Coulomb-oscillator duality in spaces of constant curvature

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November 23, 1998

PACS: 02.20.+b, 03.65.Fd

Abstract

In this paper we construct generalizations to spheres of the well known Levi Civita and Kustaanheimo-Steifel regularizing transformations in Euclidean spaces of dimension 2, 3 and 5. The corresponding classical and quantum mechanical analogues of the coulomb problem on these spheres are discussed.

*Work supported in part by the National Science Foundation under grant DMS 94–00533

1 Introduction.

It is well known that the problem of a body moving under the influence of a central force field with potential μ/r has a singularity at the origin. We refer to this as the *Kepler problem.* This problem is usually posed in 3 dimensions, but since the motion is always constrained to a plane perpendicular to the constant angular momentum vector we can reduce it to 2 dimensions with Newtonian equations of motion and energy integral

$$\frac{d^2}{dt^2}\boldsymbol{r} = -\frac{\mu}{r^3}\boldsymbol{r}, \quad \frac{1}{2}(\frac{dr}{dt})^2 - \frac{\mu}{r} + \frac{1}{2r^2} = h, \tag{1}$$

where $r^2 = \boldsymbol{r}\cdot\boldsymbol{r}$, $r^2\frac{d\theta}{dt} = c$ and $\boldsymbol{r} = (x, y) = (r\cos\theta, r\sin\theta)$. As is well known [1], in two dimensions the Levi Civita transformation effectively removes the singularity and rewrites this problem in terms of the classical harmonic oscillator. In this process the original problem has been regularized. To achieve the regularization, instead of t we use the variable s defined by $\frac{ds}{dt} = \frac{1}{r}$. With $x' = \frac{dx}{ds}$ etc., the original equations (1) are

$$\boldsymbol{r}'' - \frac{r'}{r}\boldsymbol{r}' + \frac{\mu}{r}\boldsymbol{r} = \boldsymbol{0}, \quad \frac{1}{2r^2}\boldsymbol{r}'\cdot\boldsymbol{r}' - \frac{\mu}{r} = h. \tag{2}$$

Instead of using the variables (x, y) it is convenient to make the transformation

$$\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{vmatrix}\begin{vmatrix} u_1 \\ u_2 \end{vmatrix} \quad \text{or} \quad \boldsymbol{r} = L(\boldsymbol{u})\boldsymbol{u}. \tag{3}$$

From the explicit form of these relations it follows that $\boldsymbol{r}' = 2L(\boldsymbol{u})\boldsymbol{u}'$. The equations of motion are equivalent to

$$\boldsymbol{u}'' + \frac{\frac{\mu}{2} - \boldsymbol{u}'\cdot\boldsymbol{u}'}{\boldsymbol{u}\cdot\boldsymbol{u}}\boldsymbol{u} = \boldsymbol{u}, \quad \frac{\mu}{2} = \boldsymbol{u}'\cdot\boldsymbol{u}' - \frac{h}{2}\boldsymbol{u}\cdot\boldsymbol{u}. \tag{4}$$

Consequently we have the regularized equation of motion

$$\boldsymbol{u}'' - \frac{h}{2}\boldsymbol{u} = \boldsymbol{0}.$$

This is essentially the equation for the harmonic oscillator if $h < 0$. The solution $u_1 = \alpha\cos(\omega s)$, $u_2 = \beta\sin(\omega s)$, $\omega^2 = -h/2$ is equivalent to elliptical motion.

The relationship between the harmonic oscillator and the corresponding Kepler problem can also be easily seen from the point of view of Hamilton-Jacobi theory. Indeed the Hamiltonian can be written in the two equivalent forms

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\mu}{\sqrt{x^2+y^2}} = \frac{1}{8(u_1^2+u_2^2)}[p_{u_1}^2 + p_{u_2}^2 + 8\mu]. \qquad (5)$$

If we now write down the corresponding Hamilton-Jacobi equation via the substitutions

$$p_{u_1} \to \partial_{u_1} S = S_{u_1}, \qquad p_{u_2} \to \partial_{u_2} S = S_{u_2}$$

we obtain

$$S_{u_1}^2 + S_{u_2}^2 + 8\mu - 8E(u_1^2 + u_2^2) = 0. \qquad (6)$$

This is just the Hamilton-Jacobi equation for a mechanical system with Hamiltonian

$$H' = p_{u_1}^2 + p_{u_2}^2 - 8E(u_1^2 + u_2^2)$$

and energy -8μ. (This is the pseudo-Coulomb problem, see [2, 3]. Reference [2] also obtains (6) as an an application of Stäckel transform theory.)

