

THE EXISTENCE OF NON-TOPOLOGICAL MULTIVORTEX SOLUTIONS IN THE RELATIVISTIC SELF-DUAL CHERN-SIMONS THEORY

DONGHO CHAE AND OLEG YU. IMANUVILOV

ABSTRACT. We construct a general type of multivortex solutions of the self-duality equations (the Bogomol'nyi equations) of (2+1) dimensional relativistic Chern-Simons model with the non-topological boundary condition near infinity. For such construction we use a modified version of the Newton iteration method developed by Kantorovich.

INTRODUCTION

The Lagrangian density of the (2+1)-dimensional relativistic Chern-Simons gauge field theory is given by

$$\mathcal{L} = \frac{\kappa}{4} \varepsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho + (D_\mu \phi) \overline{(D^\mu \phi)} - \frac{1}{\kappa^2} |\phi|^2 (1 - |\phi|^2)^2, \quad (1)$$

where A_μ ($\mu = 0, 1, 2$) is the gauge field on \mathbb{R}^3 , $F_{\mu\nu} = \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu$ is the corresponding curvature tensor, $\phi = \phi_1 + i\phi_2$ ($i = \sqrt{-1}$) is a complex field on \mathbb{R}^3 , called the Higgs field, $D_\mu = \frac{\partial}{\partial x^\mu} - iA_\mu$ is the gauge covariant derivative associated with A_μ , $\varepsilon_{\mu\nu\rho}$ is the totally skewsymmetric tensor with $\varepsilon_{012} = 1$, and finally $\kappa > 0$ is the Chern-Simons coupling constant. Our metric on \mathbb{R}^3 is $(g_{\mu\nu}) = \text{diag}(1, -1, -1)$. This model was suggested by Hong-Kim-Pac[8] and Jackiw-Weinberg[10] to study vortex solutions of the Abelian Higgs model which carry both electric and magnetic charges(See [5] for a general survey of the model). This feature of the model is important in the

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physics of high critical temperature superconductivity. The Gauss equation (variational equation for A_0) of (1) is given by

$$\kappa F_{12} = -2|\phi|^2 A_0. \quad (2)$$

Using this relation, and by integration by part the static energy corresponding to (1) can be written as ([8],[10])

$$E = \int_{\mathbb{R}^2} \left\{ \frac{\kappa^2 F_{12}^2}{4|\phi|^2} + \sum_{j=1}^2 |D_j \phi|^2 + \frac{1}{\kappa^2} |\phi|^2 (1 - |\phi|^2)^2 \right\} dx \quad (3-a)$$

$$= \int_{\mathbb{R}^2} \left\{ |(D_1 \pm iD_2)\phi|^2 + \left| \frac{\kappa F_{12}}{2\phi} \pm \frac{1}{\kappa} \bar{\phi} (|\phi|^2 - 1) \right|^2 \right\} dx \pm \int_{\mathbb{R}^2} F_{12} dx, \quad (3-b)$$

where $+$ ($-$) sign are chosen if the integral $\int_{\mathbb{R}^2} F_{12} dx$ has nonnegative(nonpositive) sign. Below we choose the upper sign. We have thus

$$E \geq \left| \int_{\mathbb{R}^2} F_{12} dx \right|$$

and the minimum of the energy is saturated if and only if (ϕ, A) , $A = (A_1, A_2)$ satisfies the self-duality equations, or the Bogomol'nyi equations:

$$(D_1 + iD_2)\phi = 0, \quad (4)$$

$$F_{12} + \frac{2}{\kappa^2} |\phi|^2 (|\phi|^2 - 1) = 0. \quad (5)$$

The system (4)-(5) is equipped with the following natural boundary conditions

$$|\phi(x)| \rightarrow 1 \text{ as } |x| \rightarrow \infty \quad (6)$$

or

$$|\phi(x)| \rightarrow 0 \text{ as } |x| \rightarrow \infty \quad (7)$$

in order to make the energy (3-a) finite. The solutions (ϕ, A) of (4)-(5) satisfying (6) are called topological solutions, while the solutions of (4)-(5) satisfying the boundary condition (7) are called nontopological solutions. Following Jaffe-Taubes [11], we can reduce system (4)-(5) with (6) or (7)

to the more simplified form of partial differential equations as follows . We introduce new variable (u, θ) by

$$\phi = e^{\frac{1}{2}(u+i\theta)}, \quad \theta = 2 \sum_{j=1}^N \arg(z - z_j), \quad z = x_1 + ix_2 \in \mathbb{C}^1 = \mathbb{R}^2, \quad (8)$$

where z_j , allowing multiplicities, $j = 1, 2, \dots, N$ are the zeros, called the centres of the vorticities, of $\phi(z)$. Then, we can rewrite (4) -(5) with (6) or (7) as (Hereafter, we set $\kappa = 2$ for simplicity.)

$$\Delta u = e^u(e^u - 1) + 4\pi \sum_{j=1}^N \delta(z - z_j), \quad (9)$$

$$u(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty \quad (\text{topological boundary condition})$$

or

$$u(x) \rightarrow -\infty \text{ as } |x| \rightarrow \infty \quad (\text{non-topological boundary condition}). \quad (10)$$

For topological boundary condition Wang[16] proved existence of general multivortex solutions, using the variational method similar to Jaffe-Taubes[11]. Later, Spruck-Yang [13] proved existence of topological solutions, using a more constructive iteration method, and generated even shapes of vortices by numerical simulations. (See [15] also for the study of (4)-(5) in a periodic bounded domain, and [3] for the study of topological solitons of the Chern-Simons model coupled with the Maxwell fields in a self-dual fashion.) The non-topological solutions, however, has not been well understood yet compared to the topological ones. In [14] Spruck-Yang proved existence of radially symmetric non-topological solutions which correspond to solutions of (4)-(5) with a single center(See [4] also for related studies of the radial solutions). In this paper we prove existence of general type of non-topological multivortex solutions. Our method of existence proof is quite constructive using an iteration scheme. Moreover, we establish precise decay estimates near infinity of our solutions. More specifically we prove the following theorem:

Main Theorem. Let $\{z_j\}_{j=1}^N \subset \mathbb{C}^1$ be arbitrarily given. Then, there exists a solution (ϕ, A) to (4)-(5) (with $\kappa = 2$), (7) such that the function $\phi(z)$ has the zeros $\{z_j\}_{j=1}^N$ with possible multiplicities, and the pair (ϕ, A) make the energy functional (3-a) finite. Moreover, our constructed solutions satisfy:
(i) The decay estimates; there exists $\beta > 0$ such that

$$|\phi|^2 + |F_{12}| + |D_1\phi|^2 + |D_2\phi|^2 = O\left(\frac{1}{|x|^{2N+4+\beta}}\right) \text{ as } |x| \rightarrow \infty \quad (11)$$

(ii) Flux integral; there exists a positive constant γ , which depends on the solution, and can be fixed to be arbitrarily small, such that

$$\Phi = \int_{\mathbb{R}^2} F_{12} dx = 4\pi(N+1) + \gamma. \quad (12)$$

Remark 1.

Since $|\phi|^2 = e^u$, where u is a solution of (9), we can deduce that if

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\ln|x|} = -B, \quad (13)$$

then necessarily we have $B > 2N + 4$. It would be interesting to compare this result with the previously known one for the (See Theorem 2.1 of [14], and Theorem 3.2 of [4].) for the radially symmetric solutions of the following equation for vorticities concentrated on the origin of \mathbb{R}^2 .

$$\Delta u = e^u(e^u - 1) + 4\pi N\delta(z), \quad (14)$$

equipped with the nontopological boundary condition (10). They proved existence of radially symmetric non-topological solution u of (14) satisfying (13) with $B > 2N + 4$. Thus our existence result is a ‘‘proper’’ generalization of the previous ones for the radially symmetric ones in [4] and [14].

Remark 2. Equation (12) implies that the integral $\int_{\mathbb{R}^2} F_{12} dx$, which corresponds to the minimum of the total static energy given by (3.a)-(3.b), is not ‘‘quantized’’ contrary to the case of topological solutions. This phenomena, which was discussed in the physics literature (See e.g. [5] and references therein) for radially symmetric solutions, is to the authors’ knowledge rigorously verified first time in this paper for a nonradial solution. We also note that the estimate $\Phi > 4(N+1)\pi$ was proved by Spruck-Yang[14] for

any radially symmetric solution. The fact that γ can be chosen arbitrarily small implies that their estimate is sharp for radially symmetric solution. For arbitrary nonradial solutions such estimate is currently not available.

Our idea of proof is the following: We formulate our problem of construction of nontopological solutions as one of finding roots of a functional equation defined in suitable function spaces. To find the roots we use the theorem on the convergence of a modified Newton's iteration scheme developed by Kantorovich [12]. Basic observation is that after suitable scale transformation our equation can be a perturbed one from a radially symmetric equation, and the iteration starting from a radial solution of the Liouville equation superposed with appropriate "small" nonradial function works well. The organization of the paper is the following. In Section 1 we introduce basic function spaces, and derive some properties of them useful in the following sections. In Section 2 we formulate our problem in terms of root finding of some mapping between function spaces introduced in Section 1. In Section 3 we prove various estimates on the mapping introduced in Section 2 in order to apply our iteration scheme. In Section 4 based on the estimates established in the previous section, we prove our main theorem, using the Newton-Kantorovich iteration scheme. We postpone all the proofs of the auxiliary lemmas to Appendix in order to help the readers to keep on the main stream of argument.

1. INTRODUCTION OF FUNCTION SPACES

In this section we define the class of functional spaces which are necessary in sequel and establish some properties of functions from these spaces. Let $\alpha \in (0, 1)$ be given. We introduce the Hilbert spaces X_α and Y_α as follows

$$X_\alpha = \left\{ u(x) \in L^2_{loc}(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha}) u^2 dx < \infty \right\},$$

equipped with the inner product $(u, v)_{X_\alpha} = \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha}) uv dx$, and

$$Y_\alpha = \left\{ u \in W^{2,2}_{loc}(\mathbb{R}^2) \mid \|\Delta u\|_{X_\alpha}^2 + \left\| \frac{u}{1 + |x|^{1+\frac{\alpha}{2}}} \right\|_{L^2(\mathbb{R}^2)}^2 < \infty \right\}$$

equipped with the inner product

$$(u, v)_{Y_\alpha} = (\Delta u, \Delta v)_{X_\alpha} + \int_{\mathbb{R}^2} \frac{uv}{1 + |x|^{2+\alpha}} dx.$$

These spaces are equipped with the natural Banach space norms;

$$\|u\|_{X_\alpha} = \sqrt{(u, u)_{X_\alpha}}, \quad \|u\|_{Y_\alpha} = \sqrt{(u, u)_{Y_\alpha}}$$

respectively. Thanks to the inequality

$$\int_{\mathbb{R}^2} |u| dx \leq \left(\int_{\mathbb{R}^2} \frac{1}{1 + |x|^{2+\alpha}} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} (1 + |x|^{2+\alpha}) u^2 dx \right)^{\frac{1}{2}}$$

there is a continuous imbedding

$$X_\alpha \hookrightarrow L^1(\mathbb{R}^2) \quad \forall \alpha \in (0, 1). \quad (1.1)$$

Also, by the local regularity of the Laplace operator (See [7]) we have

$$Y_\alpha \subset C_{loc}^0(\mathbb{R}^2) \quad \forall \alpha \in (0, 1). \quad (1.2)$$

We start from the following elementary proposition

Proposition 1.1. *Let $\alpha \in (0, 1)$ and $v \in Y_\alpha$ be a harmonic function. Then $v \equiv \text{const}$.*

Proof. In the polar coordinates the Fourier expansion in θ yields

$$v(x) = \sum_{k=0}^{\infty} v_k(r) e^{ik\theta}$$

with the rapidly decreasing coefficients, $v_k(r)$ in k for each fixed $r \geq 0$. Since $v(x)$ is the harmonic function, each $v_k(r)$ satisfies the ordinary differential equation:

$$\frac{d^2 v_k}{dr^2} + \frac{1}{r} \frac{dv_k}{dr} - \frac{k^2}{r^2} v_k = 0 \quad \forall k \in \mathbb{Z}_+ \cup \{0\}.$$

The general solution of these ordinary differential equations is well known:

$$v_k(r) = M_{1,k} r^k + \frac{M_{2,k}}{r^k} \quad \text{for } k \in \mathbb{Z}_+, \quad v_0(r) = M_{1,0} + M_{2,0} \ln r,$$

where $M_{1,k}, M_{2,k}$ are the constants. Noting $\int_{-\pi}^{\pi} e^{ik\theta} d\theta$ equals 0 for $k \neq 0$ and 2π for $k = 0$, we deduce

$$v_k \in X_\alpha \quad \forall k \in \mathbb{Z}_+ \cup \{0\}.$$

Hence $M_{1,k} = 0$ for all $k \in \mathbb{Z}_+$. Then taking into account that v is smooth function in neighborhood of zero we obtain $M_{2,k} = 0$ for all $k \in \mathbb{Z}_+ \cup \{0\}$. ■

Denote $\ln^+ |x| = \max\{0, \ln |x|\}$ below. The next lemma provides the pointwise estimate for an arbitrary function from Y_α . In particular it imply that on infinity functions from this space have at most logarithmic growth.

