EVALUATING THE DIMENSION OF AN INERTIAL MANIFOLD FOR THE KURAMOTO-SIVASHINSKY EQUATION

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ABSTRACT. The Kuramoto-Sivashinsky equation is a dissipative evolution equation in one space dimension which, despite its apparent simplicity, gives rise to a very rich dynamical behavior, as evidenced for instance by the study in [16], of its complicated set of stationary solutions and stationary and Hopf bifurcations. The large time behavior of the solutions is usually embodied by the attractor and the inertial manifolds which have been the object of many studies.

In the present article, explicit expressions which can be completely evaluated are obtained for the dimension of an inertial manifold for the Kuramoto-Sivashinsky equation. This involves reworking the analysis in [1] to estimate the radius of the absorbing ball. From there, the choice of phase space, spectral gap condition, and preparation of the equation outside the absorbing ball are varied and the results compared over a moderate domain length. A new preparation of the equation is introduced which leads to the smallest dimension of those compared. The dimension is also obtained for the equation prepared using radii smaller than that of the known absorbing ball, down to the radius of the global attractor suggested by computations.

1. INTRODUCTION

The Kuramoto-Sivashinsky equation (KSE) is a well-known dissipative evolution equation introduced by Kuramoto [13] to model pattern formation in thermohydraulics (formation of rolls in Bénard convection), and by Sivashinsky [24] to model the propagation of a front flame in space dimension two. Despite its apparent simplicity and some similarity with the Burgers equation, this equation displays many interesting aspects, from the point of view of partial differential equations. We recall these equations: the KSE reads

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = f, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.1}$$

and is subjected to appropriate boundary and initial condition, while the Burgers equation reads

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = f, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{1.2}$$

An important difference between the KSE and the Burgers equation is that the dynamics of the latter is known to be trivial ([9], [Foias-Temam, unpublished, based on the method in [5]]): for a given stationary force f, the Burgers equation, say with a Dirichlet boundary condition, possesses a unique stationary solution, and all solutions of (1.2) converge to this stationary solution, as $t \to \infty$, for any initial value $u(0) = u_0$. On the contrary, equation (1.1) possesses a very rich dynamics and has served as a prototype for the study of dynamics of partial differential equations; see for instance [16] for a thorough study of the rich set of stationary solutions of (1.1), and of stationary and Hopf bifurcations for this equation.

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Another surprising property of (1.1) (supplemented with suitable boundary conditions), is the stability of the solutions as time goes to infinity. Equation (1.1) is indeed linearly unstable in the sense that, for $f = 0$, the solution $u = 0$ is linearly unstable; equation (1.1) linearized around $u = 0$ possesses a finite number of unstable modes. On the contrary equation (1.1) is nonlinearly stable, in the sense that all solution of (1.1) remain bounded, as $t \to \infty$, for a given f independent of t, and with, say, space-periodic boundary conditions. See below more details on the proof of dissipativity related to nonlinear stability.

Finally, due to its exceptionally rich dynamics, equation (1.1) has served, as we said, as a prototype for infinite dynamical systems, and has been the object of much studies concerning its attractors (see e.g. [18]), and its exact and approximate inertial manifolds that are (roughly speaking) finite dimensional manifolds to which all solutions of (1.1) converge exponentially fast, exactly or approximately. Beside the theoretical issue, there are two other issues motivating these studies: on the one hand the computational issue; and on the other hand a conjecture of Pomeau and Maunneville [20], that the large time behavior of the solutions of the equation is governed by a finite number of parameters proportional to L, the nondimensional length of the interval $\left(-\frac{L}{2} \leq x \leq \frac{L}{2}\right)$, which in turn, is also proportional to the number of unstable modes of the linearized form of equation (1.1). The results closest to this conjecture are recalled below, and we return in this work to these estimates.

A number of computational papers on the Kuramoto-Sivashinsky equation invoke the fact that it has an inertial manifold. In some instances this is key to the particular computational approach taken, as when approximate inertial manifolds (AIMs) are employed. It is true that most AIMs are well-motivated even in the absence of an inertial manifold, and furthermore that in practice, the dimension of an AIM need not have any connection to the dimension of an inertial manifold, should one exist. The reason, in essence, is that the accuracy of many AIMs, as measured by the distance to the global attractor, is improved by increasing the dimension. Recently however, a new breed of AIM has been introduced [2],[22] which converges with fixed dimension to not only the global attractor, but also to an inertial manifold, when one exists. Thus it has become more useful to know the dimension of a legitimate inertial manifold. Up to now estimates for this dimension have typically been of the form dim $\leq cL^b$, where c is a universal constant and L is the length of the domain. Over the last decade there has been a dramatic reduction of the exponent b, yet c remains an elusive quantity, the end result of a series of transformations of other universal constants, involving considerable analysis. The purpose of this paper is to calculate (and to some extent reduce) such universal constants and thereby arrive at a means to completely determine the dimension of an inertial manifold.

The first ingredient in any recipe for inertial manifolds is dissipativity, embodied by the existence of an absorbing ball, which is itself related to nonlinear stability, as we said before. For the Kuramoto-Sivashinsky equation, this property was first proved only in the invariant subspace of odd functions in [18]. Dissipativity for the general periodic case was later established independently by Collet et al. [1], and Goodman [6] (see also Il'yashenko [7] and Pinto [19]). We will consider both the odd case as well as the general periodic case. We follow closely the estimates in [1] to obtain the radius of an absorbing ball, which is expressed in terms of L alone, i.e. without universal constants. In doing so, some minor modifications are made to reduce the radius by a modest factor. The approach in [1] involves the translation by a carefully chosen gauge function, constructed in Fourier space. An alternative construction in physical space by Cheskidov is included in an appendix to this paper.

The study of the dynamics does not depend on the behavior of the solutions outside the absorbing ball and, usually, the equation is "prepared" which means that the nonlinear term is modified outside a sufficiently large ball to make the nonlinear term globally Lipschitz. This does not change the trajectories within the absorbing ball, in particular, those on the global attractor, the largest bounded invariant set for the KSE. We employ two new preparations based on a Lipschitz extension result of Valentine (c.f. [8]). Both lead to optimal Lipschitz constants, for the function spaces involved. The standard preparation, which is scalar multiplication by a cut-off function of some norm of the solution, results in a larger Lipschitz constant.

The final ingredient for inertial manifolds, is a spectral gap condition which requires that the gap between eigenvalues of the linear part of the equation dominate in some sense the Lipschitz constant for the prepared nonlinearity. For this stage we consider both the framework in [23], and that in Miklavcic [17]. The minimum dimensions needed to satisfy the corresponding gap conditions are then calculated by numerically evaluating both sides of each relation. This is done using a rigorous absorbing ball over $L \in [2\pi, 6\pi]$. The resulting dimension seems much higher than that suggested by the form of the global attractor found by computational means [10]. Computational evidence also suggests that the global attractor is contained in a ball that is much smaller than that of the rigorous absorbing ball. For this reason, we fix $L = L^* = 4\sqrt{2}\pi$, and find the minimal dimension for which a gap condition is satisfied as the radius of preparation is varied, from that of a rigorous absorbing ball, to one which would just contain computed elements of the global attractor.

The dimension of the inertial manifold for the KSE with periodic *and odd* boundary conditions was first estimated in [3]. Following the extension of dissipativity to the general periodic case and improved estimate for the absorbing ball in [1], revised estimates for the dimension were obtained in [25] and [21]. As far as we know, the best estimate for the power b is $b = 2.46$, obtained in [21] by reapplying the approach in [3]. Again, while the constant c is not explicitly determined in that estimate, the methods used have an inflationary effect on it. The evaluation of the minimal dimension used here is effectively $O(L^{2.46})$, matching the order in [21].

2. Preliminaries

First consider the general framework of an evolutionary equation of the form

$$\frac{du}{dt} + Au = f(u), \quad u \in E, \quad u(0) = u_0, \tag{2.1}$$

where A is a linear operator and E is a Banach space. We will find an inertial manifold as the graph of a function $\Phi : PE \to QE$, where P is a finite-dimensional projection, and $Q = I - P$. The restriction of the flow to an inertial manifold yields the inertial form

$$\frac{dp}{dt} + Ap = Pf(p + \Phi(p)), \quad p(0) = p_0 \in PE, \tag{2.2}$$

a finite set of ordinary differential equations (ODEs) which has the same long-time dynamics as (2.1).

In the first approach we will need to satisfy the following assumptions.

A1 The nonlinear term f in (2.1) is globally Lipschitz continuous from a Banach space E into another Banach space F,

$$|f(u) - f(v)|_F \le M_1 |u - v|_E, \qquad \forall\, u, v \in E, \tag{2.3}$$

with

$$E \subset F \subset \mathcal{E},$$

the injections being continuous, and each space dense in the following one. It follows that

$$|f(u)|_F \le M_0 + M_1 |u|_E, \qquad \forall\, u \in E,$$

for some $M_0 \ge 0$. (Actually $M_0 = |f(0)|_F$ is optimal.)

A2 The linear operator $-A$ generates a strongly continuous semigroup $\{e^{-tA}\}_{t\ge 0}$ of bounded operators on $\mathcal{E}$ such that

$$e^{-tA} F \subset E, \qquad \forall\, t > 0.$$

A3 There exist two sequences of numbers $\{\lambda_n\}_{n=n_0}^{n_1}$, $\{\Lambda_n\}_{n=n_0}^{n_1}$, for some $n_0 \in \mathbb{N}$, $n_1 \in \mathbb{N} \cup \infty$, with $0 < \lambda_n \le \Lambda_n$ for $n_0 \le n \le n_1$, and a sequence $\{P_n\}_{n=n_0}^{n_1}$ of finite dimensional projectors, such that if $Q_n = I - P_n$, then the following exponential dichotomies hold:

$P_n\mathcal{E}$ is invariant under e^{-tA}, for $t \geq 0$, and $\{e^{-tA}|_{P_n\mathcal{E}}\}_{t\geq 0}$ can be extended to a strongly continuous group $\{e^{-tA}P_n\}_{t\in\mathbb{R}}$ of bounded operators on $P_n\mathcal{E}$ with

$$\begin{aligned} &\|e^{-tA}P_n\|_{\mathcal{L}(E)} \leq K_1 e^{-\lambda_n t}, \qquad \forall\ t \leq 0, \\ &\|e^{-tA}P_n\|_{\mathcal{L}(F,E)} \leq K_1 \lambda_n^\alpha e^{-\lambda_n t}, \qquad \forall\ t \leq 0; \end{aligned} \tag{2.4}$$

$Q_n\mathcal{E}$ is positively invariant under e^{-tA}, for $t \geq 0$, with

$$\begin{aligned} &\|e^{-tA}Q_n\|_{\mathcal{L}(E)} \leq K_2 e^{-\Lambda_n t}, \qquad \forall\ t \geq 0, \\ &\|e^{-tA}Q_n\|_{\mathcal{L}(F,E)} \leq K_2(t^{-\alpha} + \Lambda_n^\alpha) e^{-\Lambda_n t}, \qquad \forall\ t > 0, \end{aligned} \tag{2.5}$$

where $K_1, K_2 \geq 1$, and $0 \leq \alpha < 1$. We will at times drop the subscripts on the projectors and write P and Q for simplicity.

A4 Equation (2.1) has a continuous semiflow $\{S(t)\}_{t\geq 0}$ in E, given by $S(t)u_0 = u(t)$, where $u = u(t)$ is the mild solution of (2.1) defined through the variation of constants formula:

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(u(s))ds, \qquad \forall\ t \geq 0.$$

A5 There exists $K_3 \geq 0$ independent of n such that

$$\|AP_n\|_{\mathcal{L}(E)} \leq K_3 \lambda_n.$$

A6 For simplicity we assume that A is invertible.

