ROBUSTNESS OF EXPONENTIAL DICHOTOMIES IN INFINITE DIMENSIONAL DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we examine the issue of the robustness, or stability, of an exponential dichotomy, or an exponential trichotomy, in a dynamical system on a Banach space W. These two hyperbolic structures describe longtime dynamical properties of the associated time-varying linearized equation $\partial_t v + Av = B(t)v$, where the linear operator $B(t)$ is the evaluation of a suitable Fréchet derivative along a given solution in the set K in W. Our main objective is to show, under reasonable conditions, that if $B(t) = B(\lambda, t)$ depends continuously on a parameter $\lambda \in \Lambda$ and there is an exponential dichotomy, or exponential trichotomy, at a value $\lambda_0 \in \Lambda$, then there is an exponential dichotomy, or exponential trichotomy, for all λ near λ_0. We present several illustrations indicating the significance of this robustness property.

1. INTRODUCTION

An exponential dichotomy is one of the most basic concepts arising in the theory of dynamical systems. This topic, for example, plays a central role in the study of stable and unstable manifolds, and in many aspects of the theory of stability. Even in the context of bifurcation theory, the exponential dichotomy has a role. However in this context, the exponential dichotomy is represented by its younger sibling, the exponential trichotomy. In particular, topics such as the reduction principle and the center manifold theorem, the robustness of periodic solutions and invariant manifolds, and the chaotic behavior, as seen in the Poincaré-Melnikov scenario, are based on the theory of exponential trichotomies.

Because of the central role played by the hyperbolic structures of an exponential dichotomy and/or an exponential trichotomy in the perturbations theory of dynamical systems, one oftentimes would like to have specific information on how these hyperbolic structures depend on the coefficients, or the parameters, of the

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problem. For example, can one show that these hyperbolic structures, which are properties of the linear equation

$$\partial_t u + Au = B(\lambda, t)u, \qquad u \in W, \tag{1.1}$$

depend on the parameter $\lambda \in \Lambda$, where the phase space W is a Banach space. In other words, if one assumes that equation (1.1) has an exponential dichotomy for $\lambda = \lambda_0$, when can one conclude that equation (1.1) has an exponential dichotomy for all λ near λ_0, and in this case, how do the dynamical features of the dichotomies depend on λ?

We will say much more about the linear operators A and $B(\lambda, t)$ appearing in equation (1.1) later. Suffice it to say here that equation (1.1) arises as the linearization of a nonlinear problem

$$\partial_t u + Au = F(u)$$

along some globally defined solution $\phi(t)$, where $B(t) = B(\lambda_0, t) = DF(\phi(t))$, for $t \in R$, and DF is a suitable Fréchet derivative of F. The dependence of B on a parameter λ typically arises in two situations: (1) the function $F(u) = F(\lambda, u)$ depends on a parameter, and/or (2) the solution ϕ lies in a set $K_\lambda \subset W$, where K_λ depends on some dynamical parameter. A heuristic version of the theorem we seek to prove here is the following.

Main Theorem. *Assume that the linear operator $B(\lambda, t)$ varies continuously for $(\lambda, t) \in \Lambda \times R$. If a compactness condition is satisfied, and equation (1.1) has an exponential dichotomy (or an exponential trichotomy) for some value $\lambda_0 \in \Lambda$, then there is a neighborhood $O = O(\lambda_0)$ of λ_0 in Λ, such that equation (1.1) has an exponential dichotomy (or an exponential trichotomy) for all $\lambda \in O$. Furthermore the characteristics of these hyperbolic structures are well-behaved over O.*

The proof of this result is based on two Robustness Theorems and several applications of these theorems, see Sections 5 and 6.

We are motivated by a desire to apply this theory to the study of a wide range of differential equations, including ordinary differential equations, partial differential equations, and functional differential equations, see for example, Hale (1977, 1988) Pliss (1966, 1977) Pliss and Sell (1991, 1998), Sell and You (1998), and Temam (1988). In the partial differential equations realm alone, we are seeking a theory which applies to parabolic problems (such as reaction diffusion equations and the Navier-Stokes equations), on the one hand, and to hyperbolic problems (such as fully integrable systems and nonlinear wave equations with dissipation), on the other hand. The most challenging of these problems is represented by the Navier-Stokes equations. In this setting the phase space $W = H$ is a Hilbert space, and the time-varying operator $B(t)$ is an unbounded linear operator on H, because the nonlinarities $F(u)$ arising in the Navier-Stokes equations are not Fréchet differentiable on the phase space H. Instead, one needs to restrict $F(u)$ to a subspace V, where $V \hookrightarrow H$ is compactly imbedded in H. For the applications to the Navier-Stokes equations, and related problems, we need to exploit the dynamical significance of the fact that the linear operator A "dominates" $B(t)$.

In Section 2 we present the basic theory of linear skew product semiflows, as it will be used in this paper. The definitions of an exponential dichotomy and an exponential trichotomy are given in Section 3. In addition to the continuous time case, where $t \in R$, we are interested in the discrete version where the time t is restricted to Z, the integers. In Section 4, we will examine the connections between a discrete dynamical system and an associated linear inhomogeneous problem

$$w_{n+1} = T_n w_n + f_n, \qquad \text{for } n = 0, \pm 1, \pm 2, \dots, \tag{1.3}$$

where $T = \{T_n\}$ is a sequence with $T \in \ell_\infty(Z, \mathcal{L}(W))$, and $\mathcal{L}(W)$ is the space of bounded linear operators on W. Our objective in this section is to present a new proof of a theorem of Henry (1981), which gives a necessary and sufficient condition for an exponential dichotomy for the discrete dynamical system generated by

$$w_k = \hat{\Phi}(T, k) w_0 = T_{k-1} \cdot \dots \cdot T_1 \cdot T_0 \, w_0,$$

in terms of the bounded solutions of equation (1.3). As we will show, the Henry Theorem on discrete dynamics introduces a powerful technique for the study of exponential dichotomies, or exponential trichotomies, for the continuous time problem generated by the solutions of equation (1.1). The Robustness Theorems are formulated and proved in Section 5. Finally in Section 6 we examine several applications of the Robustness Theorems, with special emphasis on the Navier-Stokes equations and the nonlinear wave equation.

2. Linear Skew Product Semiflows

We will let W denote a metric space. The distance between two points u and v in W will be denoted by $d(u, v) = d_W(u, v)$, where $d = d_W$ is a metric on W. Recall that if W is a Banach space, then the standard metric on W is given by $d(u, v) = \|u - v\|$, where $\| \cdot \|$ is the norm on W. In the case of a Fréchet space W, we will use an invariant metric d, where $d(u, v) = d(u - v, 0)$, for all $u, v \in W$. Let $R = (-\infty, \infty)$ and $R^+ = [0, \infty)$. A mapping $\sigma = \sigma(u, t)$, where $\sigma : W \times [0, \infty) \to W$, is said to be a **semiflow** on W, provided the following hold:

(1) $\sigma(w, 0) = w$, for all $w \in W$.

(2) The **semigroup property** holds, i.e.,

$$\sigma(\sigma(w, s), t) = \sigma(w, t + s), \qquad \text{for all } w \in W, \text{ and } s, t \in R^+. \tag{2.0}$$

(3) The mapping $\sigma : W \times (0, \infty) \to W$ is continuous.

If it happens that the semiflow satisfies the stronger continuity property, where $\sigma : W \times [0, \infty) \to W$ is continuous, then we say that the semiflow is **continuous at** $t = 0$. It should be noted that we do not assume, in this definition, the continuity of σ at $t = 0$, even though some semiflows have this property. Possible singularities at $t = 0$ arise naturally in some infinite dimensional problems. More information on this issue can be found in Sell and You (1998).

It is convenient, for some purposes, to write σ in an alternate form $\sigma(u, t) = S(t)u$. In this notation, the semigroup property (2.0) then takes on the form

$$S(t)S(s)u = S(s + t)u, \qquad \text{for all } s, t \geq 0. \tag{2.0a}$$

It may happen that for each $t \geq 0$, the mapping $S(t)$ is a one-to-one mapping of M onto M with a continuous inverse $S(t)^{-1}$. In this case, we set $S(-t) \stackrel{\text{def}}{=} S(t)^{-1}$, for $t > 0$. As a result, (2.0a) holds for all $u \in M$ and all $s, t \in R$. In this case, the dynamical system will be referred to as a **flow**. In the case of a flow $S(t)$ on W, we define the **trajectory** $\gamma(A)$ and the **hull** $H(A)$ of a set A in W by

$$\gamma(A) = \{S(t)w : w \in A \text{ and } t \in R\} \quad \text{and} \quad H(A) = \text{Cl}\ \gamma(A),$$

where the closure is taken in W.

There do exists discrete versions of these concepts. For a fixed number $\tau > 0$, we define

$$\tau Z^+ = \{n\tau : n = 0, 1, \cdots\} \quad \text{and} \quad \tau Z = \{n\tau : n = 0, \pm 1, \cdots\}.$$

A mapping $\sigma(\cdot, t) : W \to W$, for $t \in \tau Z^+$, is said to be a **discrete semiflow** on W, provided that σ is continuous, $\sigma(w, 0) = w$, for all $w \in W$, and equation (2.0) holds for all $w \in W$ and $s,\ t \in \tau Z^+$. A **discrete flow** is defined similarly, but now with τZ^+ replaced by τZ. The prototype of a discrete semiflow arises when one begins with a semiflow $\sigma(w, t)$ on W, where $t \in [0, \infty)$, and then restricts time t to satisfy $t \in \tau Z^+$, for some $\tau > 0$. In this case, the semigroup property implies that

$$\sigma(w, n\tau) = E^n(w), \qquad \text{for } n = 0, 1, 2, \cdots, \tag{2.0c}$$

where $E(w) = \sigma(w, \tau)$. Note that $E : W \to W$ is a continuous mapping on W. Conversely, if one begins with a continuous mapping E on W and uses (2.0c) to define σ, then σ is a discrete semiflow on W. An illustration of a discrete semiflow is the period map generated by a system of ordinary differential equations with periodic coefficients.

In this paper we will study a class of linear skew product semiflows on a product space $\mathcal{E} = W \times \mathcal{W}$, where W is a Banach space, and $\mathcal{W}$ is a metric space. In our main applications, $\mathcal{W}$ is a Fréchet space consisting of time-varying functions $B = B(t)$, where $t \in R$. In this application, the mapping $(B, \tau) \to B_\tau$, where $B_\tau(t) = B(\tau + t)$, generates a flow on $\mathcal{W}$. In anticipation of this application, which is described in more detail later, we let $\mathcal{W}$ denote <u>any</u> metric space with a given <u>flow</u> on σ on $\mathcal{W}$, and we will write this flow in the form $\sigma(B, \tau) = B_\tau$, for $B \in \mathcal{W}$ and $\tau \in R$.

A **linear skew-product semiflow** on $\mathcal{E} = W \times \mathcal{W}$ is a mapping $\pi = (\Phi, \sigma)$ of the form

$$\pi(w, B, t) = (\Phi(B, t)w,\ B_t), \qquad \text{for } t \geq 0,$$

with the following properties:

(1) The mapping $\sigma(B, \tau) = B_\tau$ is a flow on $\mathcal{W}$;
(2) $\Phi(B, 0) = I$, the identity operator, for all $B \in \mathcal{W}$;
(3) $\Phi(B, t)$ is an element of $\mathcal{L}(W)$ that satisfies the cocycle identity:

$$\Phi(B, s + \tau) = \Phi(B_\tau, s)\Phi(B, \tau), \qquad \text{for } B \in \mathcal{W},\ \tau \in R \text{ and } s \geq 0.$$

(4) The mapping from $\mathcal{E} \times (0, \infty)$ into W given by $(w, B, t) \to \Phi(B, t)w$ is continuous.

(5) For each $(w, B) \in \mathcal{E}$ the mapping $t \to \Phi(B,t)w$ is continuous at $t = 0$, and for each $w \in W$ the limit $\lim_{t\to 0^+} \Phi(B,t)w = w$ is uniform for B in compact sets, i.e., for every compact set $\mathcal{K}_0 \subset \mathcal{W}$, $w \in W$, and $\epsilon > 0$, there is a $\delta > 0$ such that $\|\Phi(B,t)w - w\| \le \epsilon$, for all $B \in \mathcal{K}_0$ and $t \in [0, \delta]$.

The prototype of a semiflow is a C_0-semigroup of linear operators on a Banach space W. In this case, $\mathcal{W} = \{A\}$ reduces to a single point, where $A_\tau = A$, for all $\tau \in R$, and $\Phi(A,t) = e^{At}$ is the C_0-semigroup generated by A.

For many applications one is interested in the case where $\mathcal{K}$ is an **invariant** set in $\mathcal{W}$, i.e., $\sigma(\mathcal{K}, t) = \mathcal{K}$, for all $t \ge 0$. If instead, one has $\sigma(\mathcal{K}, t) \subset \mathcal{K}$, for all $t \ge 0$, then we say that $\mathcal{K}$ is a **positively invariant** set. Note that if $\mathcal{K}$ is an invariant set for $S(t)$, then it follows from the continuity property for a semiflow that $\sigma : K \times [0,\infty) \to K$ is continuous. For such applications, the following feature is useful and easy to verify.

Lemma 2.1 (Continuity Lemma). *Let $S(t)$ be a semiflow on W, and let K be a compact, invariant set for $S(t)$. Then for any T with $0 < T < \infty$ and any $\epsilon > 0$, there is a $\delta > 0$, such that if $v \in K$ and if u satisfies $d(u, v) \le \delta$, then one has*

$$d(S(t)u, K) \le d(S(t)u, S(t)v) < \epsilon \qquad \textit{for } 0 \le t \le T,$$

where $d(u, K) = \inf\{d(u, w) : w \in K\}$.

There are two prototypical examples of a linear skew-product semiflows in the finite dimensional setting. In the first example, we begin with a flow $(\theta, t) \to \theta \cdot t$ on a compact metric space M. For example, M might be an invariant manifold for a given ordinary differential equation on R^N. Next we let $B : M \to \mathcal{L}(R^n)$ denote a continuous mapping into the space $\mathcal{L}(R^n)$ of bounded linear operators on R^n. (We do not insist here that $N = n$.) Next we let $\Phi(\theta, t)$ be the fundamental solution operator (or solution matrix) of the linear ordinary differential equation

$$\partial_t w = B(\theta \cdot t)w = A(t)w, \qquad \text{for } \theta \in M,$$

that satisfies $\Phi(\theta, 0) = I$. In this example one obtains a linear skew product flow on $R^n \times M$ given by $\pi(w, \theta, t) = (\Phi(\theta,t)w, \theta \cdot t)$.

For the second example, we consider the time varying problem $A = A(t)$, where A is a $n \times n$ matrix valued function that satisfies

$$A(\cdot) \in \mathcal{W} \stackrel{\text{def}}{=} L^\infty(R; \mathcal{L}(R^n)) \cap C(R; \mathcal{L}(R^n)),$$

where the translational flow $(B, \tau) \to B_\tau$ is given on $\mathcal{W}$. In this example, $\mathcal{W}$ is a Fréchet space with the topology of uniform convergence on bounded sets. We let $\Phi(A, t)$ denote the fundamental solution operator of the ordinary differential equation $\partial_t w = A(t)w$, see Miller (1965), Sell (1967ab, 1971), Miller and Sell (1970), Sacker and Sell (1974, 1976ab), Daletski and Krein (1974), Johnson (1986), Shen and Yi (1995abc, 1996ab), and Yi (1996). In this case, $\pi(w, A, t) = (\Phi(A,t)w, A_t)$ is a linear skew product flow on $R^n \times \mathcal{W}$.

