Asymptotic formuli for steady state voltage potentials in the presence of conductivity imperfections of small area.

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Abstract

We derive asymptotic formuli for two dimensional steady state voltage potentials associated with thin, "curve-like" conductivity imperfections. Our derivation is formal, and based on asymptotic matching of terms in an appropriate set of integral equations. In combination with the formuli (rigorously) derived in [3] for imperfections of small diameter, these new formuli cover the generic imperfections of small area in two dimensions.

1 Introduction

The aim of this paper is to advance the development of asymptotic formuli for steady state voltage potentials associated with a finite number of small imperfections inside an otherwise uniform conductor. Our interest in such formuli owes to the fact, that they provide extremely powerful tools to solve the inverse problem of identifying the conductivity imperfections, given electric boundary measurements (cf. [3], [7]). We have already in [3] derived formuli of this kind for conductivity imperfections of the form

$$\omega_\epsilon = \cup_{i=1}^N (\bar{x}_i + \epsilon B_i) \quad , \tag{1}$$

where the B_i are smooth (star shaped) domains in R^n, $n = 2, 3$, and the "diameter" of the imperfections, ϵ, is a small parameter. In this paper we restrict attention to the two dimensional case, and we develop asymptotic formuli for the other generic form of imperfections of small area, namely thin imperfections (of finite length). To be quite precise, we consider imperfections of the form

$$\omega_\epsilon = \cup_{i=1}^N \{x + \eta h^i(x) n(x) : \ x \in \sigma_0^i \ , \quad \eta \in (-\epsilon, \epsilon)\} \quad , \tag{2}$$

where the "thickness", ϵ, is now the small parameter. Each σ_0^i is a simple curve, $n(x)$ is a unit normal vectorfield to σ_0^i, and the function $h^i(x) \geq 0$ represents the thickness variation along σ_0^i. We assume the curves σ_0^i are smooth; they may be either open or closed (but are otherwise non-selfintersecting). We also assume the curves are separated apart from each other and apart from the boundary, $\partial\Omega$. The "background" is homogeneous (with conductivity = 1) and the imperfections have conductivities k_i, $1 \leq i \leq N$. Let u_ϵ denote the steady state voltage potential in the presence of the conductivity imperfections, *i.e.*, the solution to

$$\nabla \cdot (\gamma_\epsilon \nabla u_\epsilon) = 0 \ \text{ in } \Omega \ , \quad \gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} = g \ \text{ on } \partial\Omega \quad ,$$

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with γ_ϵ given by

$$\begin{aligned}\gamma_\epsilon(\cdot) &= 1 \ \text{ in } \Omega \setminus \omega_\epsilon \ , \\ \gamma_\epsilon(\cdot) &= k_i \ \text{ in } \{x + \eta h^i(x) n(x) : \ x \in \sigma_0^i \ , \ \ \eta \in (-\epsilon, \epsilon)\} \ , \quad 1 \le i \le N \ .\end{aligned}$$

Let U_0 denote the "background" potential, that is, the solution to

$$\triangle U_0 = 0 \ \text{ in } \Omega \ , \ \ \frac{\partial U_0}{\partial \nu} = g \ \text{ on } \partial\Omega \ .$$

The function g represents the applied boundary current; it satisfies $\int_{\partial\Omega} g(x)\, d\sigma_x = 0$, and, unless otherwise specified, we always assume it is smooth. The potentials, u_ϵ and U_0, are normalized by $\int_{\partial\Omega} u_\epsilon(x)\, d\sigma_x = \int_{\partial\Omega} U_0(x)\, d\sigma_x = 0$. The main achievement of this paper is the formal derivation of the following

Asymptotic Formuli for u_ϵ:

$$u_\epsilon(y) = U_0(y) - \epsilon \sum_{i=1}^{N} \int_{\sigma_0^i} \nabla_x \Phi_0(x,y) \cdot A^i(x) \nabla U_0(x) h^i(x)\, ds_x \ + o(\epsilon) \ \ , \ \ y \in \bar{\Omega} \setminus \left(\cup_{i=1}^N \sigma_0^i \right) \ \ , \quad (3)$$

and

$$\begin{aligned} u_\epsilon(y) - U_0(y) + 2 \int_{\partial\Omega} (u_\epsilon(x) - U_0(x)) \frac{\partial \Phi}{\partial \nu(x)}(x,y)\, d\sigma_x \\ = -2\epsilon \sum_{i=1}^{N} \int_{\sigma_0^i} \nabla_x \Phi(x,y) \cdot A^i(x) \nabla U_0(x) h^i(x)\, ds_x \ + o(\epsilon) \ \ , \ \ y \in \partial\Omega \ . \end{aligned} \quad (4)$$

Here $\Phi(x,y) = \Phi(y,x)$ is any fundamental solution for the Laplacian, defined for $x, y \in \Omega'$, $x \neq y$, with Ω compactly contained in Ω'. For instance

$$\Phi(x,y) = -\frac{1}{2\pi} \log |x - y| \ .$$

The function $\Phi_0(x,y) = \Phi_0(y,x)$ $(x, y \in \Omega,\ x \neq y)$ is the particular (Ω-) fundamental solution, which additionally satisfies

$$\frac{\partial \Phi_0(x,y)}{\partial \nu(x)} = -\frac{1}{|\partial\Omega|} \ \ x \in \partial\Omega \ , \ \ y \in \Omega \ .$$

Let $\tau(x)$ and $n(x)$ denote a unit tangential and a unit normal field to σ_0^i. The symmetric matrix $A^i(x)$, $x \in \sigma_0^i$, is defined by

$$A^i(x) \text{ has eigenvectors } \tau(x) \text{ and } n(x) \ \ ,$$

$$\text{the eigenvalue corresponding to } \tau(x) \text{ is } 2(k_i - 1) \ \ ,$$

and

$$\text{the eigenvalue corresponding to } n(x) \text{ is } 2(1 - \frac{1}{k_i}) \ \ .$$

Note that A^i is positive definite if $k_i > 1$, and negative definite if $k_i < 1$. The above formuli have some analogy with those established in [3], however their derivation is entirely different and considerably more complicated. Both of these formuli, but in particular the second, have important applications to the inverse problem of estimating the locations and thicknesses of the imperfections, given boundary current and voltage measurements. As indicated earlier, our derivation of the formuli (3) and (4) does not represent a rigorous proof, rather it is based on formal power series expansions and the matching of terms of equal powers of ϵ. From the very nature of this derivation it follows, that one cannot expect the remainder in (3) to be $o(\epsilon)$ uniformly in $\bar{\Omega} \setminus \left(\cup_{i=1}^N \sigma_0^i\right)$ (this is also evident from its form). Rather, one might expect the remainder to be uniformly $o(\epsilon)$ a fixed distance away from $\cup_{i=1}^N \sigma_0^i$, but with this estimate degenerating as y approaches $\cup_{i=1}^N \sigma_0^i$. With this single proviso, and given sufficient regularity of $\partial\Omega$, we are convinced that the resulting formuli may be rigorously established (as was indeed done for the corresponding formuli in [3] and [7]). However, we have not attempted to do this. As a partial compensation we do provide quite detailed computational evidence concerning the accuracy of these formuli.

A brief outline of the paper is as follows. Using the energy argument found in [3] we start by showing that u_ϵ converges to U_0 as ϵ tends to zero. As discussed towards the end of section 2 this convergence is not uniform with respect to the values of the conductivities of the imperfections (this appears to contrast with [3]). We then represent the voltage potential u_ϵ in terms of single layer potentials and derive the integral equations for the associated densities. By formally expanding the densities in powers of ϵ, and matching terms of equal powers in the integral equations we derive formuli for the first two terms in the expansion for u_ϵ. All the details of this derivation are provided in section 3 for the case of a single "closed" imperfection of uniform thickness (an imperfection with a closed "center" curve, and with $h(x) \equiv 1$). The same derivation could be carried out for a "closed" imperfection with variable h, however, instead of doing this we perform, in section 4, a simple conformal change of variables, which does provide an intuitive justification of the formuli for the first two terms for a general h. The use of h, that vanish in an entire interval, immediately yields the formuli corresponding to open curves as well. Fairly simple arguments, introduced at the end of section 4, transform the formuli for the first two terms in the expansion for u_ϵ into the asymptotic statements (3) and (4). Section 5 is devoted to the presentation of numerical results that practically demonstrate the smallness of the remainder terms $o(\epsilon)$ in (3) and (4). Finally, section 6 contains a very short discussion concerning the application of our asymptotic formuli for the purpose of identifying certain aspects of the conductivity imperfections.

2 Energy estimates

We start by identifying the principal term of the voltage potential, u_ϵ, provided the imperfections have small area, and conductivities $0 < k \leq k_i \leq K$, $1 \leq i \leq N$. As already mentioned in the introduction, the principal term (sometimes referred to as the background potential) is given by

$$\triangle U_0 = 0 \quad \text{in } \Omega\ , \qquad \frac{\partial}{\partial \nu} U_0 = g \quad \text{on } \partial\Omega\ .$$

Indeed (just supposing $g \in H^{-1/2}(\partial\Omega)$) u_ϵ converges strongly in $H^1(\Omega)$ towards U_0, as $\epsilon \to 0$ ($|\omega_\epsilon| \to 0$). This is a simple consequence of the following estimate.

Proposition 2.1 *Let u_ϵ and U_0 be defined as above. There exists a contant, C, only depending on Ω, such that*

$$\int_\Omega (|\nabla(U_0 - u_\epsilon)|^2 + |U_0 - u_\epsilon|^2)\, dx \leq C \left(\frac{\max\{1,K\}}{\min\{1,k\}}\right)^2 \|\nabla U_0\|^2_{L^\infty(\omega_\epsilon)} |\omega_\epsilon| \quad .$$

Proof : Since $\int_\Omega |U_0 - u_\epsilon|^2\, dx \leq C(\int_\Omega (|\nabla(U_0 - u_\epsilon)|^2\, dx + |\int_{\partial\Omega}(U_0 - u_\epsilon)\, d\sigma|^2)$, and since $\int_{\partial\Omega}(U_0 - u_\epsilon)\, d\sigma = 0$, it suffices to show that

$$\int_\Omega |\nabla(U_0 - u_\epsilon)|^2\, dx \leq \left(\frac{\max\{1,K\}}{\min\{1,k\}}\right)^2 \|\nabla U_0\|^2_{L^\infty(\omega_\epsilon)} |\omega_\epsilon| \quad . \tag{5}$$

Simple manipulations give

$$\begin{aligned}
\int_\Omega \gamma_\epsilon |\nabla(U_0 - u_\epsilon)|^2\, dx &= \int_\Omega \gamma_\epsilon \nabla U_0 \nabla(U_0 - u_\epsilon)\, dx - \int_\Omega \gamma_\epsilon \nabla u_\epsilon \nabla(U_0 - u_\epsilon)\, dx \\
&= \int_\Omega (\gamma_\epsilon - 1)\nabla U_0 \nabla(U_0 - u_\epsilon)\, dx + \int_\Omega \nabla U_0 \nabla(U_0 - u_\epsilon)\, dx \\
&\qquad - \int_\Omega \gamma_\epsilon \nabla u_\epsilon \nabla(U_0 - u_\epsilon)\, dx \\
&= \int_{\omega_\epsilon} (\gamma_\epsilon - 1)\nabla U_0 \nabla(U_0 - u_\epsilon)\, dx \quad .
\end{aligned}$$

For the last identity we have used integration by parts in combination with the fact that $\frac{\partial U_0}{\partial \nu} = \gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} = g$ on $\partial\Omega$, and the fact that $\gamma_\epsilon = 1$ in $\Omega \setminus \omega_\epsilon$. Introducing $\gamma_\epsilon = k_i$ in $\omega_\epsilon^i = \{x + \eta h^i(x) n(x),\ x \in \sigma_0^i\ ,\ \ \eta \in (-\epsilon, \epsilon)\}$ we arrive at

$$\begin{aligned}
\int_\Omega \gamma_\epsilon |\nabla(U_0 - u_\epsilon)|^2\, dx &= \sum_{i=1}^N \int_{\omega_\epsilon^i} (k_i - 1)\nabla U_0 \nabla(U_0 - u_\epsilon)\, dx \\
&\leq \max\{1,K\} \|\nabla U_0\|_{L^\infty(\omega_\epsilon)} |\omega_\epsilon|^{1/2} (\int_\Omega |\nabla(U_0 - u_\epsilon)|^2\, dx)^{1/2} \quad . \qquad (6)
\end{aligned}$$

