A Characterization of Hard-Threshold Boolean Functions

*Fernanda Botelho**
botelhof@msci.memphis.edu
Department of Mathematics
The University of Memphis
Memphis, TN 38152

Abstract

This paper characterizes a class of boolean functions, designated of hard-threshold type, in terms of their action on the underlying space graph.

1 Introduction

Boolean functions are discrete valued functions that provide iterative models with a large range of applicability to many different areas of science, namely theoretical computer science (automata networks and cellular automata), biomathematics (Hopfield networks that model interaction between neurons), and physics (discrete models for disorded matter such as the spin glass problem).

In this paper we consider a class of boolean functions, designated hard-threshold, that are examples of discrete type recurrent neural networks and perform efficiently certain functions of the human brain. Recurrent Neural Networks are mathematical models inspired in the human brains and that aim to mimick some specified function. They consist of a large number of processing units that operate in parallel. Each unit of the net makes decisions based on the information received from other units. This way of operating allows a high speed performance and a more economical machine implementation. In fact, information is distributed throughout the units which avoids the expensive maintenance of a central database unit. Also information is encoded and stored in the processing units as binary sequences. One such sequence is designated a configuration state. The activation of the network consists on an interchange of information stored in the units composing the net. Based on the information flow and the present state of each unit (the unit's input), the network makes a simultaneous update of the state of each unit, the new configuration is called an output.

Hard-threshold boolean functions appeared as input output functions of networks that mimick functions such as associative memories. Examples of associative memories are networks that associate to the picture of an animal its name or to a picture of a car its model.

*This work was done while the author was visiting the Institute For Mathematics and Its Applications (IMA) at the University of Minnesota. IMA is supported by the National Sciences Foundation.

Given pairs of binary vectors ($\mathbf{x}$, $\mathbf{y}$), the action of an associative memory network on a given input, $\mathbf{x}$, should result in the corresponding output $\mathbf{y}$. Moreover, given a corrupted or incomplete version of the input, the network should still recover the expected output $\mathbf{y}$.

In this paper we study conditions that assure that a given function is of hard-threshold type and therefore can be implemented as an associative memory network. This study provides a characterization based on the action of the map on the vertices of the underlying graph space.

In section 2 we present and motivate the notation and basic definitions used throughout the paper, in section 3 we state and prove the main result.

2 Basic Definitions and Notation

A basic structure of a Hopfield network with n units is shown in Fig.1. Each unit or cell is connected to each one of the other units composing the net with a channel, that represents a real synapsis between two neural cells. Each cell may also be connected to itself via a feedback connection. Signals passing through a network connection change linearly by a multiplcative factor, called weight. The weight matrix collects all these multiplicative factors and is denoted by $\mathbf{W}$. The incoming signal into a given cell is just the sum of all linearly altered signals sent from any cell connected to that specific one. As soon as a signal reaches a cell, it is modified by the cell external input (constant vector $\Theta = \{\theta_i\}_{\{i=1\ldots n\}}$) and its transfer function. The transfer function captures the cell's response to an incoming signal. A cell may send a signal down its axon or remain passive, depending on the intensity of the incoming signal. This is represented by the function σ that at each real number x assigns 1 if $x \geq 0$ and 0 otherwise. A configuration is a binary n-sequence whose components represent the states of the network cells.

We set notation to be followed throughout the entire paper. Boldface lower case letters represent configurations of the network which are vertices of the unit hypercube $[0, 1]^2$. A point $\mathbf{x}$ is an n-tuple of 0's and 1's, $\{x_i\}_{i=1}^n$, with x_i representing the state of cell i. The connecting weight ω_{ij} is attached to the connection from cell j to cell i and is the ij entry of the connecting matrix, $\mathbf{W}$. The action of $\mathbf{W}$ on some point $\mathbf{x}$ is given by the standard matrix multiplication and denoted by $\mathbf{W}(\mathbf{x})$ or just $\mathbf{W}\mathbf{x}$. For simplicity of notation we define $\sigma(\mathbf{x})$ to be the column vector $[\sigma(x_i)]_i$. The map π_j represents the standard projection onto the j-th component.

