Nonlocal Dynamics of Passive Tracer Dispersion with Random Stopping

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Abstract

We investigate the nonlocal behavior of passive tracer dispersion with random stopping at various sites in fluids. This kind of dispersion processes is modeled by an integral partial differential equation, i.e., an advection-diffusion equation with a memory term. We have shown the exponential decay of the passive tracer concentration, under suitable conditions for the velocity field and the probability distribution of random stopping time.

Key words: passive tracer, transport and dispersion, long-time behavior, random stopping, nonlocal system

1 Introduction

The discharge of pollutants into coastal seas or rivers is common but harmful. The pollutants may be held up or trapped at some sites during the process of their dispersion. For the benefit of better environment, it is important to understand the dynamics of such passive tracers.

The Eulerian approach for studying passive tracer dispersion attempts to understand the evolution of tracer concentration profile as a continuous field quantity [3] [12], which ultimately, at large times, satisfies the advection-diffusion equation.

When there are side branches, traps or some special sites in a shear flow, where passive tracers may be held-up or arrested and the stopping times of tracers at these traps are random, the tracer concentration profile $C(x,t)$ then satisfies a nonlocal transport equation derived by Young [16]

$$C_t + U(x)C_x = DC_{xx} - \partial_t \int_0^t k(t-s)C(x,s)ds \tag{1}$$

where $D > 0$ is the diffusion constant and $U(x)$ is the smooth fluid velocity field. The kernel $k(t) > 0$ is proportional to the probability distribution of tracer stopping times at traps.

Young [16] has shown that the anomalous diffusion phenomenon can occur under certain conditions on velocity field $U(x)$ and kernel $k(t)$.

In this paper, we study the long-time behavior of the concentration profile. We have shown the exponential decay of the passive tracer concentration, under suitable conditions for the velocity field and the probability distribution of random stopping time.

2 Exponential Decay with Increasing Random Stopping Distribution

We rewrite equation (1) as

$$C_t + U(x)C_x = DC_{xx} - k(0)C - \int_0^t k'(t-s)C(x,s)ds, \tag{2}$$

where x varies on the real line $R = (-\infty, \infty)$. We impose an appropriate initial condition

$$C(x,0) = C_0(x). \tag{3}$$

We denote $L^2(R)$ as the standard space of square-integrable functions. The usual mean-square norm in this space is denoted as $\|\cdot\|$. Moreover, $H^1(R)$ denotes the usual Sobolev space. Note that from [1], p. 57, $H^1(R) = H_0^1(R)$. We also denote $L^1(R^+)$ as the space of integrable functions on positive real line $R^+ = (0,\infty)$. Moreover, $C^1(R^+)$ is the space of continuously differentiable (smooth) functions on R^+.

In order to study the evolution of the concentration $C(x,t)$, we should estimate the nonlocal convolution integral in (2), and thus we need the following result [14].

Lemma. *If $k \in L^1(R^+)$ is a positive kernel and satisfies $k' \in L^1(R^+)$, then for any $y \in L^1(R^+)$,*

$$\int_{t_0}^{t_1} |k * y(\tau)|^2 d\tau \leq \beta_0 K \int_{t_0}^{t_1} k * y \;\; y(\tau) d\tau,$$

where $K = |k|_1^2 + 4|k'|_1^2$, $0 \leq t_0 < t_1 < \infty$, and $\beta_0 > 0$ is such that $k(t) - \beta_0 e^{-t} > 0$. Moreover, if $k(t) - \beta_0 e^{-\gamma t} > 0$ for some constant $\gamma > 0$, then the conclusion also holds with $\beta_0 K$ replaced by $\frac{1}{\gamma}\beta_0 K$.

Now we estimate the norm of the concentration $C(x,t)$.