Lemma 1.1. *Let $\alpha \in (0, 1)$, then there exists $C_1 > 0$ such that for all $v \in Y_\alpha$*

$$|v(x)| \leq C_1 \|v\|_{Y_\alpha} (\ln^+ |x| + 1) \quad \forall x \in \mathbb{R}^2. \quad (1.3)$$

Proof. For given $v \in Y_\alpha$, we set $\Delta v = g$. By definition of the space Y_α the function $g \in X_\alpha$. We consider the function

$$\tilde{v}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x - \tau| g(\tau) d\tau.$$

It is well-known that $\Delta \tilde{v} = g$ in \mathbb{R}^2 . Using the Cauchy-Bynakovskii inequality we obtain

$$\begin{aligned} |\tilde{v}(x)| &\leq \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \ln |x - \tau| g(\tau) d\tau \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} |\ln |x - \tau|| |g(\tau)| d\tau \\ &\leq \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} (1 + |\tau|^{2+\alpha}) |g(\tau)|^2 d\tau \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \frac{\ln^2 |x - \tau|}{1 + |\tau|^{2+\alpha}} d\tau \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2\pi} \|v\|_{Y_\alpha} \left(\int_{\mathbb{R}^2} \frac{\ln^2 |\tau|}{1 + |\tau|^{2+\alpha}} d\tau \right)^{\frac{1}{2}}. \end{aligned} \quad (1.4)$$

Let us estimate the last integral in (1.4)

$$\begin{aligned} I^2 &= \int_{\mathbb{R}^2} \frac{\ln^2 |\tau|}{1 + |\tau|^{2+\alpha}} d\tau \\ &= \int_{|\tau| \leq 1} \frac{\ln^2 |\tau|}{1 + |\tau|^{2+\alpha}} d\tau + \int_{|\tau| \geq 1} \frac{\ln^2 |\tau|}{1 + |\tau|^{2+\alpha}} d\tau \\ &= A_1 + A_2. \end{aligned}$$

Obviously,

$$A_1 = \int_{|\tau| \leq 1} \frac{\ln^2 |\tau|}{1 + |\tau|^{2+\alpha}} d\tau \leq \int_{|\tau| \leq 1} \ln^2 |\tau| d\tau \leq C. \quad (1.5)$$

Note that for $|\tau| \geq 2|x|$ we have

$$|\tau - x| \geq |\tau| - |x| = \frac{1}{2}|\tau| + \frac{1}{2}|\tau| - |x| \geq \frac{1}{2}|\tau|.$$

Thus,

$$\begin{aligned} \int_{|\tau| \geq 1} \frac{\ln^2 |\tau|}{1 + |x - \tau|^{2+\alpha}} d\tau &= \int_{\{|\tau| \geq 1\} \cap \{|\tau| \geq 2|x|\}} \frac{\ln^2 |\tau|}{1 + |x - \tau|^{2+\alpha}} d\tau \\ &+ \int_{\{|\tau| \geq 1\} \cap \{|\tau| < 2|x|\}} \frac{\ln^2 |\tau|}{1 + |x - \tau|^{2+\alpha}} d\tau \leq C(1 + (\ln^+ 2|x|)^2), \end{aligned} \quad (1.6)$$

where the constants C depends on α only. Then inequalities (1.4)-(1.6) imply

$$|\tilde{v}(x)| \leq C\|v\|_{Y_\alpha} (\ln^+ |x| + 1) \quad \forall x \in \mathbb{R}^2. \quad (1.7)$$

Hence $\tilde{v} \in Y_\alpha$ and by Proposition 1.1 there exists $k \in \mathbb{R}$ such that $\tilde{v}(x) = v(x) + k$. Thus the pointwise estimate (1.7) yields

$$\|v + k\|_{Y_\alpha} \leq C\|v\|_{Y_\alpha}.$$

Hence

$$|k| \leq C\|k\|_{Y_\alpha} \leq C(\|v + k\|_{Y_\alpha} + \|v\|_{Y_\alpha}) \leq C\|v\|_{Y_\alpha}.$$

Then by (1.7) and this estimate we obtain (1.3). \square

2. FUNCTIONAL FORMULATION OF THE PROBLEM

The aim of this section is two-fold. First we wish to transform equation (9) to more convenient form, and second we would like to outline the strategy of the proof of the main theorem.

As the first step in transforming equation (9) we get rid of delta-functions in its right hand side, using solutions for the Liouville equation.

Throughout this paper we denote $z = x_1 + ix_2$, $a = a_1 + ia_2 \in \mathbb{C}^1 = \mathbb{R}^2$. Let us define

$$f(z) = (N + 1) \prod_{k=1}^N (z - z_k),$$

and

$$F(z) = \int_0^z f(\xi) d\xi.$$

Let us introduce a functions $\rho_{\varepsilon, a}(z)$, $\rho(r)$ by

$$\rho_{\varepsilon, a}(z) = \frac{8\varepsilon^{2N+2} |f(z)|^2}{\left(1 + \varepsilon^{2N+2} \left|F(z) + \frac{a}{\varepsilon^{N+3}}\right|^2\right)^2}, \quad \rho(r) = \frac{8(N+1)^2 r^{2N}}{(1 + r^{2N+2})^2}. \quad (2.1)$$

We note that for any $\varepsilon > 0$ and $a \in \mathbb{C}^1$, $\ln \rho_{\varepsilon,a}(z) = \Theta(z)$ is a solution of the Liouville equation

$$\Delta \Theta + e^\Theta = 4\pi \sum_{j=1}^N \delta(z - z_j)$$

in \mathbb{R}^2 .

Defining $v(z) = u(z) - \ln \rho_{\varepsilon,a}(z)$, we obtain from (9)

$$\Delta v + \rho_{\varepsilon,a} e^v - \rho_{\varepsilon,a}^2 e^{2v} - \rho_{\varepsilon,a} = 0. \quad (2.2)$$

Then, making change of variables $z \rightarrow \frac{z}{\varepsilon}$ and denoting $\tilde{v}(z) = v\left(\frac{z}{\varepsilon}\right)$, we have

$$\begin{aligned} \Delta \tilde{v} + \frac{8\varepsilon^{2N} |f\left(\frac{z}{\varepsilon}\right)|^2 e^{\tilde{v}}}{\left(1 + \varepsilon^{2N+2} \left|F\left(\frac{z}{\varepsilon}\right) + \frac{a}{\varepsilon^{N+3}}\right|^2\right)^2} - \frac{64\varepsilon^{4N+2} |f\left(\frac{z}{\varepsilon}\right)|^4 e^{2\tilde{v}}}{\left(1 + \varepsilon^{2N+2} \left|F\left(\frac{z}{\varepsilon}\right) + \frac{a}{\varepsilon^{N+3}}\right|^2\right)^4} \\ - \frac{8\varepsilon^{2N} |f\left(\frac{z}{\varepsilon}\right)|^2}{\left(1 + \varepsilon^{2N+2} \left|F\left(\frac{z}{\varepsilon}\right) + \frac{a}{\varepsilon^{N+3}}\right|^2\right)^2} = 0. \end{aligned} \quad (2.3)$$

We denote

$$g_\varepsilon(z, a) = \frac{8\varepsilon^{2N} |f\left(\frac{z}{\varepsilon}\right)|^2}{\left(1 + \left|\varepsilon^{N+1} F\left(\frac{z}{\varepsilon}\right) + \frac{a}{\varepsilon^2}\right|^2\right)^2}. \quad (2.4)$$

Then, we can write (2.3) as

$$\Delta \tilde{v} + g_\varepsilon e^{\tilde{v}} - \varepsilon^2 g_\varepsilon^2 e^{2\tilde{v}} - g_\varepsilon = 0. \quad (2.5)$$

To transform (2.5) further we construct a solution to the ordinary differential equation

$$L_1 w = \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} + \rho w = f \quad \text{in } \mathbb{R}_+. \quad (2.6)$$

We first observe that that the function $\varphi_0(r)$ defined by

$$\varphi_0(r) = \frac{1 - r^{2N+2}}{1 + r^{2N+2}} \quad (2.7)$$

solves (2.6) with $f = 0$. (For rational derivation of this fact please see the proof of Lemma 2.4.) From this we establish the following two lemmas, the proofs of which are in Appendix.

Lemma 2.1. *Given $f = f(r)$ with $rf(r) \in L^1(\mathbb{R}_+)$, $f(r) \in C^1(\mathbb{R}_+)$, the ordinary differential equation (2.6) has an explicit solution*

$$w(r) = \varphi_0(r) \left\{ \int_0^r \frac{\phi_f(s) - \phi_f(1)}{(1-s)^2} ds + \frac{\phi_f(1)r}{1-r} \right\} \quad (2.8)$$

with

$$\phi_f(r) = \left(\frac{1+r^{2N+2}}{1-r^{2N+2}} \right)^2 \frac{(1-r)^2}{r} \int_0^r \varphi_0(t) t f(t) dt,$$

where $\phi_f(1)$ and $w(1)$ are defined as limits of $\phi_f(r)$ and $w(r)$ as $r \rightarrow 1$.

Lemma 2.2. *Let $w_0(r)$ be a solution of the equation*

$$L_1 w_0 - \rho^2 = 0 \quad \text{in } \mathbb{R}_+,$$

obtained by substituting $f = \rho^2$ in the solution formula (2.7)-(2.8). Then, the pointwise estimates hold true

$$\sup_{r>0} e^{w_0(r)} < \infty, \quad |w_0(r)| \leq C(\ln^+ r + 1) \quad \forall r > 0, \quad (2.9)$$

and the asymptotic formula

$$w_0(r) = -\tilde{C} \ln r + o(\ln r) \quad \text{as } r \rightarrow \infty, \quad (2.10)$$

where C and \tilde{C} are positive constants independent of r .

Let us introduce the mapping $P_\varepsilon(\cdot, \cdot) : Y_\alpha \times \mathbb{R}^2 \mapsto X_\alpha$ given by

$$P_\varepsilon(u, a) = \Delta u + g_\varepsilon(z, a) e^{u+\varepsilon^2 w_0} - \varepsilon^2 g_\varepsilon^2(z, a) e^{2(u+\varepsilon^2 w_0)} - g_\varepsilon(z, a) + \varepsilon^2 \Delta w_0, \quad (2.11)$$

where $w_0(x) = w_0(|x|)$ is the function defined by Lemma 2.2. Then, the problem of solving equation (9) is reduced to that of finding a root, $(u_\varepsilon^*, a_\varepsilon^*) \in Y_\alpha \times \mathbb{R}^2$ of the functional equation

$$P_\varepsilon(u, a) = 0. \quad (2.12)$$

The transformation of equation (9) is finished.

Finally we note that once a solution of (2.12), $(u_\varepsilon^*, a_\varepsilon^*)$ is found, then our solution u of the equation (9) is recovered by the formula

$$u(x) = \ln \rho_{\varepsilon, a_\varepsilon^*}(x) + u_\varepsilon^*(\varepsilon x) + \varepsilon^2 w_0(\varepsilon x). \quad (2.13)$$

Of course, after that one should check if this solution really satisfies the nontopological boundary conditions.

The proof of our main theorem is based on the following theorem due to Kantorovich [12, p. 532]:

Theorem 2.1. *Let B_1 and B_2 be Banach spaces, and $\Omega \subset B_1$ be an arbitrary domain, $P : \Omega \rightarrow B_2$ be a given mapping which has a continuous second derivative in Ω_0 , where $\Omega_0 = \{v \in \Omega \mid \|v - v_0\|_{B_1} \leq \tilde{r}\}$. Suppose, in addition, that*

- (i) $\Gamma_0 = [P'(v_0)]^{-1}$ exists and continuous linear operator;
 - (ii) $\|\Gamma_0(P(v_0))\| \leq \eta$;
 - (iii) $\|\Gamma_0 P''(v)\| \leq K \quad \forall v \in \Omega_0$;
- Then, provided

$$h = K\eta \leq \frac{1}{2}, \quad \text{and } \tilde{r} \geq r_0 = \frac{1 - \sqrt{1 - 2h}}{h}\eta,$$

The functional equation

$$P(v) = 0 \quad \text{in } \Omega$$

has a solution v^* such that

$$\|v^* - v_0\|_{B_1} \leq r_0.$$

Later, we apply the Newton-Kantorovich iteration method to solve (2.12), starting from the origin of some subspace of the space $Y_\alpha \times \mathbb{R}^2$. For this we need to estimate the distance in X_α from origin to

$$P_\varepsilon(0, 0) = g_\varepsilon(z, 0)e^{\varepsilon^2 w_0} - \varepsilon^2 g_\varepsilon^2(z, 0)e^{2\varepsilon^2 w_0} - g_\varepsilon(z, 0) + \varepsilon^2 \Delta w_0, \quad (2.14)$$

and prove the invertibility of the first derivative of P_ε at the origin,

$$\begin{aligned} P'_\varepsilon(0, 0)[v, b] &= \Delta v + g_\varepsilon(z, 0)e^{\varepsilon^2 w_0} v - 2\varepsilon^2 g_\varepsilon^2(z, 0)e^{2\varepsilon^2 w_0} v \\ &+ \sum_{j=1}^2 \frac{\partial g_\varepsilon(z, 0)}{\partial a_j} e^{\varepsilon^2 w_0} b_j - 2\varepsilon^2 \sum_{j=1}^2 \frac{\partial g_\varepsilon(z, 0)}{\partial a_j} g_\varepsilon(z, 0)e^{2\varepsilon^2 w_0} b_j \\ &- \sum_{j=1}^2 \frac{\partial g_\varepsilon(z, 0)}{\partial a_j} b_j, \end{aligned} \quad (2.15)$$

and finally we need to estimate the norm of the second derivative,

$$\begin{aligned}
P'_\varepsilon(u, a)[v, b; \tilde{v}, \tilde{b}] &= g_\varepsilon(z, a)e^{u+\varepsilon^2 w_0} v \tilde{v} + \sum_{j=1}^2 \frac{\partial g_\varepsilon(z, a)}{\partial a_j} e^{u+\varepsilon^2 w_0} v \tilde{b}_j \\
&+ \sum_{j=1}^2 \frac{\partial g_\varepsilon(z, a)}{\partial a_j} e^{u+\varepsilon^2 w_0} b_j \tilde{v} + \sum_{j,k=1}^2 \frac{\partial^2 g_\varepsilon(z, a)}{\partial a_j \partial a_k} e^{u+\varepsilon^2 w_0} b_j \tilde{b}_k \\
&- 4\varepsilon^2 \sum_{j=1}^2 \frac{\partial g_\varepsilon(z, a)}{\partial a_j} g_\varepsilon(z, a) e^{2u+2\varepsilon^2 w_0} b_j \tilde{v} \\
&- 2\varepsilon^2 \sum_{j,k=1}^2 \left(\frac{\partial^2 g_\varepsilon(z, a)}{\partial a_j \partial a_k} g_\varepsilon(z, a) + \frac{\partial g_\varepsilon(z, a)}{\partial a_j} \frac{\partial g_\varepsilon(z, a)}{\partial a_k} \right) b_j \tilde{b}_k e^{2u+2\varepsilon^2} \\
&- 4\varepsilon^2 g_\varepsilon^2(z, a) e^{2u+2\varepsilon^2 w_0} v \tilde{v} \\
&- 4\varepsilon^2 \sum_{j=1}^2 \frac{\partial g_\varepsilon(z, a)}{\partial a_j} g_\varepsilon(z, a) e^{2u+2\varepsilon^2 w_0} v \tilde{b}_j \\
&- \sum_{j,k=1}^2 \frac{\partial^2 g_\varepsilon(z, a)}{\partial a_j \partial a_k} b_j \tilde{b}_k
\end{aligned} \tag{2.16}$$

at the origin.