The set of all n-sequences of 0's and 1's is denoted by $\mathcal{V}$. A boolean map is a transformation on $\mathcal{V}$.

Definition 2.1 *A boolean map T is of hard-threshold type if there exists a matrix $\mathbf{W}$ and a constant n-vector Θ such that*

$$T(\mathbf{x}) = \sigma(\mathbf{W}\mathbf{x} - \Theta),$$

Since $\mathcal{V}$is a finite set we can always choose Θ so that $\mathbf{W}\mathbf{x} - \Theta \neq 0$ for all $\mathbf{x} \in \mathcal{V}$. This is an assumption made throughout the entire paper.

Definition 2.2 *Given $\mathbf{x}$ and $\mathbf{y}$ points in $\mathcal{V}$, $\mathbf{y}$ is called an immediate neighbor of $\mathbf{x}$ if and only*

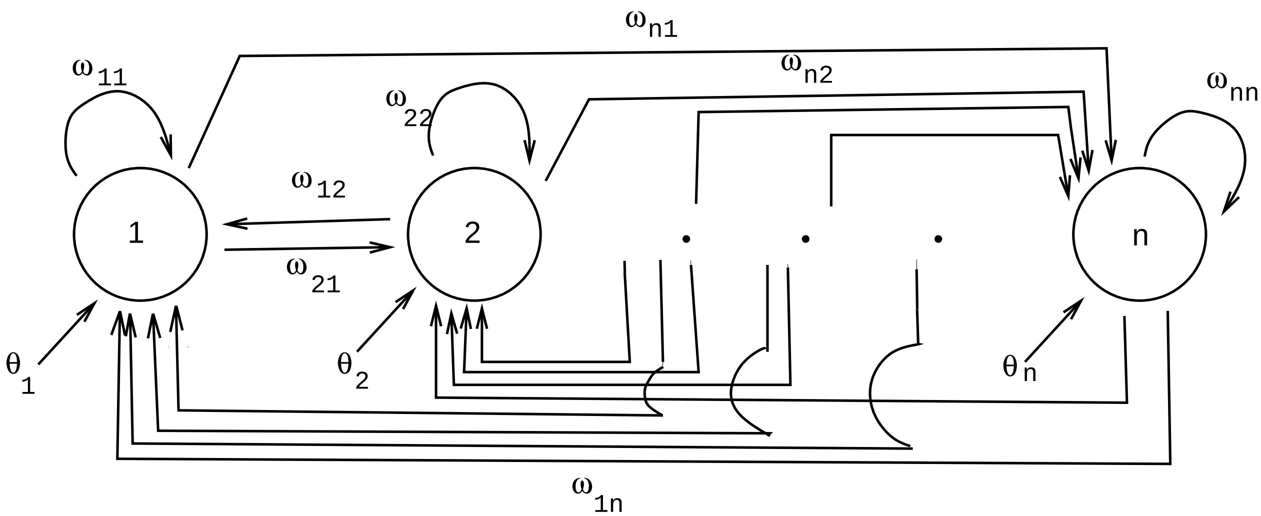

Figure 1: Hopfield Network with n Cells

if there exists $i \in \mathbf{N}$ such that

$$\mathbf{x}_k = \mathbf{y}_k \quad \text{for } k \neq i \quad \text{and} \quad \mathbf{x}_i \neq \mathbf{y}_i.$$

For simplicity of notation and to emphasize the dependence on the i^{th}−coordinate, we represent the immediate neighbor of $\mathbf{x}$ altered at site i by $\mathbf{x}^i$. The set of all immediate neighbors of $\mathbf{x}$ is denoted by $\mathcal{N}_{\mathbf{x}}$.

