$

We have therefore the equation

$$u_0^2\,|c|_t-\mu^2\,|c|_{ss}+\widetilde{f_1}(u_0)\,|c|=\frac{1}{\varepsilon}\left|\widetilde{f_2}(u_0)\right|\ ,$$

which is identical in structure to equation (7.16). As we commented there, $|c|<\overline{c}$ where $\overline{c}$ is a solution of

$$\begin{aligned}u_0^2\overline{c}_t-\mu^2\overline{c}_{ss} &= \frac{1}{\varepsilon}\left|\widetilde{f_2}(u_0)\right|\ ,\\ \overline{c}(s,0) &= \frac{1}{\varepsilon}\left|f_2(u_0)\right|\ .\end{aligned}$$

Hence

$$|c|\le\int_{-\infty}^{\infty}\widetilde{K}(s,t;\sigma,0)\frac{1}{\varepsilon}\,|f_2(u_0)|\,d\sigma+\int_{-\infty}^{\infty}\int_0^t\widetilde{K}(s,t;\sigma,\tau)\frac{1}{\varepsilon}\left|\widetilde{f_2}(u_0)\right|d\sigma d\tau\equiv I_1+I_2\ ,$$

where $\widetilde{K}(s,t;\sigma,\tau)$ is the fundamental solution of the parabolic operator $u_0^2\overline{c}_t-\mu^2\overline{c}_{ss}$ (see next subsection). By the result of Lemma 7.3, we can estimate

$$\begin{aligned}|I_1| &\le C\frac{1}{\varepsilon}\sup|u_0|^3\,\frac{C_1}{\mu\sqrt{t}}\int_{-\infty}^{\infty}e^{-\frac{(s-\xi)^2}{C_2\mu^2t}}e^{-\xi^2}d\xi d\tau\ ,\\ &\le C\frac{1}{\varepsilon\mu}\sup|u_0|^3\,\frac{1}{\sqrt{t}\sqrt{1+\frac{1}{\mu^2t}}}\le\frac{C}{\varepsilon}\sup|u_0|^3\ .\end{aligned}$$

We can get for I_2 identical estimate as we obtained for $v(s,t)$ and this ends the proof.□

We next turn our attention to system (7.12)-(7.13). We can prove the following lemma:

Lemma 7.2 *If ε is small enough, then the solution of system (7.12)-(7.14) satisfies*

$$|\widehat{w}|+|\widetilde{u}|\le\frac{1}{\varepsilon\mu}h(\sup u_0,T)\ ,$$

where $h(\sup u_0,T)$ is a bounded function (for $T<T^*$) independent of ε.

Proof. Let us drop by the moment the $O(\varepsilon^2)$ terms and find estimates for the simpler problem

$$u^2\widehat{w}_t - \mu^2\widehat{w}_{ss} + f_1(u_0)\widehat{w} - \widetilde{f_3}(s,t)\widetilde{u} = 0 \,, \tag{7.21}$$

$$\widetilde{u}_t - f_4(u_0)\widetilde{u} - f_5(u_0)\widehat{w} = f_5(u_0)w_1 \,, \tag{7.22}$$

where $\widetilde{f_3}(s,t) = f_3(u_0) - 2\varepsilon u_0 w_{1,t} > 0$.

As $w_1 < 0$ and $f_5 > 0$ then we can construct local solutions (in a similar way as we did in Section 3) such that $\widetilde{u}_t \sim f_5(u_0)w_1$, $u^2\widehat{w}_t \sim \widetilde{f_3}(s,t)\widetilde{u}$ for a short time. A simple integration shows then that $|\widehat{w}| \le |w_1|$ for times small enough. Therefore, from equation (7.22) we conclude $\widetilde{u} < 0$. As $\widetilde{f_3} > 0$ then $\widehat{w} < 0$ by equation (7.21). Hence, both $\widehat{w}$ and $\widetilde{u}$ are negative for every time.

Keeping in mind that $1 \le u_0 \le \sup u_0$, equations (7.21),(7.22), (7.9), and the estimate for $|w_{1,t}|$ given in Lemma 7.1 ,we obtain from (7.21) that, in the interval $t \in [0,T]$,

$$\begin{aligned} 0 &= u^2\left|\widehat{w}\right|_t - \mu^2\left|\widehat{w}\right|_{ss} + f_1(u_0)\left|\widehat{w}\right| - \widetilde{f_3}(u_0)\left|\widetilde{u}\right| \\ &\ge u^2\left|\widehat{w}\right|_t - \mu^2\left|\widehat{w}\right|_{ss} + \frac{1}{4}\left|\widehat{w}\right| - \left(\frac{13\sup u_0^2}{8} + 2C\sup u_0^4\right)\left|\widetilde{u}\right| \\ &\ge u^2\left|\widehat{w}\right|_t - \mu^2\left|\widehat{w}\right|_{ss} - \left(\frac{13\sup u_0^2}{8} + 2C\sup u_0^4\right)\left|\widetilde{u}\right| \\ &\ge \frac{1}{2}\left|\widehat{w}\right|_t - \mu^2\left|\widehat{w}\right|_{ss} - \left(\frac{13\sup u_0^2}{8} + 2C\sup u_0^4\right)\left|\widetilde{u}\right| \,. \end{aligned}$$

We have assumed that $u > \frac{1}{2}$ for $t < T$ as it is the case for u_0 and well be the case for u when taking μ large enough. Analogously, we obtain from (7.22),

$$\begin{aligned} \sup u_0^4 \sup|w_1| &\ge f_5(u_0)\left|w_1\right| \ge \left|\widetilde{u}\right|_t + \frac{1}{2}\left|\widetilde{u}\right| - \sup u_0^4\left|\widehat{w}\right| \\ &\ge \left|\widetilde{u}\right|_t - \sup u_0^4\left|\widehat{w}\right| \,. \end{aligned}$$

We will compare $\widehat{w}, \widetilde{u}$ with $x(t), y(t)$ solution of the system of ordinary differential equations

$$\left\{ \begin{array}{l} \frac{1}{2}\frac{dx}{dt} - (\frac{13a}{8} + c)y = 0 \,, \\ \frac{dy}{dt} - a^2 x = b \,, \end{array} \right. \tag{7.23}$$

where $a = \sup u_0^2$, $b = \sup u_0^4 \sup|w_1| \le \frac{1}{\varepsilon\mu}\sup u_0^7$ (by Lemma 7.1) and $c = 2C\sup u_0^4$; with initial conditions $x(0) = y(0) = 0$. The solutions $x(t), y(t)$ are always positive as can be easily verified by writing them in the form of series expansions (the coefficients are positive). Moreover, we can write

$$|x(t)| + |y(t)| \le \frac{1}{\varepsilon\mu}h(\sup u_0, T) \quad \text{for } t \in [0,T] \,.$$

Notice that $h(\sup u_0, T)$ is independent of ε and μ.