While the constructions given in the last two paragraphs differ, there is a close connection between them. In particular the flow $(\theta, \tau) \to \theta \cdot \tau$ on M has a continuous imbedding into the flow $(B,\tau) \to B_\tau$ by defining $A(t) = B(\theta \cdot t)$, for $(\theta, t) \in M \times R$. The collection $\mathcal{K}$ of all such A, is the continuous image of a compact set, and thus $\mathcal{K}$ is compact. Furthermore, $\mathcal{K}$ is invariant, and one has the commutivity property

$$A_\tau(t) = A(\tau + t) = B(\theta \cdot (\tau + t)), \qquad \text{for } t, \tau \in R,$$

since $(\theta \cdot \tau) \cdot t = \theta \cdot (\tau + t)$, for τ, $t \in R$. As we show later, one can construct similar examples in the infinite dimensional setting.

In order to prepare for the applications of linear skew product semiflows in the infinite dimensional setting, it is convenient to build a little more structure into the theories given above. In particular, we return to the linear skew product flow $\pi = \pi(w, B, t)$ on $\mathcal{E} = W \times \mathcal{W}$ with continuous time $t \in R^+$. We now assume that $\mathcal{W} = \mathcal{W}(R, \mathcal{L})$ is a Fréchet space, where each $B \in \mathcal{W}$ is a function $B = B(t)$ defined on R and assuming values in $\mathcal{L}$, i.e., $B(t) \in \mathcal{L} = \mathcal{L}(V, W)$, for all $t \in R$, where $\mathcal{L} = \mathcal{L}(V, W)$ is the Banach space of all bounded linear operators from V into W. When $V = W$, we will write $\mathcal{L}(W) = \mathcal{L}(W, W)$. The norm $\|B(t)\|_{\mathcal{L}}$ is the operator norm, i.e.,

$$\|B(t)\|_{\mathcal{L}} = \sup\{\|B(t)v\|_W : \|v\|_V \le 1\}.$$

We assume that $\mathcal{W}$ is a Fréchet space, with an invariant metric given by

$$d(B - C) = d_{\mathcal{W}}(B - C) = \sum_{n=1}^{\infty} 2^{-n} \min(\|B - C\|_n, 1), \tag{2.1e}$$

where $\|\cdot\|_n$, for $n = 1, 2, \ldots$, is a nested[1] family of pseudonorms with $\|B\|_n \le \|B\|_{n+1}$, for all $n = 1, 2, \ldots$ and $B \in \mathcal{W}$, see Kelley and Namioka (1976).

Basic Axioms. *We make the following two assumptions concerning this abstract setup:*

(1) *the mapping $(B, \tau) \to B_\tau$ defines a continuous-time flow on $\mathcal{W}$, and*
(2) *for each $B \in \mathcal{W}$ and $t \ge 0$, there is an operator $\Phi(B, t) \in \mathcal{L}(W)$ with the property that $\pi(w, B, t) = (\Phi(B, t)w, B_t)$ is a linear skew product semiflow on $\mathcal{E} = W \times \mathcal{W}$.*

In the following definitions of distinguished subsets of $\mathcal{E}$, we will make use of the concept of a negative continuation, which we now define. A mapping $\phi(t) = (w(t), B_t) : R \to \mathcal{E}$ is said to be a **globally defined solution** through (w_0, B) provided that ϕ is continuous, that $w(0) = w_0$, and that

$$\Phi(B_\tau, t)w(\tau) = w(\tau + t), \qquad \text{for all } \tau \in R \text{ and } t \ge 0. \tag{2n}$$

In this case, the restriction of ϕ to $(-\infty, 0]$, which we will denote by $\phi^{w_0, B}$, is said to be a **negative continuation** of (w_0, B). We do not assume the uniqueness of the negative continuations, but we will show later that, when one has an exponential dichotomy, then in a qualified sense, some negative continuations are unique.

[1] The nesting is not essential, but it is convenient.

Let us now restrict to the linear skew product semiflows $\pi = (\Phi, \sigma)$ on $\mathcal{E}(\mathcal{K}) = W \times \mathcal{K}$, where $\mathcal{K}$ is an invariant set in $\mathcal{W}$. Thus one has $\sigma(\mathcal{K}, t) = \mathcal{K}$, for all $t \geq 0$. We define next the following six subsets of $\mathcal{E}(\mathcal{K})$:

(1) $\mathcal{U}$: The set of points $(w, B) \in \mathcal{E}(\mathcal{K})$ such that there is a negative continuation $\phi^{w,B}(t) = (w(t), B_t)$ that satisfies $\|w(t)\| \to 0$, as $t \to -\infty$.
(2) $\mathcal{B}^-$: The set of points $(w, B) \in \mathcal{E}(\mathcal{K})$ such that there is a negative continuation $\phi^{w,B}(t) = (w(t), B_t)$ that satisfies $\sup_{t\leq 0} \|w(t)\| < \infty$.
(3) $\mathcal{B}_u^-$: The set of points $(w, B) \in \mathcal{E}(\mathcal{K})$ such that there is a unique bounded negative continuation $\phi^{w,B}$.
(4) $\mathcal{B}^+$: The set of points $(w, B) \in \mathcal{E}(\mathcal{K})$ such that $\sup_{t\geq 0} \|\Phi(B, t)w\| < \infty$.
(5) $\mathcal{S}$: The set of points $(w, B) \in \mathcal{E}(\mathcal{K})$ such that $\|\Phi(B, t)w\| \to 0$, as $t \to \infty$.
(6) $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}^- \cap \mathcal{B}^+$.

One refers to $\mathcal{U}$ as the **unstable set**, to $\mathcal{S}$ as the **stable set** and to $\mathcal{B}$ as the **bounded set**. Note that the set $\mathcal{B}$ is an invariant set, and it is the set of all points $(w_0, B) \in \mathcal{E}(\mathcal{K})$ such that there is a globally defined solution ϕ through (w_0, B) with $\sup_{t\in R} \|\phi(t)\|_W < \infty$.

A **discrete** linear skew product semiflow $\hat{\pi}$ on $\mathcal{E}(\mathcal{K}) = W \times \mathcal{K}$ is defined by using the definition of a linear skew product flow, but now restricting the time t to assume values in the discrete semigroup Z^+. Thus $\hat{\sigma}(B, \tau) = B_\tau$ is a discrete semiflow on $\mathcal{K}$, where $B \in \mathcal{K}$, and for $\hat{\pi}(w, B, \tau) = (\hat{\Phi}(B, \tau)w, B_\tau)$ one has

(1) $\hat{\Phi}(B, 0) = I$, the identity operator, for all $B \in \mathcal{K}$;
(2) $\hat{\Phi}(B, \tau)$ is an element of $\mathcal{L}(W)$ that satisfies the cocycle identity:

$$\hat{\Phi}(B, \sigma + \tau) = \hat{\Phi}(B_\tau, \sigma)\hat{\Phi}(B, \tau), \qquad \text{for } B \in \mathcal{K} \text{ and all } \sigma, \tau \in Z^+.$$

(3) For each $\tau \in Z^+$, the mapping from $\mathcal{E}(\mathcal{K})$ into W given by $(w, B, \tau) \to \hat{\Phi}(B, \tau)w$ is continuous.

In other words, properties (1), (2) hold, and (3) for a linear skew product semiflow hold for the discrete times $\tau \in Z^+$. Property (4) is vacuous in this setting.

The discrete linear skew product semiflows arise in two standard ways in our study of the dynamics of evolutionary equations. First, one may begin with a linear skew product semiflow $\pi(w, B, t) = (\Phi(B, t)w, B_t)$ defined for continuous time $t \in R^+$. A discrete counterpart then arises by restricting the time to assume values $t = n\tau$ in the discrete semigroup τZ^+, where $n = 0, 1, \cdots$ and $\tau > 0$ is fixed.

A second construction arises when $T = \{T_n\} = \{T_n : -\infty < n < \infty\}$ is a given sequence with $T_n \in \mathcal{L}(W)$, for all $n \in Z$, with

$$\|T\|_\infty \stackrel{\text{def}}{=} \sup_{n\in Z} \|T_n\| < \infty, \tag{2.2a}$$

that is $T \in \mathcal{Z} \stackrel{\text{def}}{=} \ell_\infty(Z, \mathcal{L}(W))$, where W is a given Banach space. The base space $\mathcal{Z}$ is then the collection of all such sequences, and the dynamics on $\mathcal{Z}$ is the translational flow $(T, \tau) \to T \cdot \tau$, where $T \cdot \tau = \{T_{\tau+n}\}$, for $\tau \in Z$. In this case, the space $\mathcal{Z}$ is blessed with two topologies. First $\mathcal{Z}$ is a Banach space in the norm $\|\cdot\|_\infty$. But more important for us, $\mathcal{Z}$ is a Fréchet space under the topology of uniform convergence on bounded sets in Z. This second topology is generated by the family of pseudo norms

$$\|T\|_n \overset{\text{def}}{=} \sup\{\|T_k\|_{\mathcal{L}(W)} : |k| \le n\}, \qquad \text{for } n = 1, 2, \cdots,$$

and the corresponding invariant metric

$$d(T - S) \overset{\text{def}}{=} \sum_{n=1}^{\infty} 2^{-n} \min(\|T - S\|_n, 1), \qquad T,\, S \in \mathcal{Z}. \tag{2.2c}$$

The linear mapping $\hat{\Phi}$ is then defined, for $k \ge 0$, by using induction and setting $\hat{\Phi}(T, 0) = I$ and $\hat{\Phi}(\theta, k)w = T_{k-1} \cdot \ldots \cdot T_1 \cdot T_0\, w$, for $k \ge 1$. As a result,

$$\hat{\pi}(w, T, k) \overset{\text{def}}{=} (\hat{\Phi}(T, k)w, T \cdot k), \qquad \text{for } k \ge 0, \tag{2.2e}$$

is a discrete linear skew product semiflow on $W \times \mathcal{Z}$.

3. Exponential Dichotomies and Exponential Trichotomies

Our objective in this section is to present the basic theory of exponential dichotomies and exponential trichotomies on a Banach space. The definitions of these concepts are similar to those used in the theory of ordinary differential equations, see Pliss and Sell (1991, 1998). However, since we are considering semiflows here, it is essential that we exercise special care in dealing with the negative continuations.

Let W be a given Banach space and let $\mathcal{W}$ be a metric space. For any set $\mathcal{V}$ in the product space $\mathcal{E} = W \times \mathcal{W}$, we define the **fiber** $\mathcal{V}(B_0)$ over the point $B_0 \in \mathcal{W}$ by

$$\mathcal{V}(B_0) \overset{\text{def}}{=} \{(w, B) \in \mathcal{E} : (w, B) \in \mathcal{V} \text{ and } B = B_0\}.$$

Similarly for any set $\mathcal{K} \subset \mathcal{W}$ we define the **restriction of** $\mathcal{V}$ **to** $\mathcal{K}$ as

$$\mathcal{V}(\mathcal{K}) \overset{\text{def}}{=} \{(w, B) \in \mathcal{E} : B \in \mathcal{K}\} = \bigcup_{B \in \mathcal{K}} \mathcal{V}(B).$$

Notice that $\mathcal{E}(B) = W \times \{B\}$, and $\mathcal{E}(B)$ is a Banach space, with the same structure as W. A mapping $P : \mathcal{E} \to \mathcal{E}$ is said to be a **projector** if P is continuous and has the form $P(w, B) = (P(B)w, B)$, where $P(B)$ is a continuous (linear) projection[2] on the fiber $\mathcal{E}(B)$. This means that $P(B) : \mathcal{E}(B) \to \mathcal{E}(B)$ is a bounded linear mapping that satisfies $P(B)P(B) = P(B)^2 = P(B)$. For any projector $P : \mathcal{E} \to \mathcal{E}$ we define the **range** and **null space** by

$$\mathcal{R} = \mathcal{R}(P) = \{(w, B) \in \mathcal{E} : P(B)w = w\}$$

and

$$\mathcal{N} = \mathcal{N}(P) = \{(w, B) \in \mathcal{E} : P(B)w = 0\}.$$

Note that the fibers $\mathcal{R}(B)$ and $\mathcal{N}(B)$ are closed linear subspaces of $\mathcal{E}(B)$, since $P(B)$ is a continuous linear mapping. Furthermore, these fibers vary continuously in B, which implies that $P(B)$ varies continuously in the operator norm in $\mathcal{L}(W)$. The following result is easily proven

[2] There is a small ambiguity in this notation since we use both $P(w, B)$ and $P(B)w$ to represent P. These two formulations really offer two ways of viewing the concept of a projector.

Lemma 3.1. *Let P be a projector on $\mathcal{E}$. Then $\mathcal{R}$ and $\mathcal{N}$ are closed subsets in $\mathcal{E}$, and one has*

$$\mathcal{R}(B) \cap \mathcal{N}(B) = \{0\} \quad \textit{and} \quad \mathcal{R}(B) + \mathcal{N}(B) = \mathcal{E}(B), \qquad \textit{for all } B \in \mathcal{W}. \tag{3.1}$$

If P is a projector on $\mathcal{E}$, then the mapping $Q = I - P$, where $Q : \mathcal{E} \to \mathcal{E}$ is defined by $Q(w, B) = ((I - P(B))w, B)$, is also a projector on $\mathcal{E}$. The projector Q is called the **complementary projector** to P, and one has $\mathcal{R}(Q) = \mathcal{N}(P)$ and $\mathcal{N}(Q) = \mathcal{R}(P)$.

The range and nullspace of a projector are subbundles of $\mathcal{E}$. More precisely, a subset $\mathcal{X}$ in $\mathcal{E}$ is said to be a **subbundle** of $\mathcal{E}$ if there is a projector P on $\mathcal{E}$ with the property that $\mathcal{X} = \mathcal{R}(P)$. In this case, $\mathcal{Y} = \mathcal{N}(P)$ is a **complementary** subbundle, and one has $\mathcal{E} = \mathcal{X} + \mathcal{Y}$, in the sense that (3.1) is valid. The equation $\mathcal{E} = \mathcal{X} + \mathcal{Y}$ is sometimes referred to as a **Whitney sum** of subbundles. The **trivial** subbundle $\mathcal{E}_0 = \{0\} \times \mathcal{Y}$ plays a role in the theory of exponential dichotomies.

Now let $\pi = (\Phi, \sigma)$ be a linear skew product semiflow on $\mathcal{E} = W \times \mathcal{W}$, and let $\mathcal{K}$ be an invariant set in $\mathcal{W}$. In this setting, a projector P on $\mathcal{E}(\mathcal{K}) = W \times \mathcal{K}$ is said to be **invariant** if one has

$$P(B_t)\Phi(B, t) = \Phi(B, t)P(B), \qquad \text{for all } t \geq 0 \text{ and } B \in \mathcal{K}. \tag{3.1i}$$

The invariance of a projector is equivalent to the assertion that both subbundles, $\mathcal{R}$ and $\mathcal{N}$, are positively invariant under the linear skew product semiflow π. Note that P is invariant if and only if the complementary projector Q is invariant.

