

ON THE EXISTENCE OF GLOBAL ATTRACTORS

GEORGE R. SELL AND YUNCHENG YOU

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ABSTRACT. Let σ be a semiflow on a complete metric space W . Assume that σ is κ -contracting (in the sense of Kuratowski) and point dissipative. We show here that σ has a global attractor in W . This is an improvement over earlier results in which a further assumption, wherein the orbit of each bounded set is bounded, is made for σ . Furthermore, this result includes the Billotti-LaSalle Theorem, where σ is assumed to be compact for $t \geq r > 0$, as a special case.

1. INTRODUCTION

A semiflow is a mechanism for describing the solutions of a given differential equation. While this concept had its birth in the theory of dynamical systems and ordinary differential equations, our interest here is in the infinite dimensional theory with applications to partial differential equations and functional differential equations. About 20 to 30 years ago, researchers began to observe that some of these infinite dimensional systems had finite dimensional dynamical structures. In many cases, the derivation of the finite dimensional structures was based on properties of the global attractor for the given dynamical system. It is the issue of the existence of a global attractor in the infinite dimensional setting that is the main topic of this paper.

Among the earliest papers on the existence of a global attractor are those of Billotti and LaSalle (1971) and Ladyzhenskaya (1972). The objective of the Billotti and LaSalle paper involves the study of a semiflow σ on a complete metric space W , where the semiflow was assumed to be compact, i.e., σ maps each bounded set in W into a compact set, for some time $t > 0$. In addition, they assumed the semiflow to be (point) dissipative, i.e., each solution of the given differential equation is ultimately bounded, where the bound does not depend on the initial condition. Under these conditions, they then showed the existence of a global attractor for the given semiflow. The focus of the Ladyzhenskaya paper, on the other hand, is on the two dimensional (2D) Navier-Stokes equations (NSE). In particular, she

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showed that the semiflow generated by the solutions of the 2D NSE is compact on a suitable Hilbert space, and that this semiflow is (point) dissipative. She then showed the existence of a global attractor for this problem.

The Billotti-LaSalle Theorem is well suited for applications to parabolic-type partial differential equations (such as the 2D NSE and various reaction diffusion equations) and to functional differential equations where the time derivatives are not evaluated in the past. The reason for this is that the semiflows generated by these problems are compact.

However, this theorem is not applicable in the setting of hyperbolic-type partial differential equations, such as the nonlinear wave equation, nor in the setting of neutral functional differential equations. The semiflows generated by these problems, even when they are (point) dissipative, are not compact, see for example, Babin and Vishik (1983), Ball (1978), Feireisl and Zuazua (1993), Ghidaglia and Temam (1987), Hale (1988), Hale and Raugel (1992), Haraux (1988), Ladyzhenskaya (1987), Marion (1989), Massatt (1983), Teman (1988, Chap 2), and You (1994). The strategy used by Hale, LaSalle, and Slemrod (1972), in addressing the dynamics of such problems, is based on a concept of asymptotic smoothness, also see Hale and Lopes (1973). Another approach, used by Ladyzhenskaya (1987) and Ball (1996), is based on a related concept of asymptotic compactness. As a practical matter, a simple verification of the asymptotic smoothness or asymptotic compactness properties is based on the concept of a κ -contracting semiflow, see Lopes and Ceron (1984), Hale (1988), Temam (1988), and Sell and You (1998). A basic theorem, one finds in the literature, see Hale (1988), for example, for the existence of a global attractor is the following: Assume that σ is a κ -contracting semiflow on a complete metric space W and

- (1) that σ is (point) dissipative, and
- (2) that the positive trajectory of every bounded set is bounded,

then there is a global attractor \mathfrak{A} , and \mathfrak{A} attracts every bounded set B in W .

In this paper, we will prove the following extension of the last result:

Main Theorem. *Let σ be a κ -contracting semiflow on a complete metric space W , and assume that σ is point dissipative. Then there is a global attractor \mathfrak{A} for σ , and \mathfrak{A} attracts every compact set K in W .*

Assume, in addition, that σ satisfies one of the following properties:

- (1) σ is compact, or
- (2) σ is ultimately bounded,

then \mathfrak{A} attracts every bounded set B in W .

The very definition of an attractor is a subtle issue, which we have dealt with carefully. Some authors have used various minimality, or maximality, or connectedness, properties in their definitions. As a result, this can lead to difficulties in comparing theories in different papers. Since these concepts are, in fact, consequences of more basic features of an attractor, we feel that they should not be included as a part of the basic definition. Even though the definition of an attractor we use here is weaker than that used by others, we are able to develop a theory with the same richness as found in other works. See for example, Babin and Vishik (1992), Ball (1996), Conley (1978), Hale (1988), Ladyzhenskaya (1987, 1991), Sell

(1971), Temam (1988), and Vishik (1992). An extensive bibliography on dynamical systems appears in Sell (1997) on the Web.

In Section 2 we present the basic theory of semiflows for finite and infinite dimensional systems. In addition, we present the concepts of compact and κ -contracting semiflows. The basic theory of attractors is presented in Section 3, and the proof of the Main Theorem is given in Section 4. Comparisons between our theory and the theories of other researchers are presented in Section 5.

2. BASIC THEORY OF SEMIFLOWS

We will let W denote a complete metric space. The distance between two points u and v in W will be denoted by $d(u, v) = d_W(u, v)$, where $d = d_W$ is a metric on W . Recall that if W is a Banach space, then the standard metric on W is given by $d(u, v) = \|u - v\|$, where $\|\cdot\|$ is the norm on W . In the case of a Fréchet space W , we will use an invariant metric d , where $d(u, v) = d(u - v, 0)$, for all $u, v \in W$.

Let M be a subset of W , and let $R = (-\infty, \infty)$ and $R^+ = [0, \infty)$. A mapping $\sigma = \sigma(u, t)$, where $\sigma : M \times [0, \infty) \rightarrow M$, is said to be a **semiflow** on M , provided the following hold:

- (1) $\sigma(w, 0) = w$, for all $w \in M$.
- (2) The **semigroup property** holds, i.e.,

$$(2.1) \quad \sigma(\sigma(w, s), t) = \sigma(w, t + s), \quad \text{for all } w \in M, \text{ and } s, t \in R^+.$$

- (3) The mapping $\sigma : M \times (0, \infty) \rightarrow M$ is continuous.

If in addition, the mapping $\sigma : M \times [0, \infty) \rightarrow M$ is continuous, we will say that the semiflow¹ σ is **continuous at $t = 0$** . For some applications, we have found it convenient to not include the continuity at $t = 0$ in the definition of a semiflow. Details on this issue can be found in Sell and You (1998).

The prototype of a semiflow is a C_0 -semigroup of linear operators on a Banach space W , and this semiflow is continuous at $t = 0$. In this case, the mapping $u \rightarrow \sigma(u, t)$ is a bounded linear operator on W . If the mapping $u \rightarrow \sigma(u, t)$ is not linear, then the semiflow is sometimes referred to as a **nonlinear semigroup**. On occasion we will write σ in the form $\sigma(u, t) = S(t)u$. In this notation, the semigroup property (2.1) then takes on the form

$$(2.1a) \quad S(t)S(s)u = S(s + t)u, \quad \text{for all } s, t \geq 0.$$

It may happen that for each $t \geq 0$, the mapping $S(t)$ is a one-to-one mapping of M onto M with a continuous inverse $S(t)^{-1}$. In this case, we set $S(-t) \stackrel{\text{def}}{=} S(t)^{-1}$, for $t > 0$. As a result, (2.1a) holds for all $u \in M$ and all $s, t \in R$. In this case, the dynamical system will be referred to as a **flow**. See Sell (1971, 1973) for an alternate approach.

There do exist discrete versions of these concepts. For a fixed number $\tau > 0$, define

$$\tau Z^+ = \{n\tau : n = 0, 1, \dots\} \quad \text{and} \quad \tau Z = \{n\tau : n = 0, \pm 1, \dots\}.$$

¹Notice that *continuity at $t = 0$* is a statement of joint continuity in (u, t) at each point $(u_0, 0)$.

A mapping $\sigma(\cdot, t) : M \rightarrow M$, for $t \in \tau Z^+$, is said to be a **discrete semiflow** on M , provided that σ is continuous, $\sigma(w, 0) = w$, for all $w \in M$, and equation (2.1) holds for all $w \in M$ and $s, t \in \tau Z^+$. A **discrete flow** is defined similarly, but now with τZ replacing τZ^+ . The prototype of a discrete semiflow arises when one begins with a semiflow $\sigma(w, t)$ on M , where $t \in [0, \infty)$, and then restricts time t to satisfy $t \in \tau Z^+$, for some $\tau > 0$. Another example is the Poincaré map generated by a system of ordinary differential equations with periodic coefficients. More generally, a mapping $\sigma(u, n\tau)$ is a discrete semiflow on M if and only if there is a continuous mapping $T : M \rightarrow M$ such that

$$(2.1m) \quad \sigma(u, n\tau) = T^n(u), \quad \text{for all } u \in M \text{ and } n \in Z^+.$$

Note that (2.1m) implies that $T(u) = \sigma(u, \tau)$, for all $u \in M$. The theory of global attractors we present here applies, with no other changes, to discrete semiflows. However, we will focus on the continuous time problem in the sequel.

Lemma 2.1 (Continuity Lemma). *Let $S(t)$ be a semiflow on $M \subset W$. Then the following hold:*

(1) *For any convergent sequence u_n with limit $u = \lim_{n \rightarrow \infty} u_n \in M$, one has*

$$(2.1b) \quad \sup_{\tau \leq t \leq T} d(\sigma(u_n, t), \sigma(u, t)) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

for any τ and T with $0 < \tau \leq T < \infty$.

(2) *For any compact set K in M , the set*

$$N = \{S(t)u : u \in K, \tau \leq t \leq T\}$$

is compact in M , for any τ and T with $0 < \tau \leq T < \infty$.