The most difficult part is to prove the invertibility of the operator $[P'_\varepsilon(0, 0)]$. To solve this problem let us introduce an operator $A : Y_\alpha \times \mathbb{R}^2 \rightarrow X_\alpha$ defined by

$$A(u, a_1, a_2) = Lu + Ma, \tag{2.17}$$

where the operator $L : Y_\alpha \mapsto X_\alpha$ is defined by

$$Lu = \Delta u + \frac{8(N+1)^2 r^{2N} u}{(1+r^{2N+2})^2}, \quad r = |x|, \tag{2.18}$$

and the operator $M : \mathbb{R}^2 \mapsto X_\alpha$ is defined by

$$Ma = -4(\rho w_0 - 2\rho^2) \varphi_+ a_1 - 4(\rho w_0 - 2\rho^2) \varphi_- a_2 \quad \forall a = (a_1, a_2) \in \mathbb{R}^2 \tag{2.19}$$

with the functions $\varphi_+(r, \theta), \varphi_-(r, \theta) \in Y_\alpha$ given by formulas

$$\varphi_+(r, \theta) = \frac{r^{N+1} \cos(N+1)\theta}{1+r^{2N+2}}, \quad \varphi_-(r, \theta) = \frac{r^{N+1} \sin(N+1)\theta}{1+r^{2N+2}}. \tag{2.20}$$

Later, we shall prove that the operator $P'_\varepsilon(0, 0)$ is a perturbation of the operator A in the norm $\mathcal{L}(Y_\alpha \times \mathbb{R}^2, X_\alpha)$. Now let us prove the following property of the operator A .

Theorem 2.2. *The operator A defined by (2.17)-(2.20) belongs to $\mathcal{L}(Y_\alpha \times \mathbb{R}^2, X_\alpha)$ and*

$$\text{Im } A = X_\alpha.$$

To prove this proposition first we establish facts on image and kernel of the operator L .

We start from the following:

Proposition 2.1. *Let $\alpha \in (0, \frac{1}{2})$, then the image of L , $\text{Im } L$ is closed in X_α .*

The proof of Proposition 2.1 follows immediately from the Lemma below.

Lemma 2.3. *Suppose that X, Y are Hilbert spaces, a bounded linear operator $B : X \mapsto Y$ is onto, and $K : X \mapsto Y$ is a linear compact operator. Then, $\text{Im } (B + K)$ is closed in Y .*

Proof. See the proof of Lemma 5.1 of [6], page 413. ■

Proof of Proposition 2.1. We apply Lemma 2.3 for operators $B = \Delta$, $K = \rho$. Now, it is easy to check $K : Y_\alpha \mapsto X_\alpha$ is a compact operator, by decomposing \mathbb{R}^2 into a bounded and an exterior domains, and using the Rellich-Kondrachev compactness lemma for the bounded domain. We omit details of this part. Here, we only check $\Delta : Y_\alpha \rightarrow X_\alpha$ is onto. Given $f \in X_\alpha$, let us consider a solution u of the equation

$$\Delta u = f \quad \text{in } \mathbb{R}^2$$

represented by

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| f(y) dy.$$

We will show $u \in Y_\alpha$. First $\|\Delta u\|_{X_\alpha} < \infty$ is immediate. Next, we claim

$$\int_{\mathbb{R}^2} \frac{u^2(x)}{1+|x|^{2+\alpha}} dx < \infty. \quad (2.21)$$

Indeed, we have

$$\begin{aligned} |u(x)| &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\ln|x-y||}{(1+|y|^{2+\alpha})^{\frac{1}{2}}} (1+|y|^{2+\alpha})^{\frac{1}{2}} |f(y)| dy \\ &\leq \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \frac{\ln^2|x-y|}{1+|y|^{2+\alpha}} dy \right)^{\frac{1}{2}} \|f\|_{X_\alpha} \\ &\leq C \|f\|_{X_\alpha} (\ln^+ |x| + 1), \end{aligned}$$

where we used previous estimate for I in the proof of Lemma 1.1(See (1.5)-(1.6)). Thus (2.21) follows, and $u \in Y_\alpha$. ■

Now we can compute the kernel of the operator L .

Lemma 2.4. *Let $\alpha \in (0, \frac{1}{2})$, and $L : Y_\alpha \mapsto X_\alpha$ be the differential operator defined by (2.18). Then,*

$$\ker L = \text{Span}\{\varphi_0, \varphi_+, \varphi_-\},$$

where φ_0, φ_\pm are introduced in (2.7) and (2.20) respectively.

Proof. As is well-known, for any $N, k \in \mathbb{Z}_+ \cup \{0\}$ and $a = a_1 + ia_2 \in \mathbb{C}^1$ the function

$$\Psi(a, z) = \ln \frac{8|(N+1)z^N + a(k+N+1)z^{N+k}|^2}{(1 + |z^{N+1} + az^{N+k+1}|^2)^2}.$$

satisfies the Liouville equation

$$\Delta \Psi + e^\Psi = 0 \tag{2.22}$$

except at the zeros of the polynomial $p(z) = (N+1)z^N + a(k+N+1)z^{N+k}$ in \mathbb{C}^1 . Taking derivative of (2.22) with respect to a_1 and a_2 at the point $a = 0$, we find that for each $k \in \mathbb{Z}_+ \cup \{0\}$ the functions

$$\varphi_{+,k} = \left. \frac{\partial \Psi(a, z)}{\partial a_1} \right|_{a=0} = \phi_k(r) \cos k\theta, \quad \varphi_{-,k} = \phi_k(r) \sin k\theta,$$

with

$$\phi_k(r) = \frac{(k+N+1) + (k-N-1)r^{2N+2}}{(N+1)(1+r^{2N+2})} r^k$$

satisfy the equation

$$L\varphi_{\pm,k} = 0 \quad \text{in } \mathbb{R}^2$$

and function ϕ_k satisfy to the ordinary differential equation

$$\frac{\partial^2 \phi_k}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_k}{\partial r} - \frac{k^2 \phi_k}{r^2} + \rho \phi_k = 0 \quad \forall r > 0. \tag{2.23}$$

On the other hand, as can be checked by direct computations, new functions $\tilde{\phi}_k(r) = \phi_k(1/r)$ are also solutions to the equation (2.23). Obviously the pair $\phi_k, \tilde{\phi}_k$ is the fundamental set of solutions to ordinary differential

equation (2.23). If $k \notin \{0, N+1\}$, then these solutions blow up like r^k near infinity, or like r^{-k} near 0, and do not belong to the space Y_α .

Now let $v(x) \in \ker L \cap W_{loc}^{2,2}(\mathbb{R}^2)$. Using well-known regularity results of elliptic operators we obtain that $v(x) \in C_{loc}^2(\mathbb{R}^2)$. In the polar coordinates the Fourier expansion yields

$$v(x) = \sum_{k=0}^{\infty} v_k(r) e^{ik\theta}.$$

Obviously the function $v_k(r)$ satisfies (2.23) for all $k \in \mathbb{Z}_+ \cup \{0\}$. Thus $v_k(r)$ is the linear combination of the functions $\phi_k(r), \tilde{\phi}_k(r)$

$$v_k(r) = c_{k,1} \phi_k(r) + c_{k,2} \tilde{\phi}_k(r).$$

Since v is the smooth function $c_{k,2} = 0$. On the other hand $v(x) \in Y_\alpha$, thus $c_{k,1} = 0$ for all $k \in \mathbb{Z}_+ \setminus \{N+1\}$. This proves the lemma. ■

The image of the operator L is completely described by the following proposition.

Proposition 2.2. *Let $\alpha \in (0, \frac{1}{2})$, then for the image of the operator L defined by (2.1) and (2.18) we have*

$$\text{Im } L = \{f \in X_\alpha \mid \int_{\mathbb{R}^2} f \varphi_\pm dx = 0\}.$$

Proof. Since the image $\text{Im } L$ is closed in X_α by Proposition 2.1, we can decompose X_α as follows:

$$X_\alpha = \text{Im } L \oplus (\text{Im } L)^\perp.$$

Let $\xi \in (\text{Im } L)^\perp$, then

$$(Lu, \xi)_{X_\alpha} = 0 \quad \forall u \in Y_\alpha.$$

Thus, denoting $(1 + |x|^{2+\alpha})\xi(x) = \psi(x)$, we have

$$(Lu, \psi)_{L^2(\mathbb{R}^2)} = 0 \quad \forall u \in Y_\alpha.$$

Since $C_0^\infty(\mathbb{R}^2) \subset Y_\alpha$, this implies immediately that

$$L\psi = 0 \quad \text{in } \mathbb{R}^2 \quad (2.24)$$

by integration by parts and standard density argument.

Since $\xi \in X_\alpha$, we have $\int_{\mathbb{R}^2} \frac{|\psi|^2}{1+|x|^{2+\alpha}} dx < \infty$, and thus by (2.24)

$$\psi \in \ker L \cap Y_\alpha$$

By Lemma 2.4 the the function ψ is a linear combination of φ_0, φ_\pm ,

$$\psi = C_0\varphi_0 + C_+\varphi_+ + C_-\varphi_-,$$

where C_0, C_\pm are constants. Let us consider a function $f(r) = \lambda(r)\varphi_0(r)$, where $\lambda(r)$ is a smooth cut-off function which is nonnegative, and whose non empty support is in $(\frac{1}{2}, 1)$. Let w be the solution of $L_1w = f$ given by (2.8). Now suppose C_0 is not equal to 0, then this leads us to the contradiction,

$$\int_{\frac{1}{2}}^1 C_0\lambda(r)\varphi_0^2(r)rdr = \int_0^\infty L_1w(r)C_0\varphi_0(r)rdr = C_0 \int_{\mathbb{R}^2} Lw\varphi_0 dx = 0.$$

Let $m(r) \in C^\infty(\mathbb{R}_+^1)$, $m(r) = 1$ for $r \in [0, 1]$ and $m(r) = 0$ for $r > 2$. Denote $m_\varepsilon(x) = m(\varepsilon x)$. Obviously

$$\varphi_\pm m_\varepsilon \rightarrow \varphi_\pm \quad \text{in } X_\delta \quad \text{as } \varepsilon \rightarrow +0,$$

for all $\delta < -2$. Thus, using (1.3), we have

$$\begin{aligned} (Lu, \varphi_\pm)_{L^2(\mathbb{R}^2)} &= \lim_{\varepsilon \rightarrow +0} (Lu, \varphi_\pm m_\varepsilon)_{L^2(\mathbb{R}^2)} \\ &= \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}^2} \{ \varepsilon^2 (\Delta m)(\varepsilon r) \varphi_\pm u + 2\varepsilon ((\nabla m)(\varepsilon r), \nabla \varphi_\pm u) \} dx = 0 \quad \forall u \in Y_\alpha. \end{aligned}$$

This completes the proof. ■

In (2.17)-(2.20) the operator A was introduced by $A(u, a) = Lu + Ma$. Since variables u and a are independent, and ImL is the linear space of codimension we can prove Theorem 2.2 by showing that the space $(ImL)^\perp$ does not contain any vector orthogonal to ImM . This goal will be achieved with the help of the following auxiliary Lemma, whose proof is given in Appendix.

Lemma 2.5. *The following inequality holds true:*

$$\int_{\mathbb{R}^2} (\rho w_0 - 2\rho^2) \varphi_{\pm}^2 dx < 0.$$

Now we can prove Theorem 2.2.

Proof of Theorem 2.2. Given $f \in X_{\alpha}$, we want to show that there exists $u \in Y_{\alpha}$, $a_1, a_2 \in \mathbb{R}$ such that $A(u, a_1, a_2) = f$. Let us define

$$C_{\pm} = \int_{\mathbb{R}^2} f \varphi_{\pm} dx, \quad \tilde{C}_{\pm} = 4 \int_{\mathbb{R}^2} (\rho w_0 - 2\rho^2) \varphi_{\pm}^2 dx.$$

We note that $\tilde{C}_{\pm} \neq 0$ due to Lemma 2.5. Thus, the function, \tilde{f} introduced below, is well defined.

$$\tilde{f} = f - \frac{4C_+}{\tilde{C}_+} (\rho w_0 - 2\rho^2) \varphi_+ - \frac{4C_-}{\tilde{C}_-} (\rho w_0 - 2\rho^2) \varphi_-.$$

Then, from $\int_0^{2\pi} \sin(N+1)\theta \cos(N+1)\theta d\theta = 0$ we obtain

$$\int_{\mathbb{R}^2} \tilde{f} \varphi_{\pm} dx = 0.$$

Hence, by Proposition 2.2 there exists $u \in Y_{\alpha}$ such that $\Delta u + \rho u = \tilde{f}$, and we have

$$A\left(u, -\frac{C_+}{\tilde{C}_+}, -\frac{C_-}{\tilde{C}_-}\right) = f.$$

This completes the proof of the proposition. ■

3. PRELIMINARY ESTIMATES

Our aim in this section is the proof of invertibility of $P'_{\varepsilon}(0, 0)$, and appropriate norm estimates for $P'_{\varepsilon}(0, 0)^{-1}$, $P_{\varepsilon}(0, 0)$, and $P''_{\varepsilon}(0, 0)$ respectively. For those purposes we need the following lemmas.