This transformation also achieves a regularization of the corresponding quantum mechanical problem, which we call the *quantum Coulomb* problem. Indeed, the Schrödinger equation in the presence of the potential μ/r in two dimensions has the form

$$-\frac{1}{2}(\partial_x^2 + \partial_y^2)\Psi + \frac{\mu}{\sqrt{x^2+y^2}}\Psi = E\Psi. \qquad (7)$$

In the coordinates (u_1, u_2), (7) becomes

$$(\partial_{u_1}^2 + \partial_{u_2}^2)\Phi + \{-8\mu + 8E(u_1^2 + u_2^2)\}\Phi = 0. \qquad (8)$$

Here, (8) has all the appearances of the Schrödinger equation in a potential $4E(u_1^2+u_2^2)$ and energy 4μ. The corresponding bound state energy spectrum can be easily computed from this reformulation of the Coulomb problem, although the weight function for the inner product is no longer the same [2, 4]. (Indeed, the Virial Theorem states that for the Coulomb problem the change in weight function does not alter the bound state spectrum, [4].) The wave functions have the form $\Phi = \varphi_1(u_1)\varphi_2(u_2)$ where the functions φ_λ satisfy

$$(\partial_\lambda^2 + \kappa_\lambda + 8Eu_\lambda^2)\varphi_\lambda = 0, \quad \lambda = 1, 2, \quad \kappa_1 + \kappa_2 = 8\mu.$$

The bound state eigenvalues are quantised according to

$$\kappa_\lambda = 2\sqrt{-2E}(2n_\lambda + 1), \quad \lambda = 1, 2, \quad E_{n_1+n_2} = -2\mu^2/(n_1 + n_2 + 1)^2,$$

where n_1, n_2 are integers, see e.g., [5].

In the following sections we will extend this and the Kustaanheimo-Steifel construction to spheres in 2, 3 and 5 dimensions.

2 The transformation on the 2-sphere

We can pose an analogue for the Kepler problem in which the space is the two dimensional sphere. The potential, which is the analogue of the coulomb potential in the case of quantum mechanics and the gravitational potential in the case for the Kepler problem, is taken to be

$$V = \frac{\mu s_3}{\sqrt{s_1^2 + s_2^2}}, \tag{9}$$

where

$$\boldsymbol{s} \cdot \boldsymbol{s} \equiv s_1^2 + s_2^2 + s_3^2 = 1.$$

(Note that $V = \mu \cot \alpha$ where α is the arc length distance from $\boldsymbol{s}$ to the north pole of the sphere. Furthermore, the leading term in the Laurent series expansion in α about the north pole is $\frac{\mu}{\alpha}$.) Just as in the case of Euclidean space, the corresponding classical equations of motion under the influence of such a potential can be simplified. The classical equations are

$$\ddot{\boldsymbol{s}} = -(\dot{\boldsymbol{s}} \cdot \dot{\boldsymbol{s}})\boldsymbol{s} - \boldsymbol{\nabla} V, \tag{10}$$

where the first term on the right hand side is the centripetal force term, corresponding to the constraint of the motion to the unit sphere, and the potential satisfies

$$\boldsymbol{s} \cdot \boldsymbol{\nabla} V = 0. \tag{11}$$

Here, $\dot{\boldsymbol{s}} = \frac{d}{dt}\boldsymbol{s}$. (In studying (10) and (11) we initially regard the coordinates $\boldsymbol{s}$ as unconstrained and then restrict our attention to solutions on the sphere.) In the case of the particular potential (9) these equations become

$$\frac{d^2}{dt^2} s_j = -s_j(\dot{\boldsymbol{s}} \cdot \dot{\boldsymbol{s}}) + \frac{\mu s_j s_3}{(s_1^2 + s_2^2)^{\frac{3}{2}}}, \quad j = 1, 2$$

$$\frac{d^2}{dt^2}s_3 = -s_3(\dot{\boldsymbol{s}}\cdot\dot{\boldsymbol{s}}) - \frac{\mu}{(s_1^2+s_2^2)^{\frac{1}{2}}},$$

subject to the constraints

$$\boldsymbol{s}\cdot\boldsymbol{s} = 1 \tag{12}$$

and its differential consequences

$$\boldsymbol{s}\cdot\dot{\boldsymbol{s}} = 0, \quad \boldsymbol{s}\cdot\ddot{\boldsymbol{s}} + \dot{\boldsymbol{s}}\cdot\dot{\boldsymbol{s}} = 0.$$