In addition we will consider a *spectral gap condition* (SGC) of the form

$$\Lambda_n - \lambda_n > 3M_1K_1K_2[\lambda_n^\alpha + (1+\gamma_\alpha)\Lambda_n^\alpha], \quad \text{for some } n \in \mathbb{N}, \tag{2.6}$$

where

$$\gamma_\alpha = \begin{cases} \int_0^{+\infty} e^{-r} r^{-\alpha} dr, & \text{if } 0 < \alpha < 1, \\ 0, & \text{if } \alpha = 0, \end{cases}$$

or alternatively the weaker gap condition

$$\Lambda_n - \lambda_n > 2M_1K_1\lambda_n^\alpha + 2M_1K_2(1+\gamma_\alpha)\Lambda_n^\alpha, \tag{2.7}$$

We will apply the following existence result.

Theorem 2.1. *Assume that* **A1** *through* **A6** *hold along with (2.6) (resp. (2.7)). Then there exists an inertial (resp. invariant) manifold of dimension equal to that of P_nE.*

The inertial manifold case was proved in [23] and the extension to the invariant manifold case was discussed in [12].

Under the SGC in (2.6) the manifold enjoys the property of *asymptotic completeness*, (also referred to as *exponential tracking* [4]). This means that corresponding to any initial condition $u_0 \in E$ there exists a particular solution *on* the manifold, to which the trajectory through u_0 is attracted at an exponential rate, forward in time.

We will compare the minimal dimension provided by Theorem 2.1 with that given by an alternative approach taken by Miklavcic [17]. While the latter also applies to the general Banach space, there is included in [17] a simplification to the following special case

H1 A is a self-adjoint operator in a Hilbert space X.

H2 The spectrum of A, $\mathrm{Sp}(A)$, satisfies

$$\mathrm{Sp}(A) \subset (a, \lambda] \cup [\Lambda, \infty)$$

for some $a < \lambda < \Lambda$.

H3 The nonlinear term $f : D(A^{\alpha+\beta}) \to D(A^{\beta})$ with $\beta \in \mathbb{R}$, $\alpha \in [0,1)$ satisfies

$$|A_b^{\beta}(f(u) - f(v))|_X \leq \tilde{M}|A_b^{\alpha+\beta}(u - v)|_X, \tag{2.8}$$

where $A_b = A + bI$ and $b = -a$.

Theorem 2.2. *Assume* **H1-H3** *hold. If in addition* λ *and* Λ *satisfy*

$$\Lambda - \lambda \geq \tilde{M}[(\Lambda - a)^{\alpha} + (\lambda - a)^{\alpha}]. \tag{2.9}$$

then there exists an inertial manifold of dimension equal to that of PX*, where* P *is the projector associated with* $\mathrm{Sp}(A) \cap (a, \lambda]$.

In the case of the KSE the global Lipschitz conditions in (2.3) and (2.8) do not hold, so one must prepare the equation. The prepared equation is written as

$$\frac{du}{dt} + Au = f_{\rho}(u), \tag{2.10}$$

where $f_{\rho} : E \to F$ which agrees with f for $|u|_E \leq \rho$, and is globally Lipshitz. One choice for this prepared nonlinearity is

$$f_{\rho}(u) = \theta_{\rho}(|u|_E)f(u), \tag{2.11}$$

with

$$\theta_{\rho}(r) = \theta(\frac{r^2}{\rho^2}), \tag{2.12}$$

$$\theta(s) = 2(s-1)^3 - 3(s-1)^2 + 1, \qquad \text{for } s \in [1,2], \tag{2.13}$$

which is easily seen to satisfy $|\theta'(s)| \leq 3/2$. Thus the flow outside of the ball of radius $\sqrt{2}\rho$ reduces to the linear flow.

It is possible to quantify the relation between the global Lipschitz constant of f_{ρ} in E with that of f restricted to a ball in E of arbitrary radius.

Lemma 2.3. *If for some functions* $d_0, d_1 : (0, \infty) \to (0, \infty)$ *one has for all* $|u|_E, |v|_E < r$ *that*

$$\begin{aligned} |f(u)|_F &\leq d_0(r), \\ |f(u) - f(v)|_F &\leq d_1(r)|u - v|_E, \end{aligned} \tag{2.14}$$

then

$$\mathrm{Lip}(f_{\rho}) \leq \frac{3\sqrt{2}}{\rho} d_0(\sqrt{2}\rho) + d_1(\sqrt{2}\rho). \tag{2.15}$$

We see that $M_{\rho} = \mathrm{Lip}(f_{\rho})$ for f_{ρ} as in (2.11) will be much larger than $\mathrm{Lip}(f|_{B(0,\rho)})$, for large ρ. Since M_{ρ} figures prominently in the gap condition, we seek to minimize it with an alternative choice for f_{ρ}, provided by the following remarkable result due to Valentine (and earlier Kirszbraun in the finite dimensional case, see [8]) which can also be used when we want to truncate f outside a set other than a ball in the phase space..

Theorem 2.4. *Let* E, F *be Hilbert spaces and* S *be an arbitrary subset of* E*. For any Lipschitz function* $g : S \to F$ *there exists a function* $\overline{g} : E \to F$ *such that* $\overline{g}|_S = g$ *and* $\mathrm{Lip}(\overline{g}) = \mathrm{Lip}(g)$.

Note that the preparation

$$f_{\rho} = \overline{f|_{B(0,\rho)}}. \tag{2.16}$$

will always have a smaller Lipschitz constant than that in (2.11) and can be used when we want to truncate f outside a set other than a ball in the phase space. While we will not consider the preparation in (2.11) further in the analysis which follows, it is worth mentioning that the function in (2.11) is easily implemented in computational applications [11] and it can be used when E is simply a Banach space.

The function in (2.16) is, in general, nonconstructive, but in some cases it can be easily computed, as well. In fact, if S is a closed, convex (nonempty) subset, then we can consider the orthogonal projector P_S onto S, which has Lipschitz constant equal to one, and take $\overline{g} = g \circ P_S$. In some cases, P_S can be computed via some minimization algorithm, and in the case that S is a closed ball of radius ρ, then P_S is simply $P_S(u) = u$, inside the ball, and $P_S(u) = (\rho/|u|_E)u$, outside. Notice then that in the case that S is a ball, this preparation is easier to implement and gives a better Lipschitz constant than the preparation (2.11) usually found in the literature. In addition to the straightforward extension used in (2.16), however, we will also consider in section 4.3 a more complicated extension for which the generality of Theorem 2.4 comes into play. It is the latter preparation which will yield an inertial manifold of lowest dimension.

The difference in the dynamic behavior of the original equation and the prepared equation depends on the choice of ρ. For a dissipative equation such as the KSE, we have an *absorbing ball*, i.e. there exists $\rho_E > 0$ such that

$$|S(t)u_0|_E \leq \rho_E, \quad \forall t \geq t_0(|u_0|_E) \geq 0.$$

The long-time dynamics of (2.1) is described by the global attractor $\mathcal{A}$ of (2.1), defined by

$$\mathcal{A} = \bigcap_{t\geq 0} S(t)B_\rho.$$

This is the largest bounded invariant set for the system, and contains all steady states, periodic solutions, invariant tori, as well as the unstable manifolds associated with those objects. The global attractor must be contained in any inertial manifold, and thus provides a lower bound on the minimal dimension for all inertial manifolds. In essence we wish to compare the global attractor $\mathcal{A}$ of the original equation with the global attractor $\mathcal{A}_\rho$ of the prepared equation. If ρ exceeds the radius of an absorbing ball, then certainly $\mathcal{A} \subset \mathcal{A}_\rho$. It is conceivable, however, that the preparation could introduce additional elements to the long time dynamics of the prepared equation (2.10) outside the ball of radius ρ. Nevertheless, when one speaks of an inertial manifold for the KSE, one is in fact referring to a portion inside the absorbing ball of an inertial manifold for the KSE prepared outside of that ball. Recall that since the preparation is not unique, neither is the manifold. Thus the estimate of the radius of an absorbing ball has impact on the dimension estimate of an inertial manifold through Lemma 2.3 and the (SGC), through (2.16) or alternately, through (2.11).

3. Radius of an absorbing ball

The Kuramoto-Sivashinsky equation (KSE) is often written as

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{3.1}$$

subject to periodic boundary conditions $u(x,t) = u(x+L,t)$, $L > 0$.

The key to the dissipativity analysis is the translation of the solution u by a carefully chosen gauge function. Such an approach was first taken in [18] in the case of odd functions, and later modified in [1] to extend to the general periodic case. The modification actually resulted in a better estimate for the radius of the absorbing ball in the odd case. In both works the estimate takes the form cL^b, where c is some constant independent of L. For the radius measured in the L^2-norm [18] has $b = 5/2$, while [1] has $b = 8/5$. In this section we rework the latter estimate to obtain a bound which can be *completely* evaluated in terms of L.

We will use H^1_{per} to denote the subspace of the Sobolev space $H^1((-L/2, L/2))$ consisting of functions which are periodic with period L, and $\dot{H}^1_{\text{per}}$ ($\dot{L}^2_{\text{per}}$) to denote the subset of H^1_{per} (L^2) consisting of functions with zero mean. In addition, we use H^1_{odd} (L^2_{odd}) for the subspace of odd, periodic functions in H^1 (L^2), and $\dot{H}^1_{\text{even}}$ for the subspace of even, periodic functions with zero mean. We will use the following norm and seminorm

$$|\cdot| = |\cdot|_{L^2} \qquad |\cdot|_1 = |\nabla \cdot|_{L^2}$$

in L^2 and H^1 respectively.

We will also have occasion to work in the fractional Sobolev spaces H^s_{odd}, $s \notin \mathbb{Z}$. For $s > 0$ this space can be defined by Fourier series, with H^{-s} interpreted as the dual of H^s with respect to the L^2 inner product (cf. [14]). We take the norm in H^s_{odd} of an element $u = \sum_{k=1}^{\infty} u_k w_k$ to be

$$|u|_s = \left\{ \sum_{k=1}^{\infty} |u_k|^2 \left(\frac{2\pi k}{L} \right)^{2s} \right\}^{1/2},$$

where $\{w_k\}$ is the complete set of orthonormal eigenfunctions in L^2_{odd} given by

$$w_k(x) = \sqrt{\frac{2}{L}} \sin(\frac{2\pi}{L} kx). \tag{3.2}$$

3.1. **An absorbing ball in L^2.** At a certain point in the estimate for the radius in [1] we find we can make a slight adjustment that results in a reduction factor of about a half. To explain we must refer to various elements of the argument in [1]. As a consequence we will adopt in this section, much of their notation to make the comparison easier, even though it may conflict with the use of some symbols elsewhere in this work. The KSE is rewritten as

$$\partial_t u = \mathcal{L}u - uu', \quad \mathcal{L} = -\partial_x^4 - \partial_x^2 \tag{3.3}$$

$(u' = \partial_x u)$ so that $v(x,t) = u(x,t) - \Phi(x)$ satisfies

$$\partial_t v = [\mathcal{L} - \Phi']v - vv' + \mathcal{L}\Phi - \Phi v' - \Phi\Phi'. \tag{3.4}$$

The idea is to choose $\Phi \in L^2_{\text{odd}}$ so that $[\mathcal{L} - \Phi']$ is a negative definite operator. In fact, applying $\int v\cdot = \int_{-L/2}^{L/2} v\cdot$ to (3.4) and integrating by parts, Collet *et al.* [1] obtain

$$\frac{1}{2}\frac{d}{dt}|v|^2 = -(v,v)_{\Phi/2} - (v,\Phi)_\Phi \tag{3.5}$$

where

$$(v_1, v_2)_{\gamma\Phi} = -\int v_1[\mathcal{L} - \gamma\Phi']v_2. \tag{3.6}$$

They then define the quadratic form

$$R_{\gamma\Phi}(v) = (v,v)_{\gamma\Phi},$$

and observe that integration by parts gives

$$R_0(\Phi) \equiv R_{\gamma\Phi}(\Phi) = -\int \Phi\mathcal{L}\Phi, \quad \forall\gamma \in [\frac{1}{4}, 1]. \tag{3.7}$$

We now recall a lemma from [1] which plays a critical role in the estimate of the radius of an absorbing ball.