Simple manipulations of (6) now lead to

$$\int_\Omega |\nabla(U_0 - u_\omega)|^2\, dx \leq \left(\frac{\max\{1,K\}}{\min\{1,k\}}\right)^2 \|\nabla U_0\|^2_{L^\infty(\omega_\epsilon)} |\omega_\epsilon| \quad ,$$

which is exactly the desired estimate (5). □

As indicated by the constant $C\left(\frac{\max\{1,K\}}{\min\{1,k\}}\right)^2$ appearing in the above lemma, the convergence of u_ϵ towards U_0 is not necessarily uniform in the size of the conductivities, k_i. When ω_ϵ is of the form $\omega_\epsilon = \cup_{i=0}^N (\bar{x}_i + \epsilon B_i)$ (as studied in [3]) then the convergence is most likely uniform. But when ω_ϵ is of the form

$$\omega_\epsilon = \cup_{i=0}^N \{x + \eta h^i(x) n(x) :\ x \in \sigma_0^i\ ,\ \eta \in (-\epsilon, \epsilon)\} \quad ,$$

as studied in this paper, then the convergence of u_ϵ towards U_0 is only uniform in k, K under very exceptional circumstances. This may be explained as follows. For a fixed $\epsilon > 0$ the potential u_ϵ has limits u_ϵ^0 and u_ϵ^∞ as all the k_i tend to 0 and all the k_i tend to ∞, respectively. The potentials, u_ϵ^0 and u_ϵ^∞, solve

$$\triangle u_\epsilon^0 = \triangle u_\epsilon^\infty = 0 \quad \text{in } \Omega \setminus \omega_\epsilon \quad , \qquad \frac{\partial}{\partial \nu} u_\epsilon^0 = \frac{\partial}{\partial \nu} u_\epsilon^\infty = g \quad \text{on } \partial\Omega \quad ,$$

with

$$\frac{\partial}{\partial \nu} u_\epsilon^0 = 0 \quad , \quad \text{and } \frac{\partial}{\partial \tau} u_\epsilon^\infty = 0 \quad \text{on } \partial\omega_\epsilon \quad ,$$

respectively. As ϵ tends to zero (so $|\omega_\epsilon|$ tends to zero) u_ϵ^0 and u_ϵ^∞ have limits U_0^0 and U_0^∞, which solve

$$\triangle U_0^0 = \triangle U_0^\infty = 0 \quad \text{in } \Omega \setminus \cup_{i=1}^N \sigma_0^i \quad , \qquad \frac{\partial}{\partial \nu} U_0^0 = \frac{\partial}{\partial \nu} U_0^\infty = g \quad \text{on } \partial\Omega \quad ,$$

with

$$\frac{\partial}{\partial \nu} U_0^0 = 0 \quad , \quad \text{on } \sigma_0^i \quad , \quad 1 \le i \le N \quad , \quad \text{and } \frac{\partial}{\partial \tau} U_0^\infty = 0 \quad \text{on } \sigma_0^i \quad , \quad 1 \le i \le N \quad ,$$

respectively. Since the potentials U_0^0 and U_0^∞ are generally different from U_0, it follows immediately that the convergence of u_ϵ towards U_0 generally will not be uniform in k, K.

3 A two term expansion for a uniform, "closed" layer

We now focus on the case where the imperfections ω_ϵ consist of a single, uniformly thin domain

$$\omega_\epsilon = \{x + \eta n(x) : \ x \in \sigma_0 \ , \ \eta \in (-\epsilon, \epsilon)\} \quad ,$$

and σ_0 is a smooth, **closed**, simple curve inside Ω. The unit vectorfield $n(\cdot)$ is smooth and normal to σ_0 (we may for instance suppose it points outward from the domain enclosed by σ_0). We denote by $\sigma_\epsilon^\pm$ the two smooth, closed curves

$$\sigma_\epsilon^\pm = \{x \pm \epsilon n(x) : \ x \in \sigma_0\} \quad ,$$

which form the boundary of ω_ϵ. The notation ν has previously been used for the unit outward normal vectorfield on $\partial\Omega$ – we shall use the same notation for the outward vectorfield on $\partial\omega_\epsilon$, in other words

$$\nu(x + \epsilon n(x)) = n(x) \quad , \text{ and } \quad \nu(x - \epsilon n(x)) = -n(x) \quad , \ x \in \sigma_0 \quad .$$

In the remainder of this paper Φ shall always refer to the particular fundamental solution

$$\Phi(x, y) = -\frac{1}{2\pi} \log |x - y| \quad .$$

The potential $u_\epsilon(y)$ may be represented as a classical single layer potential on $\partial\Omega \cup \partial\omega_\epsilon$

$$u_\epsilon(y) = \int_{\partial\Omega} \Phi(x, y) \psi_\epsilon(x) \, d\sigma_x + \int_{\partial\omega_\epsilon} \Phi(x, y) \psi_\epsilon(x) \, d\sigma_x, \quad y \in \Omega \quad , \tag{7}$$

where $\psi_\epsilon(\cdot)$ is an appropriately determined density on $\partial\Omega \cup \partial\omega_\epsilon = \partial\Omega \cup \sigma_\epsilon^+ \cup \sigma_\epsilon^-$. For any $x \in \sigma_0$ let $\tau(x)$ denote a tangential field, chosen such that the 90° rotation from τ to n is counterclockwise. Let $\frac{d}{dt_x}$ denote the tangential derivative in the direction of $\tau(x)$. We now define the signed curvature

$$\kappa(x) = \left(\frac{d}{dt_x}\tau(x)\right) \cdot n(x) \qquad x \in \sigma_0 \ .$$

If ds denotes surface measure on σ_0 (at the point x) then the coressponding surface measures (at the points $x \pm \epsilon n(x)$) on the curves $\sigma_\epsilon^\pm$ are given by

$$d\sigma = (1 \mp \epsilon\kappa(x))ds + O(\epsilon^2) \ . \tag{8}$$

The smooth density ψ_ϵ (of the representation (7)) is determined by the two conditions

$$\frac{\partial u_\epsilon}{\partial \nu} = g \quad \text{on } \partial\Omega \ , \tag{9}$$

and

$$\frac{\partial u_\epsilon^e}{\partial \nu} = k\frac{\partial u_\epsilon^i}{\partial \nu} \quad \text{on } \partial\omega_\epsilon \ ,$$

where u_ϵ^e and u_ϵ^i refer to the potential in $\Omega \setminus \omega_\epsilon$ and in ω_ϵ, respectively, and k is the conductivity inside ω_ϵ. The latter condition may also be rewritten

$$\frac{\partial u_\epsilon^e}{\partial n} = k\frac{\partial u_\epsilon^i}{\partial n} \quad \text{on } \partial\omega_\epsilon \ , \tag{10}$$

with the field n substituted for ν. Using the well known limiting formula for the normal derivative of a single layer potential (see for instance [4] or [5]) and combining the representation (7) with the two conditions (9) and (10), we arrive at the following integral equations for the density ψ_ϵ

$$\frac{1}{2}\psi_\epsilon(y) + \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi_\epsilon(x)\, d\sigma_x + \int_{\partial\omega_\epsilon} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi_\epsilon(x)\, d\sigma_x = g(y) \ , \quad y \in \partial\Omega \ , \tag{11}$$

and

$$\frac{k+1}{2k}\psi_\epsilon(y) \pm (1-\frac{1}{k})\left[\int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n(y)}\psi_\epsilon(x)\, d\sigma_x + \int_{\partial\omega_\epsilon} \frac{\partial\Phi(x,y)}{\partial n(y)}\psi_\epsilon(x)\, d\sigma_x\right] = 0 \ , \quad y \in \sigma_\epsilon^\pm \ . \tag{12}$$

To distinguish the various parts of the density ψ_ϵ, and to make the curves on which these are defined independent of ϵ, we introduce

$$\psi_\epsilon^+(x) = \psi_\epsilon(x + \epsilon n(x)) \ , \quad \text{and} \quad \psi_\epsilon^-(x) = \psi_\epsilon(x - \epsilon n(x)) \ , \quad x \in \sigma_0 \ .$$

Using this notation and the identity (8), we may formally rewrite the equations (11) and (12) as follows

$$\begin{aligned}
&\frac{1}{2}\psi_\epsilon(y) + \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi_\epsilon(x)\, d\sigma_x \\
&\quad + \int_{\sigma_0} \frac{\partial\Phi(x+\epsilon n(x),y)}{\partial\nu(y)}\psi_\epsilon^+(x)\,(1-\epsilon\kappa(x))ds_x \\
&\quad + \int_{\sigma_0} \frac{\partial\Phi(x-\epsilon n(x),y)}{\partial\nu(y)}\psi_\epsilon^-(x)\,(1+\epsilon\kappa(x))ds_x = g(y) + O(\epsilon^2) \ , \quad y \in \partial\Omega \ ,
\end{aligned} \tag{13}$$

and

$$\frac{k+1}{2k}\psi_\epsilon^\pm(y) \pm (1-\frac{1}{k})\int_{\partial\Omega}\frac{\partial\Phi(x,y\pm\epsilon n(y))}{\partial n(y)}\psi_\epsilon(x)\,d\sigma_x$$
$$\pm(1-\frac{1}{k})\Big[\int_{\sigma_0}\frac{\partial\Phi(x+\epsilon n(x),y\pm\epsilon n(y))}{\partial n(y)}\psi_\epsilon^+(x)\,(1-\epsilon\kappa(x))ds_x$$
$$+\int_{\sigma_0}\frac{\partial\Phi(x-\epsilon n(x),y\pm\epsilon n(y))}{\partial n(y)}\psi_\epsilon^-(x)\,(1+\epsilon\kappa(x))ds_x\Big] = O(\epsilon^2)\ , \quad y\in\sigma_0\ . \quad (14)$$

We may also formally rewrite the representation (7) for u_ϵ

$$\begin{aligned} u_\epsilon(y) &= \int_{\partial\Omega}\Phi(x,y)\psi_\epsilon(x)\,d\sigma_x \\ &\quad+\int_{\sigma_0}\Phi(x+\epsilon n(x),y)\psi_\epsilon^+(x)(1-\epsilon\kappa(x))\,ds_x \\ &\quad+\int_{\sigma_0}\Phi(x-\epsilon n(x),y)\psi_\epsilon^-(x)(1+\epsilon\kappa(x))\,ds_x+O(\epsilon^2)\ . \end{aligned} \quad (15)$$

We have twice used the word "formally", since we have not given (and do not intend to derive) precise estimates for the terms that we have lumped as $O(\epsilon^2)$. Rather, our goal is to determine the first terms of the asymptotic expansions

$$\psi_\epsilon(x)=\epsilon^{-1}\psi_{-1}(x)+\psi_0(x)+\epsilon\psi_1(x)+\cdots \quad x\in\partial\Omega \quad (16)$$

and

$$\psi_\epsilon^\pm(x)=\epsilon^{-1}\psi_{-1}^\pm(x)+\psi_0^\pm(x)+\epsilon\psi_1^\pm(x)+\cdots \quad x\in\sigma_0\ , \quad (17)$$

by a "formal matching" of terms of equal powers of ϵ. Knowledge of the first terms in these expansions immediately leads, via the representation (15), to knowledge of the first terms in the corresponding expansion for the potential

$$u_\epsilon=\frac{1}{\epsilon}u_{-1}+u_0+\epsilon u_1+\cdots=U_0+\epsilon u_1+\cdots\ , \quad (18)$$

as it (not unexpectedly, in light of Proposition 2.1) turns out that u_{-1} (as well as ψ_{-1} and $\psi_{-1}^\pm$) vanishes, and that u_0 equals the background potential, U_0. In order to perform the formal asymptotic matching the following observation proves crucial

Lemma 3.1
Let $\psi(x)$ be a fixed, smooth density on σ_0, and let $y\in\sigma_0$. Then, as $\epsilon\to 0$,

$$\int_{\sigma_0}\frac{\partial\Phi(x\pm\epsilon n(x),y\pm\epsilon n(y))}{\partial n(y)}\psi(x)\,ds_x\to\int_{\sigma_0}\frac{\partial\Phi(x,y)}{\partial n(y)}\psi(x)\,ds_x\ , \quad (19)$$

and

$$\int_{\sigma_0}\frac{\partial\Phi(x\mp\epsilon n(x),y\pm\epsilon n(y))}{\partial n(y)}\psi(x)\,ds_x\to\mp\frac{1}{2}\psi(y)+\int_{\sigma_0}\frac{\partial\Phi(x,y)}{\partial n(y)}\psi(x)\,ds_x\ . \quad (20)$$

Proof : Very briefly, the first statement follows from Lebesque's Dominated Convergence Theorem; the second statement is a slight variation of the classical jump condition for the normal derivative of a single layer potential. □