3.1 Exponential Dichotomy. We say that a linear skew product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = W \times \mathcal{W}$ has an **exponential dichotomy over an invariant set** $\mathcal{K} \subset \mathcal{W}$, if there is a projector P on the restriction $\mathcal{E}(\mathcal{K}) = W \times \mathcal{K}$, and constants $K \geq 1$ and $\alpha > 0$ such that the following hold:

(1) The projectors P and Q are invariant on $\mathcal{E}(\mathcal{K})$, where $Q = I - P$.
(2) One has $\mathcal{R}(P(B)) \subset \mathcal{U}(B)$, for each $B \in \mathcal{K}$. For each $w \in \mathcal{R}(P(B))$, we let

$$\phi^{w,B}(t) = (\Phi(B, t)w, B_t), \qquad \text{for } t \leq 0,$$

denote <u>any</u> negative continuation with $\|\Phi(B, t)w\| \to 0$, as $t \to -\infty$.
(3) The following inequalities are valid for all $w \in W$:

$$\|\Phi(B, t)Q(B)w\| \leq K\|w\|e^{-\alpha t}, \qquad \text{for } t \geq 0 \text{ and } B \in \mathcal{K}, \tag{3ds}$$

and

$$\|\Phi(B, t)P(B)w\| \leq K\|w\|e^{\alpha t}, \qquad \text{for } t \leq 0 \text{ and } B \in \mathcal{K}, \tag{3du}$$

where (3du) is valid for any negative continuation through $(P(B)w, B)$ that remains in $\mathcal{U}$, for all $t \leq 0$.

We will refer to (K,α) as the **characteristics** of the dichotomy, and we will call (P,Q) the **associated projectors.** Since inequalities (3ds) and (3du) hold at $t=0$, one has

$$\|P(B)\|_{\mathcal{L}(W)},\ \|Q(B)\|_{\mathcal{L}(W)} \le K, \qquad \text{for all } B \in \mathcal{K}. \tag{3do}$$

The concept of an exponential dichotomy for a discrete linear skew product semiflow is identical to that given above, with the single exception that the time t is now restricted to assume values in some discrete subgroup, or discrete semisubgroup, of R. In particular, for the discrete semiflow $\hat{\pi}$ given by equation (2.2e) on the space $W\times\mathcal{Z}$, we let $\mathcal{K}$ denote any set in the base space $\mathcal{Z}=\ell_\infty(Z,\mathcal{L}(W))$. The semiflow π is said to have an **exponential dichotomy over** $\mathcal{K}$, **with characteristics** (K,s), where $K\ge 1$ and $0<s<1$, provided that there is a projector P defined on $\mathcal{K}$ such that

(1) For each $T=\{T_n\}\in\mathcal{K}$, the projector sequence $P(T)=\{P_n\}$, where $P_n=P_n(T)$ depends on T, satisfies

$$P_{m+1}T_m = T_m P_m, \qquad \text{for all } m \in Z. \tag{3.1j}$$

(2) One has $\mathcal{R}(P(T))\subset\mathcal{U}(T)$, for each $T\in\mathcal{K}$. For each $w\in\mathcal{R}(P(T))$, we let

$$\phi^{w,T}(k) = (\Phi(T,k)w, T\cdot k), \qquad \text{for } k \le 0,$$

denote <u>any</u> negative continuation with $\|\Phi(T,k)w\|\to 0$, as $k\to-\infty$.

(3) The following inequalities are valid for all $w\in W$ and every $\tau\in Z$:

$$\|\Phi(T\cdot\tau,k)Q(T\cdot\tau)\,w\| = \|T_{\tau+k-1}\cdot\ldots\cdot T_{\tau+1}\cdot T_\tau Q_\tau\, w\| \le K\|w\|s^k, \tag{3.1s}$$

for $k\ge 0$ and $T\in\mathcal{K}$, and

$$\|\Phi(T\cdot\tau,k)P(T\cdot\tau)\,w\| \le K\|w\|s^{-k}, \qquad \text{for } k\le 0 \text{ and } T\in\mathcal{K}, \tag{3.1u}$$

where (3.1u) is valid for any negative continuation through $(P(T)w,T)$ that remains in $\mathcal{U}$, for all $k\le 0$.

It should be noted that if $\hat{\pi}$ has an exponential dichotomy over a set $\mathcal{K}\subset\mathcal{Z}$ with characteristics (K,s), then $\hat{\pi}$ has an exponential dichotomy over the invariant set $\gamma(\mathcal{K})$, with the same characteristics. In the following lemma it is shown, in addition, that $\hat{\pi}$ has an exponential dichotomy over the hull $H(\mathcal{K})=\mathrm{Cl}\ \gamma(\mathcal{K})$, with the same characteristics.

There is some additional information contained in the inequalities (3du) and (3.1u). Let us consider the continous time case, where $w\in\mathcal{R}(P(B))$. Then one has $w=P(B)w$. Also for $t\le 0$, set $\tau=-t$ and define $\hat{w}=\Phi(B,t)w$. One then has $\hat{w}\in\mathcal{R}(P(B_t))$ and $w=\Phi(B_t,\tau)\hat{w}$. Now inequalities (3du) and (3.1u) can be rewritten in the form $\|\hat{w}\|\le K\|\Phi(B_t,\tau)\hat{w}\|e^{-\alpha\tau}$, or by dropping the hats and replacing B_t by B and $\tau\ge 0$ by $t\ge 0$, one has

$$\|\Phi(B,t)w\| \ge K^{-1}\|w\|e^{\alpha t}, \qquad \text{for all } w\in\mathcal{R}(P(B)) \text{ and } t\ge 0. \tag{3duu}$$

The same inequality is valid for the discrete time problem, where $t \in Z^+$.

There are important consequences of the existence of an exponential dichotomy. It follows from (3ds) and (3du) that

$$\mathcal{N}(P(B)) \subset \mathcal{S}(B) \qquad \text{and} \qquad \mathcal{R}(P(B)) \subset \mathcal{U}(B),$$

for all $B \in \mathcal{K}$. Consequently, (3.1) implies that

$$\mathcal{S}(B) + \mathcal{U}(B) = \mathcal{E}(B), \qquad \text{for all } B \in \mathcal{K}.$$

The case $P = 0$ is allowed, provided Items (1)-(3) are valid. In this case, the linear skew product semiflow is said to be **exponentially stable**. We now have the following result wherein, among other things, we show that certain negative continuations are unique.

Lemma 3.2. *Let $\pi = (\Phi, \sigma)$ be a linear skew product semiflow on $\mathcal{E} = W \times \mathcal{W}$, where σ is a flow on $\mathcal{W}$. Assume that π has an exponential dichotomy over an invariant set $\mathcal{K} \subset \mathcal{W}$. Then the following statements are valid:*

1. *The bounded set satisfies $\mathcal{B}(\mathcal{K}) = \mathcal{E}_0(\mathcal{K}) = \{0\} \times \mathcal{K}$.*
2. *One has $\mathcal{S}(B) \cap \mathcal{U}(B) = \{0\}$, for all $B \in \mathcal{K}$.*
3. *One has $\mathcal{S}(B) = \mathcal{N}(P(B)) = \mathcal{R}(Q(B))$ and $\mathcal{U}(B) = \mathcal{R}(P(B))$, for all $B \in \mathcal{K}$, and the convergence rates in $\mathcal{S}$ and $\mathcal{U}$ are exponential over $\mathcal{K}$.*
4. *The subbundles $\mathcal{S} = \mathcal{S}(\mathcal{K}) = \mathcal{R}(Q)$ and $\mathcal{U} = \mathcal{U}(\mathcal{K}) = \mathcal{R}(P)$ satisfy*

$$\pi(\mathcal{S}, t) \subset \mathcal{S} \quad \textit{and} \quad \pi(\mathcal{U}, t) = \mathcal{U}, \qquad \textit{for } t \geq 0, \tag{3.3}$$

   that is, $\mathcal{S}$ is positively invariant and $\mathcal{U}$ is invariant.

5. *For all $B \in \mathcal{K}$, the restriction of $\Phi(B,t)$ to $\mathcal{R}(P(B)) = \mathcal{U}(B)$ is an isomorphism of $\mathcal{R}(P(B))$ onto $\mathcal{R}(P(B_t))$, for each $t \geq 0$. Moreover, for each $w \in W$, the function $\Phi(B,t)P(B)w$ has a unique negative continuation satisfying*

$$\Phi(B,t)P(B)w \in \mathcal{R}(P(B_t)), \qquad \textit{for all } t \leq 0,$$

   and the extended cocycle identity

$$\Phi(B, \tau + t)P(B) = \Phi(B_\tau, t)\Phi(B, \tau)P(B), \qquad \textit{for all } \tau, t \in R, \tag{3.3b}$$

   is valid, for all $B \in \mathcal{K}$.

6. *One has $\mathcal{U}(B) = \mathcal{B}_u^-(B)$, i.e., for each $w \in \mathcal{R}(P(B)) = \mathcal{U}(B)$, there is a unique negative continuation through (w, B), where the w-coordinate is uniformly bounded, for $t \leq 0$, and this negative continuation is $\Phi(B,t)w$.*
7. *The exponential dichotomy over $\mathcal{K}$ extends to an exponential dichotomy of the closure $Cl_{\mathcal{W}}\mathcal{K}$, with no change in the characteristics (K, α).*

In the case of a discrete linear skew product semiflow, all the assertions above remain valid for time t restricted to an appropriate discrete subgroup of R.

Proof. Item (1). Let $(w, B) \in \mathcal{B}$. It then follows from the definition of $\mathcal{B}$ that there is a global solution $\phi^{w,B}(t) = (w(t), B_t)$ such that $w(0) = w$ and

$$\|w\|_\infty \stackrel{\text{def}}{=} \sup\{\|w(t)\|_W : t \in R\} < \infty.$$

For $t \in R$, define $u(t) \stackrel{\text{def}}{=} P(B_t)w(t)$ and $v(t) \stackrel{\text{def}}{=} Q(B_t)w(t)$. It then follows from inequality (3do) that

$$\|u(t)\|_W \le K\|w\|_\infty \text{ and } \|v(t)\|_W \le K\|w\|_\infty, \qquad \text{for all } t \in R. \tag{3.3e}$$

Furthermore, the invariance identity (3.1i) and (2n) imply that for $\tau \in R$ and $t \ge 0$ one has

$$\begin{aligned}\Phi(B_\tau, t)u(\tau) &= \Phi(B_\tau, t)P(B_\tau)w(\tau) = P(B_{\tau+t})\Phi(B_\tau, t)w(\tau)\\ &= P(B_{\tau+t})w(\tau+t) = u(\tau+t).\end{aligned}$$

Hence $(u(t), B_t)$ is a global solution with $u(0) = P(B)w(0)$. Similarly, one finds that $(v(t), B_t)$ is a global solution with $v(0) = Q(B)w(0)$. Due to inequality (3.3e), we see that both of the orbits $(u(t), B_t)$ and $(v(t), B_t)$ belong to the bounded set $\mathcal{B}$, for all $t \in R$.

Now let $t \ge 0$ and set $\tau = -t$. Then equation (2n) implies that $v(0) = \Phi(B_\tau, t)v(\tau)$, for all $t \ge 0$. Next we use inequality (3ds) and $Q(B_\tau)v(\tau) = v(\tau)$ to get

$$\|v(0)\| = \|\Phi(B_\tau, t)v(\tau)\| = \|\Phi(B_\tau, t)Q(B_\tau)v(\tau)\| \le K\|v\|_\infty e^{-\alpha t}, \quad \text{for all } t \ge 0.$$

This implies that $v(0) = 0$, and $w(0) = u(0)$. Thus one has $w(t) = u(t)$, for all $t \in R$. Since $u(0) \in \mathcal{R}(P(B))$, it follows from inequality (3duu) that

$$\|u(0)\| \le K\|\Phi(B, t)u(0)\|e^{-\alpha t} \le K\|u(t)\|e^{-\alpha t} \le K\|u\|_\infty e^{-\alpha t}, \quad \text{for all } t \ge 0,$$

which implies that $w(0) = u(0) = 0$. Hence one has $\mathcal{B}(\mathcal{K}) = \mathcal{E}_0(\mathcal{K}) = \{0\} \times \mathcal{K}$.

Items (2) and (3). Since $\mathcal{S}(B) \cap \mathcal{U}(B) \subset \mathcal{B}(B)$, one has $\mathcal{S}(B) \cap \mathcal{U}(B) = \{0\}$, for all $B \in \mathcal{K}$. Since $\mathcal{R}(P(B)) \subset \mathcal{U}(B)$ and $\mathcal{R}(Q(B)) \subset \mathcal{S}(B)$, for all B, and $\mathcal{R}(P(B)) + \mathcal{R}(Q(B)) = W$, for all $B \in \mathcal{K}$, one obtains $\mathcal{S}(B) + \mathcal{U}(B) = W$, $\mathcal{S}(B) = \mathcal{N}(P(B))$, and $\mathcal{U}(B) = \mathcal{R}(P(B))$, for all $B \in \mathcal{K}$. Thus the convergence rates in $\mathcal{S}$ and $\mathcal{U}$ are exponential over $\mathcal{K}$.

Items (4) and (5). The inclusion $\pi(\mathcal{S}, t) \subset \mathcal{S}$ in (3.3) follows directly from the above and the definition of an exponential dichotomy. Next the invariance of P implies that $\Phi(B, t)$ maps $\mathcal{R}(P(B))$ into $\mathcal{R}(P(B_t))$, for each $t \ge 0$. From the definition of an exponential dichotomy, we see that $\Phi(B, t)$ maps $\mathcal{R}(P(B))$ onto $\mathcal{R}(P(B_t))$, and inequality (3duu) implies that the restriction $\Phi(B, t)\,|_{\mathcal{R}(P(B))}$ is one-to-one. Since $\Phi(B, t)$ is a bounded linear operator on W, it follows that this restriction is an isomorphism of $\mathcal{R}(P(B))$ onto $\mathcal{R}(P(B_t))$, and the inverse operator is a bounded linear operator. Thus by using this inverse operator, we see that, for each $w \in W$, the function $\Phi(B, t)P(B)w$ has a unique negative continuation with

$$\Phi(B, t)P(B)w \in \mathcal{R}(P(B_t)), \qquad \text{for all } B \in \mathcal{K} \text{ and all } t \le 0.$$

For $t \le 0$, we then define $\Phi(B, t)P(B)$ to be the unique isomorphism

$$\Phi(B, t)P(B) : \mathcal{R}(P(B_t)) \to \mathcal{R}(P(B)) \tag{3.4a}$$

with the property that $\Phi(B_t, -t)\Phi(B, t)P(B) = P(B)$. Also one has

$$\Phi(B, t)\mathcal{U}(B) = \Phi(B, t)\mathcal{R}(P(B)) = \mathcal{R}(P(B_t)) = \mathcal{U}(P(B_t)),$$

which implies that $\pi(\mathcal{U}, t) = \mathcal{U}$, for $t \geq 0$. The proof of equation (3.3b) is now straight forward, and we will omit the details.