Lemma 3.1. *There is a constant K such that, for all $L \geq 0$, there exists a function $\Phi \in L^2_{\text{odd}}$ such that for all $\gamma \in [1/4, 1]$, and all $v \in L^2_{\text{odd}}$*

$$R_{\gamma\Phi}(v) \geq \frac{1}{4}(|v''|^2 + |v|^2), \quad R_0(\Phi) \leq KL^{16/5}. \tag{3.8}$$

To estimate the radius in L^2, Collet *et al.* [1] then obtain for a parameter ϵ to be chosen

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v|^2 &\leq -(v,v)_{\Phi/2} + \frac{\epsilon}{2}(v,v)_\Phi + \frac{1}{2\epsilon}(\Phi,\Phi)_\Phi \\ &= \int v[(1-\frac{\epsilon}{2}\mathcal{L} - (\frac{1}{2} - \frac{\epsilon}{2})\Phi']v + \frac{1}{2\epsilon}R_\Phi(\Phi) \\ &= -(1-\frac{\epsilon}{2})R_{\gamma\Phi}(v) + \frac{1}{2\epsilon}R_0(\Phi),\end{aligned} \tag{3.9}$$

where

$$\gamma = \frac{\frac{1}{2} - \frac{\epsilon}{2}}{1 - \frac{\epsilon}{2}}$$

and the Cauchy-Schwarz and Young inequalities are used in the first step. They then set $\epsilon = 2/3$, so that $\gamma = 1/4$, and consequently

$$\frac{1}{2}\frac{d}{dt}|v|^2 \leq -\frac{1}{6}(|v''|^2 + |v|^2) + \frac{3}{4}R_0(\Phi). \tag{3.10}$$

Thus, by dropping the $|v''|$ term, and applying the standard Gronwall inequality it follows that

$$|v(t)|^2 \leq e^{-t/3}|v(0)|^2 + \frac{9}{2}R_0(\Phi)(1 - e^{-t/3}). \tag{3.11}$$

Adjusting for the translation, one has then that any ball in L^2_{odd}, centered at the origin, with radius $\rho > \rho_0$, where

$$\rho_0 = 3\sqrt{R_0(\Phi)/2} + |\Phi|, \tag{3.12}$$

is absorbing for (3.3). The proof for the general periodic case involves a time translation of the same gauge function, yet the resulting radius is the same as in (3.12).

To see where a reduction in the radius of the absorbing ball can be made, we consider the way Lemma 3.1 is applied. Note that this lemma is actually used twice. The first time, with $\gamma = 1$, was in order to justify the use of the Cauchy-Schwarz inequality in (3.9). For this it would suffice to have

$$R_\Phi(v, v) \geq 0. \tag{3.13}$$

It was only in the second application, with $\gamma = 1/4$, to obtain (3.10), where the full strength of estimate (3.8) was used.

To quantify the reduction (and select Φ) we must examine the proof of Lemma 3.1. Since v is in L^2_{odd}, one has for $q = 2\pi/L$

$$v = i\sum_{n\in\mathbb{Z}} v_n e^{inqx} = \sum_{n>0} 2v_n \sin nqx, \quad \text{as } v_n = -v_{-n} \in \mathbb{R},\ v_0 = 0,$$

so that

$$\frac{1}{4}|v|^2 = \frac{L}{2}\sum_{n>0} v_n^2.$$

The gauge function Φ is defined by the Fourier series of its derivative (which must be even)

$$\Phi' = -\sum_{n\in\mathbb{Z}} \psi_n e^{inqx} = -\sum_{n>0} 2\psi_n \cos nqx, \quad \text{with } \psi_n = \psi_{-n} \in \mathbb{R},\ \psi_0 = 0.$$

Using the symmetries of the Fourier coefficients, Collet *et al.* [1] ultimately obtain for all $\gamma \in [1/4, 1]$

$$R_{\gamma\Phi}(v) \geq 2L\Big[\sum_{n>0} w_n^2 + 2\gamma \sum_{k>m>0} w_k\Big(\frac{\psi_{k+m} - \psi_{k-m}}{\tau_k\tau_m}\Big)w_m\Big] \equiv (w, (I + 2\gamma\Gamma)w),$$

where

$$w_n = \tau_n v_n, \quad \tau_n = \Big(\frac{1}{2}[(nq)^4 + 1]\Big)^{1/2},$$

provided

$$\begin{cases} \psi_{2n} = 4, & \text{for } nq \leq 2 \\ \psi_{2n} \geq 0, & \text{for } nq > 2. \end{cases}$$

A sharp condition to establish (3.8) then is that

$$(w, (I + 2\gamma\Gamma)w) \geq \frac{1}{2}(w, w) = L\sum_{n>0} w_n^2 = L\sum_{n>0}\frac{1}{2}[(nq)^4 + 1]v_n^2 = \frac{1}{4}(|v''|^2 + |v|^2). \tag{3.14}$$

A sufficient condition for (3.14) to hold is provided by the Hilbert-Schmidt norm

$$\|\Gamma\|_{\mathrm{HS}}^2 \equiv \sum_{k>m>0} \left|\frac{\psi_{k+m}-\psi_{k-m}}{\tau_k \tau_m}\right|^2 \le \left(\frac{1}{4\gamma}\right)^2. \tag{3.15}$$

Note that the worst case is when $\gamma = 1$, where we actually only need that

$$(w,(I+2\gamma\Gamma)w)|_{\gamma=1} \ge 0, \tag{3.16}$$

which can be met by making

$$\|\Gamma\|_{\mathrm{HS}}^2 \le \left(\frac{1}{2\gamma}\right)^2|_{\gamma=1} = \frac{1}{4}. \tag{3.17}$$

Since (3.17) $\Rightarrow$ (3.15)$|_{\gamma=1/4}$, the estimate in (3.12) is valid provided (3.17) holds.

The gauge function is then shaped by two needs which are somewhat competitive. To satisfy (3.17) the Fourier coefficients must be nearly constant in n, at least for small n. To make $R_0(\Phi)$ small, these coefficients must decay at a rapid enough rate. With this in mind the coefficients are set in [1] to be

$$\psi_n = \begin{cases} 0, & \text{for } n \text{ odd} \\ \begin{cases} 4, & \text{when } 1 \le |n| \le 2M \\ 4h(\frac{|n|}{2M}-1), & \text{when } 2M \le |n| \end{cases}, & \text{for } n \text{ even,} \end{cases} \tag{3.18}$$

where h is some non-increasing C^1 function satisfying

$$h(0)=1, \quad h'(0)=0, \quad h\ge 0, \quad \sup|h'|<1, \quad \int_0^\infty (1+\omega^2)|h(\omega)|^2 d\omega < \infty, \tag{3.19}$$

and M is to be determined by (3.17). Collet *et al.* go on to show that under these conditions on the coefficients

$$\|\Gamma\|_{\mathrm{HS}}^2 \le \frac{128}{3} q^{-7} M^{-5} + \frac{16}{3} q^{-8} M^{-6}.$$

It is easily verified that taking their choice of $M = 4q^{-7/5}$ guarantees (3.15). The weakened condition (3.17), however, is easily shown to hold for

$$M = \eta q^{-7/5}, \tag{3.20}$$

provided η satisfies

$$f(\eta,q) = 3\eta^6 - 512\eta - 64q^{2/5} \ge 0. \tag{3.21}$$

The range of the unique positive zero of $f(\eta,q)$ for each fixed $q \in (0,1)$ is quite narrow, as evidenced in Figure 1, so we settle for $\eta = 2.82$ in our choice for M, given by (3.20). Ultimately $\rho_0 = O(q^{1/2}M^{3/2})$ so we save by modest factor of $(2.82/4)^{3/2} \approx .6$ in the size of the absorbing ball in L^2.

All that remains in order to obtain an expression for ρ_0 which can be completely evaluated, is to make an explicit choice for the function h. Indeed by (3.7) we have

$$R_0 = R_0(g) = \frac{4\pi}{q}\sum_{n=1}^{\infty}[(nl)^2-1]\psi_n^2. \tag{3.22}$$

We first consider the piecewise linear function given by

$$\tilde{h}_\delta(\omega) = \begin{cases} 1, & \text{for } 0 \le \omega \le \delta, \\ (-\omega+1+2\delta)/(1+\delta), & \text{for } \delta < \omega < 1+2\delta, \\ 0, & \text{for } 1+2\delta < \omega \end{cases}, \tag{3.23}$$

(see Figure 2). We then mollify $\tilde{h}_\delta$ over the intervals $[0,2\delta]$ and $[1,1+4\delta]$ so as to create a C^1 function h_δ satisfying all the properties in (3.19). To be specific, over both intervals we may replace $\tilde{h}_\delta$ with the cubic spline fitting the function value and derivative at each endpoint. We use here the elementary fact that for data such as this, consistent with a function that is concave as on $[0,2\delta]$) (convex as on $[1,1+4\delta]$), the spline is also concave (convex). This ensures that $\sup|h'_\delta| < 1$. The resulting gauge function is shown in Figure 3.

Expressing $R_{0,\delta}$ in terms of h_δ, and using the fact that $h_\delta(\omega) \to (1-\omega)$ as $\delta \to 0$, uniformly on $[0,1]$, allows us to write

$$
\begin{aligned}
R_{0,\delta} &= \frac{64\pi}{q}\left\{\sum_{k=1}^{M}[(2kq)^2-1] + \sum_{k=M+1}^{\infty}[(2kq)^2-1]h_\delta^2(\frac{k}{M}-1)\right\} \\
&= \frac{64\pi}{q}\left\{\sum_{k=1}^{M}[(2kq)^2-1] + \sum_{k=M+1}^{2M}[(2kq)^2-1]\left[1-\left(\frac{k}{M}-1\right)\right]^2\right\} + E_\delta,
\end{aligned}
\tag{3.24}
$$

where $|E_\delta| \to 0$ as $\delta \to 0$. Using several formulas for finite sums of powers of integers, and rearranging terms, we finally arrive at a bound on the limiting value $R_0 = \lim_{\delta\to 0} R_{0,\delta}$ given by

$$
\begin{aligned}
R_0 &= 64\pi\left\{\frac{52}{15}qM^3 - \frac{4}{3}\frac{M}{q} + \frac{2}{3}qM + \frac{1}{2}q - \frac{1}{6}\frac{1}{qM} - \frac{2}{15}\frac{q}{M}\right\} \\
&= 64\pi\left\{\frac{52}{15}\eta^3\tilde{L}^{16/5} - \frac{4}{3}\eta\tilde{L}^{12/5} + \frac{2}{3}\eta\tilde{L}^{2/5} + \frac{1}{2}\tilde{L}^{-1} - \frac{1}{6\eta}\tilde{L}^{-2/5} - \frac{2}{15\eta}\tilde{L}^{-12/5}\right\},
\end{aligned}
\tag{3.25}
$$

where $\tilde{L} = q^{-1} = L/2\pi$. We estimate the correction due to translation by

$$
\begin{aligned}
|\Phi|^2 &= \frac{L}{2q^2}\sum_{k=1}^{\infty}\frac{\psi_{2k}^2}{k^2} = \frac{8L}{q^2}\left\{\sum_{k=1}^{M}\frac{1}{k^2} + \sum_{k=M+1}^{2M}\frac{1}{k^2}(2-\frac{k}{M})\right\} \\
&< 16\pi\tilde{L}^3\left\{1 + \int_1^M \frac{1}{\xi^2}d\xi + 2\int_M^{2M}\frac{1}{\xi^2}d\xi - \frac{1}{M}\int_M^{2M}\frac{1}{\xi}d\xi\right\} \\
&= 16\pi\tilde{L}^3\left\{2 - \eta^{-1}\tilde{L}^{-7/5}\ln(2)\right\}.
\end{aligned}
\tag{3.26}
$$

We summarize with

Lemma 3.2. *With the choice of Φ given above we have, for all $v \in L^2_{\mathrm{odd}}$,*

$$R_{\Phi/4}(v) \geq \frac{1}{4}(|v''|^2 + |v|^2), \quad R_\Phi(v) \geq 0.$$

Applying Lemma 3.2 and using (3.25) and (3.26) we arrive at

Theorem 3.3. *With the above choice of Φ, the radius of the absorbing ball in L^2_{per} may be taken to be*

$$
\begin{aligned}
\rho_0 =& 12\sqrt{2\pi}\left\{\frac{52}{15}\eta^3\tilde{L}^{16/5} - \frac{4}{3}\eta\tilde{L}^{12/5} + \frac{2}{3}\eta\tilde{L}^{2/5} + \frac{1}{2}\tilde{L}^{-1} - \frac{1}{6\eta}\tilde{L}^{-2/5} - \frac{2}{15\eta}\tilde{L}^{-12/5}\right\}^{1/2} \\
&+ 4\sqrt{\pi}\tilde{L}^{3/2}\left\{2 - \eta^{-1}\tilde{L}^{-7/5}\ln(2)\right\}^{1/2}.
\end{aligned}
\tag{3.27}
$$

3.2. **An absorbing ball in H^1.** We will use the following version of Agmon's inequality in which the constant is known.