(3) *Let K be a compact set in M with the property that one has $S(t)K \subset K$, for all $t \geq 0$. Then for any τ and T with $0 < \tau \leq T < \infty$ and any $\epsilon > 0$, there is a $\delta > 0$, such that if $d(u, K) \leq \delta$, then one has*

$$d(S(t)u, K) < \epsilon, \quad \text{for } \tau \leq t \leq T.$$

If in addition, the semiflow $S(t)$ is continuous at $t = 0$, then the three properties above remain valid with $\tau = 0$.

Proof. Part (1) follows immediately from the fact that any continuous function is uniformly continuous on a compact set. For Part (2) we let $v_n = S(t_n)u_n$ be a sequence in N . By using the compactness of $[\tau, T]$ and K , we can extract subsequences, which we relabel as t_n , u_n , and v_n , so that the limits $t = \lim t_n$ and $u = \lim u_n$ exist. One then has

$$d(S(t)u, S(t_n)u_n) \leq d(S(t)u, S(t_n)u) + d(S(t_n)u, S(t_n)u_n).$$

Now $d(S(t)u, S(t_n)u) \rightarrow 0$, as $n \rightarrow \infty$, because of the continuity in t . From (2.1b), we obtain $d(S(t_n)u, S(t_n)u_n) \rightarrow 0$, as $n \rightarrow \infty$. Part (3) is an immediate consequence of the continuity property in the definition. Finally, if semiflow $S(t)$ is continuous at $t = 0$, then the argument above remains valid with $\tau = 0$. \square

2.1 Invariant Sets. Let σ be a semiflow on $M \subset W$. For any $u \in M$ the **(positive) trajectory through u** is defined as the set

$$\gamma^+(u) \stackrel{\text{def}}{=} \{S(t)u : t \geq 0\},$$

and the mapping $t \rightarrow S(t)u$ of R^+ into M is referred to as the **(positive) motion through u** . For any set $K \subset M$ we define

$$S(t)K = \sigma(K, t) \stackrel{\text{def}}{=} \{S(t)u : u \in K\},$$

and the **trajectories through K** are given by

$$\gamma^+(K) \stackrel{\text{def}}{=} \{S(t)K : t \geq 0\} = \{S(t)u : u \in K, t \geq 0\}.$$

A set $K \subset M$ is said to be **positively invariant** if $S(t)K \subset K$ for all $t \geq 0$, and K is said to be an **invariant set** if $S(t)K = K$ for all $t \geq 0$. For example, the ambient space M , as well as the trajectory $\gamma^+(B)$, where $B \subset M$, are positively invariant sets. A **stationary trajectory** (where $S(t)u = u$, for all $t \geq 0$) as well as the trajectory of a **periodic motion** (where $S(t + \tau)u = u$, for all $t \geq 0$ and some $\tau > 0$) generate examples of invariant sets. Notice that for any set B in M one has

$$(2.1c) \quad \gamma^+(S(t)B) = S(t)\gamma^+(B), \quad \text{for all } t \geq 0.$$

There is a very useful characterization of an invariant set, which is given in the next lemma. We will say that a continuous mapping $\phi^u : R \rightarrow M$ is a **globally defined motion through u with range in M** if $\phi^u(0) = u$ and one has

$$(2.3) \quad S(t)\phi^u(\tau) = \phi^u(\tau + t), \quad \text{for all } \tau \in R \text{ and } t \in R^+.$$

Notice that (2.3) implies that $S(t)u = \phi^u(t)$, for $t \geq 0$. The restriction of a globally defined motion ϕ^u to $R^- = (-\infty, 0]$ is referred to as a **negative continuation** of the motion through u . There is a related concept which arises when one has a continuous mapping $\phi^u : [-T, \infty) \rightarrow M$ that satisfies $\phi^u(0) = u$ and (2.3) is valid for $\tau \in [-T, \infty)$ and $t \geq 0$, where $T > 0$. In this case the restriction $\phi^u : [-T, 0] \rightarrow M$ is referred to as a **partial negative continuation** of the motion through u .

Not every point $u \in M$ need have a globally defined motion passing through it. Also note that we have not addressed the issue of the uniqueness of a negative continuation. However, it is known that, under certain assumptions, which hold for many partial differential equations, a negative continuation is unique whenever it exists, see Bardos and Tartar (1973) or Temam (1988).

Lemma 2.2. *Let σ be a semiflow on $M \subset W$, and let K be a set in M . Then K is an invariant set if and only if every $u \in K$ has a globally defined motion ϕ^u passing through u with $\phi^u(t) \in K$ for all $t \in R$.*

Proof. First assume that for every $u \in K$ there is a globally defined motion ϕ^u passing through u with $\phi^u(t) \in K$ for all $t \in R$. Then by using (2.3), we see that

for each $\tau \geq 0$ and any $u \in K$ one has $S(\tau)u = \phi^u(\tau) \in K$, so that $S(\tau)K \subset K$. Since $u = S(\tau)\phi^u(-\tau)$, one has $K \subset S(\tau)K$. Hence K is invariant.

Next assume that K is invariant. Since $S(1)K = K$, we see that for each $u \in K$ there is a $v^1 \in K$ with $S(t)v^1 \in K$, for $0 \leq t \leq 1$, and $S(1)v^1 = u$. By iterating this and using induction, one finds a sequence $v^k \in K$ with $S(t)v^k \in K$, for $0 \leq t \leq 1$, and $S(1)v^k = v^{k-1}$, for $k = 1, 2, \dots$, where $v^0 = u$. We now define $\phi^u : (-\infty, 0] \rightarrow K$ by $\phi^u(t) = S(t)v^k$, for $-k < t \leq -k + 1$ and $k = 1, 2, \dots$. The final step, which is left as an exercise, is to show that (2.1a) implies (2.3). \square

The following result has interesting consequences. Since the proof is rather simple, we will omit the details.

Lemma 2.3. *Let N be an invariant set for a semiflow σ on $M \subset W$, and assume that $K = \text{Cl}_M N$ is a compact set. Then K is an invariant set for σ .*

2.2 Limit Sets. The longtime dynamics of any semiflow is described in terms of the limit sets of the semiflow, i.e., the omega limit set and the alpha limit set. Let σ be a semiflow on $M \subset W$. For any set $B \subset M$ we define the the **(positive) hull** $H^+(B)$ and the **omega limit set** $\omega(B)$ as follows:

$$H^+(B) \stackrel{\text{def}}{=} \text{Cl}_M \gamma^+(B) \quad \text{and} \quad \omega(B) \stackrel{\text{def}}{=} \bigcap_{\tau \geq 0} H^+(S(\tau)B).$$

Since $\omega(B)$ is the intersection of closed sets, we see that $\omega(B)$ is always a closed set in M . (Note that $\omega(B)$ may empty, even when B is not empty.)

If $\sigma(u, t) = S(t)u$ is a flow on $M \subset W$, then the study of the alpha limit sets, i.e., the behavior as $t \rightarrow -\infty$, is very much like the study of omega limit sets, i.e., the behavior as $t \rightarrow +\infty$. Indeed, if one reverses time by defining $\hat{S}(t)$ by $\hat{S}(t)u = S(-t)u$, for $t \in \mathbb{R}$, then the alpha limit sets of S are precisely the omega limit sets of \hat{S} . The definitions and the theories are the same.

On the other hand, if $S(t)u$ is a semiflow on $M \subset W$, then several complications arise because one is not able to use the time-reversal trick. In order to describe the dynamics on the alpha limit sets in this case, we will use the globally defined motions of σ , see (2.3). Before doing this though, there are several issues which should be noted. These are:

- (1) The globally defined motions of σ need not be defined for each $u \in M$. This will mean that the alpha limit set $\alpha(u)$ need not be defined for every $u \in M$.
- (2) Since the time t -mapping, $u \rightarrow S(t)u$, need not be one-to-one, for $t > 0$, a given point $u \in M$ may have more than one globally defined motion passing through it.
- (3) Another complication arises when M is a positively invariant set for a semiflow σ on N , where $M \subset N \subset W$. One may have a point $u \in M$ with a globally defined motion $\phi^u : \mathbb{R} \rightarrow N$ with $\phi^u(t) \notin M$, for some $t < 0$. Thus globally defined motions will depend on the ambient space.

We will need here the dynamical properties of the negative hull $H^-(\phi^u)$ and the alpha limit set $\alpha(\phi^u)$, for a single negative continuation ϕ^u with $\phi^u(t) \in M$, for all $t \leq 0$. In particular, we define

$$H^-(\phi^u) = \text{Cl}_M \{\phi^u(t) : t \leq 0\} \quad \text{and} \quad \alpha(\phi^u) = \bigcap_{\tau \leq 0} \text{Cl}_M \{\phi^u(t) : t \leq \tau\}.$$

The following result, which gives a characterization of the limit sets, is very useful:

Lemma 2.4 (Characterization Lemma). *Let σ be a semiflow on $M \subset W$, and let B be a subset of M . Then the omega limit set $\omega(B)$ is characterized as the set of all $v \in M$ for which one has*

$$(2.4) \quad v = \lim_{n \rightarrow \infty} S(t_n)u_n$$

for some sequences $u_n \in B$ and $t_n \in \mathbb{R}^+$, with $t_n \rightarrow \infty$, as $n \rightarrow \infty$. The alpha limit set $\alpha(\phi^u)$, for a negative continuation, is characterized as the set of all $v \in M$ for which there is a sequence $t_n \rightarrow \infty$ with $v = \lim_{n \rightarrow \infty} \phi^u(-t_n)$.

