Passive Tracer Dispersion with Random or Periodic Source *

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Abstract

In this paper, the author investigates the impact of external sources on the pattern formation and long-time behavior of concentration profiles of passive tracers in a two-dimensional shear flow. It is shown that a time-periodic concentration profile exists for time-periodic external source, while for random source, the distribution functions of all concentration profiles weakly converge to a unique invariant measure (like a stationary state in deterministic systems) as time goes to infinity.

Key words: tracer transport, concentration, long-time behavior, periodic concentration, invariant measure

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1 Introduction

The dispersion of passive tracers (or passive scalars) occur in various geophysical and environmental systems, such as discharge of pollutants into coastal seas or rivers, and temperature or salinity evolution in oceans. Tracers are called passive when they do not dynamically affect the background fluid velocity field. For the benefit of better environment, it is important to understand the dynamics of such passive tracers.

The Eulerian approach for studying passive tracer dispersion attempts to understand the evolution of tracer concentration profile as a continuous field quantity ([2], [14]).

We consider two-dimensional passive tracer dispersion in a shear flow $(u(y), 0)$ (assume that $u(y)$ is bounded). The passive tracer concentration profile $C(x, y, t)$ then satisfies the advection-diffusion equation ([2])

$$C_t + u(y)C_x = \kappa(C_{xx} + C_{yy}) + \text{source}, \tag{1}$$

where the source (or sink) term accounts for effects of chemical reactions ([2]), external injections of pollutants ([14], [8], [15]), or heating and cooling ([1]). Here $\kappa > 0$ is the diffusivity constant.

There has been a lot of research on the advection-diffusion equation *without source*; see, for example, [2], [16], [18], [12], [4], [9], [11] and [13].

In this paper, we study the impact of the external sources on the pattern formation and long-time behavior of the concentration profile. We assume that the concentration profile satisfies double-periodic boundary conditions

$$C, C_x, C_y \;\text{ are double-periodic in } x \text{ and } y \text{ with period } 1, \tag{2}$$

and appropriate initial condition

$$C(x, y, 0) = C_0(x, y). \tag{3}$$

We will consider two classes of sources: time-periodic source $f(x, y, t)$ in Section 2 and random source in Section 3. We will summarize results in Section 4.

2 Time-periodic source

In this section we consider the advection-diffusion equation with time-periodic source (1)

$$C_t + u(y)C_x = \kappa(C_{xx} + C_{yy}) + f(x, y, t), \tag{4}$$

where $f(x,y,t)$ is periodic in t with period $T>0$.

Integrating both sides of (4) with respect to x,y on the domain $D=[0,1]\times[0,1]$, we get

$$\begin{aligned} & \frac{d}{dt}\int\int C dxdy + \int\int u(y) C_x dxdy \\ = \ & \kappa \int\int (C_{xx}+C_{yy}) dxdy + \int\int f(x,y,t) dxdy. \end{aligned} \tag{5}$$

Note that

$$\int\int u(y) C_x dxdy = 0$$

and

$$\int\int (C_{xx}+C_{yy}) dxdy = 0$$

due to the double-periodic boundary conditions (2). We thus have

$$\frac{d}{dt}\int\int C dxdy = \int\int f(x,y,t) dxdy. \tag{6}$$

Here and hereafter, all integrals are with respect to x,y over D. Thus, when there is no source, the spatial average or mean of the concentration $C(x,y,t)$ does not change with time. When there is a source, the time-evolution of the spatial average of $C(x,y,t)$ is determined only by the source term. In order to understand more delicate impact of source on the evolution of $C(x,y,t)$, it is appropriate to assume that the source has zero spatial average or mean:

$$\int\int f(x,y,t) dxdy = 0. \tag{7}$$

With such a source, the mean of $C(x,y,t)$ is a constant. Without loss of generality or after removing the non-zero constant by a translation, we may assume that $C(x,y,t)$ has zero-mean. So we study the dynamical behavior of $C(x,y,t)$ in zero-mean spaces.