We start by collecting terms of order ϵ^{-1} in (13) and (14). For $y \in \partial\Omega$ we immediately get

$$\frac{1}{2}\psi_{-1}(y) + \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi_{-1}(x)\,d\sigma_x + \int_{\sigma_0} \frac{\partial\Phi(x,y)}{\partial\nu(y)}(\psi^+_{-1} + \psi^-_{-1})(x)\,ds_x = 0 \quad . \tag{21}$$

For $y \in \sigma_0$ we get, with the help of Lemma 3.1,

$$\begin{aligned}\frac{k+1}{2k}\psi^+_{-1}(y) = -\frac{k-1}{k}\Bigg\{&\int_{\partial\Omega} \frac{\partial\Phi}{\partial n(y)}(x,y)\psi_{-1}(x)\,d\sigma_x - \frac{1}{2}\psi^-_{-1}(y) \\ &+ \int_{\sigma_0} \frac{\partial\Phi}{\partial n(y)}(x,y)(\psi^+_{-1} + \psi^-_{-1})(x)\,ds_x\Bigg\} \quad ,\end{aligned} \tag{22}$$

and

$$\begin{aligned}\frac{k+1}{2k}\psi^-_{-1}(y) = \frac{k-1}{k}\Bigg\{&\int_{\partial\Omega} \frac{\partial\Phi}{\partial n(y)}(x,y)\psi_{-1}(x)\,d\sigma_x + \frac{1}{2}\psi^+_{-1}(y) \\ &+ \int_{\sigma_0} \frac{\partial\Phi}{\partial n(y)}(x,y)(\psi^+_{-1} + \psi^-_{-1})(x)\,ds_x\Bigg\} \quad .\end{aligned} \tag{23}$$

Summation of (22) and (23) yields

$$\frac{k+1}{2k}(\psi^+_{-1} + \psi^-_{-1})(y) = \frac{k-1}{2k}(\psi^+_{-1} + \psi^-_{-1})(y) \;, \quad y \in \sigma_0 \quad ,$$

from which it immediately follows that

$$(\psi^+_{-1} + \psi^-_{-1})(y) = 0 \quad y \in \sigma_0 \quad . \tag{24}$$

By collecting terms of order ϵ^{-1} in (15) we now obtain

$$\begin{aligned} u_{-1}(y) &= \int_{\partial\Omega} \Phi(x,y)\psi_{-1}(x)\,d\sigma_x + \int_{\sigma_0} \Phi(x,y)(\psi^+_{-1} + \psi^-_{-1})(x)\,ds_x \\ &= \int_{\partial\Omega} \Phi(x,y)\psi_{-1}(x)\,d\sigma_x \quad , \quad y \in \Omega \quad , \end{aligned} \tag{25}$$

and consequently $\triangle u_{-1} = 0$ in Ω. Insertion of (24) into (21) yields

$$\frac{\partial u_{-1}}{\partial\nu} = 0 \quad \text{on } \partial\Omega \quad ,$$

and u_{-1} is therefore a constant. The normalization $\int_{\partial\Omega} u_{-1}\,ds = 0$ implies that this constant is zero, *i.e.*,

$$u_{-1} = 0 \quad .$$

From standard arguments (*cf.* [5]) it follows that

$$\psi_{-1} = 0 \quad \text{on } \partial\Omega \quad . \tag{26}$$

The identity (22) (in combination with (24) and (26)) now asserts

$$\frac{k+1}{2k}\psi_{-1}^{+}(y) = \frac{(k-1)}{2k}\psi_{-1}^{-}(y) = -\frac{(k-1)}{2k}\psi_{-1}^{+}(y) \quad y \in \sigma_0 \quad ,$$

or

$$\psi_{-1}^{+}(y) = \psi_{-1}^{-}(y) = 0 \quad , \quad y \in \sigma_0 \quad .$$

(Note that the degenerate case, $k = 1$, is slightly different, since we also need to invoke (23) in order to reach the conclusion that $\psi_{-1}^{-} = 0$). We went through the derivation for index -1, only to realize that all the corresponding terms vanish. The same would of course have happened for any index less that -1. The expansions for the ψ_ϵ's and u_ϵ thus start with terms of index 0. Furthermore (since terms of lower index vanish) the derivation for the terms of index 0 is almost identical to that, which we just went through. For $y \in \partial\Omega$ we get

$$\frac{1}{2}\psi_0(y) + \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi_0(x)\, d\sigma_x + \int_{\sigma_0} \frac{\partial\Phi(x,y)}{\partial\nu(y)}(\psi_0^{+} + \psi_0^{-})(x)\, ds_x = g(y) \quad . \tag{27}$$

For $y \in \sigma_0$ we get (again with the help of Lemma 3.1)

$$\begin{aligned}\frac{k+1}{2k}\psi_0^{+}(y) = -\frac{k-1}{k}\Bigg\{ & \int_{\partial\Omega} \frac{\partial\Phi}{\partial n(y)}(x,y)\psi_0(x)\, d\sigma_x - \frac{1}{2}\psi_0^{-}(y) \\ & + \int_{\sigma_0} \frac{\partial\Phi}{\partial n(y)}(x,y)(\psi_0^{+} + \psi_0^{-})(x)\, ds_x \Bigg\} \quad ,\end{aligned} \tag{28}$$

and

$$\begin{aligned}\frac{k+1}{2k}\psi_0^{-}(y) = \frac{k-1}{k}\Bigg\{ & \int_{\partial\Omega} \frac{\partial\Phi}{\partial n(y)}(x,y)\psi_0(x)\, d\sigma_x + \frac{1}{2}\psi_0^{+}(y) \\ & + \int_{\sigma_0} \frac{\partial\Phi}{\partial n(y)}(x,y)(\psi_0^{+} + \psi_0^{-})(x)\, ds_x \Bigg\} \quad .\end{aligned} \tag{29}$$

In the same fashion as before we therefore arrive at

$$(\psi_0^{+} + \psi_0^{-})(y) = 0 \quad y \in \sigma_0 \quad , \tag{30}$$

and

$$\begin{aligned} u_0(y) &= \int_{\partial\Omega} \Phi(x,y)\psi_0(x)\, d\sigma_x + \int_{\sigma_0} \Phi(x,y)(\psi_0^{+} + \psi_0^{-})(x)\, ds_x \\ &= \int_{\partial\Omega} \Phi(x,y)\psi_0(x)\, d\sigma_x \quad , \quad y \in \Omega \quad . \end{aligned}$$

From this last identity it follows that $\triangle u_0 =$ in Ω, and insertion of (30) into (27) gives

$$\frac{\partial u_0}{\partial\nu} = g \quad \text{on } \partial\Omega \quad .$$

We thus conclude that u_0 (with the normalization $\int_{\partial\Omega} u_0\, d\sigma = 0$) equals U_0, the background voltage potential. By subtraction of (28) and (29) we also get

$$\begin{aligned}(\psi_0^{+} - \psi_0^{-})(y) &= -2\frac{k-1}{k}\int_{\partial\Omega} \frac{\partial\Phi}{\partial n(y)}(x,y)\psi_0(x)\, d\sigma_x \\ &= -2\frac{k-1}{k}\frac{\partial U_0}{\partial n(y)}(y) \quad ,\end{aligned}$$

or

$$2\psi_0^+(y) = (\psi_0^+ - \psi_0^-)(y) = -2\frac{k-1}{k}\frac{\partial U_0}{\partial n(y)}(y) \ , \quad y \in \sigma_0 \ . \tag{31}$$

We now turn to the terms of index 1. By collecting terms of order ϵ in (13) and (15) we arrive at the following representation for u_1

$$\begin{aligned} u_1(y) &= \int_{\partial\Omega} \Phi(x,y)\psi_1(x)\, d\sigma_x \\ &\quad + \int_{\sigma_0} \Phi(x,y)(\psi_1^+ + \psi_1^- - \kappa(\psi_0^+ - \psi_0^-))(x)\, ds_x \\ &\quad + \int_{\sigma_0} \frac{\partial\Phi}{\partial n(x)}(x,y)(\psi_0^+ - \psi_0^-)(x)\, ds_x \quad , \quad y \in \Omega \setminus \sigma_0 \ , \end{aligned} \tag{32}$$

and the following equation on $\partial\Omega$

$$\begin{aligned} &\frac{1}{2}\psi_1(y) + \int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu(y)}(x,y)\psi_1(x)\, d\sigma_x \\ &\quad + \int_{\sigma_0} \frac{\partial\Phi}{\partial\nu(y)}(x,y)(\psi_1^+ + \psi_1^- - \kappa(\psi_0^+ - \psi_0^-))(x)\, ds_x \\ &\quad + \int_{\sigma_0} \frac{\partial}{\partial n(x)}\frac{\partial\Phi}{\partial\nu(y)}(x,y)(\psi_0^+ - \psi_0^-)(x)\, ds_x = 0 \ . \end{aligned} \tag{33}$$

From the above identities it is clear that $\triangle u_1 = 0$ in $\Omega \setminus \sigma_0$, and that $\frac{\partial u_1}{\partial\nu} = 0$ on $\partial\Omega$. From the above identities it is also clear, that we do not need the individual functions ψ_1^+ and ψ_1^- in order to fully characterize ψ_1 and u_1, rather we only need $\psi_1^+ + \psi_1^-$. To arrive at an expression for this linear combination we sum the two identities in (14) and collect terms of order ϵ. The resulting identity is

$$\frac{k+1}{2k}(\psi_1^+ + \psi_1^-)(y) + 2(1-\frac{1}{k})\int_{\partial\Omega} < D_y^2\Phi(x,y)n(y), n(y) > \psi_0(x) d\sigma_x + (1-\frac{1}{k})R(y) = 0 \ , \tag{34}$$

where $\epsilon R(y)$ are the terms of order ϵ in the expression

$$\begin{aligned} &r(y) = \\ &\int_{\sigma_0} \left[\frac{\partial\Phi(x+\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x+\epsilon n(x), y-\epsilon n(y))}{\partial n(y)}\right](\psi_0^+ + \epsilon\psi_1^+ - \epsilon\kappa\psi_0^+)(x)\, ds_x \\ &+ \int_{\sigma_0} \left[\frac{\partial\Phi(x-\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x-\epsilon n(x), y-\epsilon n(y))}{\partial n(y)}\right](\psi_0^- + \epsilon\psi_1^- + \epsilon\kappa\psi_0^-)(x)\, ds_x \ . \end{aligned}$$

Simple manipulations, in combination with the fact that $\psi_0^- = -\psi_0^+$, yield

$$\begin{aligned} r(y) &= \int_{\sigma_0} \left(\left[\frac{\partial\Phi(x+\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x+\epsilon n(x), y-\epsilon n(y))}{\partial n(y)}\right] \right. \\ &\quad \left. - \left[\frac{\partial\Phi(x-\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x-\epsilon n(x), y-\epsilon n(y))}{\partial n(y)}\right]\right)\psi_0^+(x)\, ds_x \\ &\quad + \epsilon\left(\int_{\sigma_0} \left[\frac{\partial\Phi(x+\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x+\epsilon n(x), y-\epsilon n(y))}{\partial n(y)}\right](\psi_1^+ - \kappa\psi_0^+)(x)\, ds_x \right. \\ &\quad \left. + \int_{\sigma_0} \left[\frac{\partial\Phi(x-\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x-\epsilon n(x), y-\epsilon n(y))}{\partial n(y)}\right](\psi_1^- - \kappa\psi_0^+)(x)\, ds_x \right) \ . \end{aligned} \tag{35}$$

From Lemma 3.1 it follows immediately that

$$\int_{\sigma_0} \left[\frac{\partial\Phi(x+\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x+\epsilon n(x), y-\epsilon n(y))}{\partial n(y)} \right] (\psi_1^+ - \kappa\psi_0^+)(x)\, ds_x$$
$$= -\frac{1}{2}(\psi_1^+ - \kappa\psi_0^+)(y) + o(1) \quad ,$$

and

$$\int_{\sigma_0} \left[\frac{\partial\Phi(x-\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x-\epsilon n(x), y-\epsilon n(y))}{\partial n(y)} \right] (\psi_1^- - \kappa\psi_0^+)(x)\, ds_x$$
$$= -\frac{1}{2}(\psi_1^- - \kappa\psi_0^+)(y) + o(1) \quad .$$

A combination of these two identities and (35) gives

$$\begin{aligned} r(y) \;=\; & \int_{\sigma_0} \Bigg(\left[\frac{\partial\Phi(x+\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x+\epsilon n(x), y-\epsilon n(y))}{\partial n(y)} \right] \\ & - \left[\frac{\partial\Phi(x-\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x-\epsilon n(x), y-\epsilon n(y))}{\partial n(y)} \right] \Bigg) \psi_0^+(x)\, ds_x \\ & -\frac{1}{2}\epsilon(\psi_1^+ + \psi_1^-)(y) + \epsilon\kappa(y)\psi_0^+(y) + o(\epsilon) \quad , \quad y \in \sigma_0 \quad . \end{aligned} \tag{36}$$

The following lemma characterizes the asymptotic behaviour of the remaining integral in the above expression for r.