Proof. We will prove this for the omega limit set only. (The argument for the alpha limit set is similar.) Assume that v satisfies (2.4) for appropriate sequences u_n and t_n , and let $v_n = S(t_n)u_n$. Since $t_n \rightarrow \infty$, one has

$$v_n \in \gamma^+(S(\tau)B) \subset H^+(S(\tau)B), \quad \text{for all } t_n \geq \tau \text{ and all } \tau \geq 0.$$

Since $H^+(S(\tau)B)$ is a closed set and since $t_n \rightarrow \infty$, one has

$$v = \lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} S(t_n)u_n \in H^+(S(\tau)B), \quad \text{for all } \tau \geq 0,$$

which implies that $v \in \omega(B)$. On the other hand, assume that $v \in \omega(B)$. Then one has $v \in H^+(S(n)B)$, for all $n \geq 1$. Hence there is a $v_n \in \gamma^+(S(n)B)$ such that $d(v, v_n) \leq \frac{1}{n}$, for all $n \geq 1$. That is to say, one has $v_n = S(n)S(\tau_n)u_n$ for some sequences $\tau_n \geq 0$ and $u_n \in B$. Consequently,

$$v = \lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} S(n)S(\tau_n)u_n = \lim_{n \rightarrow \infty} S(n + \tau_n)u_n.$$

Since $(n + \tau_n) \rightarrow \infty$, as $n \rightarrow \infty$, (2.4) holds. \square

Lemma 2.5. *Let σ be a semiflow on $M \subset W$ and let K be a nonempty, compact, invariant set in M . Then for every nonempty set B in K , the omega limit set $\omega(B)$ is a nonempty, compact, invariant set in K . Furthermore, if ϕ^u is a negative continuation, where $H^-(\phi^u)$ is a nonempty compact set, then the alpha limit set $\alpha(\phi^u)$ is a nonempty, compact, invariant set in M .*

Proof. Since the compact set K is invariant, it follows that for every point $u \in K$ there is a globally defined motion ϕ^u passing through u with range in K . It follows that for each $t \geq 0$ the set $H^+(S(t)B)$ is nonempty and compact. By using the monotonicity of $H^+(S(t)B)$, for $t \geq 0$, and the finite intersection property, we see that $\omega(B)$ is nonempty and compact in K .

In order to show that $\omega(B)$ is invariant, we first fix $\tau \geq 0$. Next let $v \in \omega(B)$ be fixed so that $v = \lim_{n \rightarrow \infty} S(t_n)u_n$, for sequences $t_n \rightarrow \infty$ and $u_n \in B$, see (2.4). We seek to find a $u \in \omega(B)$ that satisfies $v = S(\tau)u$. Since K is compact, it follows that there are subsequences of $(t_n - \tau)$ and u_n , which we relabel as $(t_n - \tau)$ and u_n , such that the limit $u = \lim_{n \rightarrow \infty} S(t_n - \tau)u_n$ exists. Since $(t_n - \tau) \rightarrow \infty$, it follows from the Characterization Lemma 2.4 that $u \in \omega(B)$. By using the continuity of the mapping $w \rightarrow S(\tau)w$ and the semigroup property (2.1a) one finds that

$$S(\tau)u = S(\tau) \lim_{n \rightarrow \infty} S(t_n - \tau)u_n = \lim_{n \rightarrow \infty} S(\tau)S(t_n - \tau)u_n = \lim_{n \rightarrow \infty} S(t_n)u_n = v.$$

The remainder of the argument is straight forward, and we omit the details. \square

2.3 Compact and κ -Contracting Semiflows. Many of the applications of the theory of semiflows to partial differential equations, or to differential equations with time delays, occur in an important setting wherein the given semiflow has some smoothing property. In this section we will examine two of these properties: compactness and κ -contracting.

The semiflow $\sigma(u, t) = S(t)u$ on W is said to be **compact**, if for every bounded set B in W there is a $r = r(B)$, $0 \leq r < \infty$, such that for every $t > r$, the set $S(t)B$ lies in a compact subset of W , i.e., $Cl_W S(t)B$ is compact. The number $r(B)$ is referred to as a **compactification time** for $S(t)B$. If $r(B) = t_0$ can be chosen independent of B , then we will say that the semiflow σ is **compact for $t > t_0$** . The following lemma gives a sufficient condition, in terms of compact imbeddings, for a semiflow σ to be compact.

The **Kuratowski measure of noncompactness** is the nonnegative, real valued function² $\kappa(B)$ defined for the bounded sets $B \subset W$ that is given by the formula

$$\kappa(B) \stackrel{\text{def}}{=} \inf\{d : B \text{ has a finite open cover of sets of diameter } < d\}.$$

If B is a nonempty, unbounded set in W , then we define $\kappa(B) = \infty$. The properties of $\kappa(B)$, which we will use here, are given in the following lemma, see Deimling (1985), Martin (1976), and Sell (1972) for more details.

Lemma 2.6. *The Kuratowski measure of noncompactness $\kappa(B)$ on a complete metric space W satisfies the following properties:*

- (1) $\kappa(B) = 0$ if and only if $Cl_W B$ is compact.
- (2) If $B_1 \subset B_2$, then $\kappa(B_1) \leq \kappa(B_2)$.
- (3) $\kappa(B_1 \cup B_2) = \max(\kappa(B_1), \kappa(B_2))$.
- (4) $\kappa(B) = \kappa(Cl_W B)$.
- (5) If B_t is a family of nonempty, closed, bounded sets defined for $t > r$ that satisfy $B_s \supset B_t$, whenever $s \leq t$, and $\kappa(B_t) \rightarrow 0$, as $t \rightarrow \infty$, then $\cap_{t>r} B_t$ is a nonempty, compact set in W .
- (6) Let B_t be given as in (5). Then for any sequences $t_n \in R^+$, with $t_n \rightarrow \infty$ as $n \rightarrow \infty$, and $u_n \in B_{t_n}$, there exist subsequences, which we relabel as t_n and u_n , such that $u = \lim_{n \rightarrow \infty} u_n$ exists and $u \in \cap_{t>r} B_t$.

If in addition, W is a Banach space, then the following are valid:

- (7) $\kappa(B_1 + B_2) \leq \kappa(B_1) + \kappa(B_2)$.
- (8) $\kappa(B) = \kappa(Cl_W coB)$, where $Cl_W coB$ is the closed convex hull of B .
- (9) If $L : W \rightarrow W$ is a bounded linear operator, then $\kappa(LB) \leq \|L\|\kappa(B)$.

Proof. The proofs of properties (1-4) and (7-9) are rather easy, and we omit the details. In order to prove Part (5), we note that it suffices to restrict t to take on integer values, $t = n > r$, only, since one has $\cap_{t>r} B_t = \cap_{n>r} B_n$. Define B by

²The Kuratowski measure of noncompactness is oftentimes denoted by the Greek letter α . We do not follow this convention, because it would be a source of confusion in the theory of dynamical systems, where one studies α -limit sets, see Birkhoff (1927), Lyapunov (1892), Nemytskii and Stepanov (1960), Poincaré (1890, 1892), and Sell (1971). Furthermore, the Greek letter κ seems to be a better way to refer to a Kuratowski-concept.

$B \stackrel{\text{def}}{=} \bigcap_{n>r} B_n$. By Part (2) one has $\kappa(B) \leq \kappa(B_n)$, for all $n > r$. Hence $\kappa(B) = 0$. Since B is closed, it is compact.

In order to show that B is nonempty, we begin by selecting a point $u_n \in B_n$, for each $n > r$. Let $A_1 = \{u_n : n > r\}$ and $A_2 = \text{Cl}_W A_1$. Then $\kappa(A_1) = \kappa(A_2)$ by Parts (2) and (4). We claim that $\kappa(A_1) = 0$. Indeed, since one has $A_1 \subset B_n \cup \{u_i : r < i \leq n-1\}$, it follows from Parts (2) and (3) and the definition of κ , that $\kappa(A_1) \leq \kappa(B_n)$, for all $n \geq 1$. Hence $\kappa(A_1) = \kappa(A_2) = 0$, and A_2 is a nonempty, compact set. Now if A_1 consists of a finite number of points, then there is a $u \in A_1$ with the property that $u \in B_n$ for all $n > r$. On the other hand, if A_1 is an infinite set, then there is a point of accumulation $u \in A_2$. Since the sets B_n are closed and monotone, one has $u \in B_n$, for all $n > r$. Thus in either case, one has $u \in B$.

For Part (6) we let t_n and u_n be sequences that satisfy $t_n \rightarrow \infty$, as $n \rightarrow \infty$, and $u_n \in B_{t_n}$. Define A_n as

$$A_n \stackrel{\text{def}}{=} \text{Cl}_W \{u_m : m \geq n\}.$$

Then A_n is bounded and closed with $A_n \supset A_{n+1}$ and $\kappa(A_n) \leq \kappa(B_{t_n}) \rightarrow 0$. Hence Part (5) is applicable, and $\bigcap_{n \geq r} A_n$ is nonempty and compact. Also for any $u \in \bigcap_{n \geq r} A_n$ there is a subsequence of u_n , which we relabel as u_n , such that $u = \lim_{n \rightarrow \infty} u_n$. \square

A semiflow σ on W is said to be a κ -**contracting**, if for every bounded set $B \subset W$, one has $\kappa(S(t)B) \rightarrow 0$, as $t \rightarrow \infty$. The semiflow σ is said to be a **uniformly κ -contracting**, if there is an $r \geq 0$ and a nonnegative function $k(t)$ with $k(t) \rightarrow 0$, as $t \rightarrow \infty$, such that for every bounded set B in W one has $\kappa(S(t)B) \leq k(t)\kappa(B)$, for all $t > r$. The prototype of a κ -contracting semiflow is a compact semiflow, since $\kappa(S(t)B) = 0$, for all $t > r(B)$ and for every bounded set B in W .

As we will see later, a κ -contracting semiflow on W has many nice properties. Two of the elementary properties are given in the following result.