Lemma 3.4. *For all $u \in \dot{L}^2_{\mathrm{per}}$*

$$|u|_\infty \leq c_1|u|^{1/2}|u|_1^{1/2}, \quad \text{for all } u \in \dot{L}^2_{\mathrm{per}}$$

where $c_1 = \sqrt{2}/2$ if u is in either the subspace of odd functions, or that of even functions with mean zero. Otherwise $c_1 = 1$ for a general element in $\dot{L}^2_{\mathrm{per}}$.

Proof. First consider the general case where $u \in \dot{L}^2_{\mathrm{per}}$. Since u has zero mean, there exists x_0 such that $u(x_0) = 0$. For all x satisfying $x_0 < x \leq x_0 + L/2$ we have

$$\begin{aligned} u^2(x) = 2\int_{x_0}^{x} uu' &\leq 2\left(\int_{x_0}^{x} u^2\right)^{1/2}\left(\int_{x_0}^{x} u'^2\right)^{1/2} \\ &\leq 2\left(\int_{x_0}^{x_0+L/2} u^2\right)^{1/2}\left(\int_{x_0}^{x_0+L/2} u'^2\right)^{1/2} \\ &= 2\left(\frac{1}{2}\int_{x_0-L/2}^{x_0+L/2} u^2\right)^{1/2}\left(\frac{1}{2}\int_{x_0-L/2}^{x_0+L/2} u'^2\right)^{1/2} \\ &= |u||u|_1. \end{aligned}$$

In the odd case, we have $u(0) = u(L/2) = 0$ so that

$$u^2(x) = 2\int_0^x uu' = \int_0^x uu' - \int_x^{L/2} uu' \tag{3.28}$$

Now let

$$v(y) = \begin{cases} -u(y), & \text{for } y \geq x, \\ u(y), & \text{for } y < x, \end{cases}$$

so that by (3.28) we have

$$\begin{aligned} u^2(x) = \int_0^{L/2} vu' &\leq \left(\int_0^{L/2} v^2\right)^{1/2}\left(\int_0^{L/2} u'^2\right)^{1/2} \\ &= \left(\frac{1}{2}\int_{-L/2}^{L/2} u^2\right)^{1/2}\left(\frac{1}{2}\int_{-L/2}^{L/2} u'^2\right)^{1/2} \\ &= \frac{1}{2}|u||u|_1. \end{aligned}$$

□

We next express the radius of an absorbing ball in $\dot{H}^1_{\mathrm{per}}$ (resp. H^1_{odd}) in terms of the radius for an absorbing ball in $\dot{L}^2_{\mathrm{per}}$ (resp. L^2_{odd}). The result below is a special case of one in [25], but here it is with a constant that is completely determined.

Theorem 3.5. *The radius of an absorbing ball in $\dot{H}^1_{\mathrm{per}}$ can be taken to be*

$$\rho_1 = (c_1)^{4/5}\rho_0^{7/5}, \tag{3.29}$$

where ρ_0 is the radius of an absorbing ball in $\dot{L}^2_{\mathrm{per}}$ and c_1 is as in Lemma 3.4.

If one restricts u to the subspace of odd functions, the reduction in the constant c_1 translates to a reduction factor of $2^{2/5} \approx 1.32$ in the estimate for ρ_1. The values of ρ_0 and ρ_1 from (3.29) are plotted against L in Figure 4 for the odd case. Over the moderate range of L considered here, smaller values of ρ_0 (and consequently ρ_1 from (3.29)) are obtained by A. Cheskidov by constructing the gauge function in physical space. This alternative approach is presented in the appendix. For large enough L the radii provided by (3.27) are smaller than those from the appendix.

Proof. As in [25], after applying $\int \cdot D^2u$ to the KSE and integrating by parts, one has

$$\frac{1}{2}\frac{d}{dt}|u|_1^2 + |u|_3^2 - |u|_2^2 - (uDu, D^2u) = 0.$$

We depart from the approach in [25], by rewriting the nonlinear term as

$$\begin{aligned}(uDu, D^2u) &= (D(uDu), Du)\\ &= -(DuDu, Du) - (uD^2u, Du)\\ &= -(DuDu, Du) - (uDu, D^2u),\end{aligned}$$

so that

$$(uDu, D^2u) = -\frac{1}{2}(DuDu, Du).$$

Applying Lemma 3.4 and interpolation, we estimate this term further by

$$\begin{aligned}|(uDu, D^2u)| &\le \frac{1}{2}||Du|_\infty |u|_1^2\\ &\le \frac{c_1}{2}|Du|^{1/2}|Du|_1^{1/2}|u|_1^2\\ &= \frac{c_1}{2}|u|_1^{5/2}|u|_2^{1/2}\\ &\le \frac{c_1}{2}|u|^{11/6}|u|_3^{7/6}.\end{aligned} \tag{3.30}$$

Interpolating once more, we have

$$\frac{d}{dt}|u|_1^2 + 2|u|_3^2 - 2|u|^{2/3}|u|_3^{4/3} - c_1|u|^{11/6}|D^3u|^{7/6} \le 0.$$

Now suppose that

$$\frac{1}{2}|u|_3^2 \ge 2|u|^{2/3}|u|_3^{4/3}, \quad \text{and} \quad \frac{1}{2}|u|_3^2 \ge c_1|u|^{11/6}|D^3u|^{7/6}, \tag{3.31}$$

or equivalently that

$$|u|_3 \ge 2^3|u|, \quad \text{and} \quad |u|_3 \ge c_1^{12/5}|u|^{11/5}. \tag{3.32}$$

It would follow then that

$$\frac{d}{dt}|u|_1^2 + |u|_3^2 \le 0,$$

and subsequently that

$$\frac{d}{dt}|u|_1^2 + \tilde{L}^{-4}|u|_1^2 \le 0. \tag{3.33}$$

By interpolation we have

$$|u|_3 \ge \frac{|u|_1^3}{|u|^2}$$

so that

$$\frac{|u|_1^3}{|u|^2} \ge 2^3|u|, \quad \text{and} \quad \frac{|u|_1^3}{|u|^2} \ge c_1^{12/5}|u|^{11/5}, \tag{3.34}$$

or equivalently

$$|u|_1 \ge 2|u|, \quad \text{and} \quad |u|_1 \ge c_1^{4/5}|u|^{7/5}, \tag{3.35}$$

implying that (3.31) holds. Thus if we have both that $|u| \le \rho_0$, and $|u|_1 \ge \rho_1$ where ρ_1 is as in (3.29), then we must have that (3.33) holds. The remainder of the proof that the ball of radius ρ_1 is in fact absorbing is as in [25]. □

We remark that a nearly identical estimate to that in (3.30) can be obtained by different means. First note that by Lemma 3.4 we find

$$|u|_{L^4}^4 \le c_1^2 |u|_\infty^2 |u|^2 \le |u|^3 |u|_1,$$

so that

$$|u|_{L^4} \le c_1^{1/2} |u|^{3/4} |u|_1^{1/4}. \tag{3.36}$$

Continuing as in [25], but using (3.36) instead of the Sobolev estimate

$$|u|_{L^4} \le c|u|_{1/4},$$

we obtain

$$\begin{aligned}
|(uDu, D^2u)| &\le |u|_{L^4} |Du|_{L^4} |D^2 u| \\
&\le c_1 |u|^{3/4} |Du| |D^2 u|^{5/4} \\
&\le c_1 |u|^{3/4} (|u|^{2/3} |D^3 u|^{1/3})(|u|^{1/3} |D^3 u|^{2/3})^{5/4} \\
&= c_1 |u|^{11/6} |D^3 u|^{7/6}.
\end{aligned}$$

This extra factor of 2, however, would translate into an inflation factor of $2^{4/5} \approx 1.74$ in the final estimate of ρ_1.

4. Dimension of an inertial manifold

There are several natural ways to express the KSE in the form of (2.1). At first we shall follow [25] by setting

$$Au = D^4 u + D^2 u + u, \qquad f(u) = -uDu + u, \tag{4.1}$$

so that the eigenvalues of the linear part are

$$\mu_k = \left(\frac{2\pi}{L} k\right)^4 - \left(\frac{2\pi}{L} k\right)^2 + 1, \quad k = 1, 2, \dots$$

corresponding to a complete set of orthonormal eigenfunctions in L^2_{odd} given by (3.2) and in $\dot{L}^2$ given by

$$w_{k1}(x) = \sqrt{\frac{2}{L}} \sin(\frac{2\pi}{L} kx), \quad w_{k2}(x) = \sqrt{\frac{2}{L}} \cos(\frac{2\pi}{L} kx).$$

We also consider

$$Au = D^4 u + D^2 u, \qquad f(u) = -uDu. \tag{4.2}$$

After selecting the separation into linear and "nonlinear" parts, there is still the selection of the relevant spaces E, F, and $\mathcal{E}$. We will consider below two choices made in [25]:

$$E = H^1_{\text{odd}}\ (\dot{H}^1_{\text{per}}), \quad F = \mathcal{E} = L^2_{\text{odd}}\ (\dot{L}^2_{\text{per}}), \tag{4.3}$$

and

$$E = L^2_{\text{odd}}\ (\dot{L}^2_{\text{per}}), \quad F = \mathcal{E} = H^{-1.5-\varepsilon}_{\text{odd}}\ (\dot{H}^{-1.5-\varepsilon}_{\text{per}}), \tag{4.4}$$

where an optimal choice of $\varepsilon \in (0, .5]$ will be determined below. While our analysis will cover all four choices above, we will actually evaluate the dimension of the inertial manifold, in the odd cases only. As we will demonstrate, the advantage of (4.4) is a smaller absorbing ball, while that of (4.3) is a smaller exponent α in the SGC. Both play a role in the eventual estimate for the dimension of the manifold. Ultimately we will compare the final result given by each choice (4.3) and (4.4), over a moderate parameter range including the particular setting where numerical experiments have been taken.