From the equations of motion we immediately deduce the energy integral

$$\frac{1}{2}\dot{\boldsymbol{s}}\cdot\dot{\boldsymbol{s}} + V = E. \tag{13}$$

In two dimensions this problem is easily transformed into a much simpler one via the transform

$$s_1 = i\frac{u_1u_2}{u_3}, \quad s_2 = i\frac{u_1^2-u_2^2}{2u_3}, \quad s_3 = u_3 + \frac{u_1^2+u_2^2}{2u_3}. \tag{14}$$

The advantage of this transform is the identity

$$s_1^2+s_2^2+s_3^2 = u_1^2+u_2^2+u_3^2,$$

from which we see that the point $\boldsymbol{u} = (u_1, u_2, u_3)$ lies on the complex unit sphere if $\boldsymbol{s}$ does. (For real spaces of constant curvature the situation is more complicated, and more interesting. Thus, for example, the upper and lower hemispheres of the real spere in u-space map to the upper and lower sheets, respectively, of the two-sheet hyperboloid in s-space. The two-sheet hyperboloid in u-space maps to the one-sheet hyperboloid in s-space.) The relationship between the infinitesimal distances is

$$d\boldsymbol{s}\cdot d\boldsymbol{s} = \frac{1}{u_3^2}(\boldsymbol{s}\cdot d\boldsymbol{s})^2 - (\frac{u_1^2+u_2^2}{u_3^2})d\boldsymbol{u}\cdot d\boldsymbol{u}. \tag{15}$$

Thus, when restricted to the unit sphere, the infinitesimal distances are related by

$$d\boldsymbol{s}\cdot d\boldsymbol{s} = -(\frac{u_1^2+u_2^2}{u_3^2})d\boldsymbol{u}\cdot d\boldsymbol{u}. \tag{16}$$

We choose a new variable τ such that

$$\frac{d\tau}{dt} = \frac{u_3^2}{u_1^2 + u_2^2}.$$

In terms of the variables τ and u_i, the equations of motion can now be written in the form

$$\begin{aligned}
(u_1')^2 + (u_2')^2 + (u_3')^2 + 2(i\mu - E) + \frac{2(E + i\mu)}{u_3^2} &= 0 && (17)\\
u_1'' + 2(E - i\mu)u_1 = 0, \quad u_2'' + 2(E - i\mu)u_2 &= 0 && (18)\\
u_3'' + 2(E - i\mu)u_3 - \frac{2(E + i\mu)}{u_3^3} &= 0, && (19)
\end{aligned}$$

subject to the constraint $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ and its differential consequences $\boldsymbol{u} \cdot \boldsymbol{u}' = 0$, $\boldsymbol{u} \cdot \boldsymbol{u}'' + \boldsymbol{u}' \cdot \boldsymbol{u}' = 0$, where $u_i' = \frac{du_i}{d\tau}$. These equations are equivalent to the equations of motion we would obtain by choosing the Hamiltonian

$$H = \frac{1}{2}(p_{u_1}^2 + p_{u_2}^2 + p_{u_3}^2) + (i\alpha - E)(u_1^2 + u_2^2 + u_3^2) + \frac{E + i\alpha}{u_3^2}, \tag{20}$$

regarding the variables u_i as independent and using the variable τ as time. In fact if we wished to solve the classical mechanical problem from the point of view of the Hamilton-Jacobi equation, we use the relation

$$\frac{1}{2}(p_{s_1}^2 + p_{s_2}^2 + p_{s_3}^2) + \frac{\mu s_3}{\sqrt{s_1^2 + s_2^2}} - E \equiv \tag{21}$$

$$-\frac{u_3^2}{u_1^2 + u_2^2}\left(\frac{1}{2}(p_{u_1}^2 + p_{u_2}^2 + p_{u_3}^2) + (i\mu - E)(u_1^2 + u_2^2 + u_3^2) + \frac{E + i\mu}{u_3^2}\right) = 0, \tag{22}$$

together with the substitutions $p_{u_i} = \frac{\partial S}{\partial u_i}$ and $p_{s_j} = \frac{\partial S}{\partial s_j}$ to obtain the Hamilton-Jacobi equations

$$(\frac{\partial S}{\partial s_1})^2 + (\frac{\partial S}{\partial s_2})^2 + (\frac{\partial S}{\partial s_3})^2 + 2\frac{\mu s_3}{\sqrt{s_1^2 + s_2^2}} - 2E = 0, \tag{23}$$

$$(\frac{\partial S}{\partial u_1})^2 + (\frac{\partial S}{\partial u_2})^2 + (\frac{\partial S}{\partial u_3})^2 + 2(i\mu - E) + 2\frac{E + i\mu}{u_3^2} = 0. \tag{24}$$