Definition 2.3 *A subset $\mathcal{S}$ of $\mathcal{V}$ is called connected if and only if for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{S}$ there exists a sequence in $\mathcal{S}$, $\{\mathbf{x^0} = \mathbf{x},\ \mathbf{x^1},\ \cdots,\ \mathbf{x^j} = \mathbf{y}\}$, such that $\mathbf{x^k}$ is an immediate neighbor of $\mathbf{x^{k-1}}$.*

Associated to a boolean function T we consider the following two sets at level i:

1. The positive set

$$\mathcal{P}_i = \{\mathbf{x} \in \mathcal{V} :\ T(\mathbf{x})_i = 1\},$$

2. The negative set

$$\mathcal{N}_i = \{\mathbf{x} \in \mathcal{V} :\ T(\mathbf{x})_i = 0\}.$$

We also denote the convex hull of $\mathcal{P}_i$ and $\mathcal{N}_i$, in R^n, by $\langle \mathcal{P}_i \rangle$ and $\langle \mathcal{N}_i \rangle$, repectively.

3 The Main Result

Proposition 3.1 *A boolean map T is of hard-threshold type if and only if for every i, $\mathcal{P}_i$ and $\mathcal{N}_i$ are connected sets in $\mathcal{V}$, and $\langle \mathcal{P}_i \rangle \cap \langle \mathcal{N}_i \rangle = \emptyset$.*

First we prove the following two Lemmas that will be used in the proof of the Proposition. These Lemmas are stated for a boolean function, T, of hard-threshold type. Given a point $\mathbf{x} \in \mathcal{V}$ we consider the set, $\mathbf{P_x} = \{i : \mathbf{x}_i = 1\}$.

Lemma 3.1 *If* $\mathbf{x} \in \mathcal{P}_i$, $\mathbf{y} \in \mathcal{N}_i$, *and* $\mathbf{y}$ *an immediate neighbor of* $\mathbf{x}$ *then* $\omega_{ik} > 0$ *if* $k \in \mathbf{P_x}$ *and* $\omega_{ik} < 0$ *if* $k \notin \mathbf{P_x}$.

Proof: We assume that $\mathbf{y} = \mathbf{x}^k$, using the notation described in the Definition 2.2. We have that

$$T(\mathbf{y})_i = \sum_{j=1}^{n} \omega_{ij}\mathbf{y}_j - \theta_i = \begin{cases} \sum_{j\in\mathbf{P_x}} \omega_{ij} - \omega_{ik} - \theta_i < 0 & \text{if} \quad k \in \mathbf{P_x} \\ \sum_{j\in\mathbf{P_x}} \omega_{ij} + \omega_{ik} - \theta_i < 0 & \text{if} \quad k \notin \mathbf{P_x}. \end{cases}$$

The statement in the Lemma follows from the assumption that $\sum_{j\in\mathbf{P_x}} \omega_{ij} - \theta_i > 0$. $\diamondsuit$

Given a subset $\mathcal{S}$ of $\mathcal{V}$ we denote by $\mathcal{N}_{\mathcal{S}}$ the set of all points that are immediate neighbors of some point in $\mathcal{S}$.

Lemma 3.2 *If* $\mathcal{S}$ *is a connected subset of* $\mathcal{P}_i$ *(or* $\mathcal{N}_i$*) such that* $\mathcal{N}_{\mathcal{S}} \setminus \mathcal{S} \subset \mathcal{N}_i$ *(or* $\mathbf{P}_i$*) then* $\mathcal{N}_{\mathcal{N}_{\mathcal{S}}\setminus\mathcal{S}} \setminus \mathcal{S} \subset \mathcal{N}_i$ *(or* $\mathcal{P}_i$*, respectively).*

Proof: Let $\mathbf{z} \in \mathcal{N}_{\mathcal{N}_{\mathcal{S}}\setminus\mathcal{S}} \setminus \mathcal{S}$, $\mathbf{y} \in \mathcal{N}_{\mathcal{S}} \setminus \mathcal{S}$ such that $\mathbf{y}^k = \mathbf{z}$, and $\mathbf{x} \in \mathcal{S}$ such that $\mathbf{y} = \mathbf{x}^t$ $(t \neq k)$. Therefore, we have that for $i \neq k$ and $i \neq t$, $\mathbf{z}_i = \mathbf{y}_i = \mathbf{x}_i$, $\mathbf{z}_k \neq \mathbf{y}_k = \mathbf{x}_k$, and $\mathbf{z}_t = \mathbf{y}_t \neq \mathbf{x}_t$. This implies that