4.1. **Preparation in H^1.** It is shown in [25] that the conditions **A1** through **A6** hold for the spaces E,F, $\mathcal{E}$ as in (4.3) if $K_1 = K_2 = (\frac{4}{3})^{1/4}$, and $\alpha = 1/4$. All that remains in order to check the gap condition in (2.6) is to estimate the global Lipschitz constant $M = M_{\rho_1}$ for the prepared equation. For the preparation in (2.16), this amounts to estimating d_1 as defined in Lemma 2.3. Proceeding as in [25], we write

$$\begin{aligned} f(u) - f(v) &= \frac{1}{2}[D(u^2 - v^2)] + u - v \\ &= \frac{1}{2}(u+v)D(u-v) + (u-v)D(u+v) + u - v, \end{aligned} \tag{4.5}$$

and estimate the true nonlinear term as

$$\begin{aligned} \frac{1}{2}|D(u^2 - v^2)| &\leq \frac{1}{2}|u+v|_\infty |u-v|_1 + \frac{1}{2}|u-v|_\infty |u+v|_1 \\ &\leq \frac{c_1}{2}|u+v|^{1/2}|u+v|_1^{1/2}|u-v|_1 + \frac{c_1}{2}|u-v|^{1/2}|u-v|_1^{1/2}|u+v|_1 \\ &\leq c_1 \tilde{L}^{1/2}|u+v|_1|u-v|_1 \\ &\leq 2c_1\tilde{L}^{1/2} r|u-v|_1, \end{aligned} \tag{4.6}$$

for all $|u|_1, |v|_1 \leq r$. Adding the linear contribution, we have

$$|f(u) - f(v)| \leq [2c_1\tilde{L}^{1/2}r + \tilde{L}]|u-v|_1.$$

Thus for E,F, $\mathcal{E}$ as in (4.3) we have

$$\operatorname{Lip}(f_{\rho_1}) = d_1(r) = 2c_1\tilde{L}^{1/2}r + \tilde{L}. \tag{4.7}$$

4.2. **Preparation in L^2.** It is shown in [25] that the assumptions **A1-A6** hold for E,F, $\mathcal{E}$ as in (4.4), if we set $K_1 = K_2 = 2$ and $\alpha = \frac{3+2\varepsilon}{8}$ so that $3/8 < \alpha \leq 1/2$. Examination of the proof of Lemma 2 from [25] reveals the constant factor provided in the restatement of that result below.

Lemma 4.1. *For all $u \in \dot{H}^{.5+\varepsilon}_{\text{per}}$*

$$|u|_\infty \leq 2c_1 c_\varepsilon \tilde{L}^\varepsilon |u|_{.5+\varepsilon}, \tag{4.8}$$

where

$$c_\varepsilon = \left(\frac{2\varepsilon+1}{4\pi\varepsilon}\right)^{1/2},$$

and c_1 is as in Lemma 3.4.

Thus if one restricts u to either the subspace of odd functions or that of even functions with mean zero, then one saves a factor of $\sqrt{2}$ in the estimate (4.8).

As in [25], for all $w \in \dot{H}^{1.5+\varepsilon}_{\text{per}}$ one has by Lemma 4.1 that

$$\begin{aligned} (f(u) - f(v), w) &= -\frac{1}{2}(u^2 - v^2, Dw) + (u - v, w) \\ &\leq \frac{1}{2}|u-v||u+v||Dw|_\infty + |u-v||w| \\ &\leq c_1 c_\varepsilon \tilde{L}^\varepsilon |u-v||u+v||w|_{1.5+\varepsilon} + \tilde{L}^{1.5+\varepsilon}|u-v||w|_{1.5+\varepsilon} \\ &\leq [2c_1 c_\varepsilon \tilde{L}^\varepsilon r + +\tilde{L}^{1.5+\varepsilon}]|u-v||w|_{1.5+\varepsilon}, \end{aligned} \tag{4.9}$$

for all $|u|, |v| \leq r$, so that by duality

$$|f(u) - f(v)|_{-1.5-\varepsilon} \leq [2c_1 c_\varepsilon \tilde{L}^\varepsilon r + +\tilde{L}^{1.5+\varepsilon}]|u-v|.$$

Thus for (4.4) we have

$$\operatorname{Lip}(f_{\rho_0}) = d_1(r) = 2c_1 c_\varepsilon \tilde{L}^\varepsilon r + \tilde{L}^{1.5+\varepsilon}. \tag{4.10}$$

4.3. **A Mixed-space Preparation.** Since the nonlinearity involves just one derivative, which corresponds roughly to $A^{1/4}$, and enabled the choice of $\alpha = 1/4$ in the H^1 case, one would expect the same in L^2, as well, rather than $\alpha = 1.5 + \varepsilon$ as above. We can recover $\alpha = 1/4$ by applying Theorem 2.4 in yet another way. First we estimate the nonlinear term as follows:

$$\begin{aligned}(f(u) - f(v), w) = \frac{1}{2}(u^2 - v^2, Dw) &\leq \frac{1}{2}|u + v|_\infty |u - v|_0 |w|_1 \\ &\leq \frac{c_1}{2}(|u|^{1/2}|u|_1^{1/2} + |v|^{1/2}|v|_1^{1/2})|u - v|_0|w|_1\end{aligned}$$

It follows that

$$|f(u) - f(v)|_{-1} \leq c_1\rho_0^{1/2}\rho_1^{1/2}|u - v|_0, \quad \forall u, v \in H^1_{\text{odd}}(\dot{H}^1_{\text{per}}) \text{ such that } |u|_1, |v|_1 \leq \rho_1.$$

Thus the nonlinear term, restricted to the absorbing ball in $H^1_{\text{odd}}(\dot{H}^1_{\text{per}})$, is Lipschitz from the L^2-topology to the H^{-1}-topology. By Theorem 2.4 we can extend this function to a globally Lipschitz function

$$\tilde{f} : L^2_{\text{odd}}(\dot{L}^2_{\text{per}}) \to H^{-1}_{\text{odd}}(\dot{H}^{-1}_{\text{per}}), \tag{4.11}$$

so that

$$\text{Lip}(\tilde{f}) = c_1\rho_0^{1/2}\rho_1^{1/2}. \tag{4.12}$$

4.4. **Using Graph Norms.** We can easily convert the Lipschitz estimates of the previous three sections for use with Theorem 2.2. This is done by determining positive constants c_2, c_3 such that

$$c_2^s|u|_{4s} \leq |u|_{D(A_b^s)} \leq c_3^s|u|_{4s}, \quad s > 0, \tag{4.13}$$

where $|\cdot|_{D(A_b^s)}$ is the graph norm of $A_b = D^4 + D^2 + bI$, defined by

$$|u|^2_{D(A_b^s)} = \sum_k |u_k|^2\mu_{b,k}^{2s},$$

with

$$\mu_{b,k} = \left(\frac{2\pi}{L}k\right)^4 - \left(\frac{2\pi}{L}k\right)^2 + b, \quad k = 1, 2, \dots .$$

For $s < 0$, the inequalities in (4.13) are reversed. Let $y = k/\tilde{L}$ so that (4.13) is equivalent to

$$c_2y^4 \leq y^4 - y^2 + b \leq c_3y^4, \quad \forall y \geq y_0 = 1/\tilde{L} > 0,$$

where $b = -a$ (assuming $A = D^4 + D^2$).

Consider

$$g(y) = (y^4 - y^2 + b)/y^4.$$

We need the maximum and minimum of $g(y)$ to be positive. If we assume $b > 1/4$, then

$$c_2 = \frac{4b - 1}{4b} \leq g(y) \leq \max\{1, g(y_0)\} = c_3$$

for all $y \geq y_0$. Hence, we have that (4.13) holds for

$$c_2 = \frac{4b - 1}{4b}, \quad c_3 = \max\{1, g(y_0)\} = \begin{cases} 1, & \text{if } 1/\tilde{L} \geq \sqrt{b} \\ g(1/\tilde{L}), & \text{if } 1/\tilde{L} < \sqrt{b}. \end{cases} \tag{4.14}$$

When working in the framework of **H1-H3**, we set A and f as in (4.2). Note that the exponential dichotomies in **A3** are not explicitly assumed in this framework, and hence the constants K_1 and K_2, which depend on the choice of A, are irrelevant. Dropping the contribution from the linear term in (4.7) we have for the preparation in H^1

$$|f_{\rho_1}(u) - f_{\rho_1}(v)| \leq 2c_1\tilde{L}^{1/2}\rho_1|u - v|_1 \leq 2c_1c_2^{-1/4}\tilde{L}^{1/2}\rho_1|u - v|_{D(A_b^{1/4})}. \tag{4.15}$$

Similarly, for the preparation in L^2, we set

$$s = \frac{-1.5 - \varepsilon}{4}$$

so that from (4.10) we have that

$$|f_{\rho_0}(u) - f_{\rho_0}(v)|_{A_b^s} \leq c_2^s |f_{\rho_0}(u) - f_{\rho_0}(v)|_{-1.5-\varepsilon} \leq c_1 c_2^s c_\varepsilon \tilde{L}^\varepsilon \rho_0 |u - v|. \tag{4.16}$$

Finally, for the mixed-space preparation, we have from (4.12) that

$$|\tilde{f}(u) - \tilde{f}(v)|_{A_b^{-1/4}} \leq c_2^{-1/4} |\tilde{f}(u) - \tilde{f}(v)|_{-1} \leq c_1 c_2^{-1/4} \rho_0^{1/2} \rho_1^{1/2} |u - v|. \tag{4.17}$$

The gap condition (2.9) can now be expressed as

$$\lambda_{n+1} - \lambda_n > M_1 (4b/(4b-1))^{1/4} [(\lambda_{n+1} + b)^{1/4} + (\lambda_n + b)^{1/4)}], \tag{4.18}$$

where

$$\lambda_n = (n/\tilde{L})^4 - (n/\tilde{L})^2.$$

We now bracket the minimizing value of b for the right hand side of (4.18). Set

$$G(b) = (4b/(4b-1))[(\lambda_{n+1} + b)^{1/4} + (\lambda_n + b)^{1/4}]^4,$$

so that

$$\begin{aligned} G'(b) = &- 1/(4b-1)^2 [(\lambda_{n+1} + b)^{1/4} + (\lambda_n + b)^{1/4}]^4 \\ &+ b/(4b-1)[(\lambda_{n+1} + b)^{1/4} + (\lambda_n + b)^{1/4}]^3 [(\lambda_{n+1} + b)^{-3/4} + (\lambda_n + b)^{-3/4}]. \end{aligned} \tag{4.19}$$

Note that $G' < 0$ if and only if

$$[(\lambda_{n+1} + b)^{1/4} + (\lambda_n + b)^{1/4}] > b(4b-1)[(\lambda_{n+1} + b)^{-3/4} + (\lambda_n + b)^{-3/4}]. \tag{4.20}$$

Replacing λ_{n+1} by λ_n on each side gives a sufficient condition for (4.20),

$$2(\lambda_n + b)^{1/4} \geq b(4b-1) 2 (\lambda_n + b)^{-3/4},$$

which is equivalent to

$$0 \geq 4b^2 - 2b - \lambda_n.$$

Solving the associated equality for b and taking the larger root, we have that

$$G'(b) < 0, \quad \text{provided} \quad 1/4 < b < b_1,$$

where

$$b_1 = \frac{1}{4}(1 + \sqrt{1 + 4\lambda_n}).$$

A similar argument shows that

$$g'(b) > 0, \quad \text{provided} \quad b_3 < b,$$

where

$$b_3 = \frac{1}{4}(1 + \sqrt{1 + 4\lambda_{n+1}}).$$

We will take as an approximation to the minimizing b

$$b_2 = \frac{1}{2}(b_1 + b_3).$$

Plot	Gap Condition	A	Preparation	Bound on Lip(f_ρ)	Ambient space E
1	(2.6)	D^4+D^2+I	(2.16)	$d_1(\rho_0)$ in (4.10)	L^2_{odd}
2	(2.6)	D^4+D^2+I	(2.16)	$d_1(\rho_1)$ in (4.7)	H^1_{odd}
3	(2.9)	D^4+D^2	(2.16)	from (4.16)	L^2_{odd}
4	(2.9)	D^4+D^2	(2.16)	from (4.15)	H^1_{odd}
5	(2.6)	D^4+D^2	(4.11)	as in (4.12)	L^2_{odd}
6	(2.9)	D^4+D^2	(4.11)	from (4.17)	L^2_{odd}

TABLE 1. Methods of estimation

4.5. **Comparing Estimates.** We now calculate the minimal dimension of an inertial manifold guaranteed by the estimates made above. This is done by numerically *evaluating* both sides of the gap condition for increasing values of n. The first n to satisfy the gap condition is denoted $n_{\min}$. The particular choices for the gap condition, the operator A, preparation, and ambient space are given in Table 1. In Figure 5 we plot this dimension versus L, *for the odd case*. The best result is obtained by combining the gap condition (2.9) of Miklavcic with the mixed-space preparation introduced here. It is not surprising that the dimension obtained with $A = D^4 + D^2 + I$ would exceed the rest due to the effect of the linear term in f on the Lipschitz estimate. Note however, that when the gap condition, and preparation are held fixed, as in the pairs of plots labeled (1,2) and (3,4) in Table 1, it is the estimates in H^1, which yield the smaller dimension. All estimates for the dimension are calculated using the absorbing balls of smaller radii obtained in the appendix.