Lemma 3.1. *Suppose $g_\varepsilon(z, a)$ is the function defined in (2.4). Then, there exists $R_0 > 1$ and $\varepsilon \in (0, 1)$ such that $|z| \geq R_0$ and $|a| < \varepsilon^2$, $\varepsilon \in (0, \varepsilon_0)$ imply*

$$|g_\varepsilon(z, a)| \leq \frac{C|z|^{2N}}{(1 + |z|^{2N+2})^2}, \quad (3.1)$$

$$\sum_{j=1}^2 \left| \frac{\partial g_\varepsilon(z, a)}{\partial a_j} \right| \leq \frac{C|z|^{2N}}{\varepsilon^2(1 + |z|^{2N+2})^2}, \quad (3.2)$$

$$\sum_{j,k=1}^2 \left| \frac{\partial^2 g_\varepsilon(z, a)}{\partial a_j \partial a_k} \right| \leq \frac{C|z|^{2N}}{\varepsilon^4(1 + |z|^{2N+2})^2}, \quad (3.3)$$

where C is independent of ε , z .

The proof of this Lemma is given in Appendix.

Lemma 3.2. *For all $\varepsilon_0 \in (0, 1)$ there exists $C = C(\varepsilon_0)$ such that*

$$e^{\varepsilon^2 w_0} = 1 + \varepsilon^2 w_0 + R(r) \quad \forall \varepsilon \in (0, \varepsilon_0), \quad r \in \mathbb{R}_+, \quad (3.4)$$

with

$$|R(r)| \leq C\varepsilon^4(\ln^+ r + 1)^2 \quad \forall r \in \mathbb{R}_+.$$

Proof. By Taylor's formula we have

$$e^{\varepsilon^2 w_0(r)} = 1 + \varepsilon^2 w_0(r) + R(r) \quad \forall r \in \mathbb{R}_+, \quad \varepsilon \in (0, 1),$$

where

$$\begin{aligned} |R(r)| &< \frac{1}{2} \sup_{\tau \in (0, 1)} e^{\tau \varepsilon^2 w_0(r)} |\varepsilon^2 w_0(r)|^2 \leq \frac{\varepsilon^4}{2} |w_0(r)|^2 \max\{1, e^{w_0(r)}\} \\ &\leq C\varepsilon^4(\ln^+ r + 1)^2, \end{aligned}$$

where we used (2.10). ■

Now we are ready to prove the existence of the operator $[P'_\varepsilon(0, 0)]^{-1}$. As was mentioned earlier, we show that $P_\varepsilon(0, 0)$ is a perturbation of the operator A . One problem is that although the operator A is onto, its kernel is not

empty (Actually Lemma 2.4 implies that $\dim\{\ker A\} = 3$). To overcome this difficulty let us decompose $Y_\alpha = \ker L \oplus (\ker L)^\perp$, and set

$$U_\alpha = (\ker L)^\perp \times \mathbb{R}^2.$$

We equip the space U_α with the norm $\|(u, a)\|_{U_\alpha} = \sqrt{\|u\|_{Y_\alpha}^2 + a_1^2 + a_2^2}$. From now on we are going to work in the space U_α instead of $Y_\alpha \times \mathbb{R}^2$. By Theorem 2.2 there exists

$$A^{-1} : X_\alpha \rightarrow U_\alpha \quad \forall \alpha \in (0, \frac{1}{2}). \quad (3.5)$$

Indeed to prove (3.5) suppose $A(v, a) = 0$. Then, this is equivalent to

$$Lv = -Ma$$

and Proposition 2.2 implies that $\int_{\mathbb{R}^2} Ma\varphi_\pm dx = 0$. By Lemma 2.5 this, in turn, is possible only if $a = 0$. So we have $Lv = 0$. Hence, $v = 0$ by definition of the space U_α .

Now we prove existence of the operator $[P'_\varepsilon(0, 0)]^{-1}$.

Proposition 3.1. *Let $\alpha \in (0, \frac{1}{2})$. There exists $\varepsilon_0 \in (0, 1)$ such that the inverse operator $\Gamma_\varepsilon = [P'_\varepsilon(0, 0)]^{-1} \in \mathcal{L}(X_\alpha, U_\alpha)$ exists for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, there exists a constant C such that the following inequality holds true.*

$$\|\Gamma_\varepsilon\|_{\mathcal{L}(X_\alpha, U_\alpha)} \leq C \quad \forall \varepsilon \in (0, \varepsilon_0), \quad (3.6)$$

where C is independent of ε .

Proof. Let us decompose

$$P'_\varepsilon(0, 0) = A + (P'_\varepsilon(0, 0) - A).$$

Since by (3.5) A is invertible operator from U_α onto X_α , it suffices to prove that there exists $\varepsilon_0 > 0$ such that

$$\|P'_\varepsilon(0, 0) - A\|_{\mathcal{L}(U_\alpha, X_\alpha)} < \frac{1}{(\|A^{-1}\|_{\mathcal{L}(X_\alpha, U_\alpha)} + 1)} \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (3.7)$$

Let $(v, b) \in Y_\alpha \times \mathbb{R}^2$ with $\|v\|_{Y_\alpha} + |b| \leq 1$. Then, from (2.15) and (2.17)-(2.20) we have

$$\begin{aligned}
& \|P'_\varepsilon(0, 0)[v, b] - A(v, b_1, b_2)\|_{X_\alpha} \\
& \leq \|g_\varepsilon(\cdot, 0)e^{\varepsilon^2 w_0} v - \rho v - 2\varepsilon^2 g_\varepsilon^2(\cdot, 0)e^{2\varepsilon^2 w_0} v\|_{X_\alpha} \\
& + \left\| \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0)e^{\varepsilon^2 w_0} b_1 - \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0)b_1 \right. \\
& \quad \left. - 2\varepsilon^2 e^{2\varepsilon^2 w_0} \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0)g_\varepsilon(\cdot, 0)b_1 + 4(\rho w_0 - 2\rho^2)\varphi_+ b_1 \right\|_{X_\alpha} \\
& + \left\| \frac{\partial g_\varepsilon}{\partial a_2}(\cdot, 0)e^{\varepsilon^2 w_0} b_2 - \frac{\partial g_\varepsilon}{\partial a_2}(\cdot, 0)b_2 \right. \\
& \quad \left. - 2\varepsilon^2 e^{2\varepsilon^2 w_0} \frac{\partial g_\varepsilon}{\partial a_2}(\cdot, 0)g_\varepsilon(\cdot, 0)b_2 + 4(\rho w_0 - 2\rho^2)\varphi_- b_2 \right\|_{X_\alpha} \\
& = I + J + K. \tag{3.8}
\end{aligned}$$

Let us estimate each term of I, J, K separately below. Using Lemma 3.2, we estimate I as follows.

$$\begin{aligned}
I^2 & \leq 2 \int_{\mathbb{R}^2} |g_\varepsilon(z, 0) - \rho(r)|^2 |v(x)|^2 (1 + |x|^{2+\alpha}) dx \\
& + 2\|\varepsilon^2 g_\varepsilon(\cdot, 0)w_0 v\|_{X_\alpha}^2 + 2\|\varepsilon^2 g_\varepsilon(\cdot, 0)Rv\|_{X_\alpha}^2 \\
& + 2\|2\varepsilon^2 g_\varepsilon^2(\cdot, 0)e^{2\varepsilon^2 w_0} v\|_{X_\alpha}^2 = I_1 + I_2 + I_3 + I_4, \tag{3.9}
\end{aligned}$$

where R is the function from (3.4).

As in the proof of Lemma 3.1 one can write

$$g_\varepsilon(z, 0) = \frac{8(N+1)^2(r^{2N} + \varepsilon R_N(\varepsilon, z))}{(1 + r^{2N+2} + \varepsilon \tilde{R}_N(\varepsilon, z))^2},$$

where $R_N(\varepsilon, z)$ and $\tilde{R}_N(\varepsilon, z)$ satisfy the estimate

$$|R_N(\varepsilon, z)| \leq C(1 + r^{2N-1}), \quad |\tilde{R}_N(\varepsilon, z)| \leq C(1 + r^{2N+1}), \quad r = |z|$$

for all $\varepsilon \in (0, 1)$ and $z \in \mathbb{C}$ with C independent of ε and z . Thus,

$$|g_\varepsilon(z, 0) - \rho(|z|)| \leq \frac{\varepsilon C(1 + r^{6N+4})}{(1 + r^{8N+8})} \quad \forall \varepsilon \in (0, 1) \quad z \in \mathbb{R}^2. \tag{3.10}$$

Hence, thanks to Lemma 1.1, I_1 is estimated as follows.

$$\begin{aligned} I_1 &\leq C\varepsilon^2 \int_{\mathbb{R}^2} \frac{(1 + |x|^{6N+4})^2}{(1 + |x|^{8N+8})^2} \|v\|_{Y_\alpha}^2 (\ln^+ |x| + 1)^2 (1 + |x|^{2+\alpha}) dx \\ &\leq C\varepsilon^2 \|v\|_{Y_\alpha}^2. \end{aligned} \quad (3.11)$$

In order to estimate I_2 one can use Lemma 3.1 and (2.10) so that

$$\begin{aligned} I_2 &\leq C\varepsilon^4 \int_{\mathbb{R}^2} \frac{|x|^{4N}}{(1 + |x|^{2N+2})^4} (\ln^+ |x| + 1)^2 \|v\|_{Y_\alpha}^2 (\ln^+ |x| + 1)^2 (1 + |x|^{2+\alpha}) dx \\ &\leq C\varepsilon^4 \|v\|_{Y_\alpha}^2. \end{aligned} \quad (3.12)$$

Now, using Lemma 3.1 and 3.2, we estimate I_3 as

$$\begin{aligned} I_3 &\leq C\varepsilon^4 \int_{\mathbb{R}^2} \frac{|x|^{4N}}{(1 + |x|^{2N+2})^4} \|v\|_{Y_\alpha}^2 (\ln^+ |x| + 1)^2 (1 + |x|^{2+\alpha}) dx \\ &\leq C\varepsilon^4 \|v\|_{Y_\alpha}^2, \end{aligned} \quad (3.13)$$

and, using (2.10),

$$\begin{aligned} I_4 &\leq C\varepsilon^4 \int_{\mathbb{R}^2} \frac{|x|^{8N}}{(1 + |x|^{2N+2})^8} \|v\|_{Y_\alpha}^2 (\ln^+ |x| + 1)^2 (1 + |x|^{2+\alpha}) dx \\ &\leq C\varepsilon^4 \|v\|_{Y_\alpha}^2 \end{aligned} \quad (3.14)$$

respectively. Combining (3.11)-(3.14), we are lead to

$$I \leq C\varepsilon \|v\|_{Y_\alpha}. \quad (3.15)$$

In order to estimate J we observe first

$$\begin{aligned}
J^2 &\leq 2 \left\| \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0)(\varepsilon^2 w_0 + \varepsilon^4 R) b_1 + 4\rho w_0 \varphi_+ b_1 \right\|_{X_\alpha}^2 \\
&\quad + 2 \left\| 2\varepsilon^2 [2\varepsilon^2 w_0 + 4\varepsilon^4 R] \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0) g_\varepsilon(\cdot, 0) b_1 \right\|_{X_\alpha}^2 \\
&\quad + 2 \left\| 2\varepsilon^2 \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0) g_\varepsilon(\cdot, 0) + 8\rho^2 \varphi_+ b_1 \right\|_{X_\alpha}^2 \\
&\leq 4b_1^2 \left\| \left(\varepsilon^2 \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0) + 4\rho \varphi_+ \right) w_0 \right\|_{X_\alpha}^2 \\
&\quad + 4\varepsilon^8 b_1^2 \left\| R \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0) \right\|_{X_\alpha}^2 \\
&\quad + 8\varepsilon^8 b_1^2 \left\| (w_0 + 2\varepsilon^2 R) \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0) g_\varepsilon(\cdot, 0) \right\|_{X_\alpha}^2 \\
&\quad + 8b_1^2 \left\| \varepsilon^2 \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0) g_\varepsilon(\cdot, 0) + 4\rho^2 \varphi_+ \right\|_{X_\alpha}^2 \\
&= J_1 + \cdots + J_4. \tag{3.16}
\end{aligned}$$

By direct computation, as in the proof of Lemma 3.1, one can write

$$\begin{aligned}
\frac{\partial g_\varepsilon(z, 0)}{\partial a_1} &= -\frac{4}{\varepsilon^2} g_\varepsilon(z, 0) \frac{\operatorname{Re} [\varepsilon^{N+1} F(\frac{z}{\varepsilon})]}{1 + \varepsilon^{2N+2} |F(\frac{z}{\varepsilon})|^2} \\
&= -\frac{4}{\varepsilon^2} g_\varepsilon(z, 0) \frac{|z|^{N+1} \cos(N+1)\theta + \varepsilon h_N(\varepsilon, z)}{1 + |z|^{2N+2} + \varepsilon \tilde{h}_N(\varepsilon, z)},
\end{aligned}$$

where $|h_N(\varepsilon, z)| \leq C(1 + |z|^N)$, and $|\tilde{h}_N(\varepsilon, z)| \leq C(1 + |z|^{2N+1})$ for all $z \in \mathbb{C}$ with C independent of ε and z .