Lemma 2.7. *Let σ be a κ -contracting semiflow on W . Let w be a point in W , where there is a negative continuation ϕ^w and a bounded set B , such that $\phi^w(t) \in B$, for all $t \leq 0$. Then the negative hull $H^-(\phi^w)$ is a compact set in $\text{Cl}_W B$, and the α -limit set $\alpha(\phi^w)$ is a nonempty, compact invariant set for σ .*

Proof. First we note that, for $\tau \geq 0$, one has

$$\{\phi^w(t) : t \leq 0\} = S(\tau)\{\phi^w(t) : t \leq -\tau\} \subset S(\tau)B.$$

The κ -contracting property then implies that $H^-(\phi^w)$ is nonempty and compact. Lemma 2.5 then implies that the alpha limit set $\alpha(\phi^w)$ is a nonempty, compact, invariant set for σ . \square

3. ATTRACTORS AND GLOBAL ATTRACTORS

The attractors of a semiflow are very important objects. The reason for this is that much (but not all) of the longtime dynamics is represented by the dynamics on and near the attractors. Of special interest is the global attractor. Not every semiflow has a global attractor. However many dissipative semiflows do, and when

this global attractor exists, it is the depository of all the longtime dynamics of the given system.

For any two bounded sets A and B in W we define

$$\delta(B, A) \stackrel{\text{def}}{=} \sup_{u \in B} \left(\inf_{v \in A} d(u, v) \right).$$

Thus one has $\delta(B, A) \leq \epsilon$ if and only if $B \subset N_\epsilon(A)$, where $N_\epsilon(A)$ is the closed ϵ -neighborhood of A . An alternate definition is

$$\delta(B, A) = \inf \{ \epsilon > 0 : B \subset N_\epsilon(A) \}.$$

The function δ is not symmetric. That is, in general one has $\delta(B, A) \neq \delta(A, B)$. If A is a set in W and u is a point in W , we define the distance from u to A by

$$\text{dist}_W(u, A) \stackrel{\text{def}}{=} \inf_{v \in A} d(u, v) = \delta(\{u\}, A).$$

Note that $\text{dist}_W(u, A) = 0$ if and only if $u \in \text{Cl}_W A$. Also note that if w_n is any convergent sequence in W with limit w_0 , then for any bounded set A in W one has $d(w_n, A) \rightarrow d(w_0, A)$, as $n \rightarrow \infty$.

We are primarily interested in the function $\delta(B, A)$ in the case where both A and B are nonempty and bounded. Nevertheless, the definition of the function δ extends in a consistent way to some other cases. For example, if A is nonempty, then $\delta(B, A) = \infty$, when $B \setminus A$ is unbounded, and $\delta(\emptyset, A) = 0$. We will define $\delta(\emptyset, \emptyset) \stackrel{\text{def}}{=} 0$ and leave $\delta(B, \emptyset)$ as being undefined when B is nonempty. Note that if A_1 is a nonempty set and $A_1 \subset A_2$, then $\delta(B, A_1) \geq \delta(B, A_2)$, and $\delta(B_1, A_1) \leq \delta(B_2, A_1)$ when $B_1 \subset B_2$.

Let σ be a semiflow on $M \subset W$, and let A and B be two sets in M . We will say that A **attracts** B if

$$(2.30) \quad \delta(S(t)B, A) \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Notice that (2.30) is equivalent to saying that for every $\epsilon > 0$ there is a $T = T(\epsilon) \geq 0$ such that

$$(2.30a) \quad \text{dist}_W(S(t)u, A) \leq \epsilon, \quad \text{for all } t \geq T \text{ and } u \in B,$$

that is, $S(t)B$ lies in $N_\epsilon(A)$, for all $t \geq T$. Since the value of $\delta(B, A)$ is unchanged if one replaces B by the closure $\text{Cl}_M B$, we see that if A attracts B , then A attracts $\text{Cl}_M B$. Also note that if A is empty, then (2.30) is valid if and only if B is empty.

3.1 Asymptotical Compactness. We now introduce a key concept which plays a pivotal role in the theory of attractors. Let σ be a semiflow on $M \subset W$. We will say that σ is **asymptotically compact on a set** $B \subset M$, if for any sequences $u_n \in B$ and $t_n \rightarrow \infty$, there exist subsequences, which we relabel as u_n and t_n , with the property that the limit $v = \lim S(t_n)u_n$ exists and $v \in M$. A semiflow σ on $M \subset W$ is said to be **ultimately bounded** if for every bounded set B in M , there is a $\tau = \tau(B) \geq 0$ such that $\gamma^+(S(\tau)B)$ is bounded. The following lemma describes some relations between the concepts of attracts, asymptotically compact, and ultimately bounded. (Also see Lemmas 4.1 and 4.2 below.)

Lemma 3.1. *Let σ be a semiflow on $M \subset W$. Then the following hold:*

- (1) *Let A and B be sets in M , where A is bounded and A attracts B . Then one has $\omega(B) \subset Cl_M A$.*
- (2) *Let σ be asymptotically compact on a nonempty set $B \subset M$. Then $\omega(B)$ is a nonempty, compact, invariant set in M , and $\omega(B)$ attracts B .*
- (3) *Let A be a nonempty, compact, invariant set in M , and assume that A attracts a nonempty set B . Then σ is asymptotically compact on B and the conclusions of Parts (1) and (2) hold.*
- (4) *Let σ be asymptotically compact on a set B in M . Then there exists a $\tau \geq 0$ such that $\gamma^+(S(\tau)B)$ is bounded.*
- (5) *If σ is asymptotically compact on every bounded set in M , the σ is ultimately bounded.*

Proof. Part (1): Let $\epsilon > 0$ be given. From (2.30a) one has $S(t)B \subset N_\epsilon(A) \cap M$, for all $t \geq T(\epsilon)$. Since $N_\epsilon(A)$ is closed, one has $H^+(S(t)B) \subset N_\epsilon(A) \cap M$, for all $t \geq T(\epsilon)$. It follows that $\omega(B) \subset \bigcap_{\epsilon > 0} (N_\epsilon(A) \cap M) \subset Cl_M A$.

Part (2): Assume now that σ is asymptotically compact on a nonempty set $B \subset M$. It follows from the definition of asymptotic compactness and the Characterization Lemma 2.4 that $\omega(B)$ is nonempty. In order to show that $\omega(B)$ is compact, we let v_n be any sequence in $\omega(B)$. Then the Characterization Lemma implies that for each $n \geq 1$ there is a $u_n \in B$ and $t_n \geq n$ such that $d(v_n, S(t_n)u_n) \leq \frac{1}{n}$. The asymptotic compactness property allows us to choose subsequences, which we relabel as v_n , u_n and t_n , so that $v = \lim S(t_n)u_n \in \omega(B) \subset M$. Clearly one then has $v = \lim v_n$, i.e., $\omega(B)$ is a compact set. The proof of the invariance of $\omega(B)$ is identical to the argument of Lemma 2.5, and we omit the details.

In order to show that $\omega(B)$ attracts B , we proceed by contradiction, and assume that for some $\epsilon > 0$ there does not exist a time $T \geq 0$ such that

$$\text{dist}_W(S(t)u, \omega(B)) \leq \epsilon, \quad \text{for all } u \in B \text{ and } t \geq T.$$

As a result there are sequences $u_n \in B$ and $t_n \in R^+$ such that $t_n \rightarrow \infty$, while

$$(2.31) \quad \text{dist}_W(S(t_n)u_n, A) > \epsilon, \quad \text{for all } n \geq 1,$$

where $A = \omega(B)$. Next we use the asymptotic compactness property to select a subsequence, which we relabel as u_n and t_n , so that the limit $v = \lim S(t_n)u_n$ exists. From the Characterization Lemma one has $v \in \omega(B)$, which contradicts (2.31).

Part (3): Now assume that A is a nonempty, compact, invariant set in M and that A attracts a nonempty set B . Let $u_n \in B$ and $t_n \rightarrow \infty$ be given sequences. One then has $\text{dist}_W(S(t_n)u_n, A) \rightarrow 0$, as $n \rightarrow \infty$, because A attracts B . This implies that there are subsequences, which we relabel as u_n and t_n , and there is a sequence $v_n \in A$ such that $d_W(S(t_n)u_n, v_n) \leq \frac{1}{n}$. Since A is compact, there are further subsequences, which we relabel as u_n , t_n , and v_n , such that the limit $v = \lim_{n \rightarrow \infty} v_n$ exists, $v \in A$, and $v = \lim_{n \rightarrow \infty} S(t_n)u_n$. Hence σ is asymptotically compact on B . The remainder of the proof now follows from Parts (1) and (2).

Parts (4) and (5): If it were not the case that $\gamma^+(S(\tau)B)$ is bounded, for some $\tau \geq 0$, then there exist sequences $u_n \in B$ and $t_n \rightarrow \infty$ such that $d(u_0, S(t_n)u_n) \rightarrow$

∞ , for any $u_0 \in M$. However, this contradicts the fact that $S(t_n)u_n$ contains a convergent subsequence. Finally, Part (5) follows immediately from the definition of ultimate boundedness and Part (4) of this Lemma. \square

3.2 Attractors and Their Properties. A set \mathfrak{A} contained in M is said to be an **attractor** for the semiflow σ on M , provided that

- (1) \mathfrak{A} is a compact, invariant set in M , and
- (2) there is a neighborhood U of \mathfrak{A} in M , such that \mathfrak{A} attracts every bounded set in U .

Equivalently, \mathfrak{A} is an **attractor** for σ provided that \mathfrak{A} is a compact, invariant set in M and there is a bounded neighborhood V of \mathfrak{A} in M with the property that \mathfrak{A} attracts V . The **basin of attraction** of \mathfrak{A} is defined as

$$B(\mathfrak{A}) \stackrel{\text{def}}{=} \{u \in M : \text{dist}_W(\sigma(u, t), \mathfrak{A}) \rightarrow 0, \text{ as } t \rightarrow \infty\}.$$

A set \mathfrak{A} is said to be a **global attractor** for σ provided that³

- (1) \mathfrak{A} is a compact, invariant set in M ;
- (2) there is a neighborhood U of \mathfrak{A} in M , such that \mathfrak{A} attracts every bounded set in U ; and
- (3) the basin of attraction satisfies $B(\mathfrak{A}) = W$.