Lemma 3.2
Let $\psi(x)$ be a fixed, smooth density on σ_0, and let $y \in \sigma_0$. Then, as $\epsilon \to 0$,

$$\begin{aligned} & \int_{\sigma_0} \Bigg(\left[\frac{\partial\Phi(x+\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x+\epsilon n(x), y-\epsilon n(y))}{\partial n(y)} \right] \\ & \quad - \left[\frac{\partial\Phi(x-\epsilon n(x), y+\epsilon n(y))}{\partial n(y)} - \frac{\partial\Phi(x-\epsilon n(x), y-\epsilon n(y))}{\partial n(y)} \right] \Bigg) \psi(x)\, ds_x \\ & \qquad = \epsilon\kappa(y)\psi(y) + o(\epsilon) \quad . \end{aligned} \tag{37}$$

Proof : By reordering the terms inside the parantheses, we may rewrite the left hand side of the identity as

$$\begin{aligned} E \;=\; & \int_{\sigma_0} \int_{-\epsilon}^{\epsilon} \frac{\partial^2\Phi}{\partial n(y)\partial n(x)}(x+\eta n(x), y+\epsilon n(y))\, d\eta\, \psi(x)\, ds_x \\ & - \int_{\sigma_0} \int_{-\epsilon}^{\epsilon} \frac{\partial^2\Phi}{\partial n(y)\partial n(x)}(x+\eta n(x), y-\epsilon n(y))\, d\eta\, \psi(x)\, ds_x \quad . \end{aligned}$$

A change of the order of integration and addition and subtraction of terms yields

$$E \;=\; \int_{-\epsilon}^{\epsilon} \int_{\sigma_0} \frac{\partial^2\Phi}{\partial n(y)\partial n(x)}(x+\eta n(x), y+\epsilon n(y))\psi(x)(1-\eta\kappa(x))\, ds_x\, d\eta$$

$$
\begin{aligned}
&- \int_{-\epsilon}^{\epsilon} \int_{\sigma_0} \frac{\partial^2 \Phi}{\partial n(y) \partial n(x)} (x + \eta n(x), y - \epsilon n(y)) \psi(x) (1 - \eta \kappa(x)) \, ds_x \, d\eta \\
&+ \int_{-\epsilon}^{\epsilon} \eta \int_{\sigma_0} \frac{\partial^2 \Phi}{\partial n(y) \partial n(x)} (x + \eta n(x), y + \epsilon n(y)) \psi(x) \kappa(x) \, ds_x \, d\eta \\
&- \int_{-\epsilon}^{\epsilon} \eta \int_{\sigma_0} \frac{\partial^2 \Phi}{\partial n(y) \partial n(x)} (x + \eta n(x), y - \epsilon n(y)) \psi(x) \kappa(x) \, ds_x \, d\eta \quad . \qquad (38)
\end{aligned}
$$

Let us consider the expressions

$$
I_\eta^\pm = \int_{\sigma_0} \frac{\partial^2 \Phi}{\partial n(y) \partial n(x)} (x + \eta n(x), y \pm \epsilon n(y)) \psi(x) (1 - \eta \kappa(x)) \, ds_x \quad , \quad -\epsilon < \eta < \epsilon \quad .
$$

We are interested in calculating the combined integral $\int_{-\epsilon}^{\epsilon} (I_\eta^+ - I_\eta^-) \, d\eta$, which accounts for the first two integrals in the right hand side of (38). Since $(1 + O(\eta^2)) d\sigma = (1 - \eta\kappa(x)) ds$ on σ_η, we obtain, by introducing $\psi^\eta(x + \eta n(x)) = \psi(x)$,

$$
I_\eta^\pm = \int_{\sigma_\eta} \frac{\partial^2 \Phi}{\partial n(y) \partial n(x)} (x, y \pm \epsilon n(y)) \psi^\eta(x) (1 + O(\eta^2)) \, d\sigma_x \quad .
$$

From the continuity of the normal derivative of a double layer potential, and its continuity with respect to the underlying curve (see Theorem 2.23 of [4]) it follows quite easily that

$$
I_{\epsilon\xi}^+ - I_{\epsilon\xi}^- \to 0 \quad ,
$$

as $\epsilon \to 0$, for any fixed $-1 < \xi < 1$, and that

$$
|I_{\epsilon\xi}^+ - I_{\epsilon\xi}^-| \leq C \quad ,
$$

with C independent of ϵ and ξ. Therefore, by Lebesque's Dominated Convergence Theorem,

$$
\int_{-\epsilon}^{\epsilon} (I_\eta^+ - I_\eta^-) \, d\eta = \epsilon \int_{-1}^{1} (I_{\epsilon\xi}^+ - I_{\epsilon\xi}^-) \, d\xi = o(\epsilon) \quad . \qquad (39)
$$

Insertion of (39) into (38) gives

$$
\begin{aligned}
E &= \int_{-\epsilon}^{\epsilon} \eta \int_{\sigma_0} \frac{\partial^2 \Phi}{\partial n(y) \partial n(x)} (x + \eta n(x), y + \epsilon n(y)) \psi(x) \kappa(x) \, ds_x \, d\eta \\
&\quad - \int_{-\epsilon}^{\epsilon} \eta \int_{\sigma_0} \frac{\partial^2 \Phi}{\partial n(y) \partial n(x)} (x + \eta n(x), y - \epsilon n(y)) \psi(x) \kappa(x) \, ds_x \, d\eta \; + o(\epsilon) \\
&= \int_{-\epsilon}^{\epsilon} \eta \frac{d}{d\eta} \Bigg[\int_{\sigma_0} \Bigg(\frac{\partial \Phi}{\partial n(y)} (x + \eta n(x), y + \epsilon n(y)) \\
&\qquad - \frac{\partial \Phi}{\partial n(y)} (x + \eta n(x), y - \epsilon n(y)) \Bigg) \psi(x) \kappa(x) \, ds_x \Bigg] d\eta \; + o(\epsilon) \quad ,
\end{aligned}
$$

so that, after integration by parts,

$$
E = - \int_{-\epsilon}^{\epsilon} \int_{\sigma_0} \left(\frac{\partial \Phi}{\partial n(y)} (x + \eta n(x), y + \epsilon n(y)) - \frac{\partial \Phi}{\partial n(y)} (x + \eta n(x), y - \epsilon n(y)) \right) \psi(x) \kappa(x) \, ds_x \, d\eta
$$

$$+\epsilon\left[\int_{\sigma_0}\left(\frac{\partial\Phi}{\partial n(y)}(x+\epsilon n(x),y+\epsilon n(y))-\frac{\partial\Phi}{\partial n(y)}(x+\epsilon n(x),y-\epsilon n(y))\right)\psi(x)\kappa(x)\,ds_x\right]$$
$$+\epsilon\left[\int_{\sigma_0}\left(\frac{\partial\Phi}{\partial n(y)}(x-\epsilon n(x),y+\epsilon n(y))-\frac{\partial\Phi}{\partial n(y)}(x-\epsilon n(x),y-\epsilon n(y))\right)\psi(x)\kappa(x)\,ds_x\right]$$
$$+o(\epsilon)\quad .$$

If we apply Lemma 3.1 to (each of the terms) in this last expression for E, we get the desired identity

$$E=\int_{-\epsilon}^{\epsilon}(\frac{1}{2}\psi(y)\kappa(y)+\frac{1}{2}\psi(y)\kappa(y))\,d\eta-\epsilon\frac{1}{2}\psi(y)\kappa(y)-\epsilon\frac{1}{2}\psi(y)\kappa(y)+o(\epsilon)=\epsilon\psi(y)\kappa(y)+o(\epsilon)\quad .$$

For the first term we have actually applied a slight variation of Lemma 3.1, noting that $y+\epsilon n(y)$ is "on the positive" side of the curve σ_η and that $y-\epsilon n(y)$ is on the "negative" side of σ_η, $-\epsilon<\eta<\epsilon$ (and used Lebesque's Dominated Convergence Theorem, as before). □

Remark

Note that the remainder term is $o(\epsilon)$, uniformly in y. For this to hold it is essential that σ_0 be a closed curve. For open curves the identity (37) degenerates as y approaches the endpoints (the "limiting" remainder itself becomes of order ϵ).

We return to the derivation of the formula for $\psi_1^+ + \psi_1^-$. Insertion of (37), with $\psi=\psi_0^+$, into (36) gives

$$r(y)=-\frac{1}{2}\epsilon(\psi_1^++\psi_1^-)(y)+2\epsilon\kappa(y)\psi_0^+(y)+o(\epsilon)\quad ,$$

or

$$R(y)=-\frac{1}{2}(\psi_1^++\psi_1^-)(y)+2\kappa(y)\psi_0^+(y)\quad .$$

From (34) we now get

$$\frac{k+1}{2k}(\psi_1^++\psi_1^-)(y)+2(1-\frac{1}{k})\int_{\partial\Omega}<D_y^2\Phi(x,y)n(y),n(y)>\psi_0(x)\,d\sigma_x$$
$$-\frac{k-1}{2k}(\psi_1^++\psi_1^-)(y)+2(1-\frac{1}{k})\kappa(y)\psi_0^+(y)=0\quad , \tag{40}$$

or, by rearrangement,

$$(\psi_1^++\psi_1^-)(y)=-2(k-1)\int_{\partial\Omega}<D_y^2\Phi(x,y)n(y),n(y)>\psi_0(x)\,d\sigma_x-2(k-1)\kappa(y)\psi_0^+(y)\,,\quad y\in\sigma_0\ .$$

In terms of the background potential U_0, this may, due to (31), be rewritten

$$(\psi_1^++\psi_1^-)(y)=-2(k-1)<D^2U_0(y)n(y),n(y)>+2\frac{(k-1)^2}{k}\kappa(y)\frac{\partial U_0}{\partial n(y)}(y)\,,\quad y\in\sigma_0\ . \tag{41}$$

From (31) and (41) we immediately conclude that

$$\begin{aligned}(\psi_1^++\psi_1^--\kappa(\psi_0^+-\psi_0^-))(y)&=-2(k-1)<D^2U_0(y)n(y),n(y)>+2(k-1)\kappa(y)\frac{\partial U_0}{\partial n(y)}(y)\\&=2(k-1)\left(\frac{d}{dt_y}\right)^2U_0(y)\,,\quad y\in\sigma_0\ .\end{aligned} \tag{42}$$

To justify the last identity in (42) we have used that

$$0 = \triangle U_0(y) = < D^2 U_0(y) n(y), n(y) > + < D^2 U_0(y) \tau(y), \tau(y) > \quad ,$$

and that

$$\begin{aligned}
\left(\frac{d}{dt_y}\right)^2 U_0(y) &= \frac{d}{dt_y}\left(\nabla U_0(y) \cdot \tau(y)\right) \\
&= < D^2 U_0(y)\tau(y), \tau(y) > + \nabla U_0(y) \cdot \frac{d}{dt_y}\tau(y) \\
&= < D^2 U_0(y)\tau(y), \tau(y) > + \frac{\partial U_0}{\partial n(y)} \frac{d}{dt_y}\tau(y) \cdot n(y) + \frac{\partial U_0}{\partial \tau(y)} \frac{d}{dt_y}\tau(y) \cdot \tau(y) \\
&= < D^2 U_0(y)\tau(y), \tau(y) > + \frac{\partial U_0}{\partial n(y)} \kappa(y) \quad .
\end{aligned}$$

We may insert (31) and (42) into the representation formula (32) for u_1 to obtain

$$\begin{aligned}
u_1(y) &= \int_{\partial\Omega} \Phi(x,y)\psi_1(x)\, d\sigma_x \\
&\quad + 2(k-1) \int_{\sigma_0} \Phi(x,y) \left(\frac{d}{dt_x}\right)^2 U_0(x)\, ds_x \\
&\quad - 2(1-\frac{1}{k}) \int_{\sigma_0} \frac{\partial \Phi}{\partial n(x)}(x,y) \frac{\partial U_0}{\partial n(x)}(x)\, ds_x \\
&= \int_{\partial\Omega} \Phi(x,y)\psi_1(x)\, d\sigma_x \\
&\quad - 2(k-1) \int_{\sigma_0} \frac{\partial \Phi(x,y)}{\partial \tau(x)} \frac{\partial U_0}{\partial \tau(x)}(x)\, ds_x \\
&\quad - 2(1-\frac{1}{k}) \int_{\sigma_0} \frac{\partial \Phi}{\partial n(x)}(x,y) \frac{\partial U_0}{\partial n(x)}(x)\, ds_x \quad , \quad y \in \Omega \setminus \sigma_0 \quad . \qquad (43)
\end{aligned}$$

The density ψ_1 is uniquely (modulo a single parameter) determined by the equation

$$\begin{aligned}
&\frac{1}{2}\psi_1(y) + \int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu(y)}(x,y)\psi_1(x)\, d\sigma_x \\
&= 2(k-1) \int_{\sigma_0} \frac{\partial}{\partial \tau(x)} \frac{\partial \Phi}{\partial \nu(y)}(x,y) \frac{\partial U_0}{\partial \tau(x)}(x)\, ds_x \\
&\quad + 2(1-\frac{1}{k}) \int_{\sigma_0} \frac{\partial}{\partial n(x)} \frac{\partial \Phi}{\partial \nu(y)}(x,y) \frac{\partial U_0}{\partial n(x)}(x)\, ds_x \quad , \quad y \in \partial\Omega \quad , \qquad (44)
\end{aligned}$$

which expresses that $\frac{\partial u_1}{\partial \nu} = 0$ on $\partial\Omega$. We note that this equation has a solution, due to the fact that the integral of the right hand side over $\partial\Omega$ vanishes.