We use $\dot{L}^2_{per}(D)$ to denote the standard function space of square-integrable double-periodic (of period 1) functions with zero mean. The usual norm in this space is denoted as $\|\cdot\|$.

Note that the linear operator $-\kappa(\partial_{xx}+\partial_{yy})+u(y)\partial_x$ is sectorial ([7], p. 19) in $\dot{L}^2_{per}(D)$. Thus if $f(x,y,t)$ has continuous derivative in time t, the linear system (4), (2), (3) has a unique strong solution for every $C_0(x,y)$ in $\dot{L}^2_{per}(D)$

([7], p. 52). We now show that this system is a dissipative system in the sense ([10] or [6]) that all solutions $C(x,y,t)$ approach a bounded set in $\dot{L}^2_{per}(D)$ as time goes to infinity. A $T-$time-periodic dissipative system in a Banach space has at least one $T-$time-periodic solution. This result follows from a Leray-Schauder topological degree argument and the Browder's principle ([10], p.235).

Multipling (4) by $C(x,y,t)$ and integrating over D, we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|C\|^2 + \int\int u(y)C_x C dxdy \\ = & -\kappa\int\int |\nabla C|^2 dxdy + \int\int f(x,y,t)C dxdy. \end{aligned} \tag{8}$$

Note that, using the double-periodic boundary conditions (2),

$$\int\int u(y)C_x C dxdy = 0. \tag{9}$$

We further assume that the square-integral of $f(x,y,t)$ with respect to x,y is bounded in time. Then, by the Young inequality,

$$\begin{aligned} \int\int f(x,y,t)C dxdy &\leq \frac{1}{2\epsilon}\int\int |f(x,y,t)|^2 dxdy + \frac{\epsilon}{2}\int\int |C|^2 dxdy \\ &\leq \frac{M}{2\epsilon} + \frac{\epsilon}{2}\int\int |C|^2 dxdy, \end{aligned} \tag{10}$$

where $M > 0$ is a constant independent of t and $\epsilon > 0$ is an arbitrary positive number.

Since C has zero mean, we can use the Poincaré inequality ([5], p. 164) to obtain

$$\|C\|^2 \leq 2\pi\|\nabla C\|^2. \tag{11}$$

Putting $(9), (10), (11)$ into (8), we obtain

$$\frac{1}{2}\frac{d}{dt}\|C\|^2 \leq (\frac{\epsilon}{2} - \frac{\kappa}{2\pi})\|C\|^2 + \frac{M}{2\epsilon}, \tag{12}$$

or

$$\frac{d}{dt}\|C\|^2 \leq (\epsilon - \frac{\kappa}{\pi})\|C\|^2 + \frac{M}{\epsilon}. \tag{13}$$

We now fix $\epsilon > 0$ so small that $\epsilon - \frac{\kappa}{\pi} < 0$. By the Gronwall inequality ([17]), we finally get

$$\|C\|^2 \leq (\|C_0\|^2 + \frac{M}{\epsilon(\epsilon - \frac{\kappa}{\pi})})e^{(\epsilon - \frac{\kappa}{\pi})t} + \frac{M}{\epsilon(\frac{\kappa}{\pi} - \epsilon)}. \tag{14}$$

Hence all solutions $C(x, y, t)$ enter a bounded set in $\dot{L}^2_{per}$,

$$\{C : \ \ \|C\| \leq \sqrt{\frac{M}{\epsilon(\frac{\kappa}{\pi} - \epsilon)}}\},$$

as time goes to infinity. The system (4) is therefore a dissipative system and hence has at least one $T-$time-periodic solution ([10], p.235).

We thus have the following conclusion.