If we define

$$\begin{aligned} \overline{w} &= |\widehat{w}| - x(t) \,, \\ \overline{u} &= |\widetilde{u}| - y(t) \,, \end{aligned}$$

then

$$u^2\overline{w}_t - \mu^2\overline{w}_{ss} - \frac{13\sup u_0^2}{8}\overline{u} \leq (\frac{1}{2} - u^2)\frac{dx}{dt} \leq 0 , \tag{7.24}$$

$$\overline{u}_t - \left(\sup u_0^4\right)\overline{w} \leq 0 . \tag{7.25}$$

From (7.24), (7.25) we obtain $\overline{u} < 0$ and $\overline{w} < 0$. Therefore

$$|\widehat{w}| + |\widetilde{u}| \leq \frac{1}{\varepsilon\mu}h(\sup u_0, T) . \tag{7.26}$$

We can incorporate the terms of order ε present in (7.12)-(7.13) and which have been dropped in the previous arguments very easily. If ε is small enough, then the signs of the inequalities above are preserved and inequality (7.26) holds.□

The conclusion of Lemmas 7.1 and 7.2 is that the solution of (7.1), (7.2) with $u(s,0) = U(s)$ ($U(s)$ given by (7.15)) and $w(s,0) = 0$ can be written in the form (7.3), (7.4) with $|\widetilde{u}|, |\widetilde{w}|$ bounded in $[0,T]$ provided $\mu > C\varepsilon^{-1}$ for some C (independent of ε).

The most remarkable fact, is that we can make $\int_{-\delta}^{\delta} u(s,t)ds$ as large as we want with just taking T sufficiently close to T^* and ε small enough (μ large enough) correspondingly. This physically has the meaning of the formation of a very long and thin thread with the mechanism described in Section 6, as the effects of the inertial terms on the dynamics are negligeable for μ large enough.

7.2 A parabolic equation with discontinuous coefficients

We consider the following parabolic Partial Differential Equation

$$c(s,t)w_t - w_{ss} = f(\mu s, t) , \tag{7.27}$$

and we will try to find a solution w such that

$$w(s,0) = 0 .$$

The function $c(s,t)$ is of the form

$$c(s,t) = \begin{cases} c_0 & \text{if } |s| > \mu^{-1} \\ a(t) & \text{if } |s| \leq \mu^{-1} \end{cases} . \tag{7.28}$$

In order to solve the problem, we are going to find a fundamental solution, i.e., a solution of the problem

$$c(s,t)K_t - K_{ss} = \delta(s-\sigma)\delta(t-\tau) . \tag{7.29}$$

Then

$$w = \int_0^t \int_{-\infty}^{\infty} K(s,t;\sigma,\tau) f(\mu\sigma,\tau) d\sigma d\tau .$$

Notice that if we make the change $s \to \mu^{-1}s$, then equation (7.27) transforms into

$$c(s,t)w_t - \mu^2 w_{ss} = f(s,t) , \tag{7.30}$$

where $c(s,t)$ is given by (7.28) with ε substituted by 1. Therefore, the solution of (7.30) is

$$w = \int_0^t \int_{-\infty}^{\infty} K(\frac{s}{\mu}, t; \sigma, \tau) f(\mu\sigma, \tau) d\sigma d\tau = \frac{1}{\mu} \int_0^t \int_{-\infty}^{\infty} K(\frac{s}{\mu}, t; \frac{\sigma}{\mu}, \tau) f(\sigma, \tau) d\sigma d\tau \ .$$

Hence, the fundamental solution of (7.30) is

$$\widetilde{K}(s,t;\sigma,\tau) \equiv \frac{1}{\mu} K(\frac{s}{\mu}, t; \frac{\sigma}{\mu}, \tau) \ ,$$

and this is the one used in the previous subsection.

The function K will depend on $(s,t;\sigma,\tau)$ but for notational simplicity we will write $K(s,t)$ in this subsection. We will also use $\varepsilon = \mu^{-1}$.

Lemma 7.3 *The fundamental solution K defined by equation (7.29) satisfies, for $|\sigma| < \varepsilon$, the estimate*

$$|K| \leq C_1 \frac{1}{\sqrt{t-\tau}} e^{-\frac{(s-\sigma)^2}{C_2(t-\tau)}} \ ,$$

where $0 < C_1,\ C_2 < C \sup a^2$ with C independent of a.

Proof. Let us, for the shake of simplicity, make the change of variables $t \to t+\tau$ so that equation (7.29) becomes

$$c(s, t+\tau) K_t - K_{ss} = \delta(s-\sigma)\delta(t) \ . \tag{7.31}$$

We can write, for $|s| < \varepsilon$, the equation

$$K_{\bar{t}} - K_{ss} = \delta(s-\sigma)\delta(\bar{t})$$

where

$$\bar{t}(t) = \int_0^t \frac{ds}{a(s+\tau)} \ ,$$

and therefore, in $|s| < \varepsilon$, a solution of (7.31) is

$$K(s,t) = \frac{1}{\sqrt{2\pi\bar{t}(t)}} e^{-\frac{(s-\sigma)^2}{2\bar{t}(t)}} \ .$$

We can compute

$$K(-\varepsilon, t) = \frac{1}{\sqrt{2\pi\bar{t}(t)}} e^{-\frac{(\varepsilon+\sigma)^2}{2\bar{t}(t)}} \ , \tag{7.32}$$

$$K_s(-\varepsilon, t) = \frac{(\varepsilon+\sigma)}{[2\pi\bar{t}(t)]^{\frac{3}{2}}} e^{-\frac{(\varepsilon+\sigma)^2}{2\bar{t}(t)}} \ . \tag{7.33}$$

The solution of (7.31) satisfies then, for $s < -\varepsilon$, the equation

$$c_0 K_t - K_{ss} = 0 \ ,$$

with conditions (7.32), (7.33). If we apply now Laplace transform

$$U(\lambda,t)=\int_{-\infty}^{-\varepsilon}e^{\lambda s}K(s,t)ds$$

we obtain

$$c_0U_t-\lambda^2U=K_s(-\varepsilon,t)-\lambda K(-\varepsilon,t)\ ,$$

with solution

$$U(\lambda,t)=\frac{1}{c_0}\int_0^t e^{\frac{1}{c_0}\lambda^2(t-\zeta)}\left[K_s(-\varepsilon,\zeta)-\lambda K(-\varepsilon,\zeta)\right]d\zeta\ .$$

Now, inverting Laplace transform, we have

$$K(s,t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{-\lambda(s+\varepsilon)}U(\lambda,t)d\lambda\ ,$$

or, introducing $\lambda=ik$,

$$K(s,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ik(s+\varepsilon)}\left[\frac{1}{c_0}\int_0^t e^{-\frac{1}{c_0}k^2(t-\zeta)}\left[K_s(-\varepsilon,\zeta)-ikK(-\varepsilon,\zeta)\right]d\zeta\right]dk\ .$$