$$\sum_{j=1}^{n} \omega_{ij} z_j - \theta_i = \begin{cases} \sum_{j=1}^{n} \omega_{ij}\mathbf{y}_j + \omega_{ik} - \theta_i & \text{if} \quad \mathbf{y}_k = 0\,(\Leftrightarrow\ k \notin \mathbf{P_x}) \\ \sum_{j=1}^{n} \omega_{ij}\mathbf{y}_j - \omega_{ik} - \theta_i & \text{if} \quad \mathbf{y}_k = 1\,(\Leftrightarrow\ k \in \mathbf{P_x}). \end{cases}$$

Moreover, $\mathbf{y} \in \mathcal{N}_i$ and Lemma 3.1 imply that $\sum_{j=1}^{n} \omega_{ij} z_j - \theta_i < 0$. The second case stated in Lemma follows by similar arguments. $\diamondsuit$

Proof of Proposition 3.1: First, we assume T is of hard-threshold type with $\mathcal{P}_i$ and $\mathcal{N}_i$, the positive and negative sets at level i. If $\mathcal{P}_i$ is not connected then Lemma 3.2 applied to a connected component leads to a contradiction. Similarly if $\mathcal{N}_i$ is not connected. This shows that both $\mathcal{P}_i$ and $\mathcal{N}_i$ are connected. Furthermore, the Definition 2.1 implies that the convex hulls are disjoint.

Conversely, we assume that for every i, $\mathcal{P}_i$ and $\mathcal{N}_i$ are connected with disjoint convex hulls. There are points $\mathbf{x} \in \langle\mathcal{P}_i\rangle$ and $\mathbf{y} \in \langle\mathcal{N}_i\rangle$ whose distance is the shortest possible ie. $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = d(\langle\mathcal{P}_i\rangle, \langle\mathcal{N}_i\rangle)$, where d represents the usual euclidean distance in R^n and $\|.\|$ the standard norm. Let $\alpha = \frac{\mathbf{x}+\mathbf{y}}{2}$, the middle point between $\mathbf{x}$ and $\mathbf{y}$. Now, we show that $\mathbf{x}$ is unique, similar arguments prove that $\mathbf{y}$ is also unique. Suppose there exists $\mathbf{x}_1 \in \langle\mathcal{P}_i\rangle$ different from $\mathbf{x}$ such that $\|\mathbf{x}_1 - \alpha\| = \|\mathbf{x} - \alpha\|$, then for $0 < \lambda < 1$ we have

$$\|\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x} - \alpha\|^2 = \lambda^2\|\mathbf{x}_1 - \alpha\|^2 + (1-\lambda)^2\|\mathbf{x} - \alpha\|^2 + 2\lambda(1-\lambda)(\mathbf{x} - \alpha, \mathbf{x}_1 - \alpha).$$

The expression $(\mathbf{x} - \alpha, \mathbf{x}_1 - \alpha)$ refers to the usual inner product in R^n. Schwartz inequality [5] implies that

$$\|\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x} - \alpha\|^2 \begin{cases} = \|\mathbf{x} - \alpha\|^2 & \text{if} \quad \mathbf{x} - \alpha = \mu(\mathbf{x}_1 - \alpha) \text{ or } \mathbf{x}_1 - \alpha = \mu(\mathbf{x} - \alpha) \\ < \|\mathbf{x} - \alpha\|^2 & \qquad \text{otherwise.} \end{cases}$$

Clearly $\|\mathbf{x}_1 - \alpha\|^2 < \|\mathbf{x} - \alpha\|^2$ cannot occur since $d(\langle \mathcal{P}_i \rangle,\ \langle \mathcal{N}_i \rangle) = 2\|\mathbf{x} - \alpha\|$. If $\mathbf{x} - \alpha = \mu(\mathbf{x}_1 - \alpha)$ or $\mathbf{x}_1 - \alpha = \mu(\mathbf{x} - \alpha)$ then $\mu = 1$ or -1. Both cases are impossible. In fact, if $\mu = 1$ then $\mathbf{x} = \mathbf{x}_1$, if $\mu = -1$ then $\alpha = \frac{\mathbf{x}+\mathbf{x}_1}{2} \in \langle \mathcal{P}_i \rangle$. By a shift of the coordinate system we assume that the origin coincides with α and we represent by $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ the vectors $\vec{\alpha\mathbf{x}}$ and $\vec{\alpha\mathbf{y}}$, respectively.