Theorem 1 *Assume that the source $f(x, y, t)$ is time-periodic with period $T > 0$ and is continuously differentiable with time t. Also assume that its mean (spatial average) is zero and its spatial square-integral is bounded in time. Then the advection-diffusion problem with time-periodic source*

$$\begin{aligned} C_t + u(y)C_x &= \kappa(C_{xx} + C_{yy}) + f(x, y, t), && (15) \\ C, C_x, C_y & \quad \textit{are double-periodic in } x \textit{ and } y \textit{ with period } 1, && (16) \\ C(x, y, 0) &= C_0(x, y), && (17) \end{aligned}$$

has a time-periodic solution with period $T > 0$.

We remark that it is generally difficult to show existence of time-periodic solutions in a spatially extended system. Our result provides such a proof of existence for an advection-diffusion system with source.

3 Random source

In this section we consider the passive tracer dispersion problem (1) with a white noise source. We want to study the long-time behavior of the distribution function of the random concentration profile $C(x, y, t)$. A white noise is usually modeled by the Ito derivative of a space-time $Q-$Wiener process

$W(x,y,t)$ which has zero mean value (expectation) for each t. Q is a symmetric non-negative linear operator in $\dot{L}^2_{per}(D)$; see [3], §4.1. In this case, (1) becomes

$$dC = (-u(y)C_x + \kappa(C_{xx} + C_{yy}))dt + dW. \tag{18}$$

This is a linear stochastic differential equation in $\dot{L}^2_{per}(D)$. As we mentioned in the last section, $\kappa(\partial_{xx} + \partial_{yy}) - u(y)\partial_x$ generates an analytic semigroup $S(t)$ in $\dot{L}^2_{per}(D)$.

Assume that Q satisfies

$$\int_0^t Tr\ S(r)QS^{\star}(r)dr < +\infty. \tag{19}$$

Then, as can be shown in [3], §5.2 and §5.4, for every initial condition $C_0(x,y)$ in $\dot{L}^2_{per}$, there exists a unique global mild solution $C(x,y,t)$ of the stochastic differential equation (18) under (2) and (3).

The corresponding deterministic equation for (18) is

$$C_t + u(y)C_x = \kappa(C_{xx} + C_{yy}). \tag{20}$$

As in (12), we have

$$\frac{d}{dt}\|C\|^2 \leq -\frac{\kappa}{\pi}\|C\|^2, \tag{21}$$

Thus, by the Gronwall inequality,

$$\|C\|^2 \leq \|C_0\|^2 e^{-\frac{\kappa}{\pi}t}. \tag{22}$$

Therefore,

$$lim_{t\to+\infty} S(t)C_0(x,y) = lim_{t\to+\infty} C(x,y,t) = 0,$$

which implies that,

$$lim_{t\to+\infty}\|S(t)\| = 0.$$

Therefore, according to Theorem 11.11 in [3], the stochastic differential equation (18) has a unique invariant measure (like a stationary state for a deterministic partial differential equation), and the distribution function of any other solution process $C(x,y,t)$ weakly converges to this invariant measure in $\dot{L}^2_{per}$ as $t \to +\infty$.

We thus have the following result.

Theorem 2 *Assume that $Q-$ Wiener process $W(x,y,t)$ satisfies*

$$\int_0^t Tr\ S(r)QS^\star(r)dr < +\infty. \tag{23}$$

Then the advection-diffusion problem with random source

$$dC = (-u(y)C_x + \kappa(C_{xx} + C_{yy}))dt + dW \tag{24}$$

has a unique invariant measure in $\dot{L}^2_{per}$, and the distribution functions of all other solutions weakly converge to this unique invariant measure as $t \to +\infty$.

4 Discussions

In this paper, we have studied the impact of external sources on the pattern formation and long-time behavior of concentration profiles of passive tracers in a two-dimensional shear flow. We have shown that a time-periodic concentration profile exists for time-periodic external source, while for random source, the distribution functions of all concentration profiles weakly converge to a unique invariant measure (like a stationary state in deterministic systems) as time goes to infinity.

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