Remark

Based on (43) and (44) it follows that u_1 is a solution to

$$\triangle u_1 = 0 \quad \text{in } \Omega \setminus \sigma_0$$

$$\begin{aligned} \frac{\partial u_1}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \\ [u_1]_{\sigma_0} &= -2(1-\frac{1}{k})\frac{\partial U_0}{\partial n} \\ [\frac{\partial u_1}{\partial n}]_{\sigma_0} &= -2(k-1)\left(\frac{d}{dt_x}\right)^2 U_0 \quad . \end{aligned}$$

Here the jump $[u_1]_{\sigma_0}$ is defined by $[u_1]_{\sigma_0} = u_1^+ - u_1^-$, where u_1^+ denotes the limiting value of u_1 approaching σ_0 from the side which n points into, and u_1^- denotes the limiting value of u_1 approaching σ_0 from the side which $-n$ points into. A similar convention applies to $[\frac{\partial u_1}{\partial n}]_{\sigma_0}$. The normalization $\int_{\partial\Omega} u_1\, d\sigma = 0$ uniquely determines u_1 (as well as ψ_1).

The preceeding derivation of the formula (43) was performed under the condition that the curve σ_0 was closed and that the domain ω_ϵ had a uniform thickness (of 2ϵ). In the following section we shall relax these conditions, and find the corresponding (very similar) formula for u_1.

4 The general, two term asymptotic formula

We shall now derive formuli for the term u_1 (and u_0) in the case when the imperfections take the form of a single thin inclusion

$$\omega_\epsilon^h = \{x + \eta h(x) n(x) : \ x \in \sigma_0 \ , \ \eta \in (-\epsilon, \epsilon)\} \quad ,$$

and σ_0 is a smooth, **open or closed**, simple curve inside Ω. The unit vectorfield $n(\cdot)$ is smooth and normal to σ_0, and the function $h(\cdot)$ is nonnegative, but **not necessarily constant**. The derivation of the formuli for u_1 (and u_0) could be carried out along the same lines as those followed in the previous section (where we required σ_0 to be closed, and $h \equiv 1$). The main added difficulty is that the the vectorfield $n(x)$ is no longer normal to σ_η at the point $x + \eta h(x) n(x)$. However, instead of presenting the technical details of this derivation, we shall arrive at these formuli (intuitively) by a "mapping" technique.

Let Ψ be a conformal mapping, defined on a set compactly containing Ω, and with $\Psi(\Omega) = \tilde{\Omega}$. Modulo a set of measure $O(\epsilon^2)$ this mapping will map ω_ϵ^1 onto

$$\Psi(\omega_\epsilon^1) = \tilde{\omega}_\epsilon^{\tilde{h}} = \{\tilde{x} + \eta \tilde{h}(\tilde{x}) \tilde{n}(\tilde{x}) : \ \tilde{x} \in \tilde{\sigma}_0 \ , \ \eta \in (-\epsilon, \epsilon)\} \quad ,$$

where $\tilde{\sigma}_0 = \Psi(\sigma_0)$, $h(\tilde{x}) = |det(D\Psi(x))|^{1/2}$, and $\tilde{n}(\tilde{x}) = D\Psi(x) n(x) / \|D\Psi(x) n(x)\|$, $\tilde{x} \in \tilde{\sigma}_0$, $x = \Psi^{-1}(\tilde{x})$. Due to the conformal nature of Ψ, the function $\tilde{u}_\epsilon = u_\epsilon \circ \Psi^{-1}$ satisfies a homogeneous conductivity problem with conductivity 1 in $\tilde{\Omega} \setminus \Psi(\omega_\epsilon^1)$, conductivity k in $\Psi(\omega_\epsilon^1)$, and boundary current $\tilde{g}(\tilde{x}) = |det(D\Psi(x))|^{-1/2} g(x)$. The formal expansion

$$\tilde{u}_\epsilon = \tilde{u}_0 + \epsilon \tilde{u}_1 + \dots \quad ,$$

is simply obtained from that for u_ϵ by composition with Ψ^{-1}, *i.e.*,

$$\tilde{u}_0 = u_0 \circ \Psi^{-1} \ , \quad \tilde{u}_1 = u_1 \circ \Psi^{-1} \ , \quad \dots$$

The first term $\tilde{u}_0$ thus solves

$$\triangle \tilde{u}_0 = 0 \quad \text{in } \tilde{\Omega} \ , \quad \frac{\partial \tilde{u}_0}{\partial \nu} = \tilde{g} \quad \text{on } \partial\tilde{\Omega} \ .$$

The second term $\tilde{u}_1$ has the form

$$\begin{aligned}
\tilde{u}_1(\tilde{y}) &= \int_{\partial\Omega} \Phi(x,y)\psi_1(x)\, d\sigma_x \\
&\quad -2(k-1)\int_{\sigma_0} \frac{\partial \Phi}{\partial \tau(x)}(x,y)\frac{\partial u_0}{\partial \tau(x)}(x)\, ds_x \\
&\quad -2(1-\frac{1}{k})\int_{\sigma_0} \frac{\partial \Phi}{\partial n(x)}(x,y)\frac{\partial u_0}{\partial n(x)}(x)\, ds_x \\
&= \int_{\partial\tilde{\Omega}} \Phi(\Psi^{-1}(\tilde{x}),y)\psi_1(\Psi^{-1}(\tilde{x}))|det(D\Psi(\Psi^{-1}(\tilde{x})))|^{-1/2}\, d\sigma_{\tilde{x}} \\
&\quad -2(k-1)\int_{\tilde{\sigma}_0} \frac{\partial \Phi}{\partial \tilde{\tau}(\tilde{x})}(\Psi^{-1}(\tilde{x}),y)\frac{\partial \tilde{u}_0}{\partial \tilde{\tau}(\tilde{x})}(\tilde{x})\, |det(D\Psi(\Psi^{-1}(\tilde{x})))|^{1/2}\, ds_{\tilde{x}} \\
&\quad -2(1-\frac{1}{k})\int_{\sigma_0} \frac{\partial \Phi}{\partial \tilde{n}(\tilde{x})}(\Psi^{-1}(\tilde{x}),y)\frac{\partial \tilde{u}_0}{\partial \tilde{n}(\tilde{x})}(\tilde{x})|det(D\Psi(\Psi^{-1}(\tilde{x})))|^{1/2}\, ds_{\tilde{x}} \ .
\end{aligned}$$

If we introduce the notation

$$\tilde{\Phi}(\tilde{x},\tilde{y}) = \Phi(\Psi^{-1}(\tilde{x}),\Psi^{-1}(\tilde{y})) = \Phi(\Psi^{-1}(\tilde{x}),y) = \Phi(x,y) \ ,$$

and

$$\tilde{\psi_1}(\tilde{x}) = \psi_1(\Psi^{-1}(\tilde{x}))|det(D\Psi(\Psi^{-1}(\tilde{x})))|^{-1/2} \ ,$$

then $\tilde{u}_1$ may be represented as

$$\begin{aligned}
\tilde{u}_1(\tilde{y}) &= \int_{\partial\tilde{\Omega}} \tilde{\Phi}(\tilde{x},\tilde{y})\tilde{\psi_1}(\tilde{x})\, d\sigma_{\tilde{x}} \\
&\quad -2(k-1)\int_{\tilde{\sigma}_0} \frac{\partial \tilde{\Phi}}{\partial \tilde{\tau}(\tilde{x})}(\tilde{x},\tilde{y})\frac{\partial \tilde{u}_0}{\partial \tilde{\tau}(\tilde{x})}(\tilde{x})\tilde{h}(\tilde{x})\, ds_{\tilde{x}} \\
&\quad -2(1-\frac{1}{k})\int_{\tilde{\sigma}_0} \frac{\partial \tilde{\Phi}}{\partial \tilde{n}(\tilde{x})}(\tilde{x},\tilde{y})\frac{\partial \tilde{u}_0}{\partial \tilde{n}(\tilde{x})}(\tilde{x})\tilde{h}(\tilde{x})\, ds_{\tilde{x}} \ . \qquad (45)
\end{aligned}$$

We observe that the function $\tilde{\Phi}(\tilde{x},\tilde{y}) = \tilde{\Phi}(\tilde{y},\tilde{x})$ is a solution to $\triangle_{\tilde{x}}\tilde{\Phi}(\tilde{x},\tilde{y}) = \delta_{\tilde{y}}$, *i.e.*, it is a fundamental solution for the Laplacian. Our choice of Φ at the beginning of the last section is convenient, but by no means the unique possibility; we could have chosen $\Phi'(x,y) = \Phi(\Psi(x),\Psi(y))$, and in that case we would obtain $\tilde{\Phi}'(\tilde{x},\tilde{y}) = \Phi(\tilde{x},\tilde{y}) = -\frac{1}{2\pi}\log|\tilde{x}-\tilde{y}|$, so that the corresponding representation for $\tilde{u}_1$ becomes

$$\begin{aligned}
\tilde{u}_1(\tilde{y}) &= \int_{\partial\tilde{\Omega}} \Phi(\tilde{x},\tilde{y})\tilde{\psi_1}(\tilde{x})\, d\sigma_{\tilde{x}} \\
&\quad -2(k-1)\int_{\tilde{\sigma}_0} \frac{\partial \Phi}{\partial \tilde{\tau}(\tilde{x})}(\tilde{x},\tilde{y})\frac{\partial \tilde{u}_0}{\partial \tilde{\tau}(\tilde{x})}(\tilde{x})\tilde{h}(\tilde{x})\, ds_{\tilde{x}} \\
&\quad -2(1-\frac{1}{k})\int_{\tilde{\sigma}_0} \frac{\partial \Phi}{\partial \tilde{n}(\tilde{x})}(\tilde{x},\tilde{y})\frac{\partial \tilde{u}_0}{\partial \tilde{n}(\tilde{x})}(\tilde{x})\tilde{h}(\tilde{x})\, ds_{\tilde{x}} \ .
\end{aligned}$$

Since $\Psi(\omega_\epsilon^1)$ and $\tilde{\omega}_\epsilon^{\tilde{h}}$ differ by a set of measure $O(\epsilon^2)$, it is reasonable to suppose that $\tilde{u}_0$ and $\epsilon\tilde{u}_1$ also represent the two first terms in an asymptotic expansion of the voltage potential corresponding to the imperfection $\tilde{\omega}_\epsilon^{\tilde{h}}$. Dropping the tildes on all variables we are thus led to the following formula for the second term in the asymptotic expansion for u_ϵ in the case of a single thin, "closed" imperfection

$$\omega_\epsilon^h = \{x + \eta h(x) n(x) : \ x \in \sigma_0 \ , \ \eta \in (-\epsilon, \epsilon)\} \quad .$$

$$\begin{aligned} u_1(y) \ = \ & \int_{\partial\Omega} \Phi(x,y)\psi_1(x)\, d\sigma_x \\ & -2(k-1)\int_{\sigma_0} \frac{\partial\Phi}{\partial\tau(x)}(x,y)\frac{\partial u_0}{\partial\tau(x)}(x)h(x)\, ds_x \\ & -2(1-\frac{1}{k})\int_{\sigma_0} \frac{\partial\Phi}{\partial n(x)}(x,y)\frac{\partial u_0}{\partial n(x)}(x)h(x)\, ds_x \quad , \end{aligned} \tag{46}$$