If we exchange the order of integration and have in mind that

$$\begin{aligned}
W_1(s,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ik(s+\varepsilon)}e^{-\frac{1}{c_0}k^2t}dk=\frac{\sqrt{c_0}}{\sqrt{2\pi t}}e^{-c_0\frac{(s+\varepsilon)^2}{2t}}\ ,\\
W_2(s,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ik(s+\varepsilon)}e^{-\frac{1}{c_0}k^2t}(ik)dk=-\frac{d}{ds}\left(\frac{\sqrt{c_0}}{\sqrt{2\pi t}}e^{-c_0\frac{(s+\varepsilon)^2}{2t}}\right)\\
&=\frac{c_0^{\frac{3}{2}}(s+\varepsilon)}{\sqrt{2\pi}t^{\frac{3}{2}}}e^{-c_0\frac{(s+\varepsilon)^2}{2t}}\ ,
\end{aligned}$$

then we obtain

$$K(s,t)=\int_0^t\left[W_1(s,t-\zeta)K_s(-\varepsilon,\zeta)+W_2(s,t-\zeta)K(-\varepsilon,\zeta)\right]d\zeta=I_1+I_2\ ,$$

where

$$I_1=\frac{(\sigma+\varepsilon)\sqrt{c_0}}{2\pi}\int_0^t\left[\frac{1}{\sqrt{t-\zeta}}e^{-c_0\frac{(s+\varepsilon)^2}{2(t-\zeta)}}\frac{1}{[\bar{t}(\zeta)]^{\frac{3}{2}}}e^{-\frac{(\varepsilon+\sigma)^2}{2\bar{t}(\zeta)}}\right]d\zeta\ ,$$

$$I_2=\frac{(s+\varepsilon)c_0^{\frac{3}{2}}}{2\pi}\int_0^t\left[\frac{1}{(t-\zeta)^{\frac{3}{2}}}e^{-c_0\frac{(s+\varepsilon)^2}{2(t-\zeta)}}\frac{1}{\sqrt{\bar{t}(\zeta)}}e^{-\frac{(\varepsilon+\sigma)^2}{2\bar{t}(\zeta)}}\right]d\zeta\ .$$

If we consider that

$$\frac{\inf a}{\zeta}\le\frac{1}{\bar{t}(\zeta)}\le\frac{\sup a}{\zeta}\ ,$$

then

$$\begin{aligned}
|I_1|&\le C(\varepsilon+\sigma)\int_0^t\left[\frac{1}{\sqrt{t-\zeta}}e^{-c_0\frac{(s+\varepsilon)^2}{2(t-\zeta)}}\frac{1}{[\bar{t}(\zeta)]^{\frac{3}{2}}}e^{-\frac{(\varepsilon+\sigma)^2\inf a}{2\zeta}}\right]d\zeta\ ,\\
&\le C\sup a^{\frac{3}{2}}(\varepsilon+\sigma)\int_0^t\left[\frac{1}{\sqrt{t-\zeta}}e^{-c_0\frac{(s+\varepsilon)^2}{2(t-\zeta)}}\frac{1}{\zeta^{\frac{3}{2}}}e^{-\frac{(\varepsilon+\sigma)^2\inf a}{4\zeta}}\right]d\zeta\ ,
\end{aligned}$$

and we arrive to an integral of the type

$$\int_0^t \left[\frac{1}{\sqrt{t-\zeta}} e^{-\frac{a}{(t-\zeta)}} \frac{1}{\zeta^{\frac{3}{2}}} e^{-\frac{b}{\zeta}} \right] d\zeta = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \equiv I_{11} + I_{12} \, , \tag{7.34}$$

where

$$\begin{aligned} I_{11} &\leq C \frac{\sqrt{2}}{\sqrt{t}} e^{-\frac{a}{t}} \int_0^{\frac{t}{2}} \frac{1}{\zeta^{\frac{3}{2}}} e^{-\frac{b}{\zeta}} d\zeta \leq C \frac{\sqrt{2}}{\sqrt{t}} e^{-\frac{a}{t}} e^{-\frac{b}{2t}} \int_0^{\frac{t}{2}} \frac{1}{\zeta^{\frac{3}{2}}} e^{-\frac{b}{2\zeta}} d\zeta \\ &= C \frac{2}{b^{\frac{1}{2}} \sqrt{t}} e^{-\frac{a}{t}} e^{-\frac{b}{2t}} \int_{\frac{b}{t}}^{\infty} \frac{1}{u^{\frac{1}{2}}} e^{-u} du \leq C \frac{1}{b^{\frac{1}{2}} \sqrt{t}} e^{-\frac{a}{2t}} e^{-\frac{b}{2t}} \, , \end{aligned}$$

$$\begin{aligned} I_{12} &= \int_{\frac{t}{2}}^t \frac{1}{\sqrt{t-\zeta}} e^{-\frac{a}{(t-\zeta)}} \frac{1}{\zeta^{\frac{3}{2}}} e^{-\frac{b}{\zeta}} d\zeta = \frac{1}{t} \int_{\frac{1}{2}}^1 \frac{1}{\sqrt{1-s}} e^{-\frac{a}{t(1-s)}} \frac{1}{s^{\frac{3}{2}}} e^{-\frac{b}{ts}} ds \\ &\leq \frac{1}{t} e^{-\frac{a}{2t}} e^{-\frac{b}{t}} \int_{\frac{1}{2}}^1 \frac{1}{\sqrt{1-s}} \frac{1}{s^{\frac{3}{2}}} ds \leq C \frac{1}{b^{\frac{1}{2}} \sqrt{t}} e^{-\frac{a}{2t}} e^{-\frac{b}{2t}} \, . \end{aligned}$$

If we now consider that

$$\begin{aligned} a + b = c_0 \frac{(s+\varepsilon)^2}{2} + \frac{\inf a(\varepsilon+\sigma)^2}{4} &\geq \frac{c_0}{2} \left[(s+\varepsilon)^2 + (\varepsilon+\sigma)^2 \right] \\ &\geq \frac{c_0}{4} (s-\sigma)^2 \, , \end{aligned}$$

then it is possible to conclude

$$|I_1| \leq C \sup a^{\frac{3}{2}} \frac{1}{\sqrt{t}} e^{-\frac{c_0}{8t}(s-\sigma)^2} \, .$$