We define a linear transformation on R^n

$$f(\vec{\mathbf{z}}) = \frac{(\vec{\mathbf{z}},\ \vec{\mathbf{x}})}{\|\vec{\mathbf{x}}\|}.$$

For simplicity of notation we represent a vector $\vec{\alpha\mathbf{z}}$ by $\mathbf{z}$. We prove that f is positive on $\langle \mathcal{P}_i \rangle$ and negative on $\langle \mathcal{N}_i \rangle$. The point $\mathbf{x} \in \mathcal{P}_i$ and $f(\mathbf{x}) = \|\mathbf{x}\| > 0$. If there exists a point $\mathbf{z} \in \mathcal{P}_i$ such that $f(\mathbf{z}) = 0$ then $\mathbf{z}$ is orthogonal to $\mathbf{x}$. Given $0 < \lambda < \frac{2\|\mathbf{x}\|^2}{\|\mathbf{z}\|^2 + \|\mathbf{x}\|^2}$, we have that

$$\|\lambda z + (1 - \lambda)\mathbf{x}\|^2 < \|\mathbf{x}\|^2$$

which implies that $d(\langle \mathcal{P}_i \rangle,\ \langle \mathcal{N}_i \rangle) < \|\mathbf{x} - \mathbf{y}\|$. Similar considerations show that f is negative on $\mathcal{N}_i$.

We define the hyperplane $\mathbf{H} = \{\mathbf{z}\ :\ \mathbf{f}(\mathbf{z}) = \mathbf{0}\}$. This hyperplane separates the sets $\mathcal{N}_i$ and $\mathcal{P}_i$. T is of hard-threshold type with connecting weights given by $\omega_{ij} = \frac{x_j - y_j}{2}$ and external input $\theta_i = \frac{x_i^2 - y_i^2}{4}$. $\diamondsuit$

Example
This example shows that the disjointness required in the Proposition 3.1 is in fact necessary. We consider a boolean map, T, defined on $\{0,\ 1\}^3$ such that

$$\mathcal{P}_1 = \{(0,\ 0,\ 1),\ (0,\ 1,\ 1),\ (0,\ 1,\ 0),\ (1,\ 1,\ 0)\}$$

and

$$\mathcal{N}_1 = \{0,\ 0,\ 0),\ (1,\ 0,\ 0),\ (1,\ 0,\ 1),\ (1,\ 1,\ 1)\}.$$

We verify that this map is not of Hard-Threshold type. Suppose otherwise that for every $\mathbf{x}$, $\pi_1(T(\mathbf{x})) = \sigma(\sum_{i=1}^{3} \omega_{1i} x_i - \theta_1)$. Since $\pi_1(T(0,\ 0,\ 0)) = 0$, we have that $\theta_1 > 0$. Furthermore, $\pi_1(T(1,\ 1,\ 1)) = 0$ and $\pi_1(T(1,\ 1,\ 0)) = 1$ imply that $\omega_{13} < 0$ and $\pi_1(T(0,\ 0,\ 1)) = 0$ which contradicts that $(0,\ 0,\ 1) \in \mathcal{P}_1$. It is easy to check that both $\mathcal{P}_1$ and $\mathcal{N}_1$ are connected subsets of $\{0,\ 1\}^3$ and that $(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}) \in \langle \mathcal{P}_1 \rangle \cap \langle \mathcal{N}_1 \rangle$.

We introduce a definition of path between two points in $\mathcal{H}$ and between two subsets of $\mathcal{V}$.