Item (6). We argue by contradiction. Assume that there is a point $(w, B) \in \mathcal{B}^-$ for which there are two negative continuations, say ϕ_1 and ϕ_2, with w-coordinate being uniformly bounded, for $t \leq 0$. We will write $\phi_1(t) = (w_1(t), B_t)$ and $\phi_2(t) = (w_2(t), B_t)$, for $t \in R$. Then w_1 and w_2 are uniformly bounded, for $t \leq 0$, and $w_1(t) = w_2(t) = \Phi(B, t)w$, for $t \geq 0$. Consequently, $w(t) \stackrel{\text{def}}{=} w_1(t) - w_2(t)$ satisfies

$$(w(t), B_t) = (w_1(t) - w_2(t), B_t) \in \mathcal{B}, \qquad \text{for all } t \in R.$$

It follows from Item (1) that $w(t) \equiv 0$. Hence one has $\mathcal{U}(B) = \mathcal{B}_u^-(B)$. Since the proof of Item (7) is straight forward, we will omit the details. □

A given linear skew product flow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = W \times \mathcal{W}$ can be imbedded into a one-parameter family $\pi_\lambda = (\Phi_\lambda, \sigma)$, for $\lambda \in R$, by defining $\Phi_\lambda(B, t) \stackrel{\text{def}}{=} e^{-\lambda t}\Phi(B, t)$. One refers to π_λ as the **shifted semiflow**. The reason for this teminology can be appreciated by assuming that $\Phi(B, t)$ is a fundamental solution operator for the nonautonomous linear evolutionary equation $\partial_t u = B(t)u$. In this case, $\Phi_\lambda(B, t)$ is a fundamental solution operator for the shifted equation $\partial_t v = (B(t) - \lambda I)v$. In the case of a discrete linear skew product semiflow, the shifted semiflow is defined as above, however the interpretation in terms of a differential equation is no longer appropriate. The set of all $\lambda \in R$ for which π_λ admits an exponential dichotomy on $\mathcal{E}$ is called the **resolvent set** for π. The **dynamical spectrum** $\Sigma(\pi)$ of π is the complement in R of the resolvent set. The unstable set, the stable set, the bounded set, etc are all defined for the shifted flow π_λ and will be denoted by $\mathcal{U}_\lambda$, $\mathcal{S}_\lambda$, $\mathcal{B}_\lambda$, etc. The two sets $\mathcal{U}_\lambda$ and $\mathcal{S}_\lambda$ are monotone in λ; $\mathcal{U}_\lambda$ is nonincreasing, and $\mathcal{S}_\lambda$ is nondecreasing, see Sacker and Sell (1978, 1980) and Chow, Lu, and Mallet-Paret (1995) for more details.

We are interested in the situation where the shifted linear skew product semiflow π_λ has an exponential dichotomy for different values of the parameter λ. More precisely, let λ and μ be given, where $\lambda < \mu$, and assume that π_λ and π_μ each has an exponential dichotomy over an invariant set $\mathcal{K}$ in $\mathcal{W}$, with invariant projectors (P_λ, Q_λ) and (P_μ, Q_μ), and characteristics $(K_\lambda, \alpha_\lambda)$ and (K_μ, α_μ). By replacing these characteristics with (K, α), where $K = \max(K_\lambda, K_\mu)$ and $\alpha = \min(\alpha_\lambda, \alpha_\mu)$, we see that (K, α) serve as characteristics for both π_λ and π_μ. As noted above, one has $\mathcal{S}_\lambda \subset \mathcal{S}_\mu$ and $\mathcal{U}_\mu \subset \mathcal{U}_\lambda$.

The existence of the exponential dichotomies for the two linear skew product semiflows π_λ and π_μ has an equivalent formulation in terms of a trichotomy. In particular, we will say that π has an **exponential trichotomy** over an invariant set $\mathcal{K} \subset \mathcal{W}$, with **characteristics** λ_1, λ_2, λ_3, λ_4, and K, where $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ and $K \geq 1$, if there exist three projectors P, Q, and R defined over $\mathcal{K}$ such that the following properties hold:

(1) Each of the projectors P, Q, and R is invariant on $\mathcal{E}(\mathcal{K})$.

(2) For each $B \in \mathcal{K}$, the projections $P(B)$, $Q(B)$, and $R(B)$ commute and one has

$I = P(B) + Q(B) + R(B)$ and $P(B)Q(B) = P(B)R(B) = Q(B)R(B) = 0$.

(3) For each $w \in \mathcal{R}(P(B))$ there is a negative continuation

$$\phi^{w,B}(t) = (\Phi(B,t)w, B_t), \qquad \text{for } t \le 0,$$

such that $\|e^{-\lambda_4 t}\Phi(B,t)w\| \to 0$, as $t \to -\infty$. (We do not require that this negative continuation be unique, and we let $(\Phi(B,t)w, B_t)$, denote any negative continuation that satisfies $\|e^{-\lambda_4 t}\Phi(B,t)w\| \to 0$, as $t \to -\infty$.)

(4) For each $w \in \mathcal{R}(R(B))$ there is a negative continuation

$$\phi^{w,B}(t) = (\Phi(B,t)w, B_t), \qquad \text{for } t \le 0,$$

such that $\|e^{-\lambda_2 t}\Phi(B,t)w\| \to 0$, as $t \to -\infty$. (We do not require that this negative continuation be unique, and we let $(\Phi(B,t)w, B_t)$, denote any negative continuation that satisfies $\|e^{-\lambda_2 t}\Phi(B,t)w\| \to 0$, as $t \to -\infty$.)

(5) The following four inequalities are valid for all $w \in W$:

$$\|\Phi(B,t)Q(B)w\| \le K\|w\|e^{\lambda_1 t}, \qquad \text{for } t \ge 0 \text{ and } B \in \mathcal{K}, \tag{3ts}$$

$$\|\Phi(B,t)P(B)w\| \le K\|w\|e^{\lambda_4 t}, \qquad \text{for } t \le 0 \text{ and } B \in \mathcal{K}, \tag{3tu}$$

$$\|\Phi(B,t)R(B)w\| \le K\|w\|e^{\lambda_3 t}, \qquad \text{for } t \ge 0 \text{ and } B \in \mathcal{K}, \tag{3tn}$$

and

$$\|\Phi(B,t)R(B)w\| \le K\|w\|e^{\lambda_2 t}, \qquad \text{for } t \le 0 \text{ and } B \in \mathcal{K}, \tag{3tm}$$

where (3tu) and (3tm) are valid for any negative continuation, as described above.

In the case of a discrete linear skew product semiflow, the definition of an **exponential trichotomy** is identical to that given above with the single exception that the time t is restricted to some discrete subgroup, or discrete semisubgroup, of R.

The following property is easily verified, and we leave the proof as an exercise:

Lemma 3.3. *The linear skew product semiflow π has an exponential trichotomy over an invariant set $\mathcal{K} \subset \mathcal{W}$ with characteristics λ_1, λ_2, λ_3, λ_4, and K if and only if both of the following two properties hold:*

(1) *For any λ with $\lambda_1 < \lambda < \lambda_2$, the semiflow π_λ has an exponential dichotomy with characteristics (K, α_λ) and associated projections (P_λ, Q_λ), where*

$$2\alpha_\lambda = \min(\lambda - \lambda_1, \lambda_2 - \lambda), \quad P_\lambda = P + R, \quad \text{and} \quad Q_\lambda = Q.$$

(2) *For any μ with $\lambda_1 < \mu < \lambda_2$, the semiflow π_μ has an exponential dichotomy with characteristics (K, α_μ) and associated projections (P_μ, Q_μ), where*

$$2\alpha_\mu = \min(\mu - \lambda_3, \lambda_4 - \mu), \quad P_\mu = P, \quad \text{and} \quad Q_\mu = R + Q.$$

As a result, we see that Lemma 3.2 is applicable to each of the exponential dichotomies, for π_λ and for π_μ. Consequently the negative continuations $(\Phi(B,t)u, B_t)$, for $t \le 0$, which are defined when $u \in \mathcal{R}(P(B)) \cup \mathcal{R}(R(B))$, are uniquely determined. Furthermore, there are unique negative continuations, denoted by $(\Phi(B,t)u, B_t)$, similarly defined for $u \in \mathcal{R}(P(B)) + \mathcal{R}(R(B))$, for $t \le 0$.

4. Discrete Inhomogeneous Equations: The Henry Theorem

We now turn our attention to the study of solutions of the discrete linear inhomogeneous equation

$$w_{n+1} = T_n w_n + f_n, \qquad \text{where } n \in Z, \tag{4.7}$$

on a Banach space W. We assume that $T = \{T_n\}$ is a given sequence with $T_n \in \mathcal{L}(W)$, for all $n \in Z$, and $\|T\|_\infty < \infty$, see (2.2a). Thus, $T = \{T_n\} \in \mathcal{Z} = \ell_\infty(Z, \mathcal{L}(W))$. We are especially interested in the case where the sequences $f = \{f_n\}$ and $w = \{w_n\}$ are in $\ell_\infty(Z, W)$. We shall say that the sequence $T = \{T_n\} \in \mathcal{Z}$ has the **Strong Boundedness Property** provided that for every sequence $\{f_n\} \in \ell_\infty(Z, W)$, there is a unique sequence $\{w_n\} \in \ell_\infty(Z, W)$ that satisfies equation (4.7). The reader should observe that we are using two topologies on the spaces $\mathcal{Z} = \ell_\infty(Z, \mathcal{L}(W))$ and $\ell_\infty(Z, W)$. For dynamical issues it is best to use the Fréchet metric topology, which is equivalent to uniform convergence on finite subsets in Z, see (2.2c). However, for the Strong Boundedness Property we use the Banach sup-norm topology, which is equivalent to uniform convergence on Z.

Assume that $T = \{T_n\}$ has the Strong Boundedness Property, and define $L = L(T) = \{L_n\}$ by

$$L_n w_n = w_{n+1} - T_n w_n, \qquad \text{for } n \in Z.$$

Then the Strong Boundedness Property implies that the linear operator $L = L(T)$ is an isomorphism of $\ell_\infty(Z, W)$ onto itself, and that the inverse mapping $G = G(T)$ is a closed linear operator with domain being $\ell_\infty(Z, W)$. It then follows from the Closed Graph Theorem that G is a bounded linear operator on $\ell_\infty(Z, W)$. Furthermore, one has

$$w_n = (Gf)_n = \sum_{k=-\infty}^{\infty} G_{n,k+1} f_k, \qquad \text{for } n \in Z, \tag{4.7c}$$

where $G_{n,m}(T) = G_{n,m} \in \mathcal{L}(W)$, for all m, $n \in Z$, see Henry (1981, pp 229-232). Now equations (4.7) and (4.7c) imply that

$$f_n = w_{n+1} - T_n w_n = \sum_{k=-\infty}^{\infty} (G_{n+1,k+1} - T_n G_{n,k+1}) f_k,$$

which in turn, implies that

$$G_{n+1,k+1} - T_n G_{n,k+1} = \begin{cases} I, & \text{if } n = k, \\ 0, & \text{if } n \neq k. \end{cases} \tag{4.7e}$$

When $T = \{T_n\}$ has the Strong Boundedness Property, the three operators T, $L(T)$, and $G(T)$ are bounded linear operators on $\ell_\infty(Z, W)$, i.e., they are elements of the Banach space $\mathcal{L}(\ell_\infty(Z, W))$. We will denote the associated norms by

$$\|G(T)\| = \|G(T)\|_{\mathcal{L}(\ell_\infty(Z,W))} \quad \text{and} \quad \|L(T)\| = \|L(T)\|_{\mathcal{L}(\ell_\infty(Z,W))}.$$

Note that one has $\|G_{n,m}(T)\| \le \|G(T)\|$, for all $m, n \in Z$. In the case of T, or more generally for any $T \in \ell_\infty(Z, \mathcal{L}(W))$, one has $T \in \mathcal{L}(\ell_\infty(Z, W))$ and $\|T\|_{\mathcal{L}(\ell_\infty(Z,W))} = \|T\|_\infty$.

Next assume that the sequence $T = \{T_n\} \in \ell_\infty(Z, \mathcal{L}(W))$ satisfies the Strong Boundedness Property and let $S = \{S_n\}$ be another sequence in $\ell_\infty(Z, \mathcal{L}(W))$. We will now show that if $\|T - S\|_\infty$ is sufficiently small, then the sequence $S = \{S_n\}$ satisfies the Strong Boundedness Property as well. Indeed one has

$$L(S) = L(T) + (T - S) = L(T)[I + G(T)(T - S)].$$

Consequently, if

$$\|G(T)\| \sup_{n \in Z} \|T_n - S_n\| = \|G(T)\| \, \|T - S\|_\infty < 1, \tag{4.7g}$$

it follows from the Neuman Property that $L(S)$ has a bounded linear inverse $G(S)$, and $G(S) = (I + G(T)(T - S))^{-1} G(T)$.

As we will now show, there is a profound connection between the Strong Boundedness Property for $T = \{T_n\}$ and the theory of exponential dichotomies for an associated discrete linear skew product semiflow $\hat{\pi}(w, T, k) = (\hat{\Phi}(T, k)w, T \cdot k)$ on the space $W \times \mathcal{Z}$, where $\mathcal{Z} = \ell_\infty(Z, \mathcal{L}(W))$, see (2.2e). Assume that $\hat{\pi}(w, T, k)$ has an exponential dichotomy, with characteristics (K, s), over a set $\mathcal{K}$ in $\mathcal{Z}$. Let $T = \{T_n\} \in \mathcal{K}$, and let $P = \{P_n\}$ denote the associated invariant projector. Set $Q = \{Q_n\} = \{I - P_n\}$. We will now show that this sequence T has the Strong Boundedness Property by using various properties described in Lemma 3.2. Let $G_{n,m}$ be defined by $G_{m,m} = Q_m$, for $n = m$, and

$$G_{n,m} \overset{\text{def}}{=} \Phi(T \cdot m, n - m) Q_m = T_{n-1} \cdot \ldots \cdot T_{m+1} \cdot T_m Q_m, \qquad \text{for } n > m. \tag{4.7h}$$

For $n < m$, we let $G_{n,m} = G_{n,m} P_m : \mathcal{R}(P_m) \to \mathcal{R}(P_n)$ denote the unique ismorphism that satisfies $-\Phi(T \cdot n, m - n) G_{n,m} = P_m$, and we write

$$G_{n,m} = -\Phi(T \cdot m, n - m) P_m, \qquad \text{for } n < m, \tag{4.7i}$$

see (3.4a). Now inequalities (3.1s) and (3.1u) imply that $\|G_{n,m}\| \le K s^{n-m}$, for $n \ge m$, and $\|G_{n,m}\| \le K s^{m-n}$, for $n < m$. Given $f = \{f_n\} \in \ell_\infty(Z, W)$ it is easily verified that $w = \{w_n\}$, where

$$w_n = \sum_{k \in Z} G_{n,k} f_{k-1} = \sum_{k \le n} \Phi(T \cdot k, n - k) Q_k f_{k-1} - \sum_{n < k} \Phi(T \cdot k, n - k) P_k f_{k-1},$$

for all $n \in Z$, is a solution of equation (4.7), and one has $\|w_n\| \le K \|f\|_\infty \frac{1+s}{1-s}$, for all $n \in Z$. This formula for w_n is the Lyapunov-Perron formula for the discrete problem (4.7). Furthermore, this is a formula for the inverse $G = G(T) = L(T)^{-1}$. A direct calculation then yields

$$\|G(T)\| \le \sup_{n \in Z} \sum_{k \in Z} \|G_{n,k}\| \le K \frac{1 + s}{1 - s} \le \frac{2K}{1 - s}. \tag{4.7k}$$

This leads us to a very useful and powerful characterization of an exponential dichotomy for $\hat{\pi} = (\hat{\Phi}, \hat{\sigma})$. In particular, we will now show that the linear skew product semiflow $\hat{\pi}$ has an exponential dichotomy over the set $\mathcal{K}_0 = \{T\}$, where $T = \{T_n\} \in \mathcal{Z}$, if and only if the given sequence T has the Strong Boundedness Property. For this purpose, for every $B > 0$, we define $\mathcal{K}_B$ denote the collection of all $T \in \mathcal{Z}$ that have the Strong Boundedness Property with $\|G(T)\| \le B$.