ψ_1 being the solution to

$$\begin{aligned} & \frac{1}{2}\psi_1(y) + \int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu(y)}(x,y)\psi_1(x)\, d\sigma_x \\ & = 2(k-1)\int_{\sigma_0} \frac{\partial}{\partial\tau(x)}\frac{\partial\Phi}{\partial\nu(y)}(x,y)\frac{\partial U_0}{\partial\tau(x)}(x)h(x)\, ds_x \\ & \quad +2(1-\frac{1}{k})\int_{\sigma_0} \frac{\partial}{\partial n(x)}\frac{\partial\Phi}{\partial\nu(y)}(x,y)\frac{\partial U_0}{\partial n(x)}(x)h(x)\, ds_x \quad , \quad y \in \partial\Omega \quad . \end{aligned} \tag{47}$$

The first term, u_0, equals U_0, the background voltage potential, and u_ϵ is formally given by

$$u_\epsilon = u_0 + \epsilon u_1 + o(\epsilon) = U_0 + \epsilon u_1 + o(\epsilon) \quad .$$

We note that the above formula has only been intuitively justified for thickness distributions, h, that may be reached from a uniform thickness distribution by a conformal mapping. As indicated earlier, the exact same formula may be derived along the lines of the derivation in the previous section, and this provides a formal justification for more general h. We also note that by permitting the thickness distribution, h, to vanish on part of σ_0 we formally obtain a representation formula corresponding to an open curve as well (namely that part of the curve, where h does not vanish). It is possibly to write (46) in a somewhat more compact form. For that purpose we introduce the symmetric-matrix valued function $A(x)$ as follows

$$A(x) \text{ has eigenvectors } \tau(x) \text{ and } n(x) \quad ,$$

$$\text{the eigenvalue corresponding to } \tau(x) \text{ is } 2(k-1) \quad ,$$

and

$$\text{the eigenvalue corresponding to } n(x) \text{ is } 2(1-\frac{1}{k}) \quad .$$

With this notation we may write

$$u_1(y) = V_1(y) - \int_{\sigma_0} \nabla_x\Phi(x,y) \cdot A(x)\nabla U_0(x) h(x)\, ds_x \quad , \tag{48}$$

where V_1 is harmonic, and additionally satisties the boundary condition

$$\frac{\partial V_1}{\partial \nu}(y) = \frac{\partial}{\partial \nu} \int_{\sigma_0} \nabla_x \Phi(x,y) \cdot A(x) \nabla U_0(x) h(x)\, ds_x \quad .$$

In terms of $u_\epsilon - U_0$ we now have the formula

$$u_\epsilon(y) - U_0(y) = \epsilon V_1(y) - \epsilon \int_{\sigma_0} \nabla_x \Phi(x,y) \cdot A(x) \nabla U_0(x) h(x)\, ds_x \ + o(\epsilon) \quad . \tag{49}$$

Let $\Phi_0(x,y)$ denote the Neumann-function for Ω, *i.e.*, the special fundamental solution that is determined by

$$-\triangle_x \Phi_0(x,y) = \delta_y \ \text{ in } \Omega \ , \ \frac{\partial \Phi_0}{\partial \nu(x)}(x,y) = -\frac{1}{|\partial\Omega|} \ \text{ on } \partial\Omega \quad .$$

It is easy to see that $\Phi_0(x,y) = \Phi_0(y,x)$ so in particular $\frac{\partial}{\partial \nu(y)}\Phi_0(x,y) = \frac{\partial}{\partial \nu(y)}\Phi_0(y,x) = -\frac{1}{|\partial\Omega|}$ for $x \in \Omega$, $y \in \partial\Omega$. It follows immediately that V_1 has the form

$$V_1(y) = \int_{\sigma_0} \nabla_x (\Phi(x,y) - \Phi_0(x,y)) \cdot A(x) \nabla U_0(x) h(x)\, ds_x \quad , \tag{50}$$

and as a consequence

$$u_\epsilon(y) - U_0(y) = -\epsilon \int_{\sigma_0} \nabla_x \Phi_0(x,y) \cdot A(x) \nabla U_0(x) h(x)\, ds_x \ + o(\epsilon) \quad ,$$

or

$$u_\epsilon(y) = U_0(y) - \epsilon \int_{\sigma_0} \nabla_x \Phi_0(x,y) \cdot A(x) \nabla U_0(x) h(x)\, ds_x \ + o(\epsilon) \quad , \quad y \in \bar{\Omega} \setminus \sigma_0 \quad . \tag{51}$$

This is indeed the first of the asymptotic formulas for u_ϵ, that we announced in the introduction to this paper (the generalization to imperfections consisting of a finite number of well separated inclusions is straighforward). The function $\Phi_0(x,y)$ cannot in general be expressed in a closed form, it is therefore practically desirable to find a "representation" formula which instead involves the function $\Phi(x,y) = -\frac{1}{2\pi}\log|x-y|$. For that purpose consider

$$\begin{aligned} &\int_{\partial\Omega} (u_\epsilon(y) - U_0(y)) \frac{\partial \Phi}{\partial \nu(y)}(z,y)\, d\sigma_y \\ &= -\epsilon \int_{\partial\sigma_0} \nabla_x \left(\int_{\partial\Omega} \Phi_0(x,y) \frac{\partial \Phi}{\partial \nu(y)}(z,y)\, d\sigma_y \right) \cdot A(x) \nabla U_0(x) h(x)\, ds_x \ + o(\epsilon) \quad , \end{aligned} \tag{52}$$

$z \in \Omega \setminus \sigma_0$. By integration by parts

$$\begin{aligned} \int_{\partial\Omega} \Phi_0(x,y) \frac{\partial \Phi}{\partial \nu(y)}(z,y)\, d\sigma_y &= \int_\Omega \nabla_y \Phi_0(x,y) \cdot \nabla_y \Phi(z,y)\, dy - \Phi_0(x,z) \\ &= \int_{\partial\Omega} \frac{\partial \Phi_0}{\partial \nu(y)}(x,y) \Phi(z,y)\, d\sigma_y + \Phi(z,x) - \Phi_0(x,z) \\ &= -\frac{1}{|\partial\Omega|} \int_{\partial\Omega} \Phi(z,y)\, d\sigma_y + \Phi(x,z) - \Phi_0(x,z) \quad , \end{aligned} \tag{53}$$

for $x, z \in \Omega$, $x \neq z$. Insertion of this formula into (52), and use of (50), yields

$$\begin{aligned}
&\int_{\partial\Omega}(u_\epsilon(y)-U_0(y))\frac{\partial\Phi}{\partial\nu(y)}(z,y)\,d\sigma_y \\
&\quad = -\epsilon\int_{\sigma_0}\nabla_x\left(-\frac{1}{|\partial\Omega|}\int_{\partial\Omega}\Phi(z,y)\,d\sigma_y+\Phi(x,z)-\Phi_0(x,z)\right)\cdot A(x)\nabla U_0(x)h(x)\,ds_x\ +o(\epsilon) \\
&\quad = -\epsilon\int_{\sigma_0}\nabla_x\left(\Phi(x,z)-\Phi_0(x,z)\right)\cdot A(x)\nabla U_0(x)h(x)\,ds_x\ +o(\epsilon) \\
&\quad = -\epsilon V_1(z)+o(\epsilon)\quad,\quad z\in\Omega\setminus\sigma_0\quad. \qquad (54)
\end{aligned}$$

From (49) and the fact that $\frac{\partial\Phi}{\partial\nu(y)}(z,y)=\frac{\partial\Phi}{\partial\nu(y)}(y,z)$, we thus get

$$\begin{aligned}
&u_\epsilon(y)-U_0(y)+\int_{\partial\Omega}(u_\epsilon(x)-U_0(x))\frac{\partial\Phi}{\partial\nu(x)}(x,y)\,d\sigma_x \\
&\qquad = -\epsilon\int_{\sigma_0}\nabla_x\Phi(x,y)\cdot A(x)\nabla U_0(x)h(x)\,ds_x\ +o(\epsilon)\quad,\quad y\in\Omega\setminus\sigma_0\quad.
\end{aligned}$$

In the limit as y converges to a location on the boundary, $\partial\Omega$, the double layer potential on the left hand side of the above identity converges to

$$-\frac{1}{2}(u_\epsilon(y)-U_0(y))+\int_{\partial\Omega}(u_\epsilon(x)-U_0(x))\frac{\partial\Phi}{\partial\nu(x)}(x,y)\,d\sigma_x\quad,$$

and so we finally obtain

$$\begin{aligned}
&u_\epsilon(y)-U_0(y)+2\int_{\partial\Omega}(u_\epsilon(x)-U_0(x))\frac{\partial\Phi}{\partial\nu(x)}(x,y)\,d\sigma_x \\
&\qquad = -2\epsilon\int_{\sigma_0}\nabla_x\Phi(x,y)\cdot A(x)\nabla U_0(x)h(x)\,ds_x\ +o(\epsilon)\quad,\quad y\in\partial\Omega\quad. \qquad (55)
\end{aligned}$$

This is indeed (an example of) the second type of asymptotic formuli for u_ϵ, that we announced in the introduction to this paper (the generalization to imperfections consisting of a finite number of well separated inclusions is straighforward). We note that if $H(x,y)=H(y,x)$, and H is harmonic (in both variables) in a set Ω', which compactly contains Ω, then it follows, along the same lines as (52)–(54), that

$$\begin{aligned}
&\int_{\partial\Omega}(u_\epsilon(x)-U_0(x))\frac{\partial H}{\partial\nu(x)}(x,y)\,d\sigma_x \\
&\qquad = -\epsilon\int_{\sigma_0}\nabla_x H(x,y)\cdot A(x)\nabla U_0(x)h(x)\,ds_x\ +o(\epsilon)\quad,\quad y\in\overline{\Omega}\setminus\sigma_0\quad.
\end{aligned}$$

This provides a very simple explanation, why we may use any fundamental solution in (55) or (4), and not just $\Phi(x,y)=-\frac{1}{2\pi}\log|x-y|$.

Remark

The formula (55) may very naturally be seen as the appropriate extension of the representation formula derived in [3]. In that paper we considered imperfections ω_ϵ of the form

$$\omega_\epsilon=\cup_{i=1}^N(\bar{x}_i+\epsilon B_i)$$

and the formula we obtained was

$$u_\epsilon(y) - U_0(y) + 2\int_{\partial\Omega}(u_\epsilon(x) - U_0(x))\frac{\partial\Phi}{\partial\nu(x)}(x,y)\,d\sigma_x$$
$$= -2\epsilon^2\sum_{i=1}^{N}\nabla_x\Phi(\bar{x}_i,y)\cdot A_i\nabla U_0(\bar{x}_i) + o(\epsilon^2) \quad , \quad y\in\partial\Omega \quad , \tag{56}$$

where the A_i are appropriately calculated polarization tensors, and everything else is as in this paper. The symmetric (positive- or negative-) definite matrices A_i are different from those appearing in (55), but as far as the structure of the representation is concerned, (56) may simply be viewed as a discrete analog of (55). This analogy applies even to the factor ϵ^2, which may be thought of as an ϵ (representing the thickness) multiplied by another ϵ (representing the element of integration along σ_0).

5 Computational experiments with the asymptotic formuli

In this section we present numerical experiments that attempt to quantify the **representation error**

$$RE_\epsilon(x) = (u_\epsilon - U_0 - \epsilon u_1)(x) \quad \text{for } x\in\partial\Omega, \tag{57}$$

between u_ϵ and our two term asymptotic formula. Here u_1 is as defined by (46) and (47), U_0 is the background potential, and u_ϵ is the steady state voltage potential in the presence of the conductivity imperfections. Note that RE_ϵ is exactly equal to the $o(\epsilon)$ term appearing in (3) (for points on $\partial\Omega$); RE_ϵ is also up to multiplicative constants equivalent to the $o(\epsilon)$ term appearing in (4).