Thus, using (3.10), we estimate

$$\begin{aligned}
J_1 &\leq 16b_1^2 \left\| (g_\varepsilon(\cdot, 0) - \rho) \frac{|z|^{N+1} \cos(N+1)\theta + \varepsilon h_N(\varepsilon, \cdot)}{1 + |z|^{2N+2} + \varepsilon \tilde{h}_N(\varepsilon, \cdot)} w_0 \right\|_{X_\alpha}^2 \\
&\quad + 16b_1^2 \left\| \left(\frac{|z|^{N+1} \cos(N+1)\theta + \varepsilon h_N(\varepsilon, \cdot)}{1 + |z|^{2N+2} + \varepsilon \tilde{h}_N(\varepsilon, \cdot)} - \varphi_+ \right) \rho w_0 \right\|_{X_\alpha}^2 \\
&\leq Cb_1^2 \|g_\varepsilon(\cdot, 0) - \rho\|_{X_\alpha}^2 + C\varepsilon^2 b_1^2 \left\| \frac{(1 + |z|^{3N+2})}{1 + |z|^{4N+4}} \rho w_0 \right\|_{X_\alpha}^2 \\
&\leq C\varepsilon^4 b_1^2 + C\varepsilon^2 b_1^2 \leq C\varepsilon^2 b_1^2. \tag{3.17}
\end{aligned}$$

Using the estimates (3.1) - (3.4), we obtain immediately

$$J_2 \leq C\varepsilon^4 b_1^2, \quad (3.18)$$

and

$$J_3 \leq C\varepsilon^4 b_1^2. \quad (3.19)$$

To estimate J_4 let us begin first

$$J_4 \leq 8b_1^2 \left(\left\| \varepsilon^2 \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0)(g_\varepsilon(\cdot, 0) - \rho) \right\|_{X_\alpha}^2 + \left\| \left(\varepsilon^2 \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0) + 4\rho\varphi_+ \right) \rho \right\|_{X_\alpha}^2 \right). \quad (3.20)$$

By (2.1), (3.2)

$$\begin{aligned} J_4 &\leq Cb_1^2 \left(\|g_\varepsilon(\cdot, 0) - \rho\|_{X_\alpha}^2 + \left\| \varepsilon^2 \frac{\partial g_\varepsilon}{\partial a_1}(\cdot, 0) + 4\rho\varphi_+ \right\|_{X_\alpha}^2 \right) \\ &\leq C\varepsilon^2 b_1^2, \end{aligned} \quad (3.21)$$

where in the last inequality we used (3.11) with $v \equiv 1$ and (3.17) with $b_1 = 1$.

Combining (3.16)-(3.19), (3.21) we obtain

$$J \leq C\varepsilon b_1. \quad (3.22)$$

Estimate of K is similar to J , and we have easily

$$K \leq C\varepsilon b_2. \quad (3.23)$$

Combining (3.15), (3.22) and (3.23), we finally estimate

$$\|P'_\varepsilon(0, 0)[v, b] - A(v, b_1, b_2)\|_{X_\alpha} \leq C\varepsilon(\|v\|_{Y_\alpha} + |b|) \leq C\varepsilon.$$

Taking $\varepsilon_0 = \frac{1}{2C(\|A^{-1}\|_{\mathcal{L}(X_\alpha, U_\alpha)} + 1)}$, (3.7) is proved. ■

As it was mentioned earlier, to apply the Newton-Kantorovich theorem we will choose $v_0 = 0 \in U_\alpha$. The following proposition provides the estimate for the parameter η of (ii) in Theorem 2.1.

Proposition 3.2. *Let $\alpha \in (0, \frac{1}{2})$, then there exists $\varepsilon_1 > 0$ such that*

$$\|P_\varepsilon(0, 0)\|_{X_\alpha} \leq C\varepsilon^3 \quad \forall \varepsilon \in (0, \varepsilon_1). \quad (3.24)$$

Proof. By virtue of Lemma 3.2, and using the definition of the function w_0 , one can rewrite (2.14) as follows.

$$\begin{aligned} P_\varepsilon(0, 0) &= g_\varepsilon(z, 0)(1 + \varepsilon^2 w_0 + \varepsilon^4 R_1(|x|)) - \varepsilon^2 g_\varepsilon^2(z, 0)(1 + 2\varepsilon^2 w_0 + \varepsilon^4 \tilde{R}_1(|x|)) \\ &\quad - g_\varepsilon(z, 0) + \varepsilon^2(-\rho w_0 + \rho^2) \\ &= \varepsilon^2(g_\varepsilon(z, 0) - \rho)w_0 + \varepsilon^2(\rho^2 - g_\varepsilon^2(z, 0)) - 2\varepsilon^4 g_\varepsilon^2(z, 0)w_0 \\ &\quad + \varepsilon^4 g_\varepsilon(z, 0)R_1(|x|) - \varepsilon^6 g_\varepsilon^2(z, 0)\tilde{R}_1(|x|), \end{aligned} \quad (3.25)$$

where $|R_1(|x|)| + |\tilde{R}_1(|x|)| \leq C(\ln^+ |x| + 1)^2$. We thus have

$$\begin{aligned} \|P_\varepsilon(0, 0)\|_{X_\alpha}^2 &\leq 2\varepsilon^4 \int_{\mathbb{R}^2} |g_\varepsilon(z, 0) - \rho|^2 |w_0|^2 (1 + |x|^{2+\alpha}) dx \\ &\quad + 4\varepsilon^4 \int_{\mathbb{R}^2} |g_\varepsilon(z, 0) - \rho|^2 (|\rho|^2 + |g_\varepsilon(z, 0)|^2) (1 + |x|^{2+\alpha}) dx \\ &\quad + 8\varepsilon^8 \int_{\mathbb{R}^2} |g_\varepsilon(z, 0)|^4 |w_0|^2 (1 + |x|^{2+\alpha}) dx \\ &\quad + C\varepsilon^8 \int_{\mathbb{R}^2} |g_\varepsilon(z, 0)|^2 (1 + |x|^{2+\alpha}) (\ln^+ |x| + 1)^4 dx \\ &\quad + C\varepsilon^{12} \int_{\mathbb{R}^2} |g_\varepsilon(z, 0)|^4 (1 + |x|^{2+\alpha}) (\ln^+ |x| + 1)^4 dx. \end{aligned} \quad (3.26)$$

Using the previous estimates (3.10) and (3.1), we obtain easily

$$\|P_\varepsilon(0, 0)\|_{X_\alpha}^2 \leq C\varepsilon^6 + C\varepsilon^6 + C\varepsilon^8 + C\varepsilon^8 + C\varepsilon^{12} \leq C\varepsilon^6 \quad (3.27)$$

for all sufficiently small ε . This completes the proof of the proposition. ■

To apply Theorem 2.1 in the next section we are going to choose $v_0 = 0$ and $\tilde{r} = \varepsilon^2$. Thus the following proposition provides the necessary estimate for the parameter K of (iii) in Theorem 2.1.

Proposition 3.3. *Let $\alpha \in (0, \frac{1}{2})$. Then, there exist $\varepsilon_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $(u, a) \in U_\alpha$ with $\|(u, a)\|_{U_\alpha} \leq \varepsilon^2$ we have the estimate*

$$\|P''_\varepsilon(u, a)\|_{\mathcal{L}(U_\alpha \times U_\alpha; X_\alpha)} \leq \frac{C}{\varepsilon^2}, \quad (3.28)$$

where constant C is independent of ε .

Proof. Let us assume $\|(u, a)\|_{U_\alpha} \leq \varepsilon^2$ below. From (2.16) we have

$$\begin{aligned} & \|P''(u, a)[v, b; \tilde{v}, \tilde{b}]\|_{X_\alpha} \\ & \leq \|g_\varepsilon e^{u+\varepsilon^2 w_0} v \tilde{v}\|_{X_\alpha} + \sum_{j=1}^2 \left\| \frac{\partial g_\varepsilon}{\partial a_j} e^{u+\varepsilon^2 w_0} b_j \tilde{v} \right\|_{X_\alpha} \\ & \quad + \sum_{j=1}^2 \left\| \frac{\partial g_\varepsilon}{\partial a_j} e^{u+\varepsilon^2 w_0} v \tilde{b}_j \right\|_{X_\alpha} \\ & \quad + \sum_{j,k=1}^2 \left\| \frac{\partial^2 g_\varepsilon}{\partial a_j \partial a_k} (e^{u+\varepsilon^2 w_0} - 1) b_j \tilde{b}_k \right\|_{X_\alpha} \\ & \quad + 4\varepsilon^2 \sum_{j=1}^2 \left\| \frac{\partial g_\varepsilon}{\partial a_j} g_\varepsilon e^{2u+2\varepsilon^2 w_0} b_j \tilde{v} \right\|_{X_\alpha} \\ & \quad + 2\varepsilon^2 \sum_{j,k=1}^2 \left\| \left(\frac{\partial^2 g_\varepsilon}{\partial a_j \partial a_k} g_\varepsilon + \frac{\partial g_\varepsilon}{\partial a_j} \frac{\partial g_\varepsilon}{\partial a_k} \right) b_j \tilde{b}_k e^{2u+2\varepsilon^2 w_0} \right\|_{X_\alpha} \\ & \quad + 4\varepsilon^2 \|g_\varepsilon^2 e^{2u+2\varepsilon^2 w_0} v \tilde{v}\|_{X_\alpha} \\ & \quad + 4\varepsilon^2 \sum_{j=1}^2 \left\| \frac{\partial g_\varepsilon}{\partial a_j} g_\varepsilon e^{2u+2\varepsilon^2 w_0} v \tilde{b}_j \right\|_{X_\alpha} = I_1 + I_2 + \dots + I_8. \end{aligned} \quad (3.29)$$

Below each term of $I_1 \dots I_8$ will be estimated separately. To estimate of I_1 we use (2.10) and (1.3) as follows.

$$\begin{aligned} I_1^2 &= \int_{\mathbb{R}^2} |g_\varepsilon(z, a)|^2 e^{2u+2\varepsilon^2 w_0} |v|^2 |\tilde{v}|^2 (1 + |x|^{2+\alpha}) dx \\ &\leq C \int_0^\infty \frac{r^{4N}}{(1 + r^{2N+2})^4} e^{2C_1 \|u\|_{Y_\alpha} (\ln^+ r + 1)} \|v\|_{Y_\alpha}^2 \|\tilde{v}\|_{Y_\alpha}^2 (\ln^+ r + 1)^4 (1 + r^{2+\alpha}) r dr \\ &\leq C \|v\|_{Y_\alpha}^2 \|\tilde{v}\|_{Y_\alpha}^2 \int_0^\infty \frac{(1 + r^{2+\alpha+2\varepsilon^2 C_1+4N+1}) (\ln^+ r + 1)^4}{(1 + r^{2N+2})^4} dr \\ &\leq C \|v\|_{Y_\alpha}^2 \|\tilde{v}\|_{Y_\alpha}^2, \end{aligned} \quad (3.30)$$

where ε_0 was chosen so that

$$4\varepsilon_0^2 C_1 < 3, \quad (3.31)$$

where C_1 is the constant from the inequality (1.3). Using (3.2), (2.9) one estimate I_2 ,

$$\begin{aligned} I_2^2 &\leq 2 \sum_{j=1}^2 \int_{\mathbb{R}^2} \left| \frac{\partial g_\varepsilon(z, a)}{\partial a_j} \right|^2 e^{2u+2\varepsilon^2 w_0} |b_j|^2 |\tilde{v}|^2 (1 + |x|^{2+\alpha}) dx \\ &\leq \frac{C}{\varepsilon^4} \int_0^\infty \frac{r^{4N}}{(1+r^{2N+2})^4} e^{2C_1 \|u\|_{Y_\alpha} (\ln^+ r + 1)} |b|^2 \|\tilde{v}\|_{Y_\alpha}^2 (\ln^+ r + 1)^2 (1+r^{2+\alpha}) r dr \\ &\leq \frac{C}{\varepsilon^4} |b|^2 \|\tilde{v}\|_{Y_\alpha}^2 \int_0^\infty \frac{(1+r^{2+\alpha+2\varepsilon^2 C_1+4N+1})(\ln^+ r + 1)^2}{(1+r^{2N+2})^4} dr \\ &\leq \frac{C}{\varepsilon^4} |b|^2 \|\tilde{v}\|_{Y_\alpha}^2, \end{aligned} \quad (3.32)$$

where we used the assumption on ε_0 in (3.31). Estimate of I_3 is similar to I_2 , and we obtain

$$I_3 \leq \frac{C}{\varepsilon^2} |\tilde{b}| \|v\|_{Y_\alpha}. \quad (3.33)$$

Thanks to (3.3) and (2.4), (2.10), (1.3) I_4 can be estimated as follows.

$$\begin{aligned} I_4^2 &\leq 2 \sum_{j,k=2}^2 \int_{\mathbb{R}^2} \left| \frac{\partial^2 g_\varepsilon(z, a)}{\partial a_j \partial a_k} \right|^2 (e^{u+\varepsilon^2 w_0} - 1)^2 |b_j|^2 |\tilde{b}_k|^2 (1 + |x|^{2+\alpha}) dx \\ &\leq \frac{C}{\varepsilon^8} \int_{\mathbb{R}^2} \frac{|x|^{4N} |u + \varepsilon^2 w_0|^2 e^{2|u|} \|e^{\varepsilon^2 w_0}\|_{C^0(\mathbb{R}_+^1)}^2 |b|^2 |\tilde{b}|^2 (1 + |x|^{2+\alpha})}{(1 + |x|^{2N+2})^4} dx \\ &\leq \frac{C}{\varepsilon^8} \int_0^\infty \frac{r^{4N} \varepsilon^4 (\ln^+ r + 1)^2 e^{2\varepsilon^2 C_1 (\ln^+ r + 1)} |b|^2 |\tilde{b}|^2 (1 + r^{2+\alpha})}{(1 + r^{2N+2})^4} r dr \\ &\leq \frac{C}{\varepsilon^4} |b|^2 |\tilde{b}|^2 \int_0^\infty \frac{(1 + r^{2+\alpha+2\varepsilon^2 C_1+4N+1})(\ln^+ r + 1)^2}{(1 + r^{2N+2})^4} dr \\ &\leq \frac{C}{\varepsilon^4} |b|^2 |\tilde{b}|^2, \end{aligned} \quad (3.34)$$

where the mean value theorem was used in the second inequality, and (3.31) was used in the last inequality. Next, we estimate I_5 and I_6 , using (3.1)-