Theorem 4.1 (Henry). *Let $\hat{\pi}(w, T, k)$ be the discrete linear skew product semiflow on $W \times \mathcal{Z}$ given by equation (2.2e). Then the following assertions are valid.*

(1) *If $\hat{\pi}$ has an exponential dichotomy, with characteristics (K, s), over a set $\mathcal{K}$ in $\mathcal{Z}$, then each $T \in \mathcal{K}$ has the Strong Boundedness Property, and the associated inverse operator $G(T)$ satisfies inequality (4.7k).*

(2) *Let $B > 0$ be fixed. Then $\hat{\pi}$ has an exponential dichotomy over $\mathcal{K}_B$ with characteristics (K, s), where*

$$K \leq 2\|G(T)\| \qquad \textit{and} \qquad \frac{2\|G(T)\|}{1 + 2\|G(T)\|} \leq s < 1. \tag{4.7m}$$

Proof. A version of this theorem can be found in Henry (1981, pp 230-232). Since we need the detailed estimates given above for our applications, and since we have a new proof of the basic theorem, we will present our argument here.

The proof of Item (1) is given in the preceeding paragraphs. For the proof of Item (2) we let $T = \{T_n\} \in \mathcal{K}_B$, and we define $P_m \stackrel{\text{def}}{=} I - G_{m,m}$, for $m \in Z$. We will make use of the following lemma. Since this lemma is an easy consequence of equations (4.7c) and (4.7e), we will omit the details.

Lemma 4.2. *Let $T \in \mathcal{K}_B$, for some $B > 0$. Then the following assertions are valid:*

(1) *Let w_n be a sequence in W that satisfies $w_{n+1} = T_n w_n$, for $n \geq m$, and set $w_n = 0$, for $n < m$. Then the sequence $w = \{w_n\}$ is in $\ell_\infty(Z, W)$ if and only if*
$$w_n = G_{n,m} w_m, \qquad \textit{for all } n \in Z,$$
where $w_m \in \mathcal{N}(P_m)$.

(2) *Let w_n be a sequence in W that satisfies $w_{n+1} = T_n w_n$, for $n < m$, and set $w_n = 0$, for $n > m$. Then the sequence $w = \{w_n\}$ is in $\ell_\infty(Z, W)$ if and only if*
$$w_n = G_{n,m+1}(-T_m w_m), \qquad \textit{for all } n \in Z,$$
where $T_m w_m \in \mathcal{N}(G_{m+1,m+1})$.

Continuing with the proof of Item (2) in Theorem 4.1, we define $\mathcal{N}_m$ to be the collection of all vectors $w_m \in W$ such that there exists a sequence $w = \{w_n\} \in \ell_\infty(Z, W)$, with $w_{n+1} = T_n w_n$, for $n \geq m$, and $w_n = 0$, for $n < m$. It follows from Lemma 4.2 that

$$\mathcal{N}_m \subset \mathcal{N}(P_m), \qquad \text{for all } m \in Z, \tag{4.7q}$$

and from the definition one has

$$T_m \mathcal{N}_m \subset \mathcal{N}_{m+1}, \qquad \text{for all } m \in Z. \tag{4.7r}$$

Let $w \in W$ and set $w_n = G_{n,m} w$, for $n \geq m$, and $w_n = 0$, for $n < m$. It is easily seen that $w_{n+1} = T_n w_n$, for $n \geq m$. Hence from Lemma 4.2 and (4.7q), one has $w_n = G_{n,m} w_m$, for all $n \in Z$, and $w_m \in \mathcal{N}(P_m)$. Hence

$$0 = P_m w_m = P_m G_{m,m} w = P_m (I - P_m) w, \qquad \text{for all } w \in W,$$

which implies that $P_m^2 = P_m$. Thus P_m and $G_{m,m}$ are complementary bounded linear projections on W, for each $m \in Z$. In addition, if $w \in W$ is chosen so that $w \in \mathcal{N}(P_m)$, one then has $P_m w = 0$, or $w = G_{m,m}w = w_m$, and $w = w_m \in \mathcal{N}_m$, i.e., $\mathcal{N}(P_m) \subset \mathcal{N}_m$. Consequently, (4.7q) implies that

$$\mathcal{N}_m = \mathcal{N}(P_m), \qquad \text{for all } m \in Z. \tag{4.7t}$$

Also note that (4.7r) and (4.7t) imply that if $P_m w = 0$, then $P_{m+1}T_m w = 0$, since $T_m w \in \mathcal{N}(P_{m+1})$. One then has

$$T_m P_m w = P_{m+1} T_m w = 0, \qquad \text{for all } w \in \mathcal{N}(P_m). \tag{4.7u}$$

Next we define $\mathcal{R}_m$ to be the collection of all vectors $w_m \in W$ such that there exists a sequence $w = \{w_n\} \in \ell_\infty(Z, W)$, with $w_{n+1} = T_n w_n$, for $n < m$, and $w_n = 0$, for $n > m$. It follows from the definition that $T_m \mathcal{R}_m = \mathcal{R}_{m+1}$, for all $m \in Z$, and Lemma 4.2 implies that the restriction of T_m to $\mathcal{R}_m$ is an isomorphism of $\mathcal{R}_m$ onto $\mathcal{R}_{m+1}$. We claim that

$$\mathcal{R}_m = \mathcal{R}(P(m)), \qquad \text{for all } m \in Z. \tag{4.7ua}$$

Let $w = \{w_n\} \in \ell_\infty(Z, W)$ be chosen so that $w_{n+1} = T_n w_n$, for $n < m$; $w_n = 0$, for $n > m$; and $w_m \in \mathcal{R}_m$. Define $\hat{w} = \{\hat{w}_n\}$ by $\hat{w}_n = w_n$, for $n < m$; and $\hat{w}_n = 0$, for $n \geq m$. Thus $\hat{w}_n = w_n$, for all $n \neq m$. It follows from Lemma 4.2 that $w_m = T_{m-1}w_{m-1} = T_{m-1}\hat{w}_{m-1} \in \mathcal{N}(G_{m,m})$, for all $m \in Z$. Since $w_m \in \mathcal{R}_m$, one has $\mathcal{R}_m \subset \mathcal{N}(G_{m,m}) = \mathcal{R}(P_m)$, for all $m \in Z$. Conversely, let $w_m \in \mathcal{N}(G_{m,m})$ and set $w_n = G_{n,m}w_m$, for $n \in Z$. One then has $w_{n+1} = T_n w_n$, for $n < m$, and $w_m = G_{m,m}w_m = 0$. It then follows from equation (4.7e) that one has $w_n = 0$, for all $n \geq m$. Hence $w_m = T_{m-1}w_{m-1} \in \mathcal{R}_m$ by Lemma 4.2. Thus one has $\mathcal{N}(G_{m,m}) \subset \mathcal{R}_m$, which implies that $\mathcal{R}_m = \mathcal{N}(G_{m,m}) = \mathcal{R}(P_m)$, for all $m \in Z$.

Next we will show that

$$T_m P_m w = P_{m+1} T_m w, \qquad \text{for all } w \in \mathcal{R}(P_m). \tag{4.7v}$$

Indeed if $w \in \mathcal{R}(P_m)$, then $T_m P_m w = T_m w \in T_m \mathcal{R}_m = \mathcal{R}_{m+1} = \mathcal{R}(P_{m+1})$, by equation (4.7ua). Since P_{m+1} is a projection, this in turn implies (4.7v). It then follows from (4.7u) and (4.7v) and the linearity of T_m and P_m that, $P_{m+1}T_m = T_m P_m$, for all $m \in Z$. This implies that the projector $P = \{P_n\}$ is invariant, see equation (3.1j).

Notice that equation (4.7e) implies that equation (4.7h) holds, for all $n > m$, with $Q_m = G_{m,m} = I - P_m$. Furthermore, since the restriction $T_m \mid_{\mathcal{R}(P_m)}$ is an isomorphism of $\mathcal{R}(P_m)$ onto $\mathcal{R}(P_{m+1})$, it follows that, for $n < m$, the operator $G_{n,m} = G_{n,m}P_m$ is the unique isomorphism of $\mathcal{R}(P_m)$ onto $\mathcal{R}(P_n)$ with the property that $-\Phi(T \cdot n, m-n)G_{n,m} = P_m$, i.e., $G_{n,m}$ satisfies (4.7i).

It remains to verify inequalities (3.1s) and (3.1u). For this purpose we let $S = \{S_n\}$ be defined by $S_n = \theta T_n$, where $\theta > 1$. Since $\|T - S\|_\infty = |\theta - 1|$, it follows from (4.7g) that if

$$\|G(T)\| \, \|T - S\|_\infty \leq B|\theta - 1| < 1,$$

then the sequence S has the Strong Boundedness Property. For the sequel we assume that θ satisfies $B|\theta - 1| \leq \frac{1}{2}$ and $\theta > 1$. Set $s = \theta^{-1}$. We then consider the equation

$$v_{n+1} = S_n v_n + g_n, \qquad n \in Z. \tag{4.7w}$$

Let $G_{n,m}(T)$ and $G_{n,m}(S)$ denote the respective inverses for $L(T)$ and $L(S)$. Let $f = \{f_n\}$ satisfy $f_{m-1} = w_m$, where $w_m \in \mathcal{N}(P_m)$, and $f_n = 0$, for $n \neq m-1$, and let $g = f$. Now assume that $v = \{v_n\}$ is the bounded solution of equation (4.7w) with this g. From Lemma 4.2, one has $v_n = G_{n,m}(S)v_m$, where $v_m \in \mathcal{N}(P_m)$. Next set $w_n = \theta^{m-n}v_n$, for $n \in Z$. Since $v_n = 0$, for $n < m$, and $\theta > 1$, it follows that $w = \{w_n\}$ is a bounded solution of equation (4.7) and $v_m = w_m$. Because of the uniqueness of the bounded solutions v and w, one has

$$\theta^{n-m} w_n = v_n = G_{n,m}(S)v_m, \qquad \text{for } n \in Z.$$

Since $v_m = w_m$, one also obtains

$$w_n = \theta^{m-n} G_{n,m}(S) w_m = G_{n,m}(T) w_m, \qquad \text{for all } m,\, n \in Z.$$

Since $G_{n,m}Q_m = G_{n,m}$, for all $n \geq m$, one has $\theta^{m-n}G_{n,m}(S) = G_{n,m}(T)$, for all $n \geq m$. As a result, one obtains

$$\|\Phi(T\cdot m, n-m)Q_m\| = \|G_{n,m}(T)\| \leq \|G(S)\|\theta^{m-n} = \|G(S)\|s^{n-m}, \qquad \text{for } n \geq m.$$

Next one sets $f_m = -T_m w_m$, where $T_m w_m \in \mathcal{N}(G_{m+1,m+1})$, and $f_n = 0$, for $n \neq m$, and let $g = f$. Now assume that $w = \{w_n\}$ is the bounded solution of equation (4.7) with this f. From Lemma 4.2, one has $w_n = G_{n,m}(T)(-T_m w_m)$, for all $n \in Z$. Next set $v_n = \theta^{n-m}w_n$, for $n \in Z$. Since $w_n = 0$, for $n > m$, and since $\theta > 1$, it follows that $v = \{v_n\}$ is a bounded solution of equation (4.7) and $v_m = w_m$. As argued above, one then has $G_{n,m}(S) = \theta^{n-m}G_{n,m}(T)$, for all $n \leq m$. Again one obtains

$$\|\Phi(T \cdot m, n-m)P_m\| = \|G_{n,m}(T)\| \leq \|G(S)\|s^{-(n-m)}, \qquad \text{for } n < m.$$

We see then that $\hat{\pi}$ has an exponential dichotomy over $\mathcal{K}_0 = \{T\}$, with characteristics (K, s), where $K = \|G(S)\|$. Since θ satisfies $\|G(T)\|(\theta - 1) \leq B(\theta - 1) \leq \frac{1}{2}$, the characteristic $s = \theta^{-1}$ satisfies

$$\frac{2\|G(T)\|}{2\|G(T)\| + 1} \leq s < 1.$$

Since $\|(I + G(T)(T-S))w\| \geq \frac{1}{2}\|w\|$, for all $w \in W$, one has

$$\|G(S)\| = \|(I + G(T)(T-S))^{-1}G(T)\| \leq 2\|G(T)\|,$$

which completes the proof of (4.7m). $\square$

There are numerous connections between the theory of exponential dichotomies for discrete and continuous-time linear skew product semiflows. In particular, let $\pi = \pi(w, B, t)$ denote the linear skew product semiflow on $W \times \mathcal{W}$, where $t \in R^+$, and let $\hat{\pi} = \hat{\pi}(w, T, k)$ denote the discrete linear skew product semiflow on $W \times \ell_\infty(Z, \mathcal{L}(W))$, where $k \in Z^+$. The proof of the following result is straight forward, and we will omit the details.

Lemma 4.3. *Let π and $\hat{\pi}$ be given as above. Let $B \in \mathcal{W}$ and define $T = \{T_n\}$ by $T_n = \Phi(B_{n\tau}, \tau)$, for $n \in Z$, where $\tau > 0$ is fixed. Then the following statements are valid:*

(1) *If π has an exponential dichotomy over the hull $H(B)$ with characteristics (K, α), then $\hat{\pi}$ has an exponential dichotomy over $\mathcal{K}_0 = \{T\}$ with characteristics (K, s), where $s = e^{-\alpha\tau}$.*

(2) *If $\hat{\pi}$ has an exponential dichotomy over $\mathcal{K}_0 = \{T\}$ with characteristics (K, s), then π has an exponential dichotomy over the hull $H(B)$ with characteristics (MK, α), where $\alpha = -\frac{1}{\tau}\log(s)$ and*

$$M = \sup\{e^{\alpha\tau}\|\Phi(B_{n\tau}, t)\| : 0 \leq t \leq \tau \text{ and } n \in Z\}.$$

5. Robustness Theorems

We now return to the linear skew product flow $\pi = \pi(w, B, t)$ on $\mathcal{E} = W \times \mathcal{W}$ with continuous time $t \in R$, where $\mathcal{W} = \mathcal{W}(R, \mathcal{L})$, see Section 2. For any set $\mathcal{A}$ in $\mathcal{W}$ and any $\epsilon > 0$, we let $N_\epsilon(\mathcal{A})$ denote the collection of all $D \in \mathcal{W}$ such that there is a $B \in \mathcal{A}$ with $d(B - D) \leq \epsilon$. The object of this section is to show that the concept of an exponential dichotomy over a compact, invariant set $\mathcal{K} \subset \mathcal{W}$ is an open condition in the sense that, if a perturbed equation remains in a prescribed neighborhood $N_\epsilon(\mathcal{K})$, for all $t \in R$, then the perturbed equation has an exponential dichotomy, with good characteristics, and the associated projectors (P, Q) vary continuously over the neighborhood $N_\epsilon(\mathcal{K})$. We remind the reader, that these neighborhoods are computed in terms of the given invariant metric on $\mathcal{W}$, see equation (2.1e).