This last equation can be solved by the separation of variables ansatz using, for example, spherical coordinates

$$u_1 = \sin\alpha\cos\varphi, \quad u_2 = \sin\alpha\sin\varphi, \quad u_3 = \cos\alpha.$$

(Note that the method described here can be applied to any potential of the form $V = f(s_3/\sqrt{s_1^2 + s_2^2})$, not just (9).) The associated quantum Coulomb problem on the sphere translates directly in this case as follows. If we write the Schrödinger equation on the sphere for the potential V, we obtain

$$(\frac{1}{2}\Delta_s - \frac{\mu s_3}{\sqrt{s_1^2 + s_2^2}} + E)\Psi \equiv -\frac{u_3^2}{(u_1^2 + u_2^2)}[\frac{1}{2}\Delta_u + \frac{i\mu - E}{u_3^2} - i\mu + E]\Psi = 0, \quad (25)$$

where

$$\Delta_s = (s_1\partial_{s_2} - s_2\partial_{s_1})^2 + (s_3\partial_{s_2} - s_2\partial_{s_3})^2 + (s_3\partial_{s_2} - s_2\partial_{s_3})^2,$$

$$\Delta_u = (u_1\partial_{u_2} - u_2\partial_{u_1})^2 + (u_3\partial_{u_2} - u_2\partial_{u_3})^2 + (u_3\partial_{u_2} - u_2\partial_{u_3})^2,$$

from which we see that the Coulomb problem on the sphere is equivalent to the corresponding quantum mechanical problem on the sphere with potential $-\frac{i\mu+E}{u_3^2}$ and energy $-i\mu + E$, but with an altered inner product. In fact, the bound energy eigenstates of this problem are quantised according to

$$E_n = \frac{1}{2}\{(n + \frac{1}{2})^2 - \frac{1}{4} - \frac{\mu^2}{(n + \frac{1}{2})^2}\},$$

for integer n, see e.g., [5].

3 The three and five dimensional Kepler and Coulomb problems

It is well known that the regularizing transformations that we have discussed for the Kepler and Coulomb problems in two dimensional Euclidean spaces are also possible in the case of three and five dimensions, [1, 6]. The only difference that occurs in these cases is that additional constraints are required. In complete analogy with the Euclidean case the corresponding regularizing transformations exist for the Kepler and Coulomb problems in spheres of dimension 3 and 5. Indeed if we consider motion on the sphere of dimension n then the classical equations of motion with the presence of a potential are just (10), (11) again, where now

$$\boldsymbol{s} = (s_1, \cdots, s_{n+1}), \tag{26}$$

subject to the constraints

$$\boldsymbol{s} \cdot \boldsymbol{s} = 1 \tag{27}$$

and its differential consequences

$$\boldsymbol{s} \cdot \dot{\boldsymbol{s}} = 0, \quad \boldsymbol{s} \cdot \ddot{\boldsymbol{s}} + \dot{\boldsymbol{s}} \cdot \dot{\boldsymbol{s}} = 0.$$

If we choose our potential to be

$$V = \frac{\mu s_{n+1}}{\sqrt{s_1^2 + .. + s_n^2}}, \tag{28}$$

these equations assume the form

$$\frac{d^2}{dt^2} s_j = -s_j \dot{\boldsymbol{s}} \cdot \dot{\boldsymbol{s}} + \frac{\mu s_j s_{n+1}}{(\boldsymbol{s} \cdot \boldsymbol{s})^{\frac{3}{2}}}, \quad j = 1, \cdots, n, \tag{29}$$

$$\frac{d^2}{dt^2} s_{n+1} = -s_{n+1} \dot{\boldsymbol{s}} \cdot \dot{\boldsymbol{s}} - \frac{\mu}{(\boldsymbol{s} \cdot \boldsymbol{s})^{\frac{1}{2}}}. \tag{30}$$

The energy integral again has the form (13).