(3.3),(2.10) as follows.

$$\begin{aligned}
I_5^2 &\leq 32\varepsilon^4 \sum_{i=1}^2 \int_{\mathbb{R}^2} \left| \frac{\partial g_\varepsilon(z, a)}{\partial a_j} \right|^2 |g_\varepsilon(z, a)|^2 e^{4u+4\varepsilon^2 w_0} |b_j|^2 |\tilde{v}|^2 (1 + |x|^{2+\alpha}) dx \\
&\leq C \int_0^\infty \frac{r^{8N}}{(1+r^{2N+2})^8} e^{4C_1 \|u\|_{Y_\alpha} (\ln^+ r + 1)} |b|^2 \|\tilde{v}\|_{Y_\alpha}^2 (\ln^+ r + 1)^2 (1+r^{2+\alpha}) r dr \\
&\leq C |b|^2 \|\tilde{v}\|_{Y_\alpha}^2 \int_0^\infty \frac{\{1+r^{2+\alpha+8N+4C_1\varepsilon_0^2+1}\} (\ln^+ r + 1)^2}{(1+r^{2N+2})^8} dr \\
&\leq C |b|^2 \|\tilde{v}\|_{Y_\alpha}^2, \tag{3.35}
\end{aligned}$$

and

$$\begin{aligned}
I_6^2 &\leq 8\varepsilon^4 \sum_{i,k=1}^2 \int_{\mathbb{R}^2} \left(\left| \frac{\partial^2 g_\varepsilon(z, a)}{\partial a_j \partial a_k} \right|^2 |g_\varepsilon(z, a)|^2 + \left| \frac{\partial g_\varepsilon(z, a)}{\partial a_j} \right|^2 \left| \frac{\partial g_\varepsilon(z, a)}{\partial a_k} \right|^2 \right) \\
&\quad \cdot |b_j|^2 |\tilde{b}_k|^2 e^{4u+4\varepsilon^2 w_0} (1 + |x|^{2+\alpha}) dx \\
&\leq \frac{C |b|^2 |\tilde{b}|^2}{\varepsilon^4} \int_0^\infty \frac{r^{8N}}{(1+r^{2N+2})^8} e^{4C_1 \|u\|_{Y_\alpha} (\ln^+ r + 1)} (1+r^{2+\alpha}) r dr \\
&\leq \frac{C}{\varepsilon^4} |b|^2 |\tilde{b}|^2 \int_0^\infty \frac{1+r^{2+\alpha+4C_1\varepsilon_0^2+8N+1}}{(1+r^{2N+2})^8} dr \leq \frac{C}{\varepsilon^4} |b|^2 |\tilde{b}|^2. \tag{3.36}
\end{aligned}$$

Estimate of I_7 is similar to I_1 , and we have

$$I_7 \leq C\varepsilon^2 \|v\|_{Y_\alpha} \|\tilde{v}\|_{Y_\alpha}. \tag{3.37}$$

Estimate of I_8 is similar to I_5 , and we have

$$I_8 \leq C |\tilde{b}| \|v\|_{Y_\alpha}. \tag{3.38}$$

Estimates (3.30), (3.32)-(3.38) provides us with (3.28). ■

4. PROOF OF THE MAIN THEOREM.

We are now equipped with all the necessary facts to apply Theorem 2.1 to prove our main theorem.

Proof of the Main Theorem. We apply Theorem 2.1 to our functional equation (2.12). Set $B_1 = U_{\frac{1}{8}}, B_2 = X_{\frac{1}{8}}$, and $v_0 = 0$. Also, we set $\alpha = \frac{1}{8}$ in

Propositions 3.1-3.3. We denote $v = (u, a)$, and define $\mathbf{P}_\varepsilon(v) = P_\varepsilon(u, a)$. Then by Proposition 3.1 there exists the operator $\Gamma_\varepsilon = [\mathbf{P}'_\varepsilon(0)]^{-1}$ with its norm estimate given by (3.6). Combining Proposition 3.1 with Proposition 3.2, we obtain

$$\|\Gamma_\varepsilon \mathbf{P}_\varepsilon(0)\|_{U_{\frac{1}{8}}} \leq \|\Gamma_\varepsilon\|_{\mathcal{L}(X_{\frac{1}{8}}, U_{\frac{1}{8}})} \|\mathbf{P}_\varepsilon(0)\|_{X_{\frac{1}{8}}} \leq C\varepsilon^3 \equiv \eta(\varepsilon) \quad (4.1)$$

for all $\varepsilon \in (0, \varepsilon_0)$. On the other hand, combining Proposition 3.1 with Proposition 3.3, we have

$$\|\Gamma_\varepsilon \mathbf{P}_\varepsilon''(v)\|_{U_{\frac{1}{8}}} \leq \|\Gamma_\varepsilon\|_{\mathcal{L}(X_{\frac{1}{8}}, U_{\frac{1}{8}})} \|\mathbf{P}_\varepsilon''(v)\| \leq C\varepsilon^{-2} \equiv K(\varepsilon)$$

for all $v \in \Omega_0 = \{v \in U_{\frac{1}{8}} \mid \|v\|_{U_{\frac{1}{8}}} \leq \varepsilon^2\}$ and $\varepsilon \in (0, \varepsilon_0)$. Hence

$$h(\varepsilon) = K(\varepsilon)\eta(\varepsilon) \leq C\varepsilon \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Taking the parameter ε sufficiently small, we have $h(\varepsilon) < \frac{1}{2}$. Moreover, since $\tilde{r} = \varepsilon^2$, and $r_0 = O(\varepsilon^3)$ as $\varepsilon \rightarrow 0$, we have $\tilde{r} > r_0$ for all sufficiently small ε . Thus all conditions of Theorem 2.1 are satisfied, and there exists a solution, $v_\varepsilon^* = (u_\varepsilon^*, a_\varepsilon^*) \in \Omega_0$ to the equation $\mathbf{P}_\varepsilon(v) = 0$ such that

$$\|v_\varepsilon^*\|_{U_{\frac{1}{8}}} \leq \frac{1 - \sqrt{1 - 2h}}{h} \eta \leq C\varepsilon^3, \quad (4.2)$$

where C is independent of ε .

We now verify the decay estimate (11) for u recovered from formula (2.13), thus showing that our solution u is really nontopological. From the explicit formula given in (2.1) we know that

$$\ln \rho_{\varepsilon, a_\varepsilon^*}(x) = -(2N + 4) \ln |x| + o(\ln |x|) \quad \text{as } |x| \rightarrow \infty. \quad (4.3)$$

From (1.3) and (4.2) we obtain

$$\begin{aligned} |u_\varepsilon^*(x)| &\leq C \|u_\varepsilon^*\|_{Y_{\frac{1}{8}}} (\ln^+ |x| + 1) \\ &\leq C \|v_\varepsilon^*\|_{U_{\frac{1}{8}}} (\ln^+ |x| + 1) \\ &\leq C\varepsilon^3 (\ln^+ |x| + 1). \end{aligned}$$

This, implies then

$$|u_\varepsilon^*(\varepsilon x)| \leq C\varepsilon^3 (\ln^+ |\varepsilon x| + 1) \leq C\varepsilon^3 (\ln^+ |x| + 1). \quad (4.4)$$

Also, from the asymptotic formula (2.10)

$$\varepsilon^2 w_0(\varepsilon x) = -\tilde{C}\varepsilon^2 \ln|x| + o(\ln|x|) \quad \text{as } |x| \rightarrow \infty, \quad (4.5)$$

where \tilde{C} is a *positive* number. Taking into account (4.3)-(4.5), we deduce from (2.13) that there is $\varepsilon_1 \in (0, \varepsilon_0)$, and a constant $\beta = \beta(\varepsilon) > 0$ such that our solution $u(x)$ of (9) satisfies

$$u(x) = -(2N + 4 + \beta) \ln|x| + o(\ln|x|) \quad \text{as } |x| \rightarrow \infty \quad (4.6)$$

for all $\varepsilon \in (0, \varepsilon_1)$. Then,

$$e^{u(x)} = O\left(\frac{1}{|x|^{2N+4+\beta}}\right) \quad \text{as } |x| \rightarrow \infty. \quad (4.7)$$

We recall that $z = x_1 + ix_2$, $\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$, $\bar{\partial}_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$ as previously, and define

$$\phi(z) = \exp\frac{1}{2}(u + i\theta) \quad \text{with } \theta = \sum_{j=1}^N 2\arg(z - z_j) \quad (4.8)$$

and

$$A_1 = -\operatorname{Re}\{2i\bar{\partial}_z \ln \phi(z)\}, \quad A_2 = -\operatorname{Im}\{2i\bar{\partial}_z \ln \phi(z)\}.$$

Then, (ϕ, A) becomes a solution of the Bogomol'nyi equations (4)-(5) satisfying (7). We now show that our solution (ϕ, A) is of finite energy, and satisfies the decay estimates (11). Since by (4.7) and (4.8)

$$|\phi(x)|^2 = O\left(\frac{1}{|x|^{2N+4+\beta}}\right) \quad \text{as } |x| \rightarrow \infty, \quad (4.9)$$

we obtain

$$|\phi|^2(1 - |\phi|^2) \in L^1(\mathbb{R}^2). \quad (4.10)$$

Then by (5) and (4.9)

$$F_{12}(x) = O\left(\frac{1}{|x|^{2N+4+\beta}}\right) \quad \text{as } |x| \rightarrow \infty, \quad \text{and } \frac{F_{12}^2}{|\phi|^2}, \quad |\phi|^2(1 - |\phi|^2) \in L^1(\mathbb{R}^2).$$

Now it suffices to show that

$$|D_1\phi|^2 + |D_2\phi|^2 = O\left(\frac{1}{|x|^{2N+4+\beta}}\right) \quad \text{as } |x| \rightarrow \infty,$$

and also that it is integrable. From (4) we have immediately that $\bar{\partial}_z \ln \phi(z) = -i\bar{\alpha}$, where $\bar{\alpha} = \frac{1}{2}(A_1 + iA_2)$, which, in turn, gives

$$\frac{1}{2} \frac{\partial \theta}{\partial x_1} - A_1 = -\frac{1}{2} \frac{\partial u}{\partial x_2}, \quad \frac{1}{2} \frac{\partial \theta}{\partial x_2} - A_2 = \frac{1}{2} \frac{\partial u}{\partial x_1}.$$

Thus, $D_1 \phi = (\frac{1}{2} \frac{\partial u}{\partial x_1} + \frac{i}{2} \frac{\partial \theta}{\partial x_1} - iA_1) \phi = \frac{1}{2} (\frac{\partial u}{\partial x_1} - i \frac{\partial u}{\partial x_2}) \phi$ and $D_2 \phi = (\frac{1}{2} \frac{\partial u}{\partial x_2} + \frac{i}{2} \frac{\partial \theta}{\partial x_2} - iA_2) \phi = \frac{1}{2} (\frac{\partial u}{\partial x_2} + i \frac{\partial u}{\partial x_1}) \phi$. Therefore,

$$|D_1 \phi|^2 + |D_2 \phi|^2 \leq \frac{1}{2} |\nabla u|^2 |\phi|^2 = \frac{1}{2} |\nabla u|^2 e^u.$$

From (9) and Proposition 1.1 we obtain

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| e^{u(y)} (e^{u(y)} - 1) dy + 2 \sum_{j=1}^N \ln|x-z_j| + C \quad (4.11)$$

for some constant C . Since $u(x) \leq 0$ for $x \in \mathbb{R}^2$ (by the maximum principle applied to (9)), taking derivative of u from (4.12), and using the inequality $(\sum_{j=1}^n a_j)^2 \leq n \sum_{j=1}^n a_j^2$, we obtain

$$|\nabla u(x)|^2 e^{u(x)} \leq C e^{u(x)} \left(\int_{\mathbb{R}^2} \frac{e^{u(y)}}{|x-y|} dy \right)^2 + C \sum_{j=1}^N \frac{e^{u(x)}}{|x-z_j|^2} = I_1(x) + I_2(x).$$

Since $u(x) = 2 \ln|x-z_j| + O(1)$ as $x \rightarrow z_j$, the function $I_2(x)$ is locally integrable, and by (4.7)

$$I_2(x) = O\left(\frac{1}{|x|^{2N+4+\beta}}\right) \text{ as } |x| \rightarrow \infty.$$

Hence $I_2(x) \in L^1(\mathbb{R}^2)$.