Taking the inner product of (1) with C in $L^2(R)$ and integrating by patrs, we obtain

$$\frac{1}{2}\frac{d}{dt}\|C\|^2 + D\|C_x\|^2 - \frac{1}{2}\int_{-\infty}^{\infty} U'(x)C^2 dx + k(0)\|C\|^2 = \\ - \int_{-\infty}^{\infty}\int_0^t k'(t-s)C(x,s)dsC(x,t)dx. \tag{4}$$

We assume that

$$\frac{1}{2}U'(x) \leq -\alpha_0 + k(0), \text{ for some positive constant } \alpha_0, \tag{5}$$

$$k' > 0, \text{and } \; k, k', k'' \in L^1(R^+), \tag{6}$$

which imply that the velocity field $U(x)$ has bounded gradient and the probability distribution of the random stopping times is strictly increasing.

Integrating (4) with respect to t from t_0 to t_1, and using (5), (6) and the Lemma above, we obtain that

$$\frac{1}{2}(\|C(t_1)\|^2 - \|C(t_0)\|^2) + \alpha_0 \int_{t_0}^{t_1} \|C(s)\|^2 ds \leq 0. \tag{7}$$

Thus we have

$$\|C(t_1)\|^2 \leq \|C(t_0)\|^2, \;\; \forall \;\; t_0 \leq t_1.$$

If we take $t_0 = t, t_1 = t+1$ in (7), then

$$\|C(t+1)\|^2 \leq \int_t^{t+1} \|C(s)\|^2 ds \leq \frac{1}{2\alpha_0}(\|C(t)\|^2 - \|C(t+1)\|^2), \tag{8}$$

and

$$\max_{t \leq s \leq t+1} \|C(s)\|^2 = \|C(t)\|^2 = \|C(t)\|^2 - \|C(t+1)\|^2 + \|C(t+1)\|^2 \\ \leq (1 + \frac{1}{2\alpha_0})(\|C(t)\|^2 - \|C(t+1)\|^2).$$

By the difference inequality of Nakao [10] [11], we finally obtain

$$\|C(t)\|^2 \leq Ae^{-Bt}, \ t \geq 0, \tag{9}$$

for some positive constants A and B. Thus $\|C(t)\|$ decays exponentially fast. Therefore we have the following result.

Theorem 1 *If the velocity field $U(x)$ has bounded gradient and the probability distribution $k(t)$ (of the passive tracer random stopping times) is strictly increasing, namely, they satisfy conditions (5) and (6), respectively, then the concentration of the passive tracer with random stopping approaches zero exponentially fast in mean-square norm.*

3 Exponential Decay with History Source Term

If there is a source term due to the past history of C from $-\infty$ to 0, more precisely, if there is a source term

$$f(x,t) = -\int_{-\infty}^{0} k'(t-s)C(x,s)dx, \ x \in R, t \geq 0,$$

then, (2) can be written as

$$\begin{aligned} C_t + U(x)C_x &= DC_{xx} - k(0)C - \int_{-\infty}^{t} k'(t-s)C(x,s)ds \\ &= DC_{xx} - k(0)C - \int_{0}^{\infty} k'(s)C(x,t-s)ds. \end{aligned} \tag{10}$$

Let

$$C^t(x,s) = C(x,t-s), \tag{11}$$

$$\eta^t(x,s) = \int_0^s C^t(x,\tau)d\tau = \int_{t-s}^{t} C(x,\tau)d\tau, \ s \geq 0. \tag{12}$$

We further assume that $\lim_{s\to\infty} k'(s) = 0$, then

$$\int_0^\infty k'(s)C(x,t-s)ds = \int_0^\infty k'(s)C^t(x,s)ds = -\int_0^\infty k''(s)\eta_t(s)ds.$$

Let $\mu(s) = k''(s)$, then (10) can be reformulated as

$$C_t + U(x)C_x = DC_{xx} - k(0)C - \int_0^\infty \mu(s)\eta^t(x,s)ds, \tag{13}$$

$$\eta_t^t(x,s) = C(x,t) - \frac{\partial}{\partial s}\eta^t. \tag{14}$$

The initial conditions are then

$$C(x,0) = C_0(x), \quad \eta^0(x,s) = \eta_0(x,s) = \int_{-s}^{0} C(x,\tau)d\tau, \ x \in R. \tag{15}$$

Let

$$H = L^2(R) \times L^2(R^+;\mu;L^2(R)),$$

where

$$L^2(R^+;\mu;L^2(R)) = \{\varphi : R^+ \to L^2(R) | \int_0^\infty \mu(s)\|\varphi(s)\|^2 ds < \infty\},$$

and $< \cdot >_\mu$, $\|\cdot\|_\mu$ denote the inner product and norm in this space, respectively.