There are several inequalities which arise in the sequel. The following statement may be helpful to the reader:

Properties of $\mathcal{W}$. *The following properties hold:*

(1) *If $B \in \mathcal{W}$ satisfies $\|B\|_K \leq \epsilon$, where $\sum_{n=K+1}^{\infty} 2^{-n} \leq \epsilon$, then $d_{\mathcal{W}}(B) \leq 2\epsilon$.*

(2) *For any set $\mathcal{K}$ in $\mathcal{W}$, let $d(B, \mathcal{K}) = \inf\{d(B - D) : D \in \mathcal{K}\}$. If $B \in \mathcal{W}$ is chosen so that for every $\epsilon > 0$ there is a $D \in \mathcal{K}$ such that $d(B - D) \leq r + \epsilon$, then one has $d(B, \mathcal{K}) \leq r$.*

(3) *Let $\mathcal{K}$ be a compact, invariant set in $\mathcal{W}$ and let $T > 0$ be given. Given $\epsilon > 0$, let $\delta > 0$ be given by the Continuity Lemma 2.1. If $B \in \mathcal{W}$ satisfies $d(B, \mathcal{K}) < \delta$, then there is a $D \in \mathcal{K}$ such that $d(B - D) \leq \delta$ and $d(B_t, \mathcal{K}) \leq d(B_t - D_t) < \epsilon$, for $0 \leq t \leq T$.*

The first of the two Robustness Theorems is the following:

Theorem 5.1 (Robustness of Dichotomies). *Assume that the linear skew product semiflow π over $\mathcal{E} = W \times \mathcal{W}$ has an exponential dichotomy over a compact invariant set $\mathcal{K} \subset \mathcal{W}$, with characteristics (K, α). Then there is an $\epsilon_0 > 0$, which depends on the characteristics, such that if $B \in \mathcal{W}$ satisfies $B_\sigma \in N_\epsilon(\mathcal{K})$, for all $\sigma \in R$ and some ϵ with $0 < \epsilon \leq \epsilon_0$, then π has an exponential dichotomy over the hull $H(B)$, with characteristics $(\hat{K}, \hat{\alpha})$, and the following hold:*

(1) *One has $\hat{K} \to K$ and $\hat{\alpha} \to \alpha$, as $\epsilon \to 0$.*

(2) *The projectors (P, Q) vary continuously over $\mathcal{K} \cup H(B)$, where $H(B)$ is the hull of B and $H(B) \subset N_{\epsilon_0}(\mathcal{K})$.*

Proof. Let $\mathcal{K}$ and B be given as in the hypotheses. As above we let $\hat{\pi} = \hat{\pi}(w, T, k)$ denote the discrete linear skew product semiflow on $W \times \mathcal{Z}$, where $k \in Z^+$, see equation (2.2e).

The strategy of the proof is rather simple. For a $\tau > 0$, which will be fixed later, we define the sequence of operators $S = \{S_n\}$ by $S_n \stackrel{\text{def}}{=} \Phi(B_{n\tau}, \tau)$, for $n \in Z$. Our objective is to show that S has the Strong Boundedness Property and then to use Theorem 4.1 and Lemma 4.3 to conclude that π has an exponential dichotomy over the hull $H(B)$. In order to show that S has the Strong Boundedness Property, we will use the exponential dichotomy of π over $\mathcal{K}$ to construct an auxiliary sequence $T = \{T_n\}$ with the following two properties: (1) the norm $\|T - S\|_\infty$ is sufficiently small, and (2) the discrete flow $\hat{\pi}$ has an exponential dichotomy over $\mathcal{K}_0 = \{T\}$, with good characteristics. (The linear transformation $T_n = M_n \Phi_n$ is constructed by setting $\Phi_n = \Phi(B^{(n-1)}, \tau)$ for a specific $B^{(n-1)} \in \mathcal{K}$ and then jumping to a suitable nearby point $B^{(n)} \in \mathcal{K}$. The mapping M_n is a small impulse which moves from $B_\tau^{(n-1)}$ to $B^{(n)}$ in $\mathcal{K}$, while realigning the stable and unstable manifolds over these points.) Having done this, we then use Theorem 4.1 to conclude that T has the Strong Boundedness Property, and next we use (4.7g) to conclude that S has the Strong Boundedness Property.

Let (K, α) be the characteristics of the exponential dichotomy, over the compact, invariant set $\mathcal{K} \subset \mathcal{W}$, for the linear skew product semiflow $\pi(w, B, t) = (\Phi(B, t)w, B_t)$. Since the associated projectors (P, Q) vary continuously over $\mathcal{K}$, it follows that for every $\delta > 0$ there is an $\eta > 0$ such that if $B_1, B_2 \in \mathcal{K}$ with $d(B_1 - B_2) \leq 2\eta$, where d is an invariant metric on $\mathcal{W}$, then there is a linear operator $M = M(B_1, B_2) \in \mathcal{L}(W)$ with $M^{-1} \in \mathcal{L}(W)$,

$$\|I - M\| \leq \delta \quad \text{and} \quad \|I - M^{-1}\| \leq \delta, \tag{5.1}$$

$M\mathcal{R}(P(B_1)) = \mathcal{R}(P(B_2))$, and $M\mathcal{R}(Q(B_1)) = \mathcal{R}(Q(B_2))$. Next we fix β so that $0 < \beta < \alpha$, and we fix $\tau > 0$ so that

$$K(1 + \delta)e^{-(\alpha-\beta)\tau} \leq 1.$$

Define (K_1, s_1) by $K_1 = 2K$ and $s_1 = e^{-\beta\tau}$.

From the Continuity Lemma 2.1 for the linear skew product semiflow π, we see that for every $\mu > 0$, there is a ν, with $0 < \nu \leq \eta$, such that if $D \in \mathcal{W}$ and $\hat{B} \in \mathcal{K}$ with $d(D - \hat{B}) \leq \nu$, then one has $\|\Phi(D, t) - \Phi(\hat{B}, t)\| \leq \mu$, for $0 \leq t \leq \tau$, and $d(D_\tau - \hat{B}_\tau) \leq \eta$. In this setting, one then has

$$\begin{aligned} \|\Phi(D, \tau) - M\Phi(\hat{B}, \tau)\| &\leq \|\Phi(D, \tau) - \Phi(\hat{B}, \tau)\| + \|\Phi(\hat{B}, \tau) - M\Phi(\hat{B}, \tau)\| \\ &\leq \mu + \|I - M\| K_0, \end{aligned} \tag{4.8g}$$

where $K_0 = \sup\{\|\Phi(\hat{B}, \tau)\| : \hat{B} \in \mathcal{K}\}$.

Now choose δ and μ so that

$$\mu \leq \frac{1 - s_1}{8K_1} \quad \text{and} \quad \delta \leq \min\left(\frac{1 - s_1}{8K_0 K_1}, 1\right). \tag{4.8h}$$

Next we let η and ν be fixed so that the implications given above are satisfied, and we set $\epsilon = \nu$. Now assume that $B_\sigma \in N_\epsilon(\mathcal{K})$, for all $\sigma \in R$. Let $B^{(n)} \in \mathcal{K}$ be fixed so that $d(B_{n\tau} - B^{(n)}) \leq \epsilon = \nu$, for all $n \in Z$. Let (P, Q) denote the invariant projectors associated with the exponential dichotomy over $\mathcal{K}$. Define $\{P_n\}$ and $\{Q_n\}$ by $P_n = P(B^{(n)})$ and $Q_n = Q(B^{(n)})$, for $n \in Z$. It then follows that

$$\|\Phi(B_{(n-1)\tau}, t) - \Phi(B^{(n-1)}, t)\| \leq \mu, \qquad \text{for } 0 \leq t \leq \tau,$$

and $d(B_{n\tau} - B_\tau^{(n-1)}) \leq \eta$. Let $B_1 = B_\tau^{(n-1)}$ and $B_2 = B^{(n)}$. One then has B_1, $B_2 \in \mathcal{K}$ and

$$d(B_1 - B_2) \leq d(B_1 - B_{n\tau}) + d(B_{n\tau} - B^{(n)}) \leq \eta + \nu \leq 2\eta,$$

since $\nu \leq \eta$. Now define $M_n = M(B_1, B_2)$, for this choice of B_1 and B_2, and set

$$T_n = M_n \Phi(B^{(n-1)}, \tau), \qquad \text{for } n \in Z.$$

It then follows from inequalities (5.1), (4.8g) and (4.8h) that

$$\|S_n - T_n\| \leq \mu + \delta K_0 \leq \frac{1 - s_1}{4K_1}, \qquad \text{for all } n \in Z. \tag{4.8n}$$

Furthermore, if $v_0 \in \mathcal{R}(Q_{n-1}) = \mathcal{R}(Q(B^{(n-1)}))$, then one has

$$\Phi(B^{(n-1)}, \tau)v_0 \in \mathcal{R}(Q(B_\tau^{(n-1)})) = \mathcal{R}(Q(B_1)).$$

Because of the choice of M_n, one then has

$$T_n v_0 = M_n \Phi(B^{(n-1)}, \tau)v_0 \in \mathcal{R}(Q_n) = \mathcal{R}(Q(B_2)).$$

Consequently one has $\|\Phi(B^{(n-1)}, t)v_0\| \leq K e^{-\alpha t}\|v_0\|$, for $0 \leq t \leq \tau$, and

$$\|T_n v_0\| \leq \|M_n\| \, \|\Phi(B^{(n-1)}, \tau)v_0\| \leq (1 + \delta)K e^{-\alpha\tau}\|v_0\| \leq e^{-\beta\tau}\|v_0\|, \qquad \text{for } n \in Z.$$

By using induction, one then finds that for $m > n$ one has

$$\|\Phi(T \cdot n, m - n)Q_n v_0\| = \|T_{m-1} \cdots T_{n+1} T_n v_0\| \leq K(1 + \delta)e^{-\beta\tau(m-n)}.$$

Similarly, for $v_0 \in \mathcal{R}(P_n)$, one obtains $\|\Phi(B_\tau^{(n-1)}, t)M_n^{-1} P_n v_0\| \leq K(1 + \delta)e^{\alpha t}\|v_0\|$, for $-\tau \leq t \leq 0$, and

$$\|\Phi(T \cdot n, m - n)P_n v_0\| \leq K(1 + \delta)e^{\beta\tau(m-n)}\|v_0\|, \qquad \text{for } m < n.$$

Since $\delta \leq 1$, we conclude that $\hat{\pi}$ has an exponential dichotomy over $\mathcal{K}_0 = \{T\}$, with characteristics (K_1, s_1).

We are now prepared to reap the harvest! It follows that T has the Strong Boundedness Property, and from inequality (4.7k), one has

$$\|G(T)\| \leq \frac{2K_1}{1 - s_1}.$$

By using this with inequality (4.8n), we then obtain

$$\|G(T)\|\,\|T-S\| \le \frac{2K_1}{1-s_1}\frac{1-s_1}{4K_1} = \frac{1}{2}.$$

This inequality, together with (4.7g) and (4.7k), imply that S has the Strong Boundedness Property, and

$$\|G(S)\| \le \|(I+G(T)(T-S))^{-1}\|\,\|G(T)\| \le 2\|G(T)\| \le \frac{4K_1}{1-s_1}.$$

It then follows from Theorem 4.1 that $\hat{\pi}$ has an exponential dichotomy over $\mathcal{K}_1 = \{S\}$ with characteristics (K_2, s_2), where

$$K_2 \le 2\|G(S)\|\frac{8K_1}{1-s_1} \quad \text{and} \quad s_2 = \frac{2\|G(S)\|}{2\|G(S)\|+1}.$$

Finally, Lemma 4.3 implies that the continuous-time semiflow π has an exponential dichotomy over the hull $H(B)$. □

The following theorem extends the robustness property to exponential trichotomies.

Theorem 5.2 (Robustness of Trichotomies). *Assume that the linear skew product semiflow π on $W \times \mathcal{W}$ has an exponential trichotomy over a compact, invariant set $\mathcal{K} \subset \mathcal{W}$. Then there is an $\epsilon_0 > 0$, which depends on the characteristics, such that if $B \in \mathcal{W}$ satisfies $B_\sigma \in N_\epsilon(\mathcal{K})$, for all $\sigma \in R$ and some ϵ with $0 < \epsilon \le \epsilon_0$, then π has an exponential trichotomy over $H(B)$ and the following hold:*

(1) *The characteristics for the exponential trichotomy are continuous at $\epsilon = 0$.*
(2) *The projectors (P, Q, R) vary continuously over $\mathcal{K} \cup H(B)$, where $H(B)$ is the hull of B and $H(B) \subset N_{\epsilon_0}(\mathcal{K})$.*

Proof. Since an exponential trichotomy is equivalent to a pair of exponential dichotomies for the shifted flows, this result is an immediate corollary of the last theorem. However, a more instructive argument arises when one uses the proof of Theorem 5.1 with a small change in the definition of the mapping $M = M(B_1, B_2)$: In place of the two equalities

$$M\mathcal{R}(P(B_1)) = \mathcal{R}(P(B_2)), \qquad M\mathcal{R}(Q(B_1)) = \mathcal{R}(Q(B_2)),$$

one uses the trio

$$M\mathcal{R}(P(B_1)) = \mathcal{R}(P(B_2)),\ M\mathcal{R}(Q(B_1)) = \mathcal{R}(Q(B_2)),\ M\mathcal{R}(R(B_1)) = \mathcal{R}(R(B_2)),$$

where (P, Q, R) are the three projectors arising in the definition of an exponential trichotomy. The remainder of the argument is unchanged. □

6. Examples

Let us now return to the Main Theorem, as stated in the Introduction. First of all, the compactness condition in the Main Theorem is the assumption in Theorems 5.1 and 5.2 that the set $\mathcal{K}$ is compact. There are two further points we wish to make in this section. First we want to illustrate the breadth our the Robustness Theorems in the realm of partial differential equations. We will do this by considering two different classes, one built upon the theory of the Navier-Stokes equations, and the other based on the nonlinear wave equation. The objective in studying these two classes is to describe the spaces W and $\mathcal{W} = \mathcal{W}(R, \mathcal{L})$ which arise in these applications, and whereby the solutions of the equations

$$\partial_t u + Au = B(\lambda, t)u, \tag{6.0}$$

where $B \in \mathcal{W}$, generate a linear skew product semiflow on $\mathcal{E} = W \times \mathcal{W}$.