If $B(\mathfrak{A}) \neq W$, then \mathfrak{A} is sometimes referred to as a **local attractor**. It happens that the empty set \emptyset is an attractor with basin of attraction $B(\emptyset) = \emptyset$. However, a global attractor is always nonempty.

Several properties of attractors, which are needed later, are given in the following two lemmas.

Lemma 3.2. *Let σ be a semiflow on $M \subset W$, and let \mathfrak{A} be an attractor for σ . Let U be a neighborhood of \mathfrak{A} with the property that \mathfrak{A} attracts every bounded set in U . Then the following are valid:*

- (1) *For every nonempty, bounded set B in U , σ is asymptotically compact on B and the omega limit set $\omega(B)$ is a nonempty, compact, invariant set with $\omega(B) \subset \mathfrak{A}$, and $\omega(B)$ attracts B .*
- (2) *The basin of attraction $B(\mathfrak{A})$ is an open set in M .*
- (3) *For every nonempty compact set K in $B(\mathfrak{A})$ one has: (a) \mathfrak{A} attracts K ; (b) $\omega(K)$ is a nonempty, compact, invariant set; (c) $\omega(K) \subset \mathfrak{A}$; and (d) $\omega(K)$ attracts K .*

Proof. Part (1) follows directly from Lemma 3.1. In order to prove Part (2), we let V be a bounded neighborhood of \mathfrak{A} with the property that \mathfrak{A} attracts V . This means that for any $\epsilon > 0$, there is a time $T = T(V, \epsilon) \geq 0$, such that $S(t)V \subset N_\epsilon(\mathfrak{A})$, for all $t \geq T$. Since we can replace V by its interior, if necessary, there is no loss in generality in assuming V to be open. Then for every $u \in B(\mathfrak{A})$, there is a time $\tau = \tau(u) \geq 0$ such that $S(\tau)u \in V$. From the continuity of σ , there is a constant $\delta > 0$ such that $S(\tau)N_\delta(u) \subset V$ as well. It then follows that \mathfrak{A} attracts $N_\delta(u)$, and

³This definition of a global attractor differs from other definitions found in the literature. For more details on these differences, see Section 5.

therefore $B(\mathfrak{A})$ is open. In particular, one has $S(t + \tau)N_\delta(u) \subset S(t)V \subset N_\epsilon(\mathfrak{A})$, for all $t \geq T = T(V, \epsilon)$. This in turn implies that

$$(2.30d) \quad S(s)N_\delta(u) \subset N_\epsilon(\mathfrak{A}), \quad \text{for all } s \geq T + \tau.$$

In order to prove Part (3) we let K be a given compact set in $B(\mathfrak{A})$, and let $\epsilon > 0$ be given so that $N_\epsilon(\mathfrak{A}) \subset U$, where U is given by Part (1). By using the compactness of K , we can find a δ -net $\{u_1, \dots, u_n\}$ in K and times $\{\tau_1, \dots, \tau_n\}$ in R^+ so that $S(\tau_i)N_\delta(u_i) \subset V$, for $1 \leq i \leq n$. Inequality (2.30d) then implies that

$$S(s)K \subset \bigcup_{i=1}^n S(s)N_\delta(u_i) \subset N_\epsilon(\mathfrak{A}), \quad \text{for all } s \geq T + \tau,$$

where $\tau = \max(\tau_1, \dots, \tau_n)$. Hence \mathfrak{A} attracts K . Finally we note that for sufficiently large τ_0 one has $S(\tau_0)K \subset U$, since $N_\epsilon(\mathfrak{A}) \subset U$, for sufficiently small $\delta > 0$. The remainder of the proof of Part (3) now follows from Part (1). \square

Lemma 3.3 (Maximality Property). *Let σ be a semiflow on $M \subset W$, and let $U \subset M$. Let B be a closed, bounded set in U , and assume that B attracts each compact set in U . Then B is maximal in the sense that every compact, invariant set K in U satisfies $K \subset B$. In particular, every compact, invariant set K in the basin of attraction of an attractor \mathfrak{A} satisfies $K \subset \mathfrak{A}$.*

Proof. Let K be a compact, invariant set in U . Let $u \in K$ and let ϕ^u be a negative continuation of the motion through u with the property that $\phi^u(t) \in K$, for all $t \leq 0$. Let $D \stackrel{\text{def}}{=} \text{Cl}_W\{\phi^u(t) : t \leq 0\}$. Then $D \subset K$ and D is a compact set in U . Since K is invariant, one has $u \in S(t)D$, for all $t \geq 0$. Since B attracts D , one has

$$\text{dist}_U(u, B) \leq \delta(S(t)D, B) \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Finally, since B is closed, this implies that $u \in B$, i.e., one has $K \subset B$. The statement concerning the basin of attraction of an attractor \mathfrak{A} follows from the previous lemma. \square

Let σ be a semiflow on W and let $A \subset W$. The set A is said to be **Lyapunov stable**, provided that A is positively invariant and for every $\tau > 0$ and every neighborhood V of A , there is a neighborhood U of A such that

$$(2.31b) \quad S(t)U \subset V, \quad \text{for all } t \geq \tau.$$

The set A is said to be **uniformly asymptotically stable** if it is Lyapunov stable and there is a neighborhood U_0 of A such that A attracts U_0 , i.e.,

$$(2.31c) \quad \delta(S(t)U_0, A) \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Notice that the definition of Lyapunov stability of a set A requires that $S(t)U \subset V$, for all $t \geq \tau$, where $\tau > 0$, see (2.31b). This differs from the classical concept where one requires $S(t)U \subset V$, for all $t \geq \tau$, where $\tau = 0$, see Hale (1988, Sect. 3.3) and Bhatia and Szegö (1970). The weaker version, with $\tau > 0$, is used here in order that the Stability Lemma and the Stability Theorem given below be valid for dynamical systems which may fail to be continuous at $t = 0$. The proofs of the following two results can be found in Sell and You (1998), also see Hale (1988).

Lemma 3.4 (Stability Lemma). *Let σ be a semiflow on $M \subset W$ and let \mathfrak{A} be an attractor in M . Then \mathfrak{A} is uniformly asymptotically stable.*

In the case of κ -contracting semiflows, the uniform asymptotic stability property is a characterization of an attractor.

Theorem 3.5 (Stability Theorem). *Let σ be a κ -contracting semiflow on $M \subset W$ and let \mathfrak{A} be a nonempty, invariant set in M . Then the following statements are equivalent:*

- (1) \mathfrak{A} is an attractor for σ .
- (2) \mathfrak{A} is uniformly asymptotically stable.
- (3) There is a neighborhood V of \mathfrak{A} , where $\gamma^+(S(\tau)V)$ is bounded, for some $\tau \geq 0$, and $\omega(V) = \mathfrak{A}$.

There are several other properties of attractors which are worth noting. Proofs of the following can be found in Sell and You (1998).

- (1) Let \mathfrak{A} be an attractor, and let V be any bounded neighborhood of \mathfrak{A} with the property that \mathfrak{A} attracts V . One then has $\mathfrak{A} = \omega(V)$.
- (2) (Minimality Property) Let \mathfrak{A} be an attractor for σ , and let U be a neighborhood of \mathfrak{A} in M with the property that \mathfrak{A} attracts every bounded set in U . Then \mathfrak{A} is minimal in the sense that if B is any closed set in U that attracts every compact set in U , then $B \supset \mathfrak{A}$.
- (3) Let \mathfrak{A} be a global attractor for a semiflow on a Banach space, or a Fréchet space. Then \mathfrak{A} is connected, see Hale (1988) and Sell and You (1998).

Furthermore, under appropriate conditions, attractors depend in an upper semi-continuous way on parameters, see Hale (1988). In addition, for many applications to partial differential equations and functional differential equations, one can prove that the global attractors have finite dimension, see Mallet-Paret (1976), Mañé (1981), Foias and Temam (1979), and Temam (1988).

4. EXISTENCE OF GLOBAL ATTRACTORS

Let σ be a semiflow on W . We will say that σ is **point dissipative**, or simply **dissipative**, if there is a nonempty bounded set A in W such that A attracts every point in W . In this case, any set B that contains an ϵ -neighborhood $N_\epsilon(A)$, for some $\epsilon > 0$, is said to be an **absorbing set** for σ . Note that the existence of an absorbing set for σ is equivalent to stating that the semiflow is point dissipative. If B is an absorbing set, then for any $u \in W$ there is a $T = T(u, \epsilon) \geq 0$ such that $S(t)u \in N_\epsilon(A) \subset B$, for all $t \geq T$. We now recall our main result.

Main Theorem. *Let σ be a κ -contracting semiflow on a complete metric space W , and assume that σ is point dissipative. Then there is a global attractor \mathfrak{A} for σ , and \mathfrak{A} attracts every compact set K in W .*

Assume in addition that σ satisfies one of the following properties:

- (1) σ is compact, or
- (2) σ is ultimately bounded,

then \mathfrak{A} attracts every bounded set B in W .

In order to prove this result, we will use the following two lemmas.

Lemma 4.1. *Let σ be a κ -contracting semiflow on a complete metric space W . Then the following holds.*

- (1) *Assume that for a nonempty set B in W , there is a $\tau \geq 0$, such that $\gamma^+(S(\tau)B)$ is bounded. Then σ is asymptotically compact on B ; the omega limit set $\omega(B)$ is a nonempty, compact, and invariant; and $\omega(B)$ attracts B .*
- (2) *Assume that B is a bounded, open set with $H^+(S(t)B) \subset B$, for some $t \geq \tau$. Then σ is asymptotically compact on B , and $\mathfrak{A} = \omega(B)$ is an attractor with $B \subset B(\mathfrak{A})$.*

Proof. Part (1): In order to show that σ is asymptotically compact on B , we let $u_n \in B$ and $t_n \rightarrow \infty$ be given sequences. By choosing subsequences, if necessary, we can assume that $\tau \leq t_n \leq t_{n+1}$, for all n . Let $v_n = S(t_n)u_n$ and set $A_n = \text{Cl}_M\{v_m : m \geq n\}$. One then has

$$\kappa(A_n) \leq \kappa(H^+(S(t_n)B)) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

By Lemma 2.6, there exists subsequences, which we relabel as u_n and t_n , such that $S(t_n)u_n$ is convergent. Hence σ is asymptotically compact on B . The remainder of Part (1) now follows from Lemma 3.1.