We first briefly describe our computational scheme for calculating (approximations to) u_ϵ (and u_1). Using Green's representation formula for $x\in\Omega\setminus\omega_\epsilon$ and passing to the limit as x approaches the boundaries $\partial\Omega$ and $\partial\omega_\epsilon$, $u_\epsilon(x)$ may be characterized as the solution to the integral equation

$$-\pi u_\epsilon(x) + C_1\int_{\partial\Omega}u_\epsilon(y)\frac{\partial\ln|x-y|}{\partial\nu(y)}d\sigma_y \quad + \quad C_2\int_{\partial\omega_\epsilon}u_\epsilon(y)\frac{\partial\ln|x-y|}{\partial\nu(y)}(x,y)ds_y$$
$$= C_3\int_{\partial\Omega}g(y)\ln|x-y|d\sigma_y \quad x\in\partial\Omega\cup\partial\omega_\epsilon. \tag{58}$$

Here we have assumed that all the imperfections have the same conductivity, k; the constants are given by $C_1 = C_3 = 1, C_2 = k-1$ when $x\in\partial\Omega$, and by $C_1 = C_3 = 1/(k+1), C_2 = (k-1)/(k+1)$ when $x\in\partial\omega_\epsilon$ (provided these boundaries are C^1 at x). In each of our numerical experiments we take only a single imperfection. To discretize (58) we use the socalled Nyström method. Let the outer boundary $\partial\Omega$ be parametrized using $r_{out}(t) = (\psi_{out}(t),\eta_{out}(t))$ for $0\le t\le 2\pi$. For our numerical experiments we consider imperfections in the form of (a) straight or curved rectangles (these correspond to σ_0 being a straight line segment or a circular arc, and $h(x)\equiv 1$), (b) "smooth" rectangles (these are straight rectangles with the two "small" sides replaced by semi-circular arcs), and (c) ellipses (these correspond to σ_0 being a straight line segment parametrized by $t\in(-1,1)$ and $h(x(t)) = \sqrt{1-t^2}$). For an ellipse, the inner boundary $\partial\omega_\epsilon$

is parametrized in a manner similar to $\partial\Omega$; in the other cases, the four line-segments (or arcs) comprising $\partial\omega_\epsilon$ are parametrized individually. Let $r_{in}(t) = (\psi_{in}(t), \eta_{in}(t))$ for $a \le t \le b$ denote the generic parameterization of $\partial\omega_\epsilon$. Using this parameterization (58) is rewritten as

$$\begin{aligned} -\pi u_\epsilon(s) &+ C_1 \int_0^{2\pi} u_\epsilon(t) K_{out}(s,t)dt + C_2 \int_a^b u_\epsilon(t) K_{in}(s,t)dt \\ &= C_3 \int_0^{2\pi} g(t)\sqrt{\psi_{out}^2(t) + \eta_{out}^2(t)}\ln|r_{out}(s) - r_{out}(t)|dt \quad s \in [0,2\pi] \cup [a,b], \end{aligned} \tag{59}$$

where $u_\epsilon(s) = u_\epsilon(r_{out}(s))$ for $s \in [0,2\pi]$, $u_\epsilon(s) = u_\epsilon(r_{in}(s))$ for $s \in [a,b]$, $g(t) = g(r_{out}(t))$, $K_{out}(s,t) = (\partial/\partial\nu(y))\ln|r_{out}(s) - r_{out}(t)|\,|r'_{out}(t)|$, and $K_{in}(s,t)$ is defined similarly. Let $t_1, t_2, \cdots, t_n$ denote a uniform partition of $[0,2\pi]$ and $t_{n+1}, t_{n+2}, \cdots, t_{n+m}$ one for $[a,b]$. Choosing a quadrature rule (we always use the trapezoidal rule) and letting $u_\epsilon^\delta(i) = u_\epsilon(r_{out}(t_i))$ for $1 \le i \le n$ and $u_\epsilon^\delta(n+i) = u_\epsilon(r_{in}(t_{n+i}))$ for $1 \le i \le m$, we replace (59) by the linear system

$$A^\delta u_\epsilon^\delta = f^\delta \quad u_\epsilon^\delta, f^\delta \in \mathcal{R}^{n+m} \text{ and } A^\delta \in \mathcal{R}^{n+m,n+m}, \tag{60}$$

obtained by collocation at the points t_i, $1 \le i \le n+m$. The superscript δ is an indicator of the size of the partitions of $[0,2\pi]$ and $[a,b]$. Entries of the matrix A^δ are calculated by evaluating the integrals on the left side of (59), using the trapezoidal rule. The entries of f^δ (apart from the first n) are calculated in the same way, through evaluation of the integrals on the right. Since the integrals involved in the first n entries of f^δ are singular (s,t may be any of the points $t_i, 1 \le i \le n$ and the kernel is singular if $s = t$), special care is excercised when evaluating these. We split the singular kernel into a smooth and non-smooth part, and use the trapezoidal quadrature rule for the smooth part, while we use a special rule based on trigonometric approximations for the non-smooth part (for a detailed discussion of this procedure see for example, pp. 328–333 in [2]). The normalization condition $\int_{\partial\Omega} u_\epsilon(x)d\sigma_x = 0$, in conjunction with (58) ensures the uniqueness of u_ϵ. This condition is imposed at the discrete level by replacing an appropriate row of the matrix A^δ by the discrete analogue of this condition before (60) is solved. In our numerical experiments we always replace the first row of A^δ. The Nyström method has been analyzed in detail when the curves $\partial\Omega$ and $\partial\omega_\epsilon$ are C^2 (see for example chapter 7 in [2] or chapter 12 in [12]).

When the imperfections are rectangles ("smooth" or not) the domain boundary does not satisfy the C^2 smoothness condition mentioned above. When the boundary $\partial\omega_\epsilon$ has a corner of "interior" angle $\pi/2$ (as is the case when we consider "non-smooth" rectangular imperfections), the constants in the formula (58) change to $C_1 = 2/(3+k), C_2 = 2(k-1)/(3+k)$, and $C_3 = 2/(3+k)$ for each of the four equations where the corner points are chosen as collocation points (for a detailed derivation of boundary integral equations similar to (58) in the presence of corners, see pp. 384–389 in [2]). Moreover, for any of our rectangles, the fact that the boundary is not C^2 also affects the computation of the entries of the matrix A^δ (for non-corner collocation points). In particular, let $\partial\omega_\epsilon = \cup_{i=1}^4 \gamma_i$, let x be any non-corner collocation point on γ_1, and let γ_2 be the part of $\partial\omega_\epsilon$ joining γ_1 at the corner nearest to x. Rewriting the second integral on the left hand side of (58) as

$$\int_{\partial\omega_\epsilon} u_\epsilon(y)\frac{\partial\ln|x-y|}{\partial\nu(y)}(x,y)ds_y = \sum_{i=1}^4 \int_{\gamma_i} u_\epsilon(y)\frac{\partial\ln|x-y|}{\partial\nu(y)}(x,y)ds_y, \tag{61}$$

we use special quadrature for the contribution from $\int_{\gamma_2}$. For the special quadrature, we use four times the usual number of quadrature points on γ_2 and use piecewise linear interpolation to evaluate u_ϵ at these new quadrature points.

The resulting matrix A^δ is full and the linear system (60) is solved using a direct solver. To arrive at u_1, we first determine $\psi_1(x)$ using (47) and then compute u_1 via (46). The integral equations involved in these computations are discretized and solved using the same techniques as described above. For our numerical experiments we always take $g = \nu_1$ or $g = \nu_2$, corresponding to the explicit formuli $U_0(x) = x_1$ or $U_0(x) = x_2$, respectively. Finally, $RE_\epsilon(x)$ is calculated by means of the formula (57).

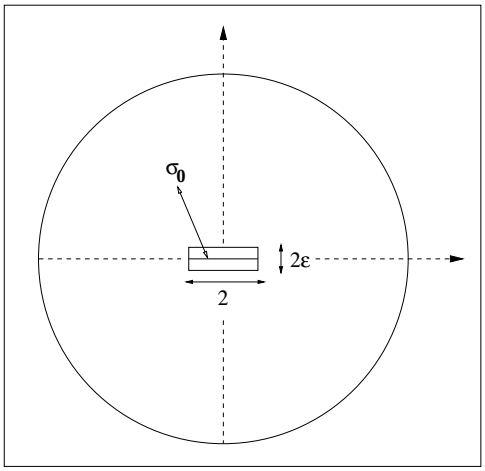

Figure 1: The typical configuration for computations involving a straight rectangle with $\sigma_0 = (-1, 1), h(x) \equiv 1$; the width of the rectangle is 2ϵ; in our actual computations the radius of the outer circle $\partial\Omega$ is 10.

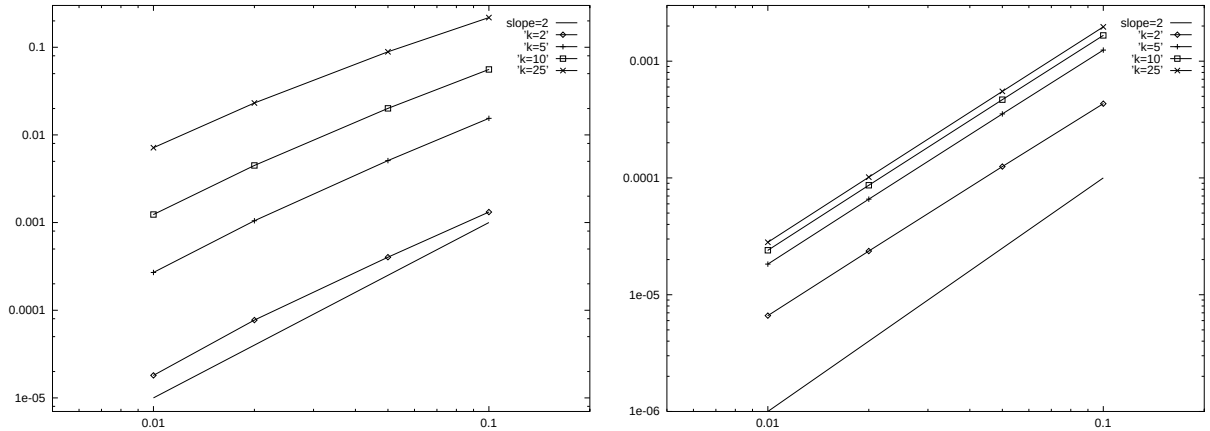

Figure 2: Approximation rates for a straight rectangular imperfection with $U_0 = x_1$ on the left and $U_0 = x_2$ on the right. $\|RE_\epsilon\|_{\ell_\infty(\partial\Omega)}$ is plotted on a log-log scale against ϵ for $0.01 \le \epsilon \le 0.1$. Different lines correspond to different conductivities in ω_ϵ ($k = 2, 5, 10, 25$). A line proportional to ϵ^2 (slope 2) is shown for easy comparison in each case.

For our computations we use the following parameters; $n = 257$ and $m = 3072$ for rectan-

gles, while $m = 1024$ for ellipses (which does represent a "smoother" situation). The region Ω is always taken to be a disk of radius 10 around the origin. For different shapes of imperfections, we show graphs of $\|RE_\epsilon\|_{\ell_\infty(\partial\Omega)} = \max_{1\le i\le n} |RE_\epsilon(t_i)|$ against ϵ on a log-log scale. We typically consider values of the thickness parameter ϵ in the range $[0.01, 0.1]$. This limitation is a consequence of the fact that all our computations are done on a workstation with limited memory. To obtain reliable numerical results for $\epsilon < 0.01$, the number of collocation points used on the boundary $\partial\omega_\epsilon$ should be increased, leading to a much larger linear system to be solved when determining u_ϵ. For the case of an elliptical imperfection, we may reliably consider values of ϵ in the range $[0.005, 0.1]$ (due to the additional "smoothness").

As a first example we take an imperfection in the form of a thin rectangle (symmetric in the origin) with length 2 and width 2ϵ as shown in figure 1. For this imperfection, we show plots for $\|RE_\epsilon\|_{\ell_\infty(\partial\Omega)}$ for different choices of k in figure 2. In all cases the approximation rate is (ultimately) of the order ϵ^2. There is, however, clearly a difference between the approximation for $U_0 = x_1$ and $U_0 = x_2$. In the latter case RE_ϵ appears to be bounded by $C\epsilon^2$, with C independent of $2 \le k$, whereas this does not appear to be true in the first case. This behaviour is to be expected since U_0^∞ is equal to U_0, when $U_0 = x_2$ (see discussion at the end of section 2) while the same identity does not hold when $U_0 = x_1$.