The second point we make is contained in a collection of dynamical examples which illustrate how the terms $B(\lambda, t)$ appearing in equation (6.0) may depend on a parameter λ. The objective here is to show that the continuous dependence on (λ, t) leads to a verification of the hypothesis that: $B_\sigma \in N_\epsilon(\mathcal{K})$, for all $\sigma \in R$, appearing in the Robustness Theorems, when λ is close to λ_0.

6.1 Navier-Stokes Equations. The Navier-Stokes equations are used to model the velocity field for an incompressible fluid motion in a smooth bounded domain Ω in R^3, or R^2. Because of the incompressibility condition, the vector field u for the velocity must satisfy $\nabla \cdot u = 0$ in Ω. The phase space for these equations is the Hilbert space H of divergent-free vector fields in $L^2(\Omega, R^d)$, where $d = 2, 3$. By projecting the Navier-Stokes equations into H, one obtains a nonlinear evolutionary equation

$$\partial_t u + Au + B(u, u) = f, \qquad \text{with } u \in H, \tag{6.2}$$

see Constantin and Foias (1988), Sell and You (1998), and Temam (1988). In equation (6.2), the linear operator A is the Stokes operator, and the nonlinearity $B(u, u)$ is a bilinear form. We assume here that f does not depend on time t.

The Stokes operator A is a sectorial operator, and $-A$ is the infinitesimal generator of an analytic semigroup e^{-At} on H. Since A has compact resolvent, the semigroup e^{-At} is compact, for $t > 0$. Furthermore, the fractional power spaces $V^{2\alpha} = \mathcal{D}(A^\alpha)$ are defined for all $\alpha \geq 0$, where $V^0 = H$. We assume here that the term f in equation (6.2) satisfies $f \in V^1 = \mathcal{D}(A^{\frac{1}{2}})$. By using the duality principle, one can extend these power spaces to negative powers, $V^{-2\alpha}$, see Constantin and Foias (1988), Pazy (1983), Sell and You (1998), and Temam (1988). It is important to note that the nonlinear term $F(u) = B(u, u)$ does not have a Fréchet derivative on the space H. However, by restricting to suitable fractional power spaces, this term is differentiable. In particular, one can show, for example, that

$$B : V^2 \times V^2 \to V^1 \quad \text{and} \quad B : V^{\frac{5}{4}} \times V^{\frac{5}{4}} \to V^0 = H$$

are Fréchet differentiable mappings, see Constantin and Foias (1988), and Sell and You (1998). In these cases, the Fréchet derivative of $F(u)$ along a global solution

$\phi(t)$ of equation (6.2) is of the form $B(t)v = DF(\phi(t))v = B(\phi(t), v) + B(v, \phi(t))$, provided that $\phi(t) \in V^2$ (or $\phi(t) \in V^{\frac{5}{4}}$), for all $t \in R$. Note that in these two cases on has $B(t) \in \mathcal{L}(V^2, V^1)$ (or $B(t) \in \mathcal{L}(V^{\frac{5}{4}}, V^0)$), for all $t \in R$.

For the study of exponential dichotomies and trichotomies for the Navier-Stokes equations, we have several choices for the Fréchet space $\mathcal{W} = \mathcal{W}(R, \mathcal{L})$. In order to accommodate these choices, as well as other possible applications, we now fix $\mathcal{L} = \mathcal{L}(V^{2\beta}, W)$, where $V^{2\alpha} = \mathcal{D}(A^\alpha)$, for $\alpha \geq 0$, is a family of fractional power spaces in a Banach space $W = V^0$ generated by a positive, sectorial operator A. For technical reasons, we require that the parameter β be fixed with $0 \leq \beta < 1$. Recall that, since A is a sectorial operator, $-A$ is the infinitesimal generator of an analytic semigroup e^{-At}.

By using the framework described in the last paragraph, the theory we describe below, will be applicable to reaction diffusion equations, the Cahn-Hilliard equations, the Kuramoto-Shivashinsky equation, and other equations, in addition to the Navier-Stokes equations, see Sell and You (1998) and Temam (1988).

For $1 \leq p < \infty$ we define $\mathcal{M}^p$ to be the space $\mathcal{M}^p \stackrel{\text{def}}{=} L^\infty(R, \mathcal{L}(V^{2\beta}, W))$, with the L^p_{loc}**-topology**. The invariant metric for this topology is given by equation (2.1e), where the family of pseudonorms $\|\cdot\|_{p;n}$ is defined by

$$\|B\|_{p;n} \stackrel{\text{def}}{=} \left(\int_{-n}^{n} \|B(t)\|^p_{\mathcal{L}}\, dt \right)^{\frac{1}{p}}, \qquad \text{for } n = 1, 2, \cdots,$$

and $\|B(t)\|_{\mathcal{L}}$ is the operator norm on $\mathcal{L} = \mathcal{L}(V^{2\beta}, W)$. One can readily show that the translational mapping $(B, \tau) \to B_\tau$ generates a flow on $\mathcal{M}^p$, for each p with $1 \leq p < \infty$.

For $p = \infty$ it is necessary to modify the definition of the underlying space, in order that the mapping $(B, \tau) \to B_\tau$ be continuous. In particular, we define $\mathcal{M}^\infty$ to be the space $\mathcal{M}^\infty \stackrel{\text{def}}{=} L^\infty(R, \mathcal{L}(V^{2\beta}, W)) \cap C(R, \mathcal{L}(V^{2\beta}, W))$, with the L^∞_{loc}-**topology**. The invariant metric for this topology is given by equation (2.1e), where the family of pseudonorms $\|\cdot\|_{\infty;n}$ is defined by

$$\|B\|_{\infty;n} \stackrel{\text{def}}{=} \sup\{\|B(t)\|_{\mathcal{L}} : |t| \leq n\}, \qquad \text{for } n = 1, 2, \cdots.$$

In this case, $(B, \tau) \to B_\tau$ is a flow on $\mathcal{M}^\infty$. Thus the first of the Basic Axioms, see Section 2, is now satisfied on each of the spaces $\mathcal{M}^p$, for $1 \leq p \leq \infty$.

The next step is the construction of the linear mapping $\Phi(B, t)$. We need to do this construction so that the second of the Basic Axioms is satisfied, i.e., we want that $\pi(w, B, t) = (\Phi(B, t)w, B_t)$ be a linear skew product semiflow on $\mathcal{E} = W \times \mathcal{M}^p$. The construction we outline here is based on the theory presented in Sell and You (1998). As we will see, in order to succeed in satisfying the second Basic Axiom, we will need to restrict p to satisfy the inequality

$$\frac{1}{1-\beta} < p \leq \infty. \tag{6.3}$$

The construction we now describe for the solution operator $\Phi(B, t)$ is valid for any B in $L^\infty(R, \mathcal{L}(V^{2\beta}, W))$, and it is based on the theory of mild solutions of the equation (6.0). We will not present all the details of the construction here, since they can be found in Sell and You (1998). However, the key ideas deserve comment.

One begins the construction by selecting a vector $v_0 \in W$ and then seeking a mild solution $\Phi(B,t)v_0$ of

$$\Phi(B,t)v_0 = e^{-At}v_0 + \int_0^t e^{-A(t-s)}B(s)\Phi(B,s)v_0\,ds. \tag{6.4}$$

If $v_0 \in V^{2\beta} \subset W$, we can apply A^β to equation (6.4), and use properties of the analytic semigroup e^{-At} to obtain

$$\begin{aligned}\|A^\beta\Phi(B,t)v_0\|_W &\le K_1 e^{-at}\|A^\beta v_0\|_W \\ &\quad + \int_0^t K_2(t-s)^{-\beta}e^{-a(t-s)}\|B(s)\|_{\mathcal{L}}\|A^\beta\Phi(B,s)v_0\|_W\,ds,\end{aligned} \tag{6.5}$$

where K_1 and K_2 are constants. The Gronwall-Henry inequality, see Henry (1981, pp 188-190) or Sell and You (1998, Appendix D), can then be applied to show the existence of a solution, and that the operator $\Phi(B,t)$ satisfies

$$\Phi(B,t) \in \mathcal{L}(V^{2\beta},V^{2\beta}) \mapsto \mathcal{L}(V^{2\beta},W), \qquad \text{for all } t \ge 0,$$

where the arrow $\mapsto$ denotes a continuous imbedding. Since $V^{2\beta}$ is dense in W, $\Phi(B,t)$ has a unique extension to operator $\Phi(B,t) \in \mathcal{L}(W,W)$, and equation (6.4) is valid, for all $v_0 \in W$.

The next step is to examine the dependence of the solution $\Phi(B,t)v_0$ on the parameters B and v_0. Once again one uses the Gronwall-Henry inequality for this purpose. When one has $1 \le p < \infty$, one needs to use the Hölder inequality in this analysis. The role of the Hölder inequality can be seen by using it on the third term in inequality (6.5). One then obtains a factor of the form

$$\int_0^t (t-s)^{-\beta q}\,ds, \qquad \text{where } q = \frac{p}{p-1}.$$

Since one wants this factor to be finite, for $t > 0$, we require that $0 \le \beta q < 1$, which is equivalent to (6.3).

The upshot of this argument is that in terms of the pseudonorms $\|\cdot\|_{p;n}$, the mapping $(v_0,B) \to \Phi(B,t)v_0$, of $W \times \mathcal{M}^p$ into W, is Lipschitz continuous in v_0 and B, for each p satisfying (6.3). Consequently, the function $\pi(v_0,B,\tau) \overset{\text{def}}{=} (\Phi(B,\tau)v_0, B_\tau)$ is continuous, and it is a linear skew product semiflow on $W \times \mathcal{M}^p$, for each p that satisfies inequality (6.3).

6.2 Nonlinear Wave Equation. The abstract setting for the nonlinear wave equation, and its relatives, is given by the hyperbolic equation

$$\partial_t^2 u + \alpha\partial_t u + Gu = D(t)u, \qquad u \in W, \tag{6.8}$$

where G is now assumed to be a positive, sectorial operator on a Banach space W, and $\alpha \in R$ is a constant. By introducing the ordered pair $v = (u, \partial_t u)^t$, we see that equation (6.8) can be written as a first order system, see (6.0), where now one has

$$A = \begin{pmatrix} 0 & -I \\ G & 0 \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} D(t) & 0 \\ 0 & -\alpha \end{pmatrix}.$$

From the definition of A, it follows that the resolvent sets $\rho(A)$ and $\rho(G)$ satisfy the implication

$$-\lambda^2 \in \rho(G) \Longrightarrow \pm\lambda \in \rho(A).$$

Since G is a positive, sectorial operator, it then follows that $R \subset \rho(A)$. Furthermore, the resolvent estimate $\|(\lambda^2 I + G)^{-1}\|_{\mathcal{L}} \leq M(a + \lambda^2)^{-1}$, where $a > 0$, implies that there is a constant M_1 such that $\|(\lambda I - A)^{-1}\|_{\mathcal{L}} \leq M_1$, for all $\lambda \in R$, and consequently, $-A$ is the infinitesimal generator of a C_0-group e^{-At} on W^2, see Pazy (1983). Hence $e^{-At}v$ is a flow on W^2.

The Banach space for equation (6.8) is now the product space W^2, with $v \in W^2$. The base space $\mathcal{W} = \mathcal{W}(R, \mathcal{L})$ is now constructed with $\mathcal{L} = \mathcal{L}(W^2)$. Typical choices for $\mathcal{W}$ are $\mathcal{W}^p = L^\infty(R, \mathcal{L}(W^2))$, with the L^p_{loc}-topology for $1 \leq p < \infty$, and $\mathcal{W}^\infty = L^\infty(R, \mathcal{L}(W^2)) \cap C(R, \mathcal{L}(W^2))$, with the L^∞_{loc}-topology. The mapping $(D, \tau) \to D_\tau$ is then a flow on $\mathcal{W}^p$, for each p with $1 \leq p \leq \infty$.

The construction of the linear operator $\Phi(D, t)$, for $D \in \mathcal{W}^p$, follows the methodology used in Section 6.1, but now the parameter β is fixed, $\beta = 0$. In particular, one seeks a mild solution of

$$\Phi(D, t)v_0 = e^{-At}v_0 + \int_0^t e^{-A(t-s)} D(s)\Phi(D, s)v_0 \, ds.$$

for $v_0 \in W^2$. In this setting the analogue of inequality (6.5) now becomes

$$\|\Phi(D, t)v_0\|_{W^2} \leq K_1 e^{at}\|v_0\|_{W^2} + \int_0^t K_1 e^{a(t-s)}\|D(s)\|_{\mathcal{L}}\|\Phi(D, s)v_0\|_{W^2} \, ds,$$

for $t \geq 0$, where $a \in R$. (A similar estimate is valid for $t \leq 0$.) By using a combination of the Hölder inequality and the Gronwall inequality, one argues that there is a mild solution for every $v_0 \in W^2$ and the operator $\Phi(D, t)$ satisfies $\Phi(D, t) \in \mathcal{L}(W^2)$, with $\|\Phi(D, t)\|_{\mathcal{L}} \leq K_3 e^{c|t|}$, for all $t \in R$. (Here c is a real constant.) Once again, the second Basic Axiom is satisfied, and in this case, $\pi(v, D, t) = (\Phi(D, t)v, D_t)$ is a flow on $W^2 \times \mathcal{W}^p$, for $1 < p \leq \infty$.

6.3 Dependence on Parameters. The next objective is to find sufficient conditions that the hypothesis:

$$B_\sigma \in N_\epsilon(\mathcal{K}), \qquad \text{for all } \sigma \in R, \tag{6.10}$$

be satisfied. The closed ϵ-neighborhood of a set $\mathcal{K}$ in $\mathcal{M}^p$ will be denoted by $N^p_\epsilon(\mathcal{K})$. Our approach is to develop these conditions through a series of illustrative examples. As we will see, the issue of satisfying (6.10) is very closely connected with the continuous dependence of the dynamics on parameters.

Example 1. Let $\mathcal{K} = H(D)$ be the hull in $\mathcal{M}^\infty$ of $D \in \mathcal{M}^\infty$ and assume that $H(D)$ is compact. In the case of the L^∞_{loc}-topology on $\mathcal{W}^\infty$, the compactness is completely determined by the Ascoli-Arzelá Theorem, see Sell (1967ab). In particular, $H(D)$ is compact if and only if $D(t)$ is uniformly continuous, for $t \in R$. Assume next that $B \in \mathcal{M}^\infty$ satisfies

$$\|B(t) - D(t)\| \leq \epsilon, \qquad \text{for all } t \in R. \tag{6.11}$$

It is easily verified that one then has $B \in N_\epsilon^\infty(\mathcal{K})$, for all $\sigma \in R$. More generally, if $B \in \mathcal{M}^p$, where $1 \le p < \infty$, satisfies

$$\int_t^{t+2} \|B - D\|^p \, ds \le \epsilon^p, \qquad \text{for all } t \in R, \tag{6.12}$$

then one has $B \in N_{a\epsilon}^p(\mathcal{K})$, for some $a > 0$. A Robustness Theorem for exponential dichotomies (with $\beta = 0$) can be found in Chow and Leiva (1995) and Leiva (1998). In this result, inequality (6.11) is assumed instead of (6.10). Also see Coppel (1978).