Part (2): Since σ is asymptotically compact on B , by Part (1), it follows from Lemma 3.1 that $\mathfrak{A} = \omega(B)$ is a compact, invariant set in W and that \mathfrak{A} attracts B . Since $\omega(B) \subset H^+(S(t)B) \subset B$, it follows that B is a neighborhood of \mathfrak{A} . From the definitions we see that \mathfrak{A} is an attractor and $B \subset B(\mathfrak{A})$. \square

Lemma 4.2. *Let σ be a κ -contracting semiflow on W , and assume that σ is point dissipative. Then the following statements are valid:*

- (1) *There is a nonempty, closed, bounded set B_0 in W that attracts all points in W .*
- (2) *Let $K_0 \stackrel{\text{def}}{=} \{u \in B_0 : \gamma^+(u) \subset B_0\}$, where B_0 satisfies Part (1). Then K_0 is a nonempty, closed, bounded, positively invariant set in W which attracts all points in W . Furthermore, σ is asymptotically compact on K_0 ; the omega limit set $A = \omega(K_0)$ is a nonempty, compact, invariant set that attracts all points in W ; and A attracts K_0 .*

Proof. Part (1): Since σ is dissipative, there is a nonempty, bounded set N in W that attracts all points in W . Then $B_0 \stackrel{\text{def}}{=} \text{Cl}_W N$ is a nonempty, closed, bounded set that attracts all points in W .

Part (2): Let K_0 be given as in the statement of the Lemma. Since B_0 attracts every point $u \in W$, it follows that the positive orbit $\gamma^+(u)$ is bounded. Consequently Lemma 4.1 implies that the omega limit set $\omega(u)$ is a nonempty, compact, invariant set, and Lemma 3.1 implies that $\omega(u) \subset B_0$. Since $\omega(u)$ is invariant, one has $\omega(u) \subset K_0$. Hence K_0 is nonempty. Clearly K_0 is bounded and positively invariant with $\gamma^+(K_0) \subset K_0 \subset B$. We claim that K_0 is closed. If this were not the case, then there is a convergent sequence $u_n \in K_0$ with $u = \lim_{n \rightarrow \infty} u_n \notin K_0$. Since B is closed, one has $u \in B$, and since $u \notin K_0$, there is a $\tau > 0$ such that $S(\tau)u$ lies in the open set B^c , the complement of B . From the continuity of σ one concludes

that $S(\tau)u_n \in B^c$, for large n , which contradicts the fact that $\gamma^+(u_n) \subset K_0 \subset B$. Hence K_0 is closed. Now Lemma 4.1 implies that $\omega(u)$ attracts u , for each $u \in W$. Since $\omega(u) \subset K_0$, we see that K_0 attracts all points in W . Since $\gamma^+(K_0) \subset K_0$, the remainder of the proof of Part (2) now follows from Lemma 4.1. \square

Proof of the Main Theorem. Let K_0 and $A = \omega(K_0)$ be given by Lemma 4.2. We then have that A is a nonempty, compact, invariant set in W , and A attracts every point in W . Let $\epsilon > 0$ be fixed and let $B = N_\epsilon(A)$ be a fixed, closed, bounded neighborhood of A . Even though $\text{Int } B$ is an absorbing set for σ , and the ω limit set $\omega(w)$ is a nonempty, compact, invariant set with $\omega(w) \subset A$, for every $w \in W$, we do not claim that A is the global attractor for σ . In general, the global attractor is a larger set. For example, the invariant set A may have an unstable manifold, which lies in B , and further, the individual solutions in the unstable manifold may leave B and remain outside B for time t in a bounded set. The global attractor \mathfrak{A} must include any unstable manifold, as well as any points on the positive trajectory of motions beginning in the unstable manifold. Our proof of this theorem, which consists in several steps, is based on a theory of negative continuations of motions of the semiflow σ . The first step is to describe a set \mathfrak{A} , which we will later show is the global attractor.

Step (1). The construction of \mathfrak{A} : In order to describe the unstable “manifold” for A , we define $N(B)$ to be the collection of all points $u \in B$ with the property that there is a negative continuation ϕ^u and a sequence t_n^u in \mathbb{R} with $t_n^u \rightarrow -\infty$ and with $\phi^u(t_n^u) \in B$, for all n . It is easily seen that one has

$$N(B) \subset S(\tau)N(B) \subset S(\tau)B, \quad \text{for each } \tau \geq 0.$$

Since $K \stackrel{\text{def}}{=} \text{Cl}_W N(B)$ satisfies $\kappa(K) = \kappa(N(B))$, it follows from the κ -contracting property and Lemma 2.6 that K is a compact set in B . Since one has $A \subset N(B)$, it follows that $A \subset K$.

Next we define a special subset Γ of K by

$$\Gamma \stackrel{\text{def}}{=} \{u \in K : d(u, A) = \epsilon\}.$$

Since K is compact, it follows that Γ is compact as well. Since A attracts all points in W , it follows that for each $u \in \Gamma$ there is a time $\tau(u)$ where $S(\tau(u))u \in \text{Int } B$, the interior of B . Due to the continuity of σ , for each $u \in \Gamma$ there is a neighborhood V_u of u with the property that $S(\tau(u))V_u \subset \text{Int } B$. Using the compactness of Γ , we choose a finite cover $\{V_{u_1}, \dots, V_{u_m}\}$ of Γ and assume that this is ordered so that $0 \leq \tau_i \leq \tau_m$, where $\tau_i = \tau(u_i)$, for $1 \leq i \leq m$. Define $T \stackrel{\text{def}}{=} \tau(u_m)$.

Note that if Γ is empty, then for every $u \in N(B)$, the negative continuation ϕ^u satisfies $\phi^u(t) \in B$, for all $t \leq 0$. The reason for this is that $\phi^u(t)$ cannot leave B , unless there is a time $t_1 < 0$ with $d(\phi^u(t_1), \mathfrak{A}) = \epsilon$, i.e., $\phi^u(t_1) \in \Gamma$. For the same reason, one has $S(t)u \in B$, for all $t \geq 0$. In other words, if Γ is empty, then $N(B)$ is an invariant set with $N(B) \subset B$. Since K is compact, it follows from Lemma 2.3, that K is invariant, which in turn implies that $K = N(B)$. When Γ is empty, we take $T = 0$.

Whether Γ is empty, or not, we define \mathfrak{A} by $\mathfrak{A} \stackrel{\text{def}}{=} \gamma^+(K)$. Since one has $S(t)\mathfrak{A} = \mathfrak{A}$, for all $t \geq 0$, we see that \mathfrak{A} is invariant and that the restriction of $S(t)$ to \mathfrak{A}

is continuous at $t = 0$. Since K is compact, it then follows from the Continuity Lemma 2.1 that the set

$$D^0 \stackrel{\text{def}}{=} \bigcup_{t=0}^{t=T} S(t)K$$

is compact. We will next show that $\mathfrak{A} = D^0$. Since the definitions of D^0 and \mathfrak{A} imply that $D^0 \subset \mathfrak{A}$, we need only to verify that for every $u \in K$ and every $t_0 \geq 0$, one has $u_0 \stackrel{\text{def}}{=} S(t_0)u \in D^0$.

Since $u \in K$, there is a sequence $u_n \in N(B)$, such that $u_n \rightarrow u$, as $n \rightarrow \infty$. From the continuity, one obtains that $S(t_0)u_n \rightarrow S(t_0)u = u_0$, as $n \rightarrow \infty$. It is convenient now to distinguish two cases. In the first case, we assume that for some sequence one has $S(t_0)u_n \in B$. By restricting to this subsequence, which we relabel as u_n , one has $S(t_0)u_n \in N(B)$, from the definition of $N(B)$. Hence one finds that $S(t_0)u_n \rightarrow S(t_0)u = u_0 \in K = \text{Cl}_W N(B)$. In the second case, we assume that there is a subsequence where $S(t_0)u_n \notin B$. We now restrict to this subsequence and relabel it as u_n . Since A attracts each point u_n , there exist two sequences t_{1n} and t_{2n} in R^+ with the following properties: (a) one has $t_{1n} < t_0 < t_{2n}$; (b) the sequences $S(t_{1n})u_n$ and $S(t_{2n})u_n$ are in Γ ; and (c) one has $S(t)u_n \notin B$, for $t_{1n} < t < t_{2n}$. It then follows from the argument above that $0 \leq t_{2n} - t_{1n} \leq T$, which implies that $t_{1n} \in [t_0 - T, t_0]$ and $t_{2n} \in [t_0, t_0 + T]$. By compactness, there exist subsequences, which we will relabel as u_n , t_{1n} , and t_{2n} , such that the following limits exist: $t_{1n} \rightarrow t_{10} \in [t_0 - T, t_0]$ and $t_{2n} \rightarrow t_{20} \in [t_0, t_0 + T]$, with $0 \leq t_{20} - t_{10} \leq T$. Since $t_{10} \leq t_0 \leq t_{20}$, this implies that $0 \leq t_0 - t_{10} \leq T$. By continuity, one then has the limit $S(t_{1n})u_n \rightarrow S(t_{10})u \in \Gamma \subset K$. Consequently, by the Continuity Lemma 2.1, one obtains

$$\lim_{n \rightarrow \infty} S(t_0)u_n = \lim_{n \rightarrow \infty} S(t_0 - t_{1n})S(t_{1n})u_n = S(t_0 - t_{10})S(t_{10})u = u_0,$$

which implies that $u_0 \in S(t_0 - t_{10})K \subset D^0$. Hence one has $\mathfrak{A} = D^0$ and we see that \mathfrak{A} is a compact, invariant set in W with $A \subset \mathfrak{A}$.