By definition, an inertial manifold in the H^1-norm implies one in the L^2-norm, so this is merely a matter of comparing two different methods of analysis, rather than finding that a lower dimensional manifold exists in one space, but not another. The values of ε in the plots labeled 1 and 3 in Table 1 were chosen independently for each parameter value L by sampling the dimension corresponding to 50 equally spaced values of ε in $(0, .5]$, and selecting the minimal dimension. The dependence of the dimension on ε for the particular parameter value $L = L^* = 4\sqrt{2}\pi$ is shown in Figure 6.

We should mention that for plot 5 in Table 1 where A is chosen as in (4.2) within the framework of **A1-A6**, one must recalculate K_1, K_2. From

$$|e^{-tA}P_n u| \le e^{-t\lambda_n}|P_n u| \le \frac{n}{\tilde{L}} e^{-t\lambda_n}|P_n u|_{-1}, \quad t \le 0,$$

we see that the first half of (2.4) in fact holds with the optimal value $K_1 = 1$ and that the second half of (2.4) holds for $K_1 = K(n)$ where $K(n)$ is defined by

$$\frac{n}{\tilde{L}} = K(n)\lambda_n^{1/4} = K(n)\left[\left(\frac{n}{\tilde{L}}\right)^4 - \left(\frac{n}{\tilde{L}}\right)^2\right]^{1/4}, \tag{4.21}$$

where it is assumed that $n \ge \tilde{L}$. Following [25] one may verify that $K_2 = K(n)$ satisfies (2.5) as well. Unlike plots 1 and 2 in Table 1, the estimate in plot 5 uses constants K_1, K_2 which vary with n, and in fact approach 1 as $n \to \infty$.

The dimension of the manifold provided by these methods seems quite high when one considers independent computational evidence. Though the elements of the global attractor found computationally at L^* include several two-dimensional unstable manifolds along with limit cycles and steady states [15], it is conceivable that all these objects could be contained in an inertial manifold of dimension three. In addition, these elements were shown in [11] to be present for certain three-dimensional approximate inertial forms for $L \in [0, L^*]$. This speculation is in the spirit of the conjecture by Pomeau and Manneville [20] that the dimension of the attractor grows like cL, for some constant c, which is sharp since the dimension of the unstable manifold of the origin grows at that rate. There are several possible sources of inflation in the rigorous estimate of the dimension. The most likely seems to be the estimate on the radius of the absorbing ball.

4.6. **Dimension Estimates in Terms of L.** In the previous subsection we numerically obtained estimates for the dimension of the inertial manifold for particular choices of L. We may also obtain a general estimate for the dimension in terms of the parameter L, in the form of dim $\leq cL^b$. We do so here using the framework in [17] and using the mixed-space preparation (4.11), and at this time we do not try to minimize for c. In this framework, the spectral gap condition is (4.18) with $M_1 = c_1\rho_0^{1/2}\rho_1^{1/2}$.

Since we are not trying to minimize for c, we may take, for simplicity, $b = 1$ in (4.18) and arrive at the condition

$$\lambda_{n+1} - \lambda_n > \left(\frac{4}{3}\right)^{1/4} c_1\rho_0^{1/2}\rho_1^{1/2}[(\lambda_{n+1}+1)^{1/4} + (\lambda_n)^{1/4}].$$

Assuming $n \geq \tilde{L}$, we find

$$\lambda_{n+1} - \lambda_n \geq \frac{n^3}{\tilde{L}^4},$$

and

$$[(\lambda_{n+1}+1)^{1/4} + (\lambda_n)^{1/4}] \leq 2(\lambda_{n+1}+1)^{1/4} \leq c_4\frac{n}{\tilde{L}},$$

for a suitable numerical constant c_4. Thus, it suffices that

$$\frac{n^3}{\tilde{L}^4} \geq c_5\rho_0^{1/2}\rho_1^{1/2}\frac{n}{\tilde{L}},$$

for a numerical constant c_5. We can rewrite this condition as

$$n^2 \geq c_5\rho_0^{1/2}\rho_1^{1/2}\tilde{L}^3.$$

Using (3.27) and (3.29), we see that $\rho_0 = O(\tilde{L}^{8/5})$ and $\rho_1 = O(\rho_0^{7/5}) = O(\tilde{L}^{56/25})$. Thus, it suffices that

$$n^2 \geq c_6\tilde{L}^{123/25}.$$

Hence, we may estimate the dimension as

$$\text{dimension } \leq c_7L^{2.46}, \tag{4.22}$$

for a suitable constant c_7. This estimate had been obtained before in [21] using a different approach and, as far as we know, is presently the best estimate in terms of the power of L.

Finally, we remark that if a better estimate for the radius of an absorbing set is found, say $\rho_0 = O(L^\alpha)$, then one may proceed as above and find the following estimate for the dimension of the inertial manifold:

$$\text{dimension } \leq c_8L^{3\alpha/5+3/2}. \tag{4.23}$$

4.7. **Semi-local Manifolds.** If we consider the KSE prepared using a smaller ball, we can expect an inertial manifold of lower dimension. This manifold might very well be a legitimate inertial manifold for the KSE, if in fact the smaller ball were absorbing. Yet even if it is not, the dynamics contained entirely within the smaller ball still coincides with that of the original KSE, within that ball. This is significant if, perhaps for geometric reasons, one wishes to work with a lower dimensional inertial form, as one can then be certain that solutions which remain within the smaller ball are indeed faithful to the PDE. All that one sacrifices in this case, is the certainty that one is seeing *all* the long-time dynamics of the original KSE, as there could be some lost between the small ball, and a legitimate absorbing ball.

We next determine how the minimal dimension for an inertial manifold varies with the radius of preparation. The results are shown in Figure 7. All of the elements of the global attractor found by computational means in [15] lie within a ball in H^1_{odd} of radius 3.2, which according to (3.35) would correspond to ball of radius 1.6 in the L^2-norm, if both were absorbing. From Figure 7 we see that the rigorous analysis for the equation prepared at these radii, provides an inertial manifold of dimension five. To achieve a dimension of three, with the same analysis, a preparation at a radius of about .7 in L^2 is needed. Finally, at a radius of .2 in L^2, the inertial manifold reduces to the two-dimensional unstable manifold of the origin.

5. Appendix

By A. CHESKIDOV[1]

In this appendix a gauge function is constructed in physical space which leads to a lower estimate for the radius of the absorbing ball over the moderate parameter range considered in this paper. Consider again the Kuramoto-Sivashinsky equation

$$\partial_t U(x,t) + (\partial_x^2 + \partial_x^4) U(x,t) + U(x,t)\partial_x U(x,t) = 0 \tag{5.1}$$

on the interval [-L/2,L/2] with periodic boundary conditions. Let

$$\mathcal{A}_L = \{U : U(x+L) = U(x), U(-x) = -U(x)\}.$$

We study the case when the initial data $U_0(x) = U(x,0)$ are in $\mathcal{A}_L$. As in section 3, the general periodic case is ultimately obtained by using an appropriate translation of the gauge function constructed below.

Since only periodic functions are considered, it can always be assumed that the functions are defined on the whole real line. In addition, when integrating over a period L, the limits of integration are usually dropped.

As before, define the operator $\mathcal{L}$ by $\mathcal{L}f := -\partial_x^2 f - \partial_x^4 f$. Then the K-S equation can be written as

$$\partial_t U = \mathcal{L}U - UU'.$$

Again, as in [1], we write $U(x,t) = V(x,t) + \Phi(x)$, where $V, \Phi \in \mathcal{A}_L$, and define the bilinear form

$$(V_1, V_2)_{\gamma\Phi} := -\int V_1(\mathcal{L} - \gamma\Phi')V_2.$$

Then we have

$$\frac{1}{2}\partial_t \|V\|^2 = -(V,V)_{\Phi/2} - (V,\Phi)_\Phi.$$

Let also $R_{\gamma\Phi}(U) := (U,U)_{\gamma\Phi}$, $Q(U) := \frac{1}{6}\|V\|^2$.

Lemma 5.1. *If there exists $\Phi \in \mathcal{A}_L$ such that*

1. $R_\Phi(V) \geq 0$, $R_{\Phi/4} \geq \frac{1}{6}\|V\|^2$, $\quad \forall V \in \mathcal{A}_L$,
2. $R_0(\Phi) = \|\Phi''\|^2 - \|\Phi'\|^2 \leq 0.072L^5$,
3. $\|\Phi\| \leq \frac{1}{\sqrt{3}}L^{1.5}$,

then

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_2 \leq 0.9\sqrt{\frac{3}{5}}L^{2.5} + \frac{1}{\sqrt{3}}L^{1.5}.$$

Proof. We have

$$\begin{aligned}
\frac{1}{2}\partial_t\|V\|^2 &= -(V,V)_{\Phi/2} - (V,\Phi)_\Phi \\
&\leq -(V,V)_{\Phi/2} + \frac{1}{3}(V,V)_\Phi + \frac{3}{4}(\Phi,\Phi)_\Phi \\
&= -\frac{2}{3}R_{\Phi/4}(V) + \frac{3}{4}R_\Phi(\Phi) \\
&\leq -\frac{1}{9}\|V\|^2 + \frac{27}{500}L^5.
\end{aligned}$$

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Thus,

$$\limsup_{t\to\infty} \|V(\cdot,t)\|^2 \leq \frac{27}{500}\cdot 9L^5 = \frac{3^5}{500}L^5.$$

Since $\|V\|^2 = \|U-\Phi\|^2$, we get

$$\limsup_{t\to\infty}(\|U\| - \|\Phi\|) \leq \limsup_{t\to\infty}\|U-\Phi\| \leq 0.9\sqrt{\frac{3}{5}}L^{2.5}.$$

Finally,

$$\limsup_{t\to\infty}\|U(\cdot,t)\|_2 \leq 0.9\sqrt{\frac{3}{5}}L^{2.5} + \frac{1}{\sqrt{3}}L^{1.5}.$$

□

Lemma 5.2. *For any $V \in \mathcal{A}_L$,*

$$\|V'\|_{L^\infty} \leq L^{\frac{1}{2}}\|V''\|\frac{1}{2\sqrt{3}}.$$

Proof.

$$\begin{aligned}
L|V'(x)| &= \left|V(L) - V(0) + \int_0^L dy \int_y^x V''(z)dz\right| \\
&= \left|\int_0^L dy \int_y^x V''(z)dz\right| \\
&= \left|\int_0^x dz \int_0^z V''(z)dy - \int_x^L dz \int_z^L V''(z)dy\right| \\
&= \left|\int_0^x zV''(z)dz - \int_x^L (L-z)V''(z)dz\right| \\
&\leq \|V''\|\left(\int_0^x z^2dz + \int_x^L (L-z)^2dz\right)^{\frac{1}{2}} \\
&= \|V''\|\left(\frac{x^3}{3} + \frac{(L-x)^3}{3}\right)^{\frac{1}{2}} \qquad \forall x.
\end{aligned}$$

So, we have that $L|V'(x)| \leq \|V''\|\left(\frac{x^3}{3} + \frac{(L-x)^3}{3}\right)^{\frac{1}{2}}$, for all x, for an arbitrary periodic function $V(x)$. Thus, $L|V'(x+a)| \leq \|V''\|\left(\frac{x^3}{3} + \frac{(L-x)^3}{3}\right)^{\frac{1}{2}}$ for all $x,\ a$. Hence,

$$L|V'(x)| \leq \inf_{x\in\Omega}\|V''\|\left(\frac{x^3}{3} + \frac{(L-x)^3}{3}\right)^{\frac{1}{2}} = L^{\frac{1}{2}}\|V''\|\frac{1}{2\sqrt{3}}.$$

□

Lemma 5.3. *For any $V \in \mathcal{A}_L$,*

$$|V(x)| \leq \min\{x, L-x\}L^{\frac{1}{2}}\|V''\|_2\frac{1}{2\sqrt{3}} \qquad \forall x.$$

Proof. Obvious, since $V(0) = V(L) = 0$. □

Here is the main result.