We are particularly interested in the dimensions $n = 3, 5$. We deal with each of these cases separately. For $n = 3$ we choose the u_j coordinates in five dimensional space according to

$$s_1 = i\frac{u_1 u_2 - u_3 u_4}{u_5}, \quad s_2 = i\frac{u_1 u_3 + u_2 u_4}{u_5}, \tag{31}$$

$$s_3 = \frac{i}{2}\frac{u_1^2 - u_2^2 - u_3^2 + u_4^2}{u_5}, \quad s_4 = u_5 + \frac{u_1^2 + u_2^2 + u_3^2 + u_4^2}{2u_5}. \tag{32}$$

The basic identity is

$$s_1^2 + s_2^2 + s_3^2 + s_4^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2,$$

and the basic relationship for the infinitesimal distances is

$$ds_1^2 + ds_2^2 + ds_3^2 + ds_4^2 = \tag{33}$$

$$\frac{-1}{u_5^2}\{(u_1^2 + u_2^2 + u_3^2 + u_4^2)[du_1^2 + du_2^2 + du_3^2 + du_4^2 + du_5^2]$$

$$+(u_4du_1 - u_3du_2 + u_2du_3 - u_1du_4)^2\},$$

where the constraint for mapping between the complex 3-sphere and the complex 4-sphere is clearly

$$u_4du_1 - u_3du_2 + u_2du_3 - u_1du_4 = 0. \tag{34}$$

In analogy with our previous analysis we choose a new variable τ according to

$$\frac{d\tau}{dt} = \frac{u_5^2}{u_1^2 + u_2^2 + u_3^2 + u_4^2}.$$

In the u coordinates the equations of motion can be written as

$$(u_1')^2 + (u_2')^2 + (u_3')^2 + (u_4')^2 + (u_5')^2 + 2(i\mu - E) + \frac{2(E + i\mu)}{u_5^2} = 0,$$

$$u_j'' + 2(E - i\mu)u_j = 0, \quad j = 1, 2, 3, 4, \tag{35}$$

$$u_5'' + 2(E - i\mu)u_5 - \frac{2(E + i\mu)}{u_5^3} = 0,$$

subject to the constraints

$$\sum_{k=1}^{5} u_k^2 = 1, \quad \sum_{k=1}^{5} u_k u_k' = 0,$$

$$\sum_{k=1}^{5}(u_k u_k'' + (u_k')^2) = 0, \quad u_4u_1' - u_3u_2' + u_2u_3' - u_1u_4' = 0.$$

Note that equations (35) are compatible with these constraints. Here, the Kepler problem on the sphere in three dimensions is equivalent to choosing a Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}(p_{u_1}^2 + p_{u_2}^2 + p_{u_3}^2 + p_{u_4}^2 + p_{u_5}^2) + \\ &\quad (i\mu - E)(u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2) + \frac{E + i\mu}{u_5^2}, \end{aligned} \tag{36}$$

regarding the variables u_j as independent and τ as time. The only difference in this case is that there is now the constraint

$$u_4 p_{u_1} - u_3 p_{u_2} + u_2 p_{u_3} - u_1 p_{u_4} = 0.$$

In terms of the Hamilton-Jacobi formulation we have the relation

$$\frac{1}{2}(p_{s_1}^2 + p_{s_2}^2 + p_{s_3}^2 + p_{s_4}^2) + \frac{\mu s_4}{\sqrt{s_1^2 + s_2^2 + s_3^2}} + E =$$

$$-\frac{u_5^2}{u_1^2 + u_2^2 + u_3^2 + u_4^2}\{\frac{1}{2}(p_{u_1}^2 + p_{u_2}^2 + p_{u_3}^2 + p_{u_4}^2 + p_{u_5}^2) +$$

$$(i\mu - E)(u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2) + \frac{E + i\mu}{u_5^2}\} = 0.$$

With the usual substitutions, the corresponding Hamilton-Jacobi equations are

$$(\frac{\partial S}{\partial s_1})^2 + (\frac{\partial S}{\partial s_2})^2 + (\frac{\partial S}{\partial s_3})^2 + (\frac{\partial S}{\partial s_4})^2 + \frac{2\mu s_4}{\sqrt{s_1^2 + s_2^2 + s_3^2}} + 2E = 0,$$

or

$$\sum_{k=1}^{5}(\frac{\partial S}{\partial u_k})^2 + 2(i\mu - E)\sum_{k=1}^{5} u_k^2 + 2\frac{E + i\mu}{u_5^2} = 0, \tag{37}$$

where the constraint has become

$$u_4\frac{\partial S}{\partial u_1} - u_3\frac{\partial S}{\partial u_2} + u_2\frac{\partial S}{\partial u_3} - u_1\frac{\partial S}{\partial u_4} = 0. \tag{38}$$