Step (2). Property A: *If u is any point in W with the property that there is a negative continuation ϕ^u and a sequence t_n^u in R with $t_n^u \rightarrow -\infty$ and with $\phi^u(t_n^u) \in B$, for all n , then one has $u \in \mathfrak{A}$.*

In order to prove Property A, we consider first the case where $u \in B$. Then one has $u \in N(B) \subset K \subset \mathfrak{A}$, from the definitions. For the case where $u \notin B$, we replace u with $v = \phi^u(t_n^u)$, for some n where $t_n^u < 0$. One then has $v \in N(B) \subset K \subset \mathfrak{A}$. Since \mathfrak{A} is invariant, one obtains $u = \phi^u(0) = S(-t_n^u)v \in \mathfrak{A}$.

Step (3). Property B: *If u is any point in W with the property that there is a negative continuation ϕ^u and a bounded set B_1 such that $\phi^u(t) \in B_1$, for all $t \leq 0$, then one has $u \in \mathfrak{A}$.*

In order to prove Property B, we let ϕ^u and B_1 be given so that B_1 is bounded and $\phi^u(t) \in B_1$, for all $t \leq 0$. It then follows from Lemma 2.7 that the α -limit set $\alpha(\phi^u)$ is a nonempty, compact, invariant set with $\alpha(\phi^u) \subset \text{Cl}_W B_1$. Let $u_0 \in \alpha(\phi^u)$. Since A attracts u_0 , the ω -limit set satisfies $\omega(u_0) = \omega(u_0) \cap \alpha(\phi^u) \subset A$. Hence there is a sequence t_n , with $t_n \rightarrow -\infty$, such that $\phi^u(t_n) \rightarrow A$, that is, there is a subsequence, which we relabel as t_n , such that $\phi^u(t_n) \in B$, for all n . It then follows from Property A that $u \in \mathfrak{A}$.

Step (4). Property C: *There is a $\tau = \tau(\epsilon) > 0$ with the following property: For any two points u and v in $N_\epsilon(\mathfrak{A})$ that satisfy:*

- (1) $S(t_0)u = v$, for some $t_0 \geq 0$;
- (2) $S(t)u \in N_\epsilon(\mathfrak{A})$, for all t with $0 \leq t \leq t_0$; and
- (3) $d(v, \mathfrak{A}) = \epsilon$;

one has $t_0 \leq \tau$.

In order to prove Property C, we define B_τ , for $\tau \geq 0$, by

$$B_\tau \stackrel{\text{def}}{=} \bigcap_{0 \leq t \leq \tau} \text{Cl}_W S(t)N_\epsilon(\mathfrak{A}).$$

One then has $B_\tau \subset N_\epsilon(\mathfrak{A}) \cap S(\tau)N_\epsilon(\mathfrak{A})$. Since $\kappa(B_\tau) \leq \kappa(S(\tau)N_\epsilon(\mathfrak{A})) \rightarrow 0$, as $\tau \rightarrow \infty$, it follows that

$$B_\infty \stackrel{\text{def}}{=} \bigcap_{\tau \geq 0} B_\tau$$

is a compact set in $N_\epsilon(\mathfrak{A})$, see Lemma 2.6.

What we need to show that for some $\tau > 0$, the set $B_\tau \cap \Gamma$ is empty. We will argue this by contradiction. Assume instead that the set $B_\tau \cap \Gamma$ is nonempty, for each $\tau \geq 0$. It then follows from Lemma 2.6 that the set $B_\infty \cap \Gamma$ is nonempty as well. Let $v \in B_\infty \cap \Gamma$. We will now show that there is a negative continuation ϕ^v with $\phi^v(t) \in N_\epsilon(\mathfrak{A})$, for all $t \leq 0$. The existence of such a negative continuation then contradicts Property B, since one would have $v \in \mathfrak{A}$ and $d(v, \mathfrak{A}) = \epsilon > 0$.

In order to construct the negative continuation ϕ^v , we begin with a sequence v_n in $B_{t_n} \cap \Gamma$, where $t_n \rightarrow \infty$ and $v_n \rightarrow v$, as $n \rightarrow \infty$. For each v_n , there is a partial negative continuation $\phi^{v_n} : [-t_n, 0] \rightarrow N_\epsilon(\mathfrak{A})$. By using Lemma 2.6 once again, there is a subsequence of $\phi^{v_n}(-1)$, which we will label as u_n^1 , such that u_n^1 converges to a point u^1 , where $S(t)u_n^1 \in N_\epsilon(\mathfrak{A})$, for $0 \leq t \leq 1$. Moreover, $S(t)u_n^1$ converges uniformly to $S(t)u^1$, for $0 \leq t \leq 1$, and $S(1)u^1 = v$. By using the fact that $-t_n \rightarrow -\infty$, along with repeated applications of Lemma 2.6, one constructs a family of subsequences of subsequences and thereby obtains the following: For each $k = 1, 2, \dots$, there is a sequence u_n^k in $N_\epsilon(\mathfrak{A})$ such that u_n^k converges to a point u^k , where $S(t)u_n^k \in N_\epsilon(\mathfrak{A})$, for $0 \leq t \leq 1$. Moreover, $S(t)u_n^k$ converges uniformly to $S(t)u^k$, for $0 \leq t \leq 1$, and $S(1)u^k = u^{k-1}$, where $u^0 = v$. The desired negative continuation ϕ^v is then defined by $\phi^v(t) = S(t+k)u^k$, for $-k < t \leq -k+1$ and $k = 1, 2, \dots$. This completes the proof of Property C.

Step (5). Property D (Lyapunov Stability): *For any $t_0 > 0$, there is a $\delta > 0$ such that $\gamma^+(S(t_0)N_\delta(\mathfrak{A})) \subset \text{Int } N_\epsilon(\mathfrak{A})$.*

In order to prove Property D, we let $\tau > 0$ be given by Property C, and we fix t_0 so that $0 < t_0$. It then follows from the Continuity Lemma 2.1 that there is a $\delta > 0$ such that if $u \in N_\delta(\mathfrak{A})$, then one has

$$d(S(t)u, \mathfrak{A}) < \epsilon, \quad \text{for } t_0 \leq t \leq t_0 + 2\tau.$$

Property C then implies that

$$d(S(t)u, \mathfrak{A}) < \epsilon, \quad \text{for all } t \geq t_0,$$

which in turn implies Property D.

Step (6). *The set \mathfrak{A} is a global attractor for σ , and \mathfrak{A} attracts each compact set in W .*

In order to prove this assertion, we first note that Property D and Lemma 4.1 imply that the omega limit set $\mathfrak{A}_0 \stackrel{\text{def}}{=} \omega(N_\delta(\mathfrak{A}))$ is a nonempty, compact, invariant set which attracts $N_\delta(\mathfrak{A})$. Since one has

$$\mathfrak{A} \subset H^+(S(t)N_\delta(\mathfrak{A})), \quad \text{for each } t \geq t_0,$$

it follows from the definition of an omega limit set that $\mathfrak{A} \subset \mathfrak{A}_0$. Since \mathfrak{A}_0 is compact and invariant, it follows from Property B that $\mathfrak{A}_0 \subset \mathfrak{A}$. Hence one has $\mathfrak{A} = \omega(N_\delta(\mathfrak{A}))$, and \mathfrak{A} is an attractor, since it attracts $N_\delta(\mathfrak{A})$, which is a neighborhood of \mathfrak{A} . Since $A \subset \mathfrak{A}$, it follows that \mathfrak{A} attracts every point in W , i.e., the basin satisfies $B(\mathfrak{A}) = W$. Hence \mathfrak{A} is the global attractor. The fact that \mathfrak{A} attracts every compact set in $B(\mathfrak{A}) = W$ now follows from Lemma 3.2.

Step (7). The Final Step: The final step is to look at the two cases:

- (1) σ is compact, or
- (2) σ is ultimately bounded.

If σ is compact, then for every bounded set B_1 in W , there is a time $t_1 > 0$, such that $K_1 = \text{Cl}_W S(t_1)B_1$ is compact. Since \mathfrak{A} attracts K_1 , it follows that it attracts B_1 , as well. If instead, σ is ultimately bounded, then we use Lemma 4.1. For each bounded set B_1 in W , the compact, invariant set $K_1 = \omega(B_1)$ attracts B_1 . From the Maximality Property (Lemma 3.3), one has $K_1 \subset \mathfrak{A}$. Hence \mathfrak{A} attracts B_1 . \square

5. COMMENTARY

1. There do exist theories of semiflows which are based upon a weakening of the continuity hypothesis,

$$(2.70) \quad \sigma : M \times (0, \infty) \rightarrow M \text{ is continuous.}$$

For example, Temam (1988) requires instead that for each $t \geq 0$, the mapping $\sigma(\cdot, t) : M \rightarrow M$ be continuous, or equivalently that $\sigma : M \times [0, \infty) \rightarrow M$ be continuous, where now $[0, \infty)$ has the discrete topology, see Ellis (1969). Of course in using the Ellis-Temam approach, one would not be able to prove, without further assumptions, some of the results given above, including the Continuity Lemma 2.1, as well as any other results, such as the Stability Lemma 3.4, and the Main Theorem, which are based on Continuity Lemma.

2. The continuity property (2.70), as well as the related properties:

$$(2.71) \quad \sigma(u, \cdot) : (0, \infty) \rightarrow M \text{ is continuous for every } u \in M,$$

and

$$(2.72) \quad \sigma(u, \cdot) : [0, \infty) \rightarrow M \text{ is continuous for every } u \in M,$$

in the context of semiflows, have been studied by several researchers. In particular, Ball (1976, 1996) has a result wherein it is shown that if a solution is measurable

in time, then it is continuous in time. Also, Ball (1974) and Chernoff and Marsden (1970) have shown that (2.71) implies (2.1b) under reasonable conditions. Finally, Ball (1996) gives an example of a semiflow on a Hilbert space with a global attractor, where property (2.71) is satisfied, but (2.72) is not. Also see Chernoff (1975).