The well-posedness of the initial boundary value problem of (13), (14) and (15) can be shown as in [7] and [9]. Here, we focus on the asymptotic behavior under the condition (5) on the velocity field $U(x)$, and the following conditions on the probability distribution $k(t)$ of the random stopping time

$$k' > 0 \text{ and } \lim_{s\to\infty} k'(s) = 0, \tag{16}$$

$$k'' \in C^1(R^+) \cap L^1(R^+) \ \text{ and } \ k''(s) \leq 0, k'''(s) \geq 0, \forall s \in R^+, \tag{17}$$

$$k'''(s) + \delta k''(s) \leq 0, \quad \forall s \in R^+, \text{ and for some constant } \delta > 0. \tag{18}$$

We now estimate the evolution of C, η^t described by (13)-(14). Multiplying (13) by C and (14) by η^t, then integrating by parts and adding the results, we get

$$\frac{1}{2}\frac{d}{dt}(\|C\|^2 + \|\eta^t\|_\mu^2) - \int_{-\infty}^{\infty} U'(x)|C|^2 dx$$

$$+k(0)\|C\|^2 + D\|C_x\|^2 + < \frac{\partial}{\partial s}\eta^t, \eta^t >_\mu = 0. \tag{19}$$

Since for a fixed $t \in (0,\infty)$ both $\mu\|\eta^t\|$ and $\mu\|\frac{\partial}{\partial s}\eta^t\|$ belong to $L^1(R^+)$, we now have

$$\lim_{s\to\infty} \mu(s)\|\eta^t(s)\|^2 = 0. \tag{20}$$

Hence, using (20) and integrating by parts, we obtain

$$< \frac{\partial}{\partial s}\eta^t, \eta^t >_\mu = -\int_0^\infty \mu'(s)\|\eta^t\|^2 ds. \tag{21}$$

By putting (5), (16)-(18), and (21) into (19), we finally get

$$\frac{d}{dt}(\|C\|^2 + \|\eta^t\|_\mu^2) + 2\alpha_0\|C\|^2 + 2\delta\|\eta^t\|_\mu^2 \leq 0, \tag{22}$$

where α_0, δ are defined in (5), (18), respectively. So by the Gronwall inequality [15], we obtain the exponential decay for $\|C\|^2 + \|\eta^t\|_\mu^2$ and thus for the concentration (in mean-square norm) $\|C\|$. We summarize this result in the following theorem.

Theorem 2 *Assume that the velocity field $U(x)$ has bounded gradient and the probability distribution $k(t)$ (of the passive tracer random stopping times) is strictly increasing, namely, they satisfy conditions (5) and (6), respectively. If $k(t)$ satisfies further growth conditions (16), (17) and (18), then the concentration of the passive tracer with random stopping and with a history source approaches zero exponentially fast in mean-square norm.*

Note that the transformation that leads to the reformulation (13)-(14) is inspired by [6]. Under this transformation, a nonautonomous system (10) with history source term becomes an autonomous sysytem (13)-(14), at the expense of increasing system components or dimension.

4 Discussions

In this paper, we have studied the nonlocal behavior of passive tracer dispersion with random stopping, using an advection-diffusion equation with a memory or history term. We have shown the exponential decay of the passive tracer concentration, under suitable conditions for the velocity field and the probability distribution of random stopping time. There are other situations where tracer dispersion is modeled by advection-diffusion type of equations with memory terms [13], [5] [8] and [4].

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