The constraint can be accommodated by using the coordinates

$$\begin{aligned}
u_1 &= \cos\alpha\cos A\cos(\frac{\psi + \eta}{2}), \quad u_2 = \cos\alpha\sin A\cos(\frac{\psi - \eta}{2}), \qquad (39)\\
u_3 &= \cos\alpha\sin A\sin(\frac{\psi - \eta}{2}), \quad u_4 = \cos\alpha\cos A\cos(\frac{\psi + \eta}{2}),\\
u_5 &= \sin\alpha.
\end{aligned}$$

The coordinates on the 3 sphere then have the form

$$\begin{aligned}
s_1 &= \frac{i\cos^2\alpha}{2\sin\alpha}(1 - 2\cos^2 A), \quad s_2 = -\frac{i\cos^2\alpha}{\sin\alpha}\sin A\cos A\cos\psi, \qquad (40)\\
s_3 &= -\frac{i\cos^2\alpha}{\sin\alpha}\sin A\cos A\sin\psi, \quad s_4 = \frac{2 - \cos^2\alpha}{\sin\alpha}
\end{aligned}$$

The constraint is equivalent to $\partial_\eta S = 0$. The associated quantum Coulomb problem on the sphere corresponding to the potential V translates directly as follows

$$(\frac{1}{2}\Delta_s + \frac{\mu s_4}{\sqrt{s_1^2+s_2^2+s_3^2}} + E)\Psi =$$

$$-u_5^{\frac{1}{2}}[\frac{u_5^2}{(u_1^2+u_2^2+u_3^2+u_4^2)}][\frac{1}{2}\Delta_u + \frac{E-i\mu-\frac{3}{4}}{u_5^2} + i\mu + E - \frac{5}{4}]u_5^{-\frac{1}{2}}\Psi,$$

where Δ_s and Δ_u have the obvious meaning. The quantised energy levels are

$$E_n = \frac{1}{2}[(n+\frac{1}{2})^2 - 1 - \frac{\alpha^2}{(n+\frac{1}{2})^2}].$$

The analogous problem in five dimensions can be realized via the variables

$$\begin{aligned}
s_1 &= \frac{i}{u_9}(u_1u_5 + u_2u_6 + u_2u_3 + u_4u_8), \quad s_2 = \frac{i}{u_9}(u_1u_6 - u_2u_5 - u_3u_8 + u_4u_7)\\
s_3 &= \frac{i}{u_9}(u_1u_7 + u_2u_8 - u_3u_5 - u_4u_6), \quad s_4 = \frac{i}{u_9}(u_1u_8 - u_2u_7 + u_3u_6 - u_4u_5),\\
s_5 &= \frac{i}{2u_9}(u_1^2+u_2^2+u_3^2+u_4^2-u_5^2-u_6^2-u_7^2-u_8^2), \quad s_6 = u_9 + \frac{1}{u_9}\sum_{k=1}^{8} u_k^2,
\end{aligned}$$

which satisfy

$$\sum_{j=1}^{6} s_j^2 = \sum_{\ell=1}^{9} u_\ell^2. \tag{41}$$

The potential has the form

$$V = \frac{\mu s_6}{\sqrt{s_1^2+s_2^2+s_3^2+s_4^2+s_5^2}}. \tag{42}$$

The relation between the infinitesimal distances on the five and eight dimensional complex spheres is

$$\sum_{j=1}^{6} ds_j^2 = \frac{-1}{u_9^2}\left[(\sum_{k=1}^{8} u_k^2)\sum_{\ell=1}^{9} du_\ell^2 + \omega_1^2 + \omega_2^2 + \omega_3^2\right], \tag{43}$$

where

$$\begin{aligned}
\omega_1 &= u_4du_1 + u_3du_2 - u_2du_3 - u_1du_4 + u_8du_5 + u_7du_6 - u_6du_7 - u_5du_8, \\
\omega_2 &= -u_3du_1 + u_4du_2 + u_1du_3 - u_2du_4 - u_7du_5 + u_8du_6 + u_5du_7 - u_6du_8, \\
\omega_3 &= u_2du_1 - u_1du_2 + u_4du_3 - u_3du_4 + u_6du_5 - u_5du_6 + u_8du_7 - u_7du_8.
\end{aligned}$$