3. The concept of an attractor, even in the finite dimensional setting, has evolved over the course of time. For example, this concept does not appear in the seminal books of Birkhoff (1927) or Nemytskii and Stepanov (1960). Even in the renaissance of dynamical systems theory which occurred in the 1950s, it is difficult to find any papers in which the attractor concept, as described herein, played an important role. For this reason, it is difficult to identify where this attractor concept first appeared. Some aspects of the concept of an attractor can be seen in the book of Bhatia and Szegö (1970), which deals with the evolution of sets in a flow. The first author recalls the early lectures of Conley in which the importance of an attractor was developed as a part of his index theory and the theory of attractor-repellor pairs. This work was subsequently published in Conley (1978).

One can find many different uses of the term “attractor” in the literature. For finite dimensional dynamics, every dynamical system is compact, and therefore, κ -contracting. As a result, the Stability Theorem 3.5 illustrates several equivalent formulations of the concept of the attractor used in the finite dimensional setting, see Conley (1978), Hale (1988), and Sell and You (1998)

It should be noted that in the past some authors have attributed the term “attractor” to a much weaker concept, *viz.* the concept of an “attracting set”. A set A in W is said to be an **attracting set** for a dynamical system σ on W , if A is a nonempty, bounded, positively invariant set, and if there is a bounded neighborhood U of A with the property that A attracts each point $u \in U$. An attracting set that is not Lyapunov stable is not a very rich dynamical object, since such sets behave badly under arbitrarily small, smooth perturbations of the vector field. An instructive example is given in polar coordinates in R^2 by

$$\partial_t r = r(1 - r), \quad \partial_t \theta = 1 - a \cos \theta,$$

where a is a constant. When $a = 1$, the fixed point $(r_0 = 1, \theta_0 = 0)$ is an attracting set that is not an attractor.

4. As noted in the Introduction, in the infinite dimensional setting, the earliest contributions to the theory of global attractors appear in Billotti and LaSalle (1971), Ladyzhenskaya (1972), and Hale, LaSalle, and Slemrod (1972). The concept of a global attractor used in this paper is somewhat weaker than the concepts appearing, for example, in Hale (1988) or Temam (1988). Recall that we defined \mathfrak{A} to be a global attractor for a dynamical system σ on W if the following three properties hold:

- (1) \mathfrak{A} is a nonempty, compact, invariant set for σ .
- (2) \mathfrak{A} attracts a bounded neighborhood of itself.
- (3) \mathfrak{A} attracts all points in W , i.e., the basin satisfies $B(\mathfrak{A}) = W$.

In Hale (1988) and Temam (1988) a set \mathfrak{A} in W is defined to be a global attractor if it satisfies (1) and the following two properties hold:

- (4) \mathfrak{A} is maximal with respect to (1), i.e., every compact, invariant set K satisfies $K \subset \mathfrak{A}$.

(5) \mathfrak{A} attracts every bounded set in W .

As seen in Lemma 2.3, the Maximality Property (4) is a consequence of (1), (2), and (3). Also properties (2) and (3) follow immediately from (1) and (5). Other authors have included the Minimality Property as a part of the definition of a global attractor, see Ladyzhenskaya (1987), for example. This property too is a consequence of (1), (2), and (3), see Sell and You (1998).

The principal advantage of using the weaker concept of a global attractor can be seen in the Main Theorem. We prove there that the set \mathfrak{A} satisfies Items (1), (2), and (3) above, but Item (5) is not resolved. The main reasons for this can be traced to Steps (5) and (6) in the proof of the Main Theorem. Since

the definitions are the handmaidens of the theorems,

our Main Theorem suggests that our concept of a global attractor is the better concept in the infinite dimensional setting. An open question is: does there exist an example of a κ -contracting semiflow that is point dissipative on a complete metric space W in which the global attractor does not attract every bounded set in W ? This issue of attracting every bounded set in W is not fully resolved by our Main Theorem, and we are unaware of any counterexample.

Bibliography

- A V Babin and M I Vishik (1983), *Regular attractors of semigroups of evolutionary equations*, J Math Pures Appl **62**, 441-491.
- A V Babin and M I Vishik (1992), *Attractors of Evolution Equations*, English translation, North Holland.
- J M Ball (1974), *Continuity properties of nonlinear semigroups*, J Functional Anal **17**, 91-102.
- J M Ball (1976), *Measurability and continuity conditions for nonlinear processes*, Proc Amer Math Soc **55**, 353-358.
- J M Ball (1978), *On the asymptotic behaviour of generalized processes, with applications to nonlinear evolutionary equations*, J Differential Equations **27**, 224-265.
- J M Ball (1996), *Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations*, Preprint, Oxford University.
- C Bardos and L Tartar (1973), *Sur l'unicité rétrograde des équations paraboliques et quelques équations voisines*, Arch Rational Mech Anal **50**, 10-25.
- N P Bhatia and G P Szegö (1970), *Stability Theory of Dynamical Systems*, Grundlehren Series Band 161, Springer Verlag, Berlin.
- J E Billotti and J P LaSalle (1971), *Dissipative periodic processes*, Bull Amer Math Soc **77**, 1082-1088.
- G D Birkhoff (1927), *Dynamical Systems*, Amer Math Soc, Providence.
- P R Chernoff (1975), *A note on the continuity of maps*, Proc Amer Math Soc **53**, 318-320.
- P R Chernoff and J E Marsden (1970), *On continuity and smoothness of group actions*, Bull Amer Math Soc **76**, 1044-1049.
- C C Conley (1978), *Isolated Invariant Sets and the Morse Index*, CBMS Regional Conference, vol 89, Amer Math Soc, Providence.
- K Deimling (1985), *Nonlinear Functional Analysis*, Springer Verlag, New York.
- R Ellis (1969), *Lectures on Topological Dynamics*, Benjamin, New York.
- E Feireisl and E Zuazua (1993), *Global attractors for semilinear wave equations ...*, Comm in PDE **18**, 1539-1555.
- C Foias and R Temam (1979), *Some analytic and geometric properties of the solutions of the Navier Stokes equations*, J Math Pures Appl **58**, 339-368.
- J M Ghidaglia and R Temam (1987), *Attractors for damped nonlinear hyperbolic equations*, J Math Pures Appl **66**, 273-319.
- J K Hale (1988), *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, Vol 25, Amer Math Soc, Providence.

- J K Hale, J P LaSalle, and M Slemrod (1972), *Theory of a general class of dissipative processes*, J Math Anal Appl **39**, 177-191.
- J K Hale and O Lopes (1973), *Fixed point theorems and dissipative processes*, J Differential Equations **13**, 391-402.
- J K Hale and G Raugel (1992), *A damped hyperbolic equation on thin domains*, Trans Amer Math Soc **329**, 185-219.
- A Haraux (1988), *Attractors of asymptotically compact processes and applications to nonlinear partial differential equations*, Comm in PDE **13**, 1383-1414.
- O A Ladyzhenskaya (1972), *On the dynamical system generated by the Navier-Stokes equations*, English translation, J Soviet Math **3**, 458-479.
- O A Ladyzhenskaya (1987), *On the determination of minimal global attractors for the Navier-Stokes and other partial differential equations*, English translation, Russian Math Surveys **42**, 27-73.
- O A Ladyzhenskaya (1991), *Attractors for Semigroups and Evolution Equations*, Cambridge Univ Press, Cambridge.
- O Lopes and S S Ceron (1984), *Existence of forced periodic solutions of dissipative semilinear hyperbolic equations and systems*, Ann Mat Pura Appl.
- A M Lyapunov (1892), *Problème général de la stabilité du mouvement*, Annals Math Studies, No 17 (1947), Princeton Univ Press, Princeton, N J.
- J Mallet-Paret (1976), *Negatively invariant sets of compact maps and an extension of a theorem of Cartwright*, J Differential Equations **22**, 331-348.
- R Mañé (1981), *On the dimension of the compact invariant sets of certain nonlinear maps*, Lecture Notes in Math, Vol 898, Springer Verlag, New York, pp. 230-242.
- M Marion (1989), *Finite dimensional attractors associated to partly dissipative reaction diffusion equations*, SIAM J Math Anal **20**, 816-844.
- R H Martin (1976), *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley, New York.
- P Massatt (1983), *Limiting behavior for strongly damped nonlinear wave equations*, J Differential Equations **48**, 334-349.
- V V Nemytskii and V V Stepanov (1960), *Qualitative Theory of Differential Equations*, Princeton Univ, Princeton NJ.
- H Poincaré (1890), *Sur le problème des trois corps et les équations de la dynamique*, Acta Math **13**, 1-270.
- H Poincaré (1892), *Les Méthodes Nouvelles de la Mécanique Céleste I-II-III*, Gauthier Villars, Paris.
- G R Sell (1971), *Topological Dynamics and Ordinary Differential Equations*, Van Nostrand, New York.
- G R Sell (1972), *A characterization of smooth α -Lipschitz mappings on a Hilbert space*, Atti Accad Naz Lincei Rend Cl Sci Fis Mat Natur **52**, 410-416.
- G R Sell (1973), *Differential equations without uniqueness and classical topological dynamics*, J Differential Equations **14**, 42-56.
- G R Sell (1996), *Global attractors for the three dimensional Navier-Stokes equations*, J Dynamics Differential Equations **8**, 1-33.
- G R Sell (1997), *References on dynamical systems*, <http://www.math.umn.edu/~sell>.
- G R Sell and Y You (1998), *Dynamics of Evolutionary Equations*, (to appear), Springer Verlag, New York.
- R Temam (1988), *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences, Vol 68, Springer Verlag, New York.
- M I Vishik (1992), *Asymptotic Behavior of Solutions of Evolution Equations*, Accademia Nazionale dei Lincei, Cambridge Univ Press.
- Y You (1994), *Global dynamics of nonlinear wave equations with nonmonotone damping*, Preprint.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA, 55455
USA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA TAMPA, FLORIDA, 33620
USA