On the other hand, we have

$$I_2 \leq C(s+\varepsilon) \sup a^{\frac{1}{2}} \int_0^t \left[\frac{1}{(t-\zeta)^{\frac{3}{2}}} e^{-c_0 \frac{(s+\varepsilon)^2}{2(t-\zeta)}} \frac{1}{\zeta} e^{-\frac{\inf a(\varepsilon+\sigma)^2}{2\zeta}} \right] d\zeta$$

and if we make the change of variables $\zeta \to t - \zeta$, then we obtain an integral of the type (7.34) that can be estimated in the same way and therefore

$$|I_2| \leq C \sup a^{\frac{1}{2}} \frac{1}{\sqrt{t}} e^{-\frac{c_0}{8t}(s-\sigma)^2} \, .$$

We obtain, by symmetry, identical expression for $s > \varepsilon$. Finally, undoing the change of variables $t \to t + \tau$ and setting $C_1 = C \sup a^{\frac{3}{2}}$, $C_2 = \frac{8}{c_0}$, the proof is complete.□

Corollary 7.4 $\widetilde{K}$ *satisfies*

$$\left| \widetilde{K} \right| \leq \frac{C_1}{\mu} \frac{1}{\sqrt{t-\tau}} e^{-\frac{(s-\sigma)^2}{C_2 \mu^2 (t-\tau)}} \, ,$$

for $|\sigma| < 1$.

8 Discussion of the solutions for very low viscosities

In [5] and [6], some singular solutions for the limit $\mu \equiv 0$ in system (2.1), (2.2) were obtained. These solutions are basically of two types:

1) solutions that develop some kind of singularity (cusps for instance) in $h(z,t)$ at finite time. This family also includes fractal-like solutions.

2) solutions that break-up in finite time, i.e. such that $h(z,t) \to 0$ in finite time at some point z_0.

We will study the first type of solutions in the context of slightly viscous fluids. More precisely, we are going to find solutions for (slightly) viscous fluids which are an infinitesimal perturbation of the solutions of the first type. Let us consider a function $\overline{u}(s,t)$ of this type. That is, a solution of the equation

$$\overline{u}_{tt} - \left(\frac{1}{\sqrt{\overline{u}}}\right)_{ss} = 0\,, \tag{8.1}$$

with initial condition

$$\begin{aligned} \overline{u}(s,0) &= u_0(s)\ ,\ u_0 \text{ analytic}\ , && (8.2)\\ \overline{u}_t(s,0) &= v_0(s)\ ,\ v_0 \text{ analytic}\ , && (8.3) \end{aligned}$$

and limit condition

$$\overline{u}(s,t) \to 1 \quad \text{as} \quad |s| \to \infty\ ;$$

such that $u(s,t)$ stops being analytic at some time $t = T^*$. In particular, we will consider a function $\overline{u}(s,t)$ containing cusps such that $\overline{u} \in C^{\alpha}(\mathbb{R}\times [0,T^*])$ but $\overline{u} \notin C^{\alpha+\varepsilon}(\mathbb{R}\times[0,T^*])$. Let us also assume that

$$\begin{aligned} \|\overline{u}(s,t)-1\|_{C^{\alpha}(\mathbb{R}\times[0,T])} &\le \nu \ \text{ for } T \le T^* \ (\nu \text{ small})\ ,\\ \left\|\left(\frac{1}{\overline{u}(s,t)}\right)_t\right\|_{C^{\alpha}(\mathbb{R}\times[0,T])} &\le C(T) \ \text{ for } T < T^*\ . \end{aligned}$$

The existence of these solutions follows immediately from the results of [5]. Nevertheless, we will describe them briefly in the Appendix.

Now we consider a solution of the viscous equation (2.14) (with $\delta = 1$, for simplicity) with initial data (8.2), (8.3) and call it $u(s,t)$. Then we have the following theorem:

Theorem 8.1 *The solution of equation (2.14) with initial data (8.2), (8.3), satisfies the inequality*

$$\|u(s,t)-\overline{u}(s,t)\|_{C^{\alpha}(\mathbb{R}\times[0,T])} \le C(T)\mu \ \textit{ for any } T < T^* \textit{ fixed}\ .$$

Proof. From (8.1) and (2.14) (with $\delta = 1$) we have

$$(u(s,t)-\overline{u}(s,t))_{tt} - \left(\frac{1}{\sqrt{u(s,t)}} - \frac{1}{\sqrt{\overline{u}(s,t)}}\right)_{ss} + \mu\left(\frac{1}{u(s,t)}\right)_{sst} = 0\,. \tag{8.4}$$

If we denote $\widetilde{u}(s,t)=u(s,t)-\overline{u}(s,t)$, we have then by (8.4),

$$\widetilde{u}_{tt}-\left(\frac{1}{\sqrt{\overline{u}+\widetilde{u}}}-\frac{1}{\sqrt{\overline{u}}}\right)_{ss}+\mu\left(\frac{1}{\overline{u}+\widetilde{u}}\right)_{sst}=0\;, \tag{8.5}$$

with $\widetilde{u}(s,0)=\widetilde{u}_t(s,0)=0$. We can formulate now the problem

$$\widetilde{u}_{tt}+\frac{1}{2}\widetilde{u}_{ss}-\mu\widetilde{u}_{sst}=\left(\frac{1}{\sqrt{\overline{u}+\widetilde{u}}}-\frac{1}{\sqrt{\overline{u}}}+\frac{1}{2}\widetilde{u}\right)_{ss}-\mu\left(\frac{1}{\overline{u}+\widetilde{u}}+\widetilde{u}\right)_{sst}\equiv f(\overline{u},\widetilde{u})$$

with homogeneous boundary conditions, and associate to it the problem

$$\begin{aligned}\widetilde{u}_{tt}+\frac{1}{2}\widetilde{u}_{ss}-\mu\widetilde{u}_{sst} &= f(\overline{u},w)\;, && (8.6)\\ \widetilde{u}(s,0) &= \widetilde{u}_t(s,0)=0\;. && (8.7)\end{aligned}$$

The solution of equation (8.5) will be a fixed point of the application W assigning to a given w the solution of (8.6), (8.7). If we apply now Laplace transform in time and Fourier transform in space to equation (8.6), we get

$$\left[\lambda^2-k^2+\mu\lambda k^2\right]U=\mu k^2F_0-k^2F_1+\mu\lambda k^2F_2\;,$$

where U, F_0, F_1, F_2 are the Laplace-Fourier transform of u and

$$\begin{aligned}f_0 &= \left(\frac{1}{\overline{u}}\right)_t\;, && (8.8)\\ f_1 &= \frac{1}{\sqrt{\overline{u}+w}}-\frac{1}{\sqrt{\overline{u}}}+\frac{1}{2}w=\frac{3}{4}(\overline{u}-1)w+\frac{3}{8}w^2+o(|\overline{u}-1|^2+w^2)\;, && (8.9)\\ f_2 &= \frac{1}{\overline{u}+w}+w-\frac{1}{\overline{u}}=2w(\overline{u}-1)+w^2+o(|\overline{u}-1|^2+w^2)\;, && (8.10)\end{aligned}$$