As before we can define a new coordinate τ such that

$$\frac{d\tau}{dt} = \frac{u_9^2}{\sum_{k=1}^{8} u_k^2}.$$

The corresponding equations of motion are given by

$$\sum_{\ell=1}^{9}(u'_\ell)^2 + 2(i\mu - E) - \frac{2(E + i\mu)}{u_9^2} = 0, \tag{44}$$

$$u''_k + 2(E - i\mu)_k = 0, \quad k = 1, \cdots, 8$$

$$u''_9 + 2(E - i\mu)u_9 - \frac{2(E + i\mu)}{u_9^3} = 0,$$

subject to the constraints

$$\begin{aligned}
\sum_{\ell=1}^{9} u_\ell^2 = 1, \quad \sum_{\ell=1}^{9} u_\ell u'_\ell = 0, \quad \sum_{\ell=1}^{9}\left(u_\ell u''_\ell + (u'_\ell)^2\right) &= 0, \\
u_4u'_1 + u_3u'_2 - u_2u'_3 - u_1u'_4 + u_8u'_5 + u_7u'_6 - u_6u'_7 - u_5u'_8 &= 0, \\
-u_3u'_1 + u_4u'_2 + u_1u'_3 - u_2u'_4 - u_7u'_5 + u_8u'_6 + u_5u'_7 - u_6u'_8 &= 0, \\
u_2u'_1 - u_1u'_2 + u_4u'_3 - u_3u'_4 + u_6u'_5 - u_5u'_6 + u_8u'_7 - u_7u'_8 &= 0.
\end{aligned}$$

These equations of motion are equivalent to what we would obtain by choosing the Hamiltonian

$$H = \frac{1}{2}\sum_{\ell=1}^{9} p_{u_\ell}^2 + (i\mu - E)\sum_{\ell=1}^{9} u_\ell^2 + \frac{E + i\mu}{u_9^2}, \tag{45}$$

regarding the variables u_i as independent and using τ as time. The associated constraints are

$$\begin{aligned}
u_4p_1 + u_3p_2 - u_2p_3 - u_1p_4 + u_8p_5 + u_7p_6 - u_6p_7 - u_5p_8 &= 0, \qquad (46) \\
-u_3p_1 + u_4p_2 + u_1p_3 - u_2p_4 - u_7p_5 + u_8p_6 + u_5p_7 - u_6p_8 &= 0, \\
u_2p_1 - u_1p_2 + u_4p_3 - u_3p_4 + u_6p_5 - u_5p_6 + u_8p_7 - u_7p_8 &= 0.
\end{aligned}$$

If we wish to solve this problem from the point of view of the Hamilton-Jacobi equation we use the relation

$$\frac{1}{2}\sum_{j=1}^{6} p_{s_j}^2 + \frac{\mu s_6}{\sqrt{s_1^2+s_2^2+s_3^2+s_4^2+s_5^2}} + E =$$

$$-\frac{u_9^2}{\sum_{k=1}^{8} u_k^2}\left\{\frac{1}{2}\sum_{\ell=1}^{9} p_{u_\ell}^2 + (i\mu - E)\sum_{\ell=1}^{9} u_\ell^2 + \frac{E+i\mu}{u_9^2}\right\} = 0.$$

The corresponding Hamilton-Jacobi equations are

$$\frac{1}{2}\sum_{j=1}^{6}(\frac{\partial S}{\partial s_j})^2 + \frac{\mu s_6}{\sqrt{s_1^2+s_2^2+s_3^2+s_4^2+s_5^2}} + E = 0, \tag{47}$$

$$\frac{1}{2}\sum_{\ell=1}^{9}(\frac{\partial S}{\partial u_\ell})^2 + (i\mu - E)\sum_{\ell=1}^{9} u_\ell^2 + \frac{E+i\mu}{u_9^2} = 0, \tag{48}$$