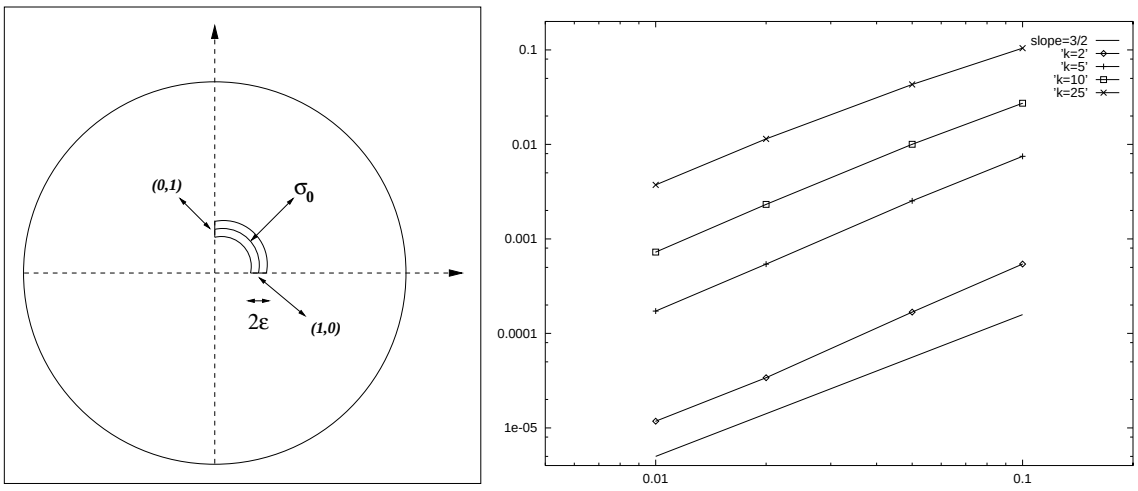

Figure 3: Typical configuration and approximation rates for a thin curved rectangle; σ_0 is arc of circle of radius 1 joining $(1, 0)$ and $(0, 1)$ and $U_0 = x_1$. $\|RE_\epsilon\|_{\ell_\infty(\partial\Omega)}$ is plotted on a log-log scale against ϵ for $0.01 \le \epsilon \le 0.1$. Different lines correspond to different conductivities in ω_ϵ ($k = 2, 5, 10, 25$). A line proportional to $\epsilon^{3/2}$ (slope 3/2) is shown for easy comparison.

The next example concerns a thin curved rectangular imperfection of width 2ϵ. The region ω_ϵ is obtained by taking σ_0 as the arc of a circle of radius 1 in the first quadrant joining the points $(1, 0)$ and $(0, 1)$ and $h(x)$ to be identically one (see figure 3). The background voltage potential U_0 is chosen to be x; the region ω_ϵ and U_0 are thus chosen in a manner so as not to exploit any symmetries. Note that the asymptotic size of $u_\epsilon - U_0 - \epsilon u_1$ is very close to $O(\epsilon^{3/2})$ (see figure 3) and not $O(\epsilon^2)$ as was the case for a straight rectangle.

Figure 4 shows the typical configuration and approximation rates for a thin "smooth" rectangle centered at $(0, 0)$. Note that the curve $\partial\omega_\epsilon$ does not have corners in the sense of discontinuous tangents, but in the sense of discontinuous curvatures (as a consequence no change

of coefficients is required in (58)); however, we have continued to use the special quadrature discussed above for collocation points near these "corners". In principle this "smooth" rectangle corresponds to a line segment σ_0^ϵ, and a thickness variation h^ϵ, that both depend on ϵ. Nonetheless, for our comparisions we have chosen to base u_1 on the ϵ-independent line segment $\sigma_0 = (-1, 1)$ and the ϵ-independent thickness variation $h \equiv 1$. We observe an approximation rate of $O(\epsilon^{3/2})$, the same as in the previous case of a curved rectangular imperfection.

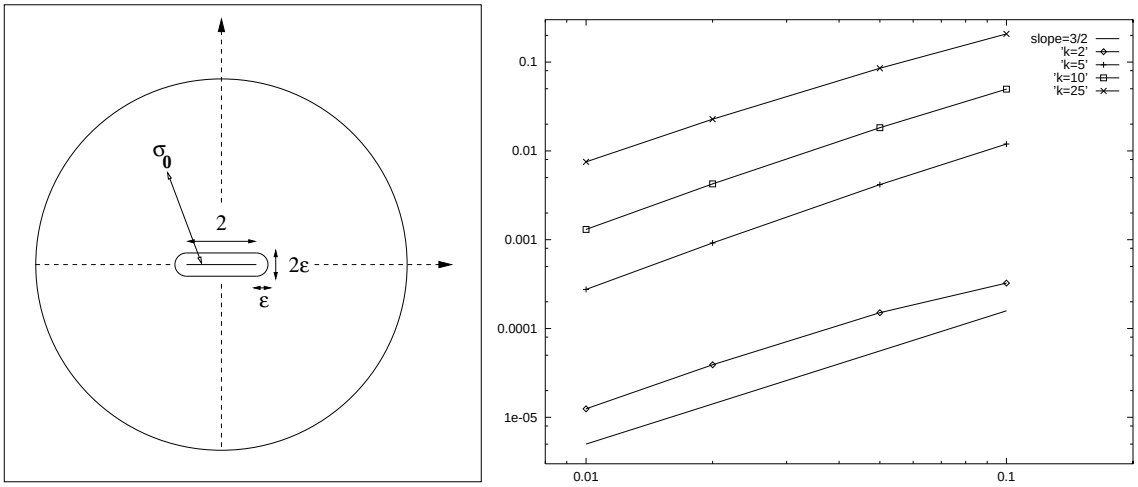

Figure 4: Typical configuration and approximation rates for a thin, "smooth" rectangle; $\sigma_0 = (-1, 1)$ is a straight line of length 2, $h(x) \equiv 1$, the ends are semi-circles of radius ϵ, and $U_0 = x_1$. $\|RE_\epsilon\|_{\ell_\infty(\partial\Omega)}$ is plotted on a log-log scale against ϵ for $0.01 \leq \epsilon \leq 0.1$. Different lines correspond to different conductivities in ω_ϵ ($k = 2, 5, 10, 25$). A line proportional to $\epsilon^{3/2}$ (slope 3/2) is shown for easy comparison.

The final example concerns a thin elliptical imperfection with major axis 2 and minor axis 2ϵ centered at $(6, 6)$ (see figure 5). The region ω_ϵ is obtained by taking $h(x(t)) = \sqrt{1 - t^2}$, where $t \in (-1, 1)$ is the parameter used to parameterize the line segment σ_0 (by arclength). We have chosen σ_0 to be aligned with a radial direction. We assume that the background voltage potential is x_1. The choice of ω_ϵ and U_0 are such as to avoid symmetries. The asymptotic magnitude of RE_ϵ appears to be $O(\epsilon^2)$. Moreover, this calculation demonstrates, that in practice our symptotic formuli remain valid even when the imperfection is very near the outer boundary $\partial\Omega$.

6 Application to electrical impedance imaging

In electrical impedance imaging we prescribe the boundary current, g, and we measure the voltage potential u_ϵ on the boundary as well. Since g determines the background potential U_0, we may assume that the quantity

$$u_\epsilon(y) - U_0(y) + 2\int_{\partial\Omega}(u_\epsilon(x) - U_0(x))\frac{\partial\Phi}{\partial\nu(x)}(x, y)\, d\sigma_x$$

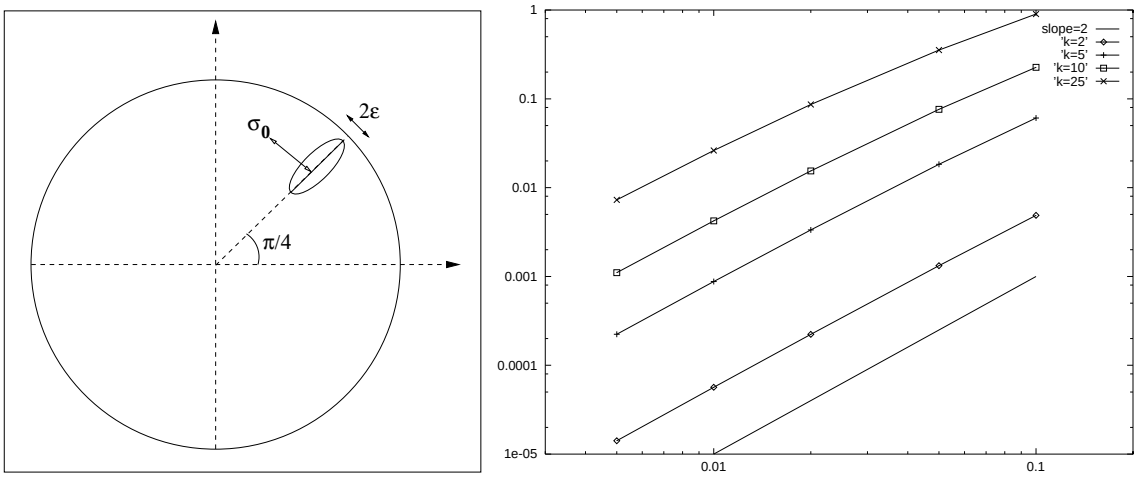

Figure 5: Typical configuration and convergence rates for thin ellipse; σ_0 is line segment joining $(6-1/\sqrt{2}, 6-1\sqrt{2})$ and $(6+1/\sqrt{2}, 6+/\sqrt{2})$ and $U_0 = x_1$. $\|RE_\epsilon\|_{\ell_\infty(\partial\Omega)}$ is plotted on a log-log scale against ϵ for $0.005 \leq \epsilon \leq 0.1$. Different lines correspond to different conductivities inside $\partial\omega_\epsilon$ ($k = 2, 5, 10, 25$) and a line proportional to ϵ^2 (slope 2) is shown for easy comparison.

is readily available for points $y \in \partial\Omega$. The inverse problem is to determine as much as possible about the set of imperfections

$$\omega_\epsilon^i = \{x + \eta h^i(x) n(x) : \ x \in \sigma_0^i \ , \quad \eta \in (-\epsilon, \epsilon)\} \quad , \qquad i = 1, ..., N \quad ,$$

(and also to determine their number N). Using the representation formula (4) we conclude that (to leading order) we are in possession of the boundary data of a function of the form

$$-2\epsilon \sum_{i=1}^{N} \int_{\sigma_0^i} \nabla_x \Phi(x, y) \cdot A^i(x) \nabla U_0(x) h^i(x) \, ds_x \quad .$$

In view of this, the inverse problem becomes one of determining, most effectively, the unknown curves σ_0^i and the unknown "thickness" functions h^i (supposing for instance that all imperfections have the same conductivity, k). This problem, and in particular its stability- and numerical aspects is the subject of current work. It is relevant to mention that impedance imaging, based on just a single set of boundary current and voltage measurements, has theoretically been shown to determine inhomogeneities of different shape, for instance convex polygons and balls (see [1], [6], [10], [9] and the references therein) whether they be small or not. The important aspect of the explicit asymptotic formuli, presented here and in [3] and [7], is that they lead to very accurate and fast reconstruction algorithms. Theoretically this comes at the cost of introducing a smallness parameter; however, our computational experience, as illustrated in the previous section, is that these formuli are quite accurate even for moderate size imperfections. It is also relevant to mention that integral representation formuli have recently been succesfully used to obtain (upper and lower) bounds for the size (volume or area) of imperfections based on a single boundary Cauchy datum (see [8] and [11]).

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References

[1] G. Alessandrini, V. Isakov and J. Powell, *Local Uniqueness in the inverse conductivity problem with one measurement*. Trans. Amer. Math. Soc., **347**, pp. 3031–3041, (1995).

[2] K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge University Press, Cambridge, (1997).

[3] D. J. Cedio-Fengya, S. Moskow and M. S. Vogelius *Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction*. Inverse Problems, **14**, pp. 553–595, (1998).

[4] D. Colton and R. Kress *Integral Equation Methods in Scattering Theory*. Krieger Publ. Co., Malabar, Florida, (1992).

[5] G. B. Folland, *Introduction to Partial Differential Equations*. Princeton University Press, Princeton, New Jersey, (1976).

[6] A. Friedman and V. Isakov, *On the uniqueness in the inverse conductivity problem with one measurement*. Indiana Univ. Math. J., **38**, pp. 563–579, (1989).

[7] A. Friedman and M. Vogelius, *Identification of small inhomogeneities of extreme conductivity by boundary measurements: a theorem on continuous dependence*. Arch. Rat. Mech. Anal., **105**, pp. 299–326, (1989).

[8] M. Ikehata, *Size estimation of inclusion*. J. Inverse and Ill-posed Problems, to appear.

[9] V. Isakov, *Inverse Problems for Partial Differential Equations*. Applied Mathematical Sciences, **127**, Springer-Verlag, New York, (1998).

[10] H. Kang and J. K. Seo, *Layer potential theory for the inverse conductivity problem*. Inverse Problems, **12**, pp. 267–278, (1996).

[11] H. Kang, J. K. Seo and D. Sheen, *The inverse conductivity problem with one measurement: stability and estimation of size*. SIAM J. Math. Anal., **28**, pp. 1389–1405, (1997).

[12] R. Kress, *Linear Integral Equations*. Springer-Verlag, Berlin, (1989).