respectively. Therefore

$$\begin{aligned}U &= \frac{\mu k^2F_0-k^2F_1+\mu\lambda k^2F_2}{[\lambda^2-k^2+\mu\lambda k^2]}=\frac{\mu k^2F_0-k^2F_1-(\lambda^2-k^2)F_2}{[\lambda^2-k^2+\mu\lambda k^2]}+F_2\\ &= \frac{\mu k^2F_0-k^2F_1-(\lambda^2-k^2)F_2}{(\lambda-\lambda_+(k))(\lambda-\lambda_-(k))}+F_2\;,\end{aligned}$$

where

$$\lambda_\pm(k)=\frac{-\mu k^2\pm\sqrt{\mu^2k^4+4k^2}}{2}\;.$$

We can write

$$\begin{aligned}u(s,t) &= f_2-\mu\int_0^t G_1(s-\xi,t-\tau;\mu)\,[f_1(\xi,\tau)-f_1(s,t)]\,d\xi d\tau\\ &\quad+\int_0^t G_1(s-\xi,t-\tau;\mu)\,[f_1(\xi,\tau)-f_1(s,t)]\,d\xi d\tau\\ &\quad+\int_0^t G_2(s-\xi,t-\tau;\mu)\,[f_2(\xi,\tau)-f_2(s,t)]\,d\xi d\tau\\ &\equiv f_2+u_0(s,t)+u_1(s,t)+u_2(s,t)\;.\end{aligned}$$

In order to analyze the integrals (8.2), (8.3), let us perform the change of variables

$$k \to \frac{k}{\sqrt{\mu(t-\tau)}} .$$

We obtain then

$$\begin{aligned} G_1 &= -\frac{1}{\sqrt{\mu(t-\tau)}} \int_{-\infty}^{\infty} e^{ik\frac{(s-\xi)}{\sqrt{\mu(t-\tau)}}} e^{\frac{-k^2-\sqrt{k^4+4\mu^{-1}(t-\tau)k^2}}{2}} \frac{k^2}{\sqrt{k^4+4\mu^{-1}(t-\tau)k^2}} dk \\ &\equiv -\frac{1}{\sqrt{\mu(t-\tau)}} K_1\left(\frac{s-\xi}{\sqrt{\mu(t-\tau)}}, \mu^{-1}(t-\tau)\right) , \end{aligned}$$

$$\begin{aligned} G_2 &= -\frac{\mu^2}{(\mu(t-\tau))^{\frac{3}{2}}} \int_{-\infty}^{\infty} e^{ik\frac{(s-\xi)}{\sqrt{\mu(t-\tau)}}} e^{\frac{-k^2-\sqrt{k^4+4\mu^{-1}(t-\tau)k^2}}{2}} \frac{\frac{1}{2}k^4 + \frac{1}{2}k^3\sqrt{k^2+4\mu^{-1}(t-\tau)}}{\sqrt{k^4+4\mu^{-1}(t-\tau)k^2}} dk \\ &\equiv -\frac{\mu^2}{(\mu(t-\tau))^{\frac{3}{2}}} K_2\left(\frac{s-\xi}{\sqrt{\mu(t-\tau)}}, \mu^{-1}(t-\tau)\right) , \end{aligned}$$

where

$$K_1(a,b) = \int_{-\infty}^{\infty} e^{ika} e^{\frac{-k^2-\sqrt{k^4+4bk^2}}{2}} \frac{k^2}{\sqrt{k^4+4bk^2}} dk , \tag{8.11}$$

$$K_2(a,b) = \int_{-\infty}^{\infty} e^{ika} e^{\frac{-k^2-\sqrt{k^4+4bk^2}}{2}} \frac{\frac{1}{2}k^4 + \frac{1}{2}k^2\sqrt{k^4+4bk^2}}{\sqrt{k^4+4bk^2}} dk . \tag{8.12}$$

Lemma 8.2 *The function $K_1(a,b)$ defined by (8.11) can be written in the form*

$$K_1(a,b) = C_1(a,b)\frac{1}{\sqrt{b}(|a|+\sqrt{b})^2} ,$$

for some function C_1 bounded and regular.

Proof. First, we see that when $0 \le a < \infty$ we have from (8.11)

$$K_1(a,b) = \int_{-\infty}^{\infty} e^{ika} e^{\frac{-k^2-\sqrt{k^4+4bk^2}}{2}} \frac{k^2}{\sqrt{k^4+4bk^2}} dk ,$$

which is bounded for every a and b.

We write next

$$K_1(a,b) = \int_0^{\infty} e^{ika} e^{\frac{-k^2-\sqrt{k^4+4bk^2}}{2}} \frac{k^2}{\sqrt{k^4+4bk^2}} dk + \int_0^{\infty} e^{-ika} e^{\frac{-k^2-\sqrt{k^4+4bk^2}}{2}} \frac{k^2}{\sqrt{k^4+4bk^2}} dk .$$

Let us now consider the complex variable function

$$f_{\pm}(z) = e^{\pm iza} e^{\frac{-z^2-z\sqrt{z^2+4b}}{2}} \frac{z}{\sqrt{z^2+4b}} ,$$

and assume $a \ge 0$. The functions $f_{\pm}(z)$ is analytic except for a branch located in the interval $|\mathrm{Im}\,\{z\}| < 2\sqrt{b}$. We can then write

$$K_1(a,b) = \int_{\mathbb{R}^+} f_+(z)dz + \int_{\mathbb{R}^+} f_-(z)dz \equiv I_1 + I_2 .$$

We can deform the line of integration for I_1 towards the line $e^{i\frac{\pi}{4}}w$ $(w > 0)$contour $\mathrm{Im}\{z\} > 0$ and the line for I_2 towards $e^{i\frac{\pi}{4}}w$ $(w < 0)$. In this way we get

$$\begin{aligned} I_1 &= e^{i\frac{\pi}{2}} \int_0^\infty e^{ie^{i\frac{\pi}{4}}wa} e^{\frac{-iw^2 - e^{i\frac{\pi}{4}}w\sqrt{iw^2+4b}}{2}} \frac{w}{\sqrt{iw^2+4b}} dw \ , \\ I_2 &= e^{-i\frac{\pi}{2}} \int_0^\infty e^{-ie^{-i\frac{\pi}{4}}wa} e^{\frac{iw^2 - e^{-i\frac{\pi}{4}}w\sqrt{-iw^2+4b}}{2}} \frac{w}{\sqrt{-iw^2+4b}} dw \ . \end{aligned}$$