Theorem 5.4. *Let $U \in \mathcal{A}_L$ be a solution of the Kuramoto-Sivashinsky equation (5.1). Then*

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_2 \le 0.9\sqrt{\frac{3}{5}}L^{2.5} + \frac{1}{\sqrt{3}}L^{1.5}.$$

Proof. We must construct $\Phi \in \mathcal{A}_L$ satisfying the conditions of the Lemma 5.1. First, let us prove that there exists $\delta \in (0, L)$ such that Φ satisfies condition 1 provided

$$\Phi'(x) = 2 \text{ on } (\delta, L-\delta),\ \Phi'(x) \le 2 \quad \forall x, \qquad \text{and } \int_{-\delta}^{\delta} \Phi'(x)x^2 dx \ge -\frac{2}{5}L\delta^2. \tag{5.2}$$

If $\gamma = \frac{1}{4}$, then using $\frac{3}{4}\|V''\|^2 - \|V\|\cdot\|V''\| + \frac{1}{3}\|V\|^2 \ge 0$ and Lemma 5.3 we obtain

$$\begin{aligned}
\|V''\|^2 - \|V'\|^2 + \gamma\int \Phi' V^2 dx &\ge \|V''\| - \|V\|\cdot\|V''\| + \frac{1}{4}\int \Phi' V^2 dx \\
&\ge \frac{1}{4}\|V''\|^2 + \frac{1}{6}\|V\|^2 + \frac{1}{4}\int (\Phi' - 2)V^2 dx \\
&\ge \frac{1}{4}\|V''\|^2 + \frac{1}{6}\|V\|^2 + \frac{1}{4}\|V''\|^2 \int (\Phi' - 2)x^2\frac{L}{12}dx \\
&\ge \left(\frac{1}{4} - \frac{L^2\delta^2}{120} - \frac{L\delta^3}{36}\right)\|V''\|^2 + \frac{1}{6}\|V\|^2.
\end{aligned}$$

Let $\delta = \frac{n}{L}$. Since $L \ge 2\pi$, it is sufficient to choose n such that

$$\frac{1}{4} - \frac{n^2}{120} - \frac{n^3}{36(2\pi)^2} \ge 0.$$

This is true for $n \le 4.64$.

Let us now consider the case $\gamma = 1$. Then since $\frac{1}{8}\|V''\|^2 - \|V\|\cdot\|V''\| + 2\|V\|^2 \ge 0$, we have

$$\begin{aligned}
\|V''\|^2 - \|V'\|^2 + \gamma\int \Phi' V^2 dx &\ge \|V''\| - \|V\|\cdot\|V''\| + \int \Phi' V^2 dx \\
&\ge \frac{7}{8}\|V''\|^2 + \int (\Phi' - 2)V^2 dx \\
&\ge \frac{7}{8}\|V''\|^2 + \|V''\|^2 \int (\Phi' - 2)x^2\frac{L}{12}dx \\
&\ge \left(\frac{7}{8} - \frac{L^2\delta^2}{30} - \frac{L\delta^3}{9}\right)\|V''\|^2.
\end{aligned}$$

Writing again $\delta = \frac{n}{L}$, we find the second condition on n:

$$\frac{7}{8} - \frac{n^2}{30} - \frac{n^3}{9(2\pi)^2} \ge 0.$$

This holds for $n \le 4.37$. So, in order to satisfy condition 1 of the lemma, it is sufficient to choose

$$\delta = \frac{4.37}{L}.$$

We construct a gauge function Φ in the following way. Let

$$f_\epsilon(x) = \frac{1}{2}\frac{L}{\delta'^3}x^3 - \frac{3}{2}\frac{L}{\delta'}x + 2x,$$

where $\delta' = \delta - \epsilon$, $0 \le \epsilon < \delta$. Let also

$$\Psi_\epsilon(x) = \begin{cases} 2x - L, & \text{for } x \in [\delta', L-\delta'], \\ f_\epsilon(x), & \text{for } x \in [0, \delta'), \\ -f_\epsilon(L-x), & \text{for } x \in (L-\delta', L], \end{cases}$$

and extend it to the whole $\mathbb{R}$ by periodicity (see Figure 3).

First we notice that

$$\int_{-\delta}^{\delta} \Psi_0'(x)x^2 dx = \int_{-\delta}^{\delta} \left(\frac{3}{2}\frac{L}{\delta^3}x^4 - \frac{3}{2}\frac{L}{\delta}x^2 + 2x^2 \right) dx > -\frac{2}{5}L\delta^2, \tag{5.3}$$

$$\int \left(\Psi_0''^2 - \Psi_0'^2 \right) dx < \int_{-\delta}^{\delta} \left(3\frac{L}{\delta^3}x \right)^2 dx = 6\frac{L^2}{\delta^3} = \frac{6}{n^3}L^5 < 0.072L^5. \tag{5.4}$$

We can choose ϵ so small that Ψ_ϵ will satisfy the same inequalities. Let now

$$\Phi_\eta = \Psi_\epsilon * \rho_\eta \quad \text{for } \eta \in (0, \epsilon],$$

where

$$\rho_\eta = \begin{cases} \dfrac{2C}{\eta} \exp\left(\dfrac{1}{4x^2/\eta^2 - 1} \right) & \text{for } x \in \left(-\frac{\eta}{2}, \frac{\eta}{2}\right), \\ 0 & \text{for } x \in \left(-\infty, -\frac{\eta}{2}\right] \cup \left[\frac{\eta}{2}, \infty\right), \end{cases}$$

and C is fixed by the condition

$$\int_{-\eta/2}^{\eta/2} \rho_\eta dx = 1.$$

Since

$$\int \left(\Phi_\eta''^2 - \Phi_\eta'^2 \right) dx \to \int \left(\Phi_0''^2 - \Phi_0'^2 \right) dx, \text{ and}$$

$$\int_{-\delta}^{\delta} \Phi_\eta' x^2 dx \to \int_{-\delta}^{\delta} \Phi_0' x^2 dx$$

as $\eta \to 0$, choosing η small enough we see that Φ_η satisfies (5.3) and (5.4) also. Let $\Phi = \Phi_\eta$. So, we have

$$\int_{-\delta}^{\delta} \Phi'(x) \cdot x^2 dx > -\frac{2}{5}L\delta^2,$$

$$\int \left(\Phi''^2 - \Phi'^2 \right) dx < 0.072L^5.$$

Finally, in order to conclude, we need to check that condition 3 of the Lemma 5.1 holds for our gauge function Φ. For this we note that

$$\|\Psi_\epsilon * \rho_\eta\| \le \|\Psi_\epsilon\|.$$

So, since

$$|\Psi_\epsilon(x)| \le |2x - L|,$$

we have

$$\|\Phi\| \le \sqrt{\int_0^L \left(2\left(x - \frac{L}{2} \right) \right)^2 dx} = \frac{1}{\sqrt{3}}L^{1.5}.$$

□

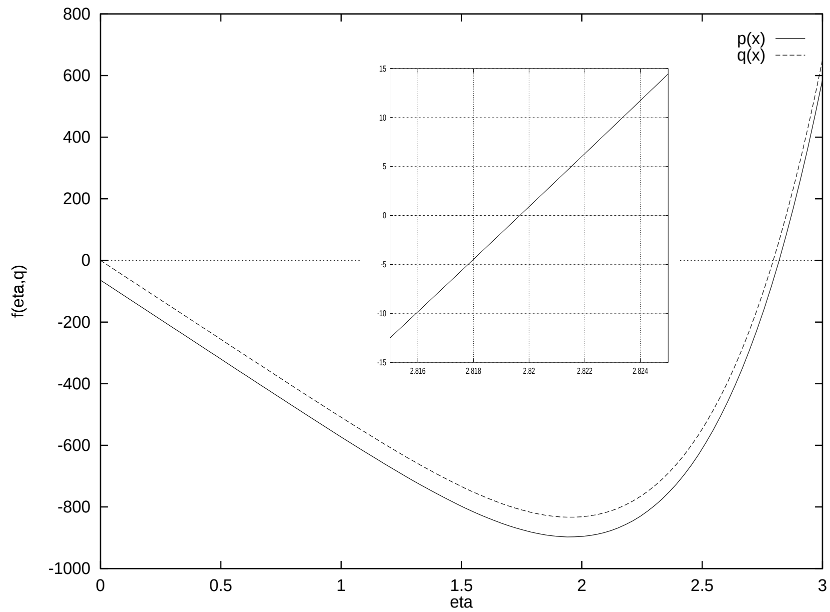

FIGURE 1. $f(\eta, q)$ from (3.21) for the extreme cases of q.

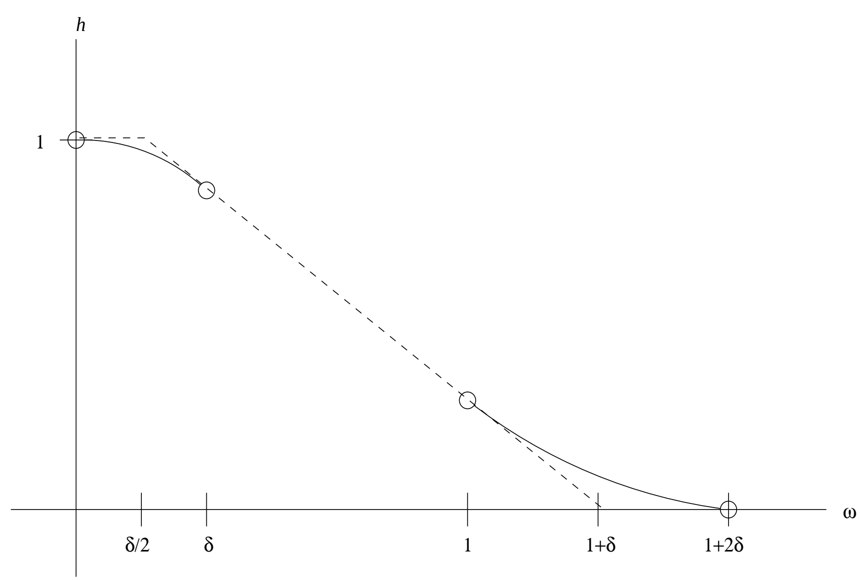

FIGURE 2. Construction of h_δ.