From the estimate

$$\int_{\mathbb{R}^2} \frac{e^{u(y)}}{|x-y|} dy \leq \int_{|x-y| \leq 1} \frac{e^{u(y)}}{|x-y|} dy + \int_{|x-y| > 1} e^{u(y)} dy \leq C,$$

and (4.7) we also find that

$$I_1(x) = O\left(\frac{1}{|x|^{2N+4+\beta}}\right) \text{ as } |x| \rightarrow \infty,$$

and is also integrable. We now prove (12). From (5) with $\kappa = 2$ we first obtain

$$\int_{\mathbb{R}^2} F_{12} dx = -\frac{1}{2} \int_{\mathbb{R}^2} e^u (e^u - 1) dx \quad (4.12)$$

Let us denote $B(z_j, \delta) = \{z \in \mathbb{C}^1 \mid |z - z_j| < \delta\}$, and

$$B_R = \{z \in \mathbb{C}^1 \mid |z| < R\}, \quad S_R = \{z \in \mathbb{C}^1 \mid |z| = R\}.$$

Let us choose δ small enough so that $B(z_j, \delta) (j = 1, \dots, N)$'s are mutually disjoint and R large enough so that B_R contains all of $B(z_j, \delta)$'s. Integrating (9) over $B_R \setminus \cup_{j=1}^N B(z_j, \delta)$, we obtain by the divergence theorem

$$\oint_{S_R} \frac{\partial u}{\partial r} ds = \int_{B_R \setminus \cup_{j=1}^N B(z_j, \delta)} e^u (e^u - 1) dx + \sum_{j=1}^N \oint_{|x-z_j|=\delta} \frac{x - z_j}{|x - z_j|} \cdot \nabla u ds. \quad (4.13)$$

Now, using the representation formula of $u(x)$ in (4.12), we deduce

$$\begin{aligned} \oint_{|x-z_j|=\delta} \frac{x - z_j}{|x - z_j|} \cdot \nabla u ds &= 2 \oint_{|x-z_j|=\delta} \frac{x - z_j}{|x - z_j|} \cdot \nabla \ln |x - z_j| ds + o(1) \\ &= 2 \oint_{|x|=\delta} \frac{\partial}{\partial r} \ln r ds + o(1) = 4\pi + o(1) \end{aligned} \quad (4.14)$$

as $\delta \rightarrow 0$. Substituting (4.14) into (4.13), and passing $\delta \rightarrow 0$, we find that

$$\oint_{S_R} \frac{\partial u}{\partial r} ds = \int_{B_R} e^u (e^u - 1) dx + 4\pi N. \quad (4.15)$$

From (4.12) and (4.15)

$$\int_{\mathbb{R}^2} F_{12} dx = -\frac{1}{2} \lim_{R \rightarrow \infty} \oint_{S_R} \frac{\partial u}{\partial r} ds + 2\pi N,$$

and for our solution, $u(x) = u_\varepsilon(x)$ given by (2.13), it suffices to show that there exists a *positive* constant \hat{C} such that

$$-\frac{1}{2} \lim_{R \rightarrow \infty} \oint_{S_R} \frac{\partial u_\varepsilon}{\partial r} ds = 2\pi N + 4 + \hat{C}\varepsilon^2 + O(\varepsilon^3) \quad (4.16)$$

as $\varepsilon \rightarrow 0$. Taking into account the formula for $u_\varepsilon(x)$ in (2.13), it suffices, in turn, to prove the followings, (4.17)-(4.19).

$$-\frac{1}{2} \lim_{R \rightarrow \infty} \oint_{S_R} \frac{\partial}{\partial r} \ln \rho_{\varepsilon, a_\varepsilon^*} ds = 2\pi N + 4, \quad (4.17)$$

$$- \lim_{R \rightarrow \infty} \oint_{S_R} \frac{\partial w_0(\varepsilon|z|)}{\partial r} ds = \hat{C}, \quad (4.18)$$

where \hat{C} is a positive constant independent of ε , and

$$\lim_{R \rightarrow \infty} \oint_{S_R} \frac{\partial u_\varepsilon^*(\varepsilon z)}{\partial r} ds = O(\varepsilon^3), \quad (4.19)$$

as $\varepsilon \rightarrow 0$. We first prove (4.17). From the formula (2.1) one can write

$$\begin{aligned} \frac{\partial}{\partial r} \ln \rho_{\varepsilon, a_\varepsilon^*} &= \frac{1}{\rho_{\varepsilon, a_\varepsilon^*}} \frac{\partial}{\partial r} \rho_{\varepsilon, a_\varepsilon^*} \\ &= \frac{8\varepsilon^{2N+2}}{\rho_{\varepsilon, a_\varepsilon^*}} \left\{ \frac{\frac{\partial}{\partial r} |f(z)|^2}{(1 + \varepsilon^{2N+2} |F(z) + \frac{a_\varepsilon^*}{\varepsilon^{N+3}}|^2)^2} \right. \\ &\quad \left. - \frac{2|f(z)|^2 \varepsilon^{2N+2} \frac{\partial}{\partial r} |F(z) + \frac{a_\varepsilon^*}{\varepsilon^{N+3}}|^2}{(1 + \varepsilon^{2N+2} |F(z) + \frac{a_\varepsilon^*}{\varepsilon^{N+3}}|^2)^3} \right\}, \end{aligned} \quad (4.20)$$

with

$$|f(z)|^2 = (N+1)^2 r^{2N} + p_{2N-1}(\theta) r^{2N-1} + \dots + p_0(\theta)$$

and

$$|F(z)|^2 = r^{2N+2} + q_{2N+1}(\theta) r^{2N+1} + \dots + q_0(\theta),$$

where $p_j(\theta)$'s and $q_k(\theta)$'s are functions only of θ . Thus, substituting

$$\frac{\partial}{\partial r} |f(z)|^2 = 2N(N+1)^2 r^{2N-1} + (2N-1)p_{2N-1}(\theta) r^{2N-2} + \dots + p_1(\theta),$$

and

$$\frac{\partial}{\partial r} |F(z)|^2 = (2N+2)r^{2N+1} + (2N+1)q_{2N+1}(\theta) r^{2N} + \dots + q_1(\theta)$$

into (4.20), we have for each $\varepsilon > 0$

$$\begin{aligned}
& \oint_{S_R} \frac{\partial}{\partial r} \ln \rho_{\varepsilon, a_\varepsilon^*} ds \\
&= \oint_{S_R} \frac{1}{|f(z)|^2} \left\{ \frac{\partial}{\partial r} |f(z)|^2 - \frac{2|f(z)|^2 \varepsilon^{2N+2} \frac{\partial}{\partial r} |F(z) + \frac{a_\varepsilon^*}{\varepsilon^{N+3}}|^2}{1 + \varepsilon^{2N+2} |F(z) + \frac{a_\varepsilon^*}{\varepsilon^{N+3}}|^2} \right\} ds \\
&= \int_0^{2\pi} \left\{ \frac{2N(N+1)^2 R^{2N-1}}{(N+1)^2 R^{2N}} - \frac{2(2N+2)\varepsilon^{2N+2} R^{2N+1}}{1 + \varepsilon^{2N+2} R^{2N+2}} \right\} R d\theta + O\left(\frac{1}{R}\right) \\
&= -4\pi N - 8\pi + O\left(\frac{1}{R}\right)
\end{aligned}$$

as $R \rightarrow \infty$. Thus (4.17) is proved. Next, in order to prove (4.18) we first note

$$r \frac{\partial w_0(r)}{\partial r} = \int_0^\infty \varphi_0(t) t \rho^2(t) dt + O\left(\frac{1}{r}\right) \quad \text{as } r \rightarrow \infty,$$

which is obtained by explicit computation, using the formula for $w_0(R)$ given by (2.7)-(2.8) with $f(t) = \rho^2(t)$. From this fact we have

$$\begin{aligned}
- \lim_{R \rightarrow \infty} \oint_{S_R} \frac{\partial w_0(\varepsilon|z|)}{\partial r} ds &= - \lim_{R \rightarrow \infty} \oint_{S_{\varepsilon R}} \frac{\partial w_0(|z|)}{\partial r} ds \\
&= - \lim_{R \rightarrow \infty} \int_0^{2\pi} \frac{\partial w_0(r)}{\partial r} \Big|_{r=R} R d\theta \\
&= -2\pi \lim_{R \rightarrow \infty} R \frac{\partial w_0(R)}{\partial R} \\
&= 2\pi \int_0^\infty \varphi_0(t) t \rho^2(t) dt > 0,
\end{aligned}$$

where the estimate on I in the proof of Lemma 2.2 was used in the last equality. The formula (4.18) is proved. Finally let us prove (4.19). By the divergence theorem

$$\begin{aligned}
\lim_{R \rightarrow \infty} \oint_{S_R} \frac{\partial u_\varepsilon^*(\varepsilon x)}{\partial r} ds &= \lim_{R \rightarrow \infty} \oint_{S_{\varepsilon R}} \frac{\partial u_\varepsilon^*(x)}{\partial r} ds \\
&= \lim_{R \rightarrow \infty} \int_{B_{\varepsilon R}} \Delta u_\varepsilon^*(x) dx = \int_{\mathbb{R}^2} \Delta u_\varepsilon^*(x) dx,
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} \Delta u_\varepsilon^*(x) dx \right| &\leq \int_{\mathbb{R}^2} |\Delta u_\varepsilon^*(x)| dx \\
&= \int_{\mathbb{R}^2} |\Delta u_\varepsilon^*(x)| (1 + |x|^{\frac{17}{8}})^{\frac{1}{2}} (1 + |x|^{\frac{17}{8}})^{-\frac{1}{2}} dx \\
&\leq \left(\int_{\mathbb{R}^2} |\Delta u_\varepsilon^*(x)|^2 (1 + |x|^{\frac{17}{8}}) dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \frac{dx}{1 + |x|^{\frac{17}{8}}} \right)^{\frac{1}{2}} \\
&\leq C \|u_\varepsilon^*\|_{Y_{\frac{1}{8}}} \leq C \|v_\varepsilon^*\|_{U_{\frac{1}{8}}} \leq C\varepsilon^3,
\end{aligned}$$

where (4.2) was used in the last inequality. We thus proved (4.19). The proof of the theorem is complete. ■

Remark 4.1. Our method of construction of solutions of the problem (9), (10) actually provides an explicit formula for approximation of those solutions. Indeed, the proof of Theorem 2.1 is based on the following modified version of Newton's iteration method:

$$v_{n+1} = v_n - [\mathbf{P}'_\varepsilon(v_0)]^{-1} \mathbf{P}_\varepsilon(v_n)$$

In our case the starting point is $v_0 = 0 \in U_{\frac{1}{8}}$ and the functions $\rho_{\varepsilon, a_\varepsilon^*}$, $w_0(x)$ are given by explicit formula. It is well known (see [12]) that the above iteration scheme has exponential rate of convergence.

APPENDIX

Proof of Lemma 2.1. Let $w(r)$ be a solution to $L_1 w = f$. Since $\varphi_0(r)$ is a solution of (2.6) with $f = 0$, we try ansatz $w(r) = \xi(r)\varphi_0(r)$, and obtain $\xi(r)$ below. By an elementary computation we have

$$f = L_1 w = \varphi_0 \xi'' + (2\varphi_0' + \frac{\varphi_0}{r}) \xi',$$

where we used the fact $L_1 \varphi_0 = 0$. Multiplying the above equations by $r\varphi_0$, we deduce that

$$\frac{d}{dr}(r\varphi_0^2 \xi') = r\varphi_0^2 \xi'' + (2\varphi_0' \varphi_0 + \varphi_0^2) \xi' = r\varphi_0 f.$$

From this we immediately have

$$\begin{aligned}\xi(r) &= \int_0^r \frac{1}{s\varphi_0^2(s)} \left\{ \int_0^s \varphi_0(t)tf(t)dt \right\} ds \\ &= \int_0^r \frac{\phi_f(s)}{(1-s)^2} ds, \quad r \in (0,1)\end{aligned}$$

where $\phi_f(\cdot)$ is defined by

$$\phi_f(s) = \frac{(s-1)^2}{s\varphi_0^2(s)} \int_0^s tf(t)\varphi_0(t)dt \quad s \in (0, \infty)$$

as in (2.8).

Thus, for $r \in (0,1)$ we find that

$$w_1(r) = \varphi_0(r) \int_0^r \frac{\phi_f(s)}{(1-s)^2} ds$$

solves (2.6) for $r \in (0,1)$. Similarly we can deduce that for any $\epsilon > 0$

$$w_2(r) = \varphi_0(r) \int_{1+\epsilon}^r \frac{\phi_f(s)}{(1-s)^2} ds$$

solves (2.6) for $r \in (1, \infty)$.

In order to extend our solution $w(r)$ to the range $r \in (0, \infty)$ we set

$$\begin{aligned}w(r) &= \varphi_0(r) \int_0^r \frac{\phi_f(s) - \phi_f(1)}{(1-s)^2} ds + \varphi_0(r) \frac{\phi_f(1)r}{1-r} \\ &= I_1(r) + I_2(r).\end{aligned}\tag{A.1}$$

We claim that $w(r)$ is a well-defined $C_{loc}^2(\mathbb{R}_+^1)$ solution of (2.6).

Obviously $I_2 \in C_{loc}^\infty(\mathbb{R}_+^1)$. In order to show that $I_1 \in C_{loc}^2(\mathbb{R}_+^1)$ we observe first $\phi_f(s) \in C_{loc}^1(\mathbb{R}_+^1)$, $\phi_f(s) \in C^2(0, k)$ for all $k > 0$. In particular, we have

$$\lim_{s \rightarrow 0^+} \frac{1}{s} \int_0^s \varphi_0(t)tf(t)dt = 0,$$

and there is no singularity near $s = 0$ of $\phi_f(s)$. Next we will show that

$$\phi_f'(1) = 0.\tag{A.2}$$

To see this we compute first

$$\begin{aligned}\phi'_f(s) &= \frac{(s-1)\eta(s)}{s^2\varphi_0^3(s)} \int_0^s tf(t)\varphi_0(t)dt + \frac{(1-s)^2f(s)}{\varphi_0(s)} \\ &= J_1(s) + J_2(s),\end{aligned}$$

where we set

$$\eta(s) = (s+1)\varphi_0(s) - 2(s-1)s\varphi'_0(s).$$

We note $J_2(1) = 0$. In order to show $J_1(1) = 0$ it suffices to prove

$$\eta(1) = \eta'(1) = \eta''(1) = 0. \quad (\text{A.3})$$

Obviously $\eta(1) = 0$. Since

$$\eta'(s) = \varphi_0(s) - 3(s-1)\varphi'_0(s) - 2(s-1)s\varphi''_0(s), \quad (\text{A.4})$$

we also find that $\eta'(1) = 0$. In order to see $\eta''(1) = 0$ we use substitution $s\varphi''_0 = -\varphi'_0 - s\rho(s)\varphi_0$ for the last term in the right hand side of (A.4), which is immediate from the ordinary differential equation for φ_0 , $L_1\varphi_0 = 0$. Thus, we obtain

$$\begin{aligned}\eta'(s) &= \varphi_0(s) - 3(s-1)\varphi'_0(s) + 2(s-1)(\varphi'_0 + s\rho\varphi_0) \\ &= \varphi_0(s) - (s-1)\varphi'_0(s) + 2s(s-1)\rho(s)\varphi_0(s),\end{aligned}$$

from which $\eta''(1) = 0$ follows easily. We thus proved (A.3), and hence (A.2).