Notice that we can write

$$I_1 = e^{i\frac{\pi}{2}} \int_0^\infty e^{R(w)+I(w)i} \frac{w}{\sqrt{iw^2+4b}} dw \ ,$$

where

$$\begin{aligned} R(w) &= \mathrm{Re}\left\{ ie^{i\frac{\pi}{4}}wa - \frac{iw^2 + e^{i\frac{\pi}{4}}w\sqrt{iw^2+4b}}{2} \right\} \ , \\ I(w) &= \mathrm{Im}\left\{ ie^{i\frac{\pi}{4}}wa - \frac{iw^2 + e^{i\frac{\pi}{4}}w\sqrt{iw^2+4b}}{2} \right\} \ . \end{aligned}$$

After some algebra, we get

$$\begin{aligned} R(w) &= -\frac{\sqrt{2}}{2}wa - \frac{\sqrt{2}}{4}w\left(w^4+16b^2\right)^{\frac{1}{4}} \left(\cos\left(\frac{1}{2}\arctan\frac{w^2}{4b}\right) - \sin\left(\frac{1}{2}\arctan\frac{w^2}{4b}\right)\right) \ , \\ I(w) &= \frac{\sqrt{2}}{2}wa - \frac{w^2}{2} - \frac{\sqrt{2}}{4}w\left(w^4+16b^2\right)^{\frac{1}{4}} \left(\cos\left(\frac{1}{2}\arctan\frac{w^2}{4b}\right) + \sin\left(\frac{1}{2}\arctan\frac{w^2}{4b}\right)\right) \ . \end{aligned}$$

$R(w)$ is negative for $w > 0$. If we write now $w = b^{\frac{1}{2}}v$ and consider that

$$\begin{aligned} R(w) &\sim -\frac{\sqrt{2}}{2}\left(a + b^{\frac{1}{2}}\right) w \ \text{ as } w \to 0 \ , \\ I(w) &\sim \frac{\sqrt{2}}{2}\left(a - b^{\frac{1}{2}}\right) w \ \text{ as } w \to 0 \ , \end{aligned}$$

then it is possible to write, using Watson's Lemma (cf. [1]),

$$\begin{aligned} I_1 &= e^{i\frac{\pi}{2}} b^{\frac{1}{2}} \int_0^\infty e^{R(b^{\frac{1}{2}}v)+I(b^{\frac{1}{2}}v)i} \frac{iv}{\sqrt{iv^2+4}} dv \\ &= i\frac{b^{\frac{1}{2}}}{2} \int_0^\delta e^{-\frac{\sqrt{2}}{2}(a+b^{\frac{1}{2}})b^{\frac{1}{2}}v} e^{\frac{\sqrt{2}}{2}(a-b^{\frac{1}{2}})b^{\frac{1}{2}}vi} v dv + Q(a,b) \ , \end{aligned} \tag{8.13}$$

with $Q(a,b)$ small compared to the first term at the right hand side of (8.13) if $(a + b^{\frac{1}{2}})b^{\frac{1}{2}} >> 1$. Analogously, one can find that

$$I_2 \sim \frac{-ib^{\frac{1}{2}}}{2} \int_0^\delta e^{-\frac{\sqrt{2}}{2}\left(a+b^{\frac{1}{2}}\right)b^{\frac{1}{2}}v} e^{-\frac{\sqrt{2}}{2}\left(a-b^{\frac{1}{2}}\right)b^{\frac{1}{2}}vi} v dv \quad \text{as } (a + b^{\frac{1}{2}})b^{\frac{1}{2}} \to \infty \ .$$

Therefore, when $(a+b^{\frac{1}{2}})b^{\frac{1}{2}} \to \infty$

$$K_1(a,b) \sim -\sqrt{b}\int_0^\delta e^{-\frac{\sqrt{2}}{2}\left(a+b^{\frac{1}{2}}\right)b^{\frac{1}{2}}v} \sin\left(\frac{\sqrt{2}}{2}\left(a-b^{\frac{1}{2}}\right)b^{\frac{1}{2}}v\right)vdv$$

$$\sim -\frac{2}{\sqrt{b}(a+\sqrt{b})^2}\int_0^{\delta\frac{\sqrt{2}}{2}\left(a+b^{\frac{1}{2}}\right)b^{\frac{1}{2}}} e^{-w}\sin\left(\frac{a-b^{\frac{1}{2}}}{a+b^{\frac{1}{2}}}w\right)wdw$$

$$\sim -\frac{2\left(a-b^{\frac{1}{2}}\right)}{\sqrt{b}(a+\sqrt{b})^3}\int_0^\infty e^{-w}w^2dw = -\frac{4\left(a-b^{\frac{1}{2}}\right)}{\sqrt{b}(a+\sqrt{b})^3}\ .$$

Thus we can globally write

$$K_1(a,b) \sim -C_1(a,b)\frac{1}{\sqrt{b}(a+\sqrt{b})}\text{-}\ , \tag{8.14}$$

where $C(a,b)$ is a bounded and regular function.

A similar estimated is obtained for $a<0$ (changing a by $|a|$ in (8.14)) and this ends the proof.□

Lemma 8.3 *The function $K_2(a,b)$ defined by (8.12) can be written in the form*

$$K_2(a,b) = C_2(a,b)\frac{1}{\left(1+\frac{|a|}{b^{\frac{1}{2}}}\right)^{\frac{3}{2}}}\ ,$$

for some function C_2 bounded and regular.

Proof. The function $K_2(a,b)$ is bounded and regular for every a,b. We can write

$$K_2(a,b) = \int_0^\infty e^{ika}e^{\frac{-k^2-\sqrt{k^4+4bk^2}}{2}}\frac{\frac{1}{2}k^3+\frac{1}{2}k^2\sqrt{k^2+4b}}{\sqrt{k^2+4b}}dk$$

$$+\int_0^\infty e^{-ika}e^{\frac{-k^2-\sqrt{k^4+4bk^2}}{2}}\frac{\frac{1}{2}k^3+\frac{1}{2}k^2\sqrt{k^2+4b}}{\sqrt{k^2+4b}}dk\ ,$$

and perform, as in the previous lemma, a change in the path of integration of the complex variable function associated to the integrand towards the lines $e^{\pm\frac{\pi}{4}i}w$. The dominant contribution as $\left(a+\sqrt{b}\right)b^{\frac{1}{2}} \to \infty$ will be

$$K_2(a,b) \sim -\frac{b^{\frac{3}{2}}}{2}\int_0^\delta e^{-\frac{\sqrt{2}}{2}\left(a+b^{\frac{1}{2}}\right)b^{\frac{1}{2}}v}\cos\left(\frac{\sqrt{2}}{2}\left(a-b^{\frac{1}{2}}\right)b^{\frac{1}{2}}v\right)v^3dv$$

$$-b^{\frac{3}{2}}\int_0^\delta e^{-\frac{\sqrt{2}}{2}\left(a+b^{\frac{1}{2}}\right)b^{\frac{1}{2}}v}\cos\left(\frac{\sqrt{2}}{2}\left(a-b^{\frac{1}{2}}\right)^{\frac{1}{2}}v+\frac{3\pi}{4}\right)v^2dv$$