Example 2. This example arises in the context of linear equations with time-varying periodic, quasi periodic, or almost periodic coefficients. Let T^k denote the k-dimensional torus with a twist flow $\sigma(\omega; \theta, t) = \theta + \omega t$, where $\theta = (\theta_1, \cdots, \theta_k) \in T^k$ and $\omega \in R^k$. Let $E = E(\theta) : T^k \to \mathcal{L}(V^{2\beta}, W)$ be a continuous function, and set $B(t) = E(\sigma(\omega; \theta_0, t))$, for some $\theta_0 \in T^k$. In this case, the hull $H(B)$, which depends on ω, is the collection of all time-varying functions $E(\sigma(\omega; \theta, t))$, where $\theta \in T^k$. Let $D(\theta, t) = D(t) = E(\sigma(\omega_0; \theta, t))$, for $t \in R$. The mapping $J : \theta \to D(\theta, \cdot)$ is a continuous mapping of T^k into $\mathcal{M}^\infty$, and the image $\mathcal{K} = J(T^k)$ is a compact invariant set in the translational flow on $\mathcal{M}^\infty$. As a matter of fact, $\mathcal{K}$ is itself a torus with dimension satisfying $\dim \mathcal{K} \le k$. Since E is a continuous mapping, it follows that for every $\epsilon > 0$ there is a $\delta > 0$ with the property that $B \in N_\epsilon^\infty(\mathcal{K})$ whenever $\|\omega - \omega_0\|_{R^k} \le \delta$. On the other hand, if the only vectors n, m in Z^k, that satisfy $m\omega + n\omega_0 = 0$, are $m = n = 0$, then one has

$$\sup_{t \in R} \|B(t) - D(t)\| = \text{Total Variation}(E) = \sup_{\theta_1, \theta_2 \in T^k} \|E(\theta_1) - E(\theta_2)\|,$$

which in general can be large. It follows then that this example is not a special case of Example 1.

Example 3. This is a generalization of Example 2, where the torus T^k is replaced by a general compact manifold M. We will let $\theta \in M$, denote a typical point in M. Assume that $\sigma(t) = \sigma(\omega; \theta_0, t)$ is a solution of the problem $\partial_t \theta = G(\omega; \theta)$, with $\sigma(0) = \theta_0$, where $G(\omega; \theta)$ is a family of vector fields on M which depends continuously on a parameter $\omega \in \Omega$ and Ω is a metric space with a metric d. Let $E = E(\theta) : M \to \mathcal{L}(V^{2\beta}, W)$ be a continuous function and set $B(t) = E(\sigma(\omega; \theta_0, t))$, for some $\theta_0 \in M$, and $D(\theta, t) = D(t) = E(\sigma(\omega_0; \theta, t))$, for $t \in R$. The mapping $J : \theta \to D(\theta, \cdot)$ is a continuous mapping of M into $\mathcal{M}^\infty$, and the image $\mathcal{K} = J(M)$ is a compact invariant set in the translational flow on $\mathcal{M}^\infty$. Since E is a continuous mapping, it follows that for every $\epsilon > 0$ there is a $\delta > 0$ with the property that $B \in N_\epsilon^\infty(\mathcal{K})$ whenever $d(\omega, \omega_0) \le \delta$. Since this application includes Example 2 as a special case, it is generally not the case that $\sup_{t \in R} \|B(t) - D(t)\|$ is small.

Example 4. (The Taylor-Couette Flow) The Taylor-Couette flow is a fluid flow in a region Ω lying between two concentric right cylinders with

$$\Omega = \{(r, \theta, z) : R_i < r < R_o \text{ and } 0 < z < H\},$$

in cylindrical coordinates, where $0 < R_i < R_o$ and $0 < H < \infty$. It is assumed that the inner cylinder is rotating with an angular velocity ω and the outer cylinder is fixed. We will assume that the fluid satisfies nonslip boundary conditions on the two cylindrical surfaces and periodic boundary conditions on the top and bottom, where $R_o - R_i < H$, see Joseph (1976), for example.

The dynamics of the fluid motion are governed by the Navier-Stokes equations and the behavior depends on the the dynamical parameter ω. As ω changes, one finds a sequence of, at least, four patterns emerging:

$$T_c^0 \longrightarrow T_t^0 \longrightarrow T^1 \longrightarrow T^2 \longrightarrow \cdots . \tag{TC}$$

For $0 < \omega < \omega_1$, one encounters a purely laminar flow T_c^0, which is referred to as the Couette flow. A bifurcation occurs at ω_1, and for $\omega_1 < \omega < \omega_2$ one finds a pattern T_t^0, which is referred to as the Taylor cells. The Taylor cells are bands which appear like donuts wrapping around the inner cylinder. The two states T_c^0 and T_t^0 represent different stationary solutions of the Navier-Stokes equations. At ω_2 a Hopf bifurcation occurs, and for $\omega_2 < \omega < \omega_3$, the Taylor cells develop a periodic behavior T^1. Based on experimental evidence, see Gollub and Swinney (1975), a secondary bifurcation occurs at ω_3, in that another fundamental frequency is found in the basic pattern. For $\omega_3 < \omega < \omega_4$, the trajectories seem to exist on a 2-dimensional torus T^2. See Haken (1983), for more details.

The local dynamical behavior of all of these states, before, during, and after the bifurcations can be described in terms of exponential trichotomies in the Navier-Stokes equations. At the experimental level, one finds a robustness in the bifurcation patterns given in (TC). The physics of the problem suggests that the exponential trichotomies are robust as one varies the physical parameters, R_i, R_o, H, and ω. Our theory offers the basis for the first rigorous proofs of the robustness property within the context of the Navier-Stokes equations.

What happens after the T^2 state, which is the onset of turbulence, is really not understood very well. Landau and Hopf conjectured that there is a sequence of toroidal bifurcations $T^n \longrightarrow T^{n+1}$, for larger and larger values of n. A mathematical theory of such bifurcations, which is based on the properties of exponential trichotomies, can be found in Chenciner and Iooss (1979) and Sell (1979, 1981). Another conjecture is due to Ruelle and Takens (1971). They claimed that the state T^2, when it bifurcates, changes into a strange attractor. However, their example of a strange attractor is based on an intermediate state, T^3, a 3-dimensional torus. It should be noted, that neither of these conjectures has been verified for the Navier-Stokes equations. As noted in Sell (1981), it is possible for both conjectures to occur together. They are not mutually exclusive.

In this illustration of the Navier-Stokes equations, with its restrictive geometry, we encountered four real parameters: R_i, R_o, H, and ω for the dynamics. Consider instead the general problem with the Navier-Stokes equations

$$\begin{gathered} \partial_t u - \nu \triangle u + (u \cdot \nabla) u + \nabla p = f \\ \nabla \cdot u = 0 \end{gathered}$$

in a smooth, bounded region Ω in R^3, or in R^2. We assume that $u(x,t) = \phi(x)$ on the boundary $\partial\Omega$, where $\int_{\partial\Omega} \phi \cdot n \, dS = 0$. For this general problem, there are three natural parameters, each describing an infinite dimensional space: the forcing term f, the boundary condition ϕ, and the region Ω. In order to extend the applications of the Robustness Theorems to handle this general problem, one needs to verify that the Fréchet derivative of the nonlinearity depends continuously on these parameters. For the study of the longtime dynamics of the Navier-Stokes equations, one need only evaluate these derivatives in the vicinity of the global attractor, see Constantin and Foias (1988), Ladyzhenskaya (1972), Sell (1996), Sell and You (1997, 1998), and Temam (1988).

7. Commentary

1. The argument, that the Strong Boundedness Property for the linear inhomogeneous equation

$$\partial_t u + Au = B(t)u + g(t) \tag{7.1}$$

implies the existence of an exponential dichotomy for the linear homogeneous equation (1.1), appears in a number of earlier works, see Perron (1930), Coppel (1965, 1978), Massera and Schaeffer (1966), Ben-Artzi, Gohberg, and Kaashoek (1993), Zhang (1995), and Latushkin, Randolph, and Schnaubelt (1996), for example. In a related study, which begins with the assumption that $\mathcal{B}(\mathcal{K}) = \{0\} \times \mathcal{K}$ (see Lemma 3.2), over a compact, invariant set $\mathcal{K}$ in $\mathcal{W}$, one obtains an Alternative Theorem,, see Sacker and Sell (1974, 1976ab, 1994) and Sacker (1978). The Alternative Theorem states that, when $\mathcal{B}(\mathcal{K}) = \{0\} \times \mathcal{K}$, then either there is an exponential dichotomy over $\mathcal{K}$, or the flow $(B, \tau) \to B_\tau$ on $\mathcal{K}$ is a gradient-like flow. What this means is that if the flow on $\mathcal{K}$ is not gradient-like, then the uniqueness[3] of bounded solutions of equation (7.1) is equivalent to the existence of an exponential dichotomy over $\mathcal{K}$.

2. One can find various versions of the Robustness Theorems in the literature. The formulation given here, which exploits the Fréchet metric toplogy on $\mathcal{W}$, does exist in the ordinary differential equations literature, see Pliss (1977), Pliss and Sell (1991, 1998), and Sacker and Sell (1978, Sect 5). However, the proof given here is new even in this case. As noted in Section 6, a weaker version of the Robustness Theorem for Exponential Dichotomies, wherein the hypothesis (6.11) is used in place of (6.10), appears in Coppel (1978), Henry (1981), Chow and Leiva (1995), and Leiva (1996). Variations based on (6.12), for example, can be found in Massera and Schaeffer (1966).

3. The parameter α appearing in equation (6.8) is the coefficient of friction, when $\alpha > 0$. In the case where G is a positive, selfadjoint operator on a Hilbert space, with compact resolvent, then equation (6.8) is a κ-contraction (in the sense of Kuratowski) when $D(t) \equiv 0$, see Sell and You (1998) or Temam (1988). More generally, if $D \in L^\infty(R, \mathcal{L}(W))$, with $\|D\|_\infty$ small, then equation (6.8) is κ-contracting. As shown in Sacker and Sell (1994), when a linear skew product semiflow is κ-contracting over a compact, invariant set $\mathcal{K}$ in $\mathcal{W}$, and it has an exponential dichotomy over $\mathcal{K}$, then the projector P onto the unstable subbundle $\mathcal{U}$ has finite rank, i.e., one has $\dim \mathcal{R}(P(B)) < \infty$, for each $B \in \mathcal{K}$. This feature is shared

[3]This fact, together with the Lyapunov-Perron formula for the existence of bounded solutions of (7.1), see Coppel (1965, 1978) for example, illustrates the phenomenon of: uniqueness implies existence.

by the Navier-Stokes equations since the induced linear skew product semiflow is compact for $t > 0$.

4. There is another very important aspect of the theory of exponential dichotomies and exponential trichotomies. This issues arises in the theory of approximation dynamics, see Pliss and Sell (1991, 1998). Assume that the nonlinear evolutionary equation

$$\partial_t u + Au = F(\lambda, u), \qquad u \in W \tag{7.2}$$

depends continuously on a parameter $\lambda \in \Lambda$. Let K_{λ_0} be a compact, invariant set for $\lambda = \lambda_0$. We say that K_{λ_0} is a **point of continuity** for equation (7.2), provided that for every $\eta > 0$, there is a neighborhood O_η of λ_0 in Λ, such that for every $\lambda \in O_\eta$, there is a homeomorphism $h = h_\lambda : K_{\lambda_0} \to W$ that satisfies the following, for each $\lambda \in O_\eta$: (1) one has $\|h(u) - u\|_W \leq \eta$, for all $u \in K_{\lambda_0}$, and (2) the set $K_\lambda = h(K_{\lambda_0})$ is a compact invariant set of (7.2).

Let K_{λ_0} be a point of continuity for equation (7.2), and assume that the Fréchet derivative $DF(\lambda, u)$ depends continuously on (λ, u), for $\lambda \in \Lambda$ and u in some neighborhood $N_\eta(K_{\lambda_0})$ of K_{λ_0}, where DF denotes the derivative with respect to u. Assume further that the linearized equation

$$\partial_t u + Au = B(\lambda_0, t)u \tag{7.3}$$

has an exponential trichotomy over K_{λ_0}, where $B(\lambda_0, t) = DF(\lambda_0, \phi(t))$ and ϕ denotes any globally defined solution of (7.2), where $\lambda = \lambda_0$ and with $\phi(t) \in K_{\lambda_0}$, for all $t \in R$. In this case, for every $\epsilon > 0$ there is an $\eta > 0$ such that, there is a neighborhood O_η of λ_0 in Λ, such that for every $\lambda \in O_\eta$, there is a homeomorphism $h = h_\lambda : K_{\lambda_0} \to W$ that satisfies the following, for each $\lambda \in O_\eta$:

(1) one has $\|h(u) - u\|_W \leq \eta$, for all $u \in K_{\lambda_0}$;
(2) the set $K_\lambda = h_\lambda(K_{\lambda_0})$ is a compact invariant set of (7.2); and
(3) the functions $B(t) = B(\lambda_0, t)$ satisfy (6.10).

Hence by the Robustness Theorem on Exponential Trichotomies, the linearized equation for (7.2) has an exponential trichotomy over K_λ, for every $\lambda \in O_\eta$. As a result, not only do the sets K_λ vary continuously over O_η, but also the hyperbolic structure on K_λ, i.e., the exponential trichotomy, varies continuously[4] as well.

The question then is: which compact, invariant sets in W are points of continnuity for the dynamics of equation (7.2)? This is a deep problem of fundamental importance in the study of approximation dynamics. Recent contributions in the context of ordinary differential equations appear in Pliss and Sell (1991, 1998), where the concept of a weakly normally hyperbolic (WNH) set is introduced. (Among other things, a WNH set is a compact, invariant set K_{λ_0} for equation (7.2), with the property that the linearized equation (7.3) has an appropriate exponential trichotomy over K_{λ_0}.) It is shown that every WNH set is a point of continuity for (7.2). Since every hyperbolic periodic orbit, every normally hyperbolic invariant manifold and every hyperbolic set are WNH sets, this result includes earlier work

[4]This is exactly the scenario one encounters in the patterns (TC) appearing in the Taylor-Couette flow.

on perturbation theory reported in Smale (1967), Sacker (1969), Fenichel (1971), Hirsch, Pugh, and Shub (1977), Arnold (1983), and Pilyugin (1992). (Also see the WNH set occuring in Meyer and Sell (1989).)

The theory in the infinite dimensional setting is still under development. In the case that the nonlinear evolutionary equation has an inertial manifold, then one can use this fact, and the corresponding reduction principle, i.e., the inertial form, to conclude that WNH sets in this setting are points of continuity. (Basically the inertial form reduces the infinite dimensional problem to a finite dimensional problem.) Recently, Bates, Lu, and Zeng (1996) have shown that in a Banach space every normally hyperbolic compact invariant manifold is a point of continuity, and Kamaev (1980) has done the same for hyperbolic sets.

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