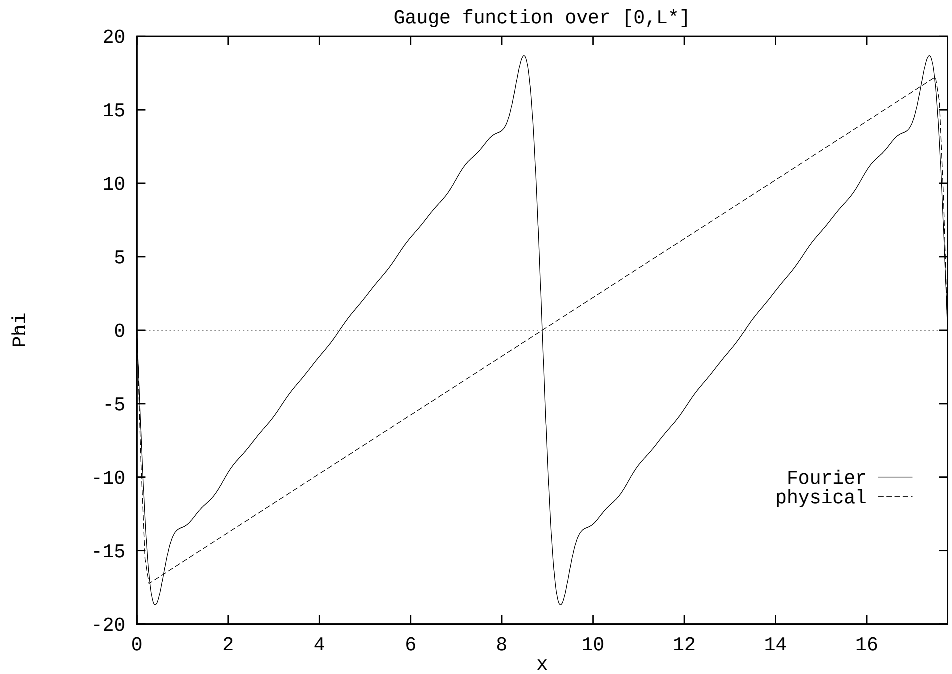

FIGURE 3. The gauge function for $L = L^*$, where (3.21) gives $M = 7$.

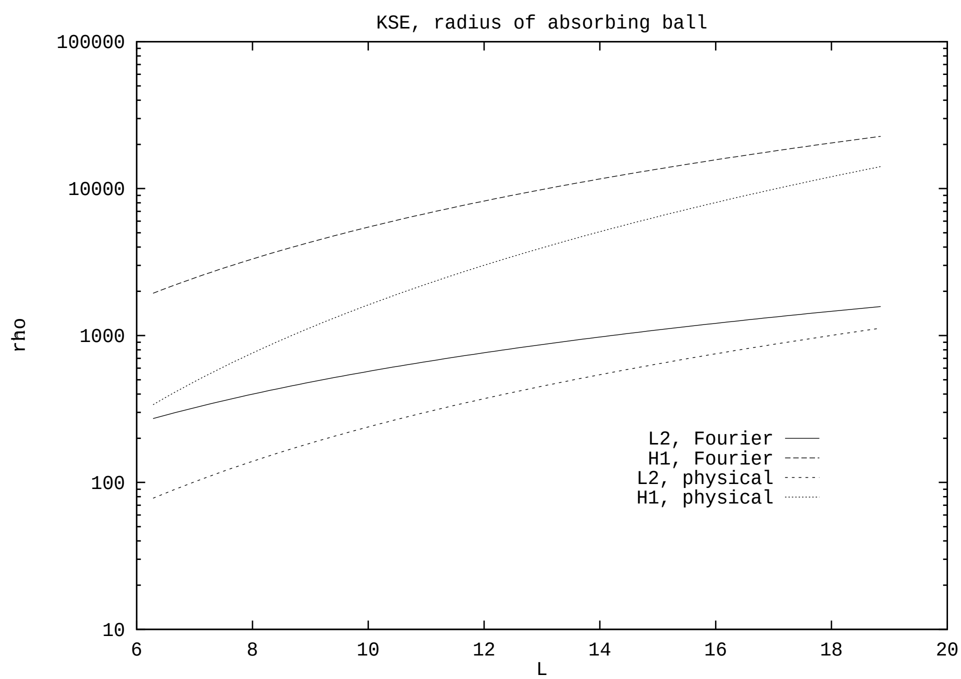

FIGURE 4. Estimate for radius of absorbing ball.

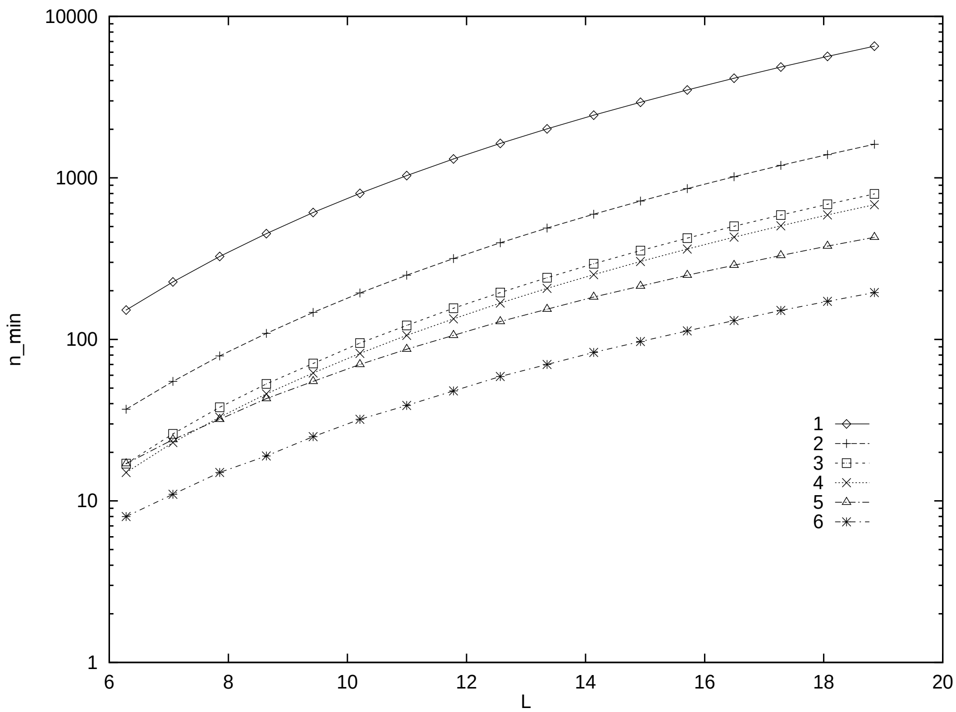

FIGURE 5. Minimal dimension obtained by the different treatments in Table 1

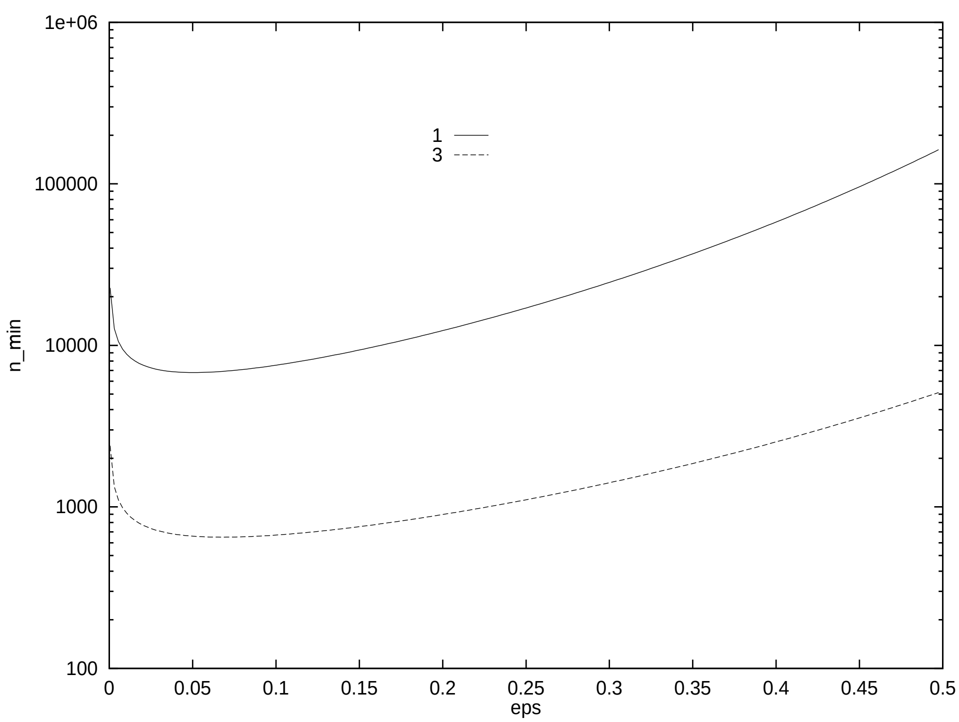

FIGURE 6. Minimal dimension at $L = L^{*}$, by two treatments in Table 1

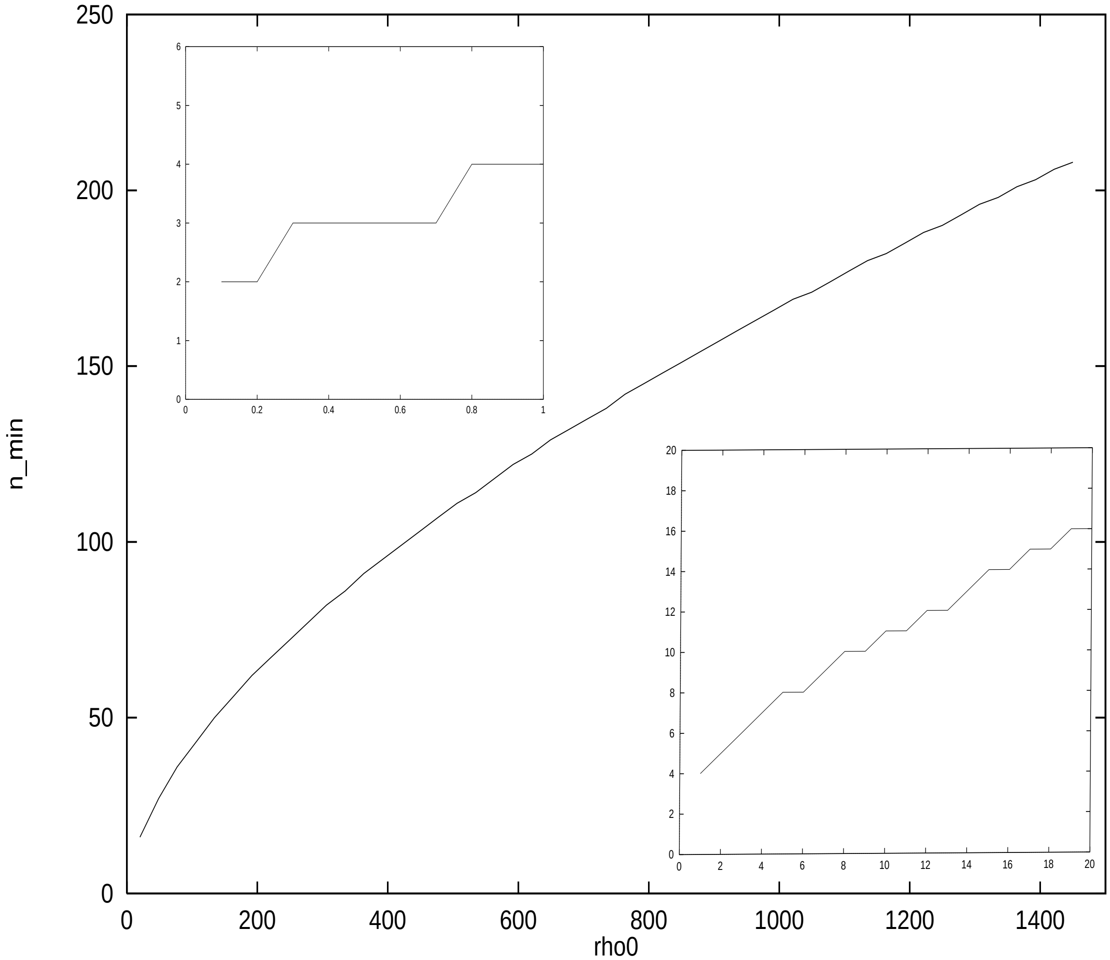

FIGURE 7. Minimal dimension for equation prepared at different radii.

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