Definition 3.1 *Given two points in $\mathcal{V}$, $\mathbf{x}$ and $\mathbf{y}$ we define a* **path** *between them to be a finite sequence $\{\mathbf{z}^0 = \mathbf{x},\ \mathbf{z}^1,\ \ldots,\ \mathbf{z}^{k-1},\ \mathbf{z}^k = \mathbf{y}\}$ such that $\mathbf{z}_i$ is an immediate neighbor of $\mathbf{z}^{i-1}$. A path between two sets is a path between two points, one point in each set. The length of a path is the number of elements in the sequence defining the path.*

Proposition 3.2 *For every i such that $\langle \mathcal{P}_i \rangle \cap \langle \mathcal{N}_i \rangle = \emptyset$, $\mathcal{P}_i$ and $\mathcal{N}_i$ are connected.*

Proof: We assume that $\mathcal{P}_i$ is not connected, then it has at least two connected components. We select two components of $\mathcal{P}_i$, $\mathcal{C}_1$ and $\mathcal{C}_2$ whose path between them has the shortest possible length. We represent such path by the sequence

$$\phi:\ \{\mathbf{x^1},\ \mathbf{x^2},\ \mathbf{x^3},\ \ldots,\ \mathbf{x^k}\},$$

where $\mathbf{x^1} \in \mathcal{C}_\mathbf{1}$, $\mathbf{x^k} \in \mathcal{C}_\mathbf{2}$, and $\mathbf{x^j} \in \mathcal{N}_\mathbf{i}$ ($\mathbf{j} \neq \mathbf{1}$ and $\mathbf{k}$). We associate to the path above an injective sequence of indices $\{i_1,\ i_2,\ \ldots,\ i_{k-1}\}$, where i_t is such that

$$\mathbf{x^t_{i_t}} \neq \mathbf{x^{t+1}_{i_t}} \text{ and } \mathbf{x^t_p} = \mathbf{x^{t+1}_p},\ \text{if } \mathbf{p} \neq \mathbf{i_t}.$$

We consider a new path between $\mathbf{x^1}$ and $\mathbf{x^k}$,

$$\mathbf{x^1},\ \bar{\mathbf{x}}^\mathbf{2},\ \bar{\mathbf{x}}^\mathbf{3},\ \ldots,\ \bar{\mathbf{x}}^{\mathbf{k-1}},\ \mathbf{x^k},$$

defined as follows:

The point $\bar{\mathbf{x}}^{t+1}$ is such that $\bar{\mathbf{x}}^{t+1}_{i_{k-t}} \neq \bar{\mathbf{x}}^{t}_{i_{k-t}}$ and $\bar{\mathbf{x}}^{t+1}_j = \bar{\mathbf{x}}^{t}_j$, for $j \neq i_{k-t}$.

The points $\bar{\mathbf{x}}^2,\ \ldots,\ \bar{\mathbf{x}}^{k-1}$ are in $\mathcal{N}_i$ since ϕ is a path of shortest length between two components of $\mathcal{P}_i$. Therefore $\frac{1}{2}(\mathbf{x^1} + \mathbf{x^k}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{x^2} + \bar{\mathbf{x}}^{\mathbf{k-1}}) \in \langle \mathcal{P}_\mathbf{i} \rangle \cap \langle \mathcal{N}_\mathbf{i} \rangle$. This leads to a contradiction which proves the statement. $\diamondsuit$

Propositions 3.1 and 3.2 are summarized in the next Theorem.

Theorem 3.1 *A boolean map is of hard-threshold type if and only if for every i, $\langle \mathcal{P}_i \rangle \cap \langle \mathcal{N}_i \rangle = \emptyset$.*

References

[1] F. Botelho and M. Garzon, *Neural Networks of rank one*, Proc. of the 13th Conf. in Applied Mathematics, **13** (1997), pp. 47-59.

[2] J. Hopfield, *Neural Networks and Physical Systems with Emergent Collective Computational Abilities*, Procc. of the National Academy of Sciences of the USA, **79** (1982), pg.2554-2558.

[3] J. Milton, *Dynamics of Small Neural Populations*, CRM Monograph Series **7** AMS Providence, Rhode Island, 1996.

[4] F. Robert, *Discrete Iterations*, Springer-Verlag, Berlin, 1986.

[5] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1976.

[6] P. de Wilde, *Neural Network Models*, Springer Verlag, 1997.