subject to the constraints

$$\begin{array}{rcl}
u_4\frac{\partial S}{\partial u_1} + u_3\frac{\partial S}{\partial u_2} - u_2\frac{\partial S}{\partial u_3} - u_1\frac{\partial S}{\partial u_4} + u_8\frac{\partial S}{\partial u_5} + u_7\frac{\partial S}{\partial u_6} - u_6\frac{\partial S}{\partial u_7} - u_5\frac{\partial S}{\partial u_8} & = & 0, \\
-u_3\frac{\partial S}{\partial u_1} + u_4\frac{\partial S}{\partial u_2} + u_1\frac{\partial S}{\partial u_3} - u_2\frac{\partial S}{\partial u_4} - u_7\frac{\partial S}{\partial u_5} + u_8\frac{\partial S}{\partial u_6} + u_5\frac{\partial S}{\partial u_7} - u_6\frac{\partial S}{\partial u_8} & = & 0, \\
u_2\frac{\partial S}{\partial u_1} - u_1\frac{\partial S}{\partial u_2} + u_4\frac{\partial S}{\partial u_3} - u_3\frac{\partial S}{\partial u_4} + u_6\frac{\partial S}{\partial u_5} - u_5\frac{\partial S}{\partial u_6} + u_8\frac{\partial S}{\partial u_7} - u_7\frac{\partial S}{\partial u_8} & = & 0.
\end{array}$$

A suitable set of coordinates on the eight dimensional sphere is

$$\begin{array}{rcl}
u_1 & = & \cos\alpha\cos\theta_1\cos\psi\cos\theta\cos\varphi, \quad u_2 = \cos\alpha\cos\theta_1\sin\psi\cos\theta\cos\varphi, \\
u_3 & = & \cos\alpha\cos\theta_1\sin\theta\cos\varphi, \quad u_4 = \cos\alpha\cos\theta_1\sin\varphi, \\
u_5 & = & \cos\alpha\sin\theta_1(-\sin\theta_2\sin\theta_2\cos\theta_4\sin\theta\cos\varphi + \cos\theta_2\cos\psi\cos\theta\cos\varphi \\
 & & -\sin\theta_2\cos\theta_3\sin\psi\cos\theta\cos\varphi - \sin\theta_2\sin\theta_3\sin\theta_4\sin\varphi), \\
u_6 & = & \cos\alpha\sin\theta_1(-\sin\theta_2\sin\theta_3\cos\theta_4\sin\varphi + \cos\theta_2\sin\psi\cos\theta\cos\varphi \\
 & & +\sin\theta_2\cos\theta_3\cos\psi\cos\theta\cos\varphi + \sin\theta_2\sin\theta_3\sin\theta_4\sin\theta\cos\varphi), \\
u_7 & = & \cos\alpha\sin\theta_1(\sin\theta_2\sin\theta_3\cos\theta_4\cos\psi\cos\theta\cos\varphi + \cos\theta_2\sin\theta\cos\varphi \\
 & & +\sin\theta_2\cos\theta_3\sin\varphi - \sin\theta_2\sin\theta_3\sin\theta_4\sin\psi\cos\theta\cos\varphi), \\
u_8 & = & \cos\alpha\sin\theta_1(\sin\theta_2\sin\theta_3\cos\theta_4\sin\psi\cos\theta\cos\varphi + \cos\theta_2\sin\varphi \\
 & & -\sin\theta_2\cos\theta_3\sin\theta\cos\varphi + \sin\theta_2\sin\theta_3\sin\theta_4\cos\varphi\cos\theta\cos\psi), \\
u_9 & = & \sin\alpha.
\end{array}$$

The corresponding quantum Coulomb problem is subject to the constraints $\partial_\varphi S = 0$, $\partial_\psi S = 0$ and $\partial_\theta S = 0$.

$$(\frac{1}{2}\Delta_s \quad + \quad \frac{\mu s_6}{\sqrt{s_1^2+s_2^2+s_3^2+s_4^2+s_5^2}} + E)\Psi \equiv \tag{49}$$

$$u_9^{\frac{3}{2}}[\frac{u_9^2}{(u_1^2+u_2^2+u_3^2+u_4^2+u_5^2+u_6^2+u_7^2+u_8^2)}[\frac{1}{2}\Delta_u + \frac{E-i\mu-\frac{15}{4}}{u_9^2}$$

$$+i\mu + E - \frac{33}{4}]u_9^{-\frac{3}{2}}\Psi = 0. \tag{50}$$

The quantised energy levels are

$$E_n = \frac{1}{2}[(n+\frac{1}{2})^2 - 4 - \frac{\alpha^2}{(n+\frac{1}{2})^2}].$$

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