From (A.2) we have

$$\frac{\phi_f(s) - \phi_f(1)}{(1-s)^2} \in C^1(0, k) \quad \forall k > 0.$$

This implies $I_1(r)$ in (A.1), and hence $w(r)$ belongs to $C^2(0, k)$ for all $k > 0$. Now, in order to show that $w(r)$ is a C^2 solution of (2.6) in $(0, \infty)$, it suffices to prove that $w(r)$ satisfies (2.6) on $(0, 1)$ and $(1, \infty)$. For $r \in (0, 1)$ we find easily

$$w(r) = \varphi_0(r) \int_0^r \frac{\phi_f(s)}{(1-s)^2} ds = w_1(r),$$

which is a solution of (2.6) by construction.

For $r > 1$, setting $\epsilon > 0$, we have

$$\begin{aligned} w(r) &= \varphi_0(r) \left\{ \int_{1+\epsilon}^r \frac{\phi_f(s) - \phi_f(1)}{(1-s)^2} ds + \int_0^{1+\epsilon} \frac{\phi_f(s) - \phi_f(1)}{(1-s)^2} ds + \frac{\phi_f(1)r}{1-r} \right\} \\ &= w_2(r) + \varphi_0(r) \left\{ -\frac{(1+\epsilon)\phi_f(1)}{\epsilon} + \int_0^{1+\epsilon} \frac{\phi_f(s) - \phi_f(1)}{(1-s)^2} ds \right\}. \end{aligned}$$

By construction the first term above is a solution of (2.6), while the second term is solution of the homogeneous equation of (2.6) (i.e. (2.6) with $f = 0$).

This completes the proof of Lemma 2.1. ■

Proof of Lemma 2.2. From the explicit solution formula (2.8) we find that $w_0(\cdot) \in C^0(\mathbb{R}_+)$, and thus (2.9) follows from (2.10). It suffices now to prove (2.10). We observe from formula (2.8) that

$$w_0(r) = \varphi_0(r) \int_2^r \left(\frac{1+s^{2N+2}}{1-s^{2N+2}} \right)^2 \frac{I(s)}{s} ds + (\text{bounded function of } r)$$

as $r \rightarrow \infty$, where

$$I(s) = \int_0^s \varphi_0(t) t \rho^2(t) dt.$$

Since $\varphi_0(r) \rightarrow -1$ as $r \rightarrow \infty$, (2.10) follows if

$$I = I(\infty) = \int_0^\infty \varphi_0(r) r \rho^2(r) dr > 0.$$

Indeed, substituting $r^2 = t$ in the integrand of I , then

$$\begin{aligned} I &= 32(N+1)^4 \int_0^\infty \left(\frac{1-t^{N+1}}{1+t^{N+1}} \right) \frac{t^{2N}}{(1+t^{N+1})^4} dt \\ &= 32(N+1)^4 \int_0^1 \frac{(1-t^{N+1})t^{2N}}{(1+t^{N+1})(1+t^{N+1})^4} dt \\ &\quad + 32(N+1)^4 \int_1^\infty \frac{(1-t^{N+1})t^{2N}}{(1+t^{N+1})(1+t^{N+1})^4} dt \\ &= 32(N+1)^4 \int_0^1 \frac{(1-t^{N+1})t^{2N}}{(1+t^{N+1})(1+t^{N+1})^4} dt \\ &\quad - 32(N+1)^4 \int_0^1 \frac{(1-s^{N+1})s^{2N+2}}{(1+s^{N+1})(1+s^{N+1})^4} ds \\ &= 32(N+1)^4 \int_0^1 \frac{(1-t^{N+1})(1-t^2)t^{2N}}{(1+t^{N+1})(1+t^{N+1})^4} dt > 0, \end{aligned}$$

where we changed variable $s = \frac{1}{t}$ in the third equality. ■

Proof of Lemma 2.5. Suppose L_1 be the differential operator introduced in (2.6). Let us first observe that the following equality holds.

$$L_1 \left[\frac{1}{16(1+r^{2N+2})^2} \right] = \frac{(N+1)^2 r^{4N+2}}{(1+r^{2N+2})^4},$$

which can be verified by an elementary computation. We prove our lemma for φ_+ only. The case for φ_- is similar. Using the above identity, we have the following

$$\begin{aligned} \int_{\mathbb{R}^2} (\rho w_0 - 2\rho^2) \varphi_+^2 dx &= \int_0^{2\pi} \int_0^\infty (\rho w_0 - 2\rho^2) \frac{r^{2N+2} \cos^2(N+1)\theta}{(1+r^{2N+2})^2} r dr d\theta \\ &= \int_0^{2\pi} \cos^2(N+1)\theta d\theta \int_0^\infty \left[\frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2} w_0 - 2\rho^2 \right] \frac{r^{2N+2}}{(1+r^{2N+2})^2} r dr \\ &= \pi \int_0^\infty \left\{ \frac{1}{2} L_1 \left[\frac{1}{(1+r^{2N+2})^2} \right] w_0 - \frac{2\rho^2 r^{2N+2}}{(1+r^{2N+2})^2} \right\} r dr \\ &= \pi \int_0^\infty \left\{ \frac{1}{2} L_1 w_0 \cdot \frac{1}{(1+r^{2N+2})^2} - \frac{2\rho^2 r^{2N+2}}{(1+r^{2N+2})^2} \right\} r dr \\ &= \pi \int_0^\infty \left\{ \frac{\rho^2}{2(1+r^{2N+2})^2} - \frac{2\rho^2 r^{2N+2}}{(1+r^{2N+2})^2} \right\} r dr \\ &= \pi \int_0^\infty \left\{ \frac{5\rho^2}{2(1+r^{2N+2})^2} - \frac{2\rho^2}{(1+r^{2N+2})} \right\} r dr \\ &= 64\pi(N+1)^4 \int_0^\infty \left\{ \frac{5r^{4N}}{2(1+r^{2N+2})^6} - \frac{2r^{4N}}{(1+r^{2N+2})^5} \right\} r dr \\ &= 32\pi(N+1)^4 \int_0^\infty \left\{ \frac{5t^{2N}}{2(1+t^{N+1})^6} - \frac{2t^{2N}}{(1+t^{N+1})^5} \right\} dt \quad (\text{By change of variable, } t = r^2) \\ &= 32\pi(N+1)^4 \int_0^\infty \left\{ \frac{-5t^N}{10(N+1)} \frac{d}{dt} \frac{1}{(1+t^{N+1})^5} + \frac{t^N}{2(N+1)} \frac{d}{dt} \frac{1}{(1+t^{N+1})^4} \right\} dt \\ &= 32\pi(N+1)^3 N \int_0^\infty \left\{ \frac{t^{N-1}}{2(1+t^{N+1})^5} - \frac{t^{N-1}}{2(1+t^{N+1})^4} \right\} dt \\ &= -16\pi(N+1)^3 N \int_0^\infty \frac{t^{2N}}{(1+t^{N+1})^5} dt < 0. \end{aligned}$$

The proof of the lemma is completed. ■

Proof of Lemma 3.1. We write $g_\varepsilon(z, a)$ as follows

$$\begin{aligned} g_\varepsilon(z, a) &= \frac{r^{2N} + \varepsilon h_1(z)}{(1 + |z^{N+1} + \varepsilon h_2(z) + \frac{a}{\varepsilon^2}|^2)^2} \\ &= \frac{\varepsilon^8 r^{2N} + \varepsilon^9 h_1(z)}{(\varepsilon^4 + \varepsilon^4 r^{2N+2} + |a|^2 + 2\varepsilon^2 \operatorname{Re}\{\bar{a}z^{N+1}\} + 2\varepsilon^3 \operatorname{Re}\{\bar{a}h_2(z)\} + \varepsilon^5 h_3(z))^2}, \end{aligned} \tag{A.5}$$

where

$$\begin{aligned} |h_1(z)| &\leq C(1 + |z|^{2N-1}), \\ |h_2(z)| &\leq C(1 + |z|^N), \\ |h_3(z)| &\leq C(1 + |z|^{2N+1}). \end{aligned}$$

There exists $R_0 > 1$, $\varepsilon_0 \in (0, 1)$ such that $|z| = r > R_0$ and $|a| < \varepsilon^2$, $\varepsilon \in (0, \varepsilon_0)$ imply

$$\begin{aligned} 2\varepsilon^2 |a| r^{N+1} &\leq 2\varepsilon^4 r^{N+1} \leq \frac{\varepsilon^4}{4} r^{2N+2}, \\ 2\varepsilon^3 |a| |h_2(z)| &\leq C\varepsilon^5 (1 + |z|^N) \leq \frac{3}{8}\varepsilon^4 + \frac{1}{4}\varepsilon^4 r^{2N+2}, \\ \varepsilon^5 |h_3(z)| &\leq C\varepsilon^5 (1 + |z|^{2N+1}) \leq \frac{3}{8}\varepsilon^4 + \frac{1}{4}\varepsilon^4 r^{2N+2}. \end{aligned}$$

Substituting these estimates into the formula (A.5), we obtain

$$\begin{aligned} |g_\varepsilon(z, a)| &\leq \frac{C\varepsilon^8 r^{2N}}{(\varepsilon^4 + \varepsilon^4 r^{2N+2} + |a|^2 - 2\varepsilon^2 |a| r^{N+1} - 2\varepsilon^3 |a| |h_2(z)| - \varepsilon^5 |h_3(z)|)^2} \\ &\leq \frac{C\varepsilon^8 r^{2N}}{(\frac{\varepsilon^4}{4} + \frac{\varepsilon^4}{4} r^{2N+2})^2} \leq \frac{Cr^{2N}}{(1 + r^{2N+2})^2} \end{aligned}$$

for all sufficiently large r . Below we put

$$H_\varepsilon(z, a) = \varepsilon^{N+1} \left(F\left(\frac{z}{\varepsilon}\right) + \frac{a}{\varepsilon^{N+3}} \right).$$

To estimate $\frac{\partial g_\varepsilon(z, a)}{\partial a_j}$ and $\frac{\partial^2 g_\varepsilon(z, a)}{\partial a_j \partial a_k}$ we compute as follows.

$$\begin{aligned}\frac{\partial g_\varepsilon(z, a)}{\partial a_1} &= -\frac{4}{\varepsilon^2} g_\varepsilon(z, a) \frac{\operatorname{Re} H_\varepsilon(z, a)}{1 + |H_\varepsilon(z, a)|^2}, \\ \frac{\partial g_\varepsilon(z, a)}{\partial a_2} &= -\frac{4}{\varepsilon^2} g_\varepsilon(z, a) \frac{\operatorname{Im} H_\varepsilon(z, a)}{1 + |H_\varepsilon(z, a)|^2}, \\ \frac{\partial^2 g_\varepsilon(z, a)}{\partial a_1^2} &= -\frac{4}{\varepsilon^4} \frac{g_\varepsilon(z, a)}{1 + |H_\varepsilon(z, a)|^2} + \frac{24}{\varepsilon^4} g_\varepsilon(z, a) \frac{(\operatorname{Re} H_\varepsilon(z, a))^2}{(1 + |H_\varepsilon(z, a)|^2)^2}, \\ \frac{\partial^2 g_\varepsilon(z, a)}{\partial a_1 \partial a_2} &= \frac{24}{\varepsilon^4} g_\varepsilon(z, a) \frac{\operatorname{Re} H_\varepsilon(z, a) \operatorname{Im} H_\varepsilon(z, a)}{(1 + |H_\varepsilon(z, a)|^2)^2}, \\ \frac{\partial^2 g_\varepsilon(z, a)}{\partial a_2^2} &= -\frac{4}{\varepsilon^4} \frac{g_\varepsilon(z, a)}{1 + |H_\varepsilon(z, a)|^2} + \frac{24}{\varepsilon^4} g_\varepsilon(z, a) \frac{(\operatorname{Im} H_\varepsilon(z, a))^2}{(1 + |H_\varepsilon(z, a)|^2)^2}.\end{aligned}$$

Since

$$\frac{|H_\varepsilon(z, a)|}{1 + |H_\varepsilon(z, a)|^2} \leq \frac{1}{2},$$

we have

$$\sum_{i=1}^2 \left| \frac{\partial g_\varepsilon(z, a)}{\partial a_j} \right| \leq \frac{4}{\varepsilon^2} |g_\varepsilon(z, a)| \leq \frac{Cr^{2N}}{\varepsilon^2(1 + r^{2N+2})^2},$$

and

$$\sum_{j, k=1}^2 \left| \frac{\partial^2 g_\varepsilon(z, a)}{\partial a_j \partial a_k} \right| \leq \frac{C}{\varepsilon^4} |g_\varepsilon(z, a)| \leq \frac{Cr^{2N}}{\varepsilon^4(1 + r^{2N+2})^2}$$

for $|z| \geq R_0$. This completes the proof of the lemma. ■

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1 DEPARTMENT OF MATHEMATECS, SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA

E-mail address: dhchae@math.snu.ac.kr

2 DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IA 50011, USA

E-mail address: vika@iastate.edu