$$\sim C(a,b)\left(\frac{1}{\sqrt{b}}\frac{A}{\left(a+b^{\frac{1}{2}}\right)^4}+\frac{B}{\left(a+b^{\frac{1}{2}}\right)^3}\right)$$

$$= C(a,b)\left(\frac{A}{b^{\frac{5}{2}}\left(1+\frac{a}{b^{\frac{1}{2}}}\right)^4}+\frac{B}{b^{\frac{3}{2}}\left(1+\frac{a}{b^{\frac{1}{2}}}\right)^3}\right), \qquad (8.15)$$

with A and B constants and $C(a,b)$ a bounded regular function. Notice that the condition $\left(a+\sqrt{b}\right)b^{\frac{1}{2}} >> 1$ implies $(1+\frac{a}{b^{\frac{1}{2}}}) >> b^{-1}$. If $a >> b^{\frac{1}{2}}$, $b << 1$, then the estimate (8.15) still holds and therefore we can write

$$C(a,b)\left(\frac{A}{b^{\frac{5}{2}}\left(1+\frac{a}{b^{\frac{1}{2}}}\right)^4}+\frac{B}{b^{\frac{3}{2}}\left(1+\frac{a}{b^{\frac{1}{2}}}\right)^3}\right) = C_2(a,b)\frac{1}{\left(1+\frac{a}{b^{\frac{1}{2}}}\right)^{\frac{3}{2}}},$$

with $C_2(a,b)$ a bounded regular function. Hence

$$K_2(a,b) = C_2(a,b)\frac{1}{\left(1+\frac{a}{b^{\frac{1}{2}}}\right)^{\frac{3}{2}}},$$

thus proving the Lemma.□

In view of the asymptotic behaviors deduced in Lemmas 8.2 and 8.3, and using the particular values of a and b in our case, it is simple to see that

$$G_1(s-\xi,t-\tau;\mu) = \mu\frac{C_1(|s-\xi|,(t-\tau))}{(|s-\xi|+(t-\tau))^2},$$

with C_1 a regular and bounded function. We recognize in G_1 the same asymptotic behaviors (up to a multiplicative constant) as

$$\frac{\partial^2}{\partial s^2}\log\left[(s-\xi)^2+(t-\tau)^2\right],$$

that is, as the second derivative of the Poisson kernel in 2 dimensions. We can apply the same theory that has been developed in order to get Hölder estimates on this kernel (cf. [8]) and obtain in our case

$$\begin{aligned}\|u_0\|_{C^\alpha} &\leq C\mu^2\|f_0\|_{C^\alpha}, \\ \|u_1\|_{C^\alpha} &\leq C\mu\|f_1\|_{C^\alpha},\end{aligned}$$

(C^α stands for $C^\alpha(\mathbb{R}\times[0,T])$). Analogously, we find that

$$G_2(s-\xi,t-\tau;\mu) = \frac{\mu^{\frac{1}{2}}}{(t-\tau)^{\frac{3}{2}}}\frac{C_2(|s-\xi|,(t-\tau))}{\left(1+\frac{|s-\xi|}{(t-\tau)}\right)^{\frac{3}{2}}} = \mu^{\frac{1}{2}}\frac{C_2(|s-\xi|,(t-\tau))}{((t-\tau)+|s-\xi|)^{\frac{3}{2}}},$$

which involves a singularity of lower order than the corresponding to G_1. Again, we can adapt the estimates in [8] to our case and conclude

$$\|u_2\|_{C^\alpha} \leq C\mu^{\frac{1}{2}}\|f_2\|_{C^\alpha}.$$

We have then

$$\|\widetilde{u}\|_{C^\alpha} \le \|f_2\|_{C^\alpha} + C\left(\mu^2 \|f_0\|_{C^\alpha} + \mu \|f_1\|_{C^\alpha} + \mu^{\frac{1}{2}} \|f_2\|_{C^\alpha}\right) ,$$

where C can be chosen independent of μ.

Having in mind the structure of (8.9), (8.10), we can prove

$$\|f_1\|_{C^\alpha} + \|f_2\|_{C^\alpha} \le C\left(\|\overline{u}-1\|_{C^\alpha} \|w\|_{C^\alpha} + \|w\|^2_{C^\alpha}\right) ,$$

with C independent of μ. Therefore

$$\|\widetilde{u}\|_{C^\alpha} \le C\left(\mu^2 \|f_0\|_{C^\alpha} + \|\overline{u}-1\|_{C^\alpha} \|w\|_{C^\alpha} + \|w\|^2_{C^\alpha}\right) .$$

We define a mapping W which assigns to a function w in a ball of radius μ the solution $\widetilde{u}$ of (8.6)-(8.7), and then, if μ is small enough, $\widetilde{u}$ belongs to that ball. If we consider two functions w_1, w_2 in the ball and the corresponding solutions $\widetilde{u}_1$, $\widetilde{u}_2$ of (8.6)-(8.7), then the differences satisfies

$$\|\widetilde{u}_1 - \widetilde{u}_2\|_{C^\alpha} \le C\|\overline{u}-1\|_{C^\alpha} \|w_1 - w_2\|_{C^\alpha} \le C\nu \|w_1 - w_2\|_{C^\alpha} \le \|w_1 - w_2\|_{C^\alpha} , \tag{8.16}$$

provided we chose ν small enough. Inequality (8.16) implies that W is a contraction and therefore, by Banach's fixed point theorem, there exists a unique solution to (8.6)- (8.7) and it satisfies

$$\|\widetilde{u}\|_{C^\alpha} \le C\mu ,$$

and the proof of Theorem 8.1 is complete.□

9 Appendix: the construction of $\overline{u}(s,t)$

In this appendix we review briefly the ideas developed in [5] in order to construct the function $\overline{u}(s,t)$ used in Section 8. The equation (8.1) can be written in form of the system:

$$\begin{aligned} u_t &= v_s , \\ v_t &= -\frac{1}{2}\frac{u_s}{u^{\frac{3}{2}}} . \end{aligned}$$

If we apply the hodograph transformation (equivalent to deducing equations for $t(u,v)$ and $s(u,v)$), we obtain:

$$\begin{aligned} t_u &= s_v , \\ s_u &= -\frac{1}{2}\frac{t_v}{u^{\frac{3}{2}}} . \end{aligned}$$

Let us introduce $w = u^{\frac{1}{4}}$ and $y = \frac{v}{2\sqrt{2}}$. Then

$$t_w = \sqrt{2}w^3 s_y , \tag{9.1}$$
$$t_y = -\sqrt{2}w^3 s_w , \tag{9.2}$$

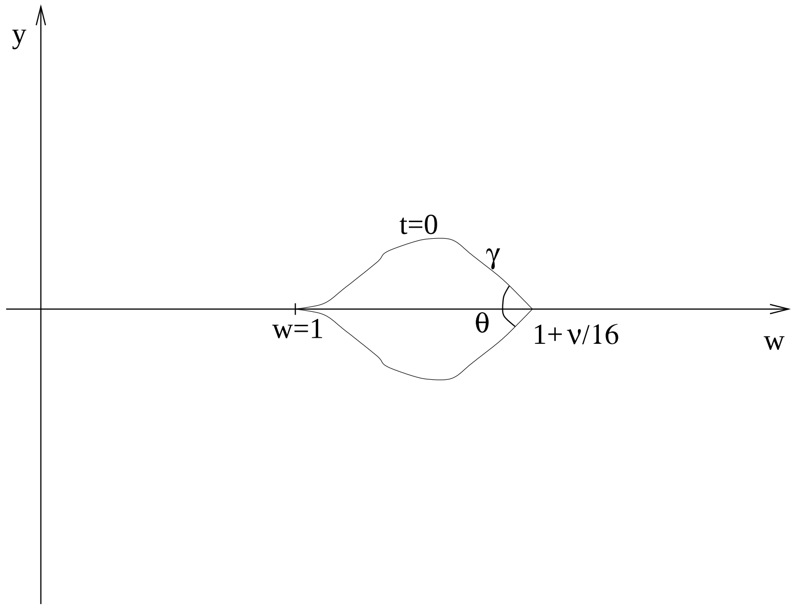

Figure 2: The function $\overline{u}(s,t)$ in the hodograph plane

or, written more compactly,

$$s_{yy} + \frac{1}{w^3}\left(w^3 s_w\right)_w = 0 \qquad \text{in } w > 0\,, \tag{9.3}$$

$$t_{yy} + w^3\left(\frac{1}{w^3} t_w\right)_w = 0 \qquad \text{in } w > 0\,. \tag{9.4}$$

The solutions of (9.3) or equivalently (9.4) represent solutions of the original equation (8.1). In [5], a technique was developed in order to construct solutions presenting some prescribed behaviors. The only thing we need is a closed curve γ in the wy- plane. It is simple to verify that, by equations (9.1), (9.2), the derivative of $t(w,y)$ along the curve is zero. Therefore, t is constant along it. We assign the value $t = 0$ to that curve. If we solve the equation (9.4) in the region enclosed by γ, subject to the Cauchy boundary condition $t = 0$, then $t < 0$ and the different level lines of $t(w,y)$ correspond to the solution of (8.1) at different times. We can use equations (9.1), (9.2) to find the value of s along the curve and in the region enclosed, and construct in this way a solution of (8.1). We can choose the curve γ in order to have different blow-up behaviors. The papers [5] and [6] are partly devoted to give a more or less exhaustive description of the possible mechanisms and we refer the reader to them for further details. In the particular case we are interested in, we have to choose the contour γ as in Figure 2.

Notice that the curve is constrained to the region $w \in \left(1 - \frac{\nu}{16}, 1 + \frac{\nu}{16}\right)$. This ensures that $u \in \left(1 - \frac{\nu}{2}, 1 + \frac{\nu}{2}\right)$ (for ν small enough). The presence of a cusp at $(w,y) = (1 + \frac{\nu}{16}, 0)$ implies the solution of (8.1) corresponding to this representation in the hodograph plane develops a cusp-like singularity when

$t \to 0^-$. The opening of the angle θ determines de degree of the cusp (cf. [5]). More precisely, the solution $h(z,t)$ will develop a cusp of the form $|z|^{\frac{\theta}{\pi}}$. This implies for $\overline{u}$ a cusp of the form $s^{\frac{1}{\frac{\pi}{\theta}-1}}$. If $\theta < \frac{\pi}{2}$, then $\overline{u} \in C_{\frac{1}{\frac{\pi}{\theta}-1}-\varepsilon}$ but $\overline{u} \notin C_{\frac{1}{\frac{\pi}{\theta}-1}+\varepsilon}$ (for ε small enough). Let us finish by remarking two facts. First, the invariance of (8.1) under translations in time allows to consider the solution in the interval $(0,T^*)$ instead of $(-T^*,0)$. Second, by the linearity of equation (9.4), we can construct a family of solutions of the form $\lambda t_0(w,y))$ where $t_0(w,y)$ is a solution of (9.4) and λ a real parameter. The larger λ is, the larger $s(w,y)$ and the smaller the Hölder seminorm of $\overline{u}-1$, so that we can make $\|\overline{u}-1\|_{C_{\frac{1}{\frac{\pi}{\theta}-1}-\varepsilon}(\mathbb{R}\times[0,T^*])} < \nu$ just by choosing λ large enough.

Acknowledgments. This work was done in its most part while the author was visiting the Institute for Mathematics and its Applications at University of Minnesota and has been partially supported by the Spanish Ministry of Education through its Postdoctoral program.

References

[1] C. M. Bender, S. A. Orszag. Advanced mathematical methods for scientists and engineers. McGraw-Hill Inc., 1978.

[2] J. Eggers. Universal pinching of 3D axisymmetric free-surface flow. Phys. Rev. Lett. **71** (1993), 3458.

[3] J. Eggers, T. F. Dupont. Drop formation in a one-dimensional approximation of the Navier-Stokes equation. J. Fluid Mech. **262** (1994), 205.

[4] J. Eggers, T. F. Dupont. Theory of drop formation. Phys. Fluids **7 (5)** (1995), 941-952 .

[5] M. A. Fontelos, J. J. L. Velázquez. On some breakup and singularity formation mechanisms for inviscid liquid jets. To appear in SIAM Journal of Applied Mathematics.

[6] M. A. Fontelos and J. J. L. Velázquez. Fractal-like singularities for the inviscid one-dimensional model of fluid jets. To appear in European Journal of Applied Mathematics.

[7] A. Friedman. Partial Differential equations of parabolic type. Prentice Hall, 1964.

[8] D. Gilbarg, N. S. Trudinger. Elliptic partial differential equations of second order. Second edition. Gründlehren del Mathematischen Wissenschaften, 224. Springer-Verlag, Berlin-New York, 1983.

[9] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva. Linear and Quasi-linear Equations of Parabolic Type. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, 1967.

[10] X. D. Shi, M. P. Brenner, S. R. Nagel. A cascade of Structure in a drop falling from a faucet. Science, Vol. **265** (1994), 219-222.

[11] Lu Ting, J.B. Keller. Slender jets and thin sheets with surface tension. SIAM J. Appl. Math., **50**,6 (1990), 1533-1546.

[12] D. T. Papageorgiou. On the breakup of viscous liquid jets. Physics of fluids, **7,** (1995), 1529-1544.