# GLOBAL BEHAVIOR OF SOLUTIONS TO A REACTION-DIFFUSION SYSTEM†0

S. B. Cui(Cui Shangbin)
Department of Mathematics, Lanzhou University,
Lanzhou, Gansu 730000, People's Republic of China

*Key words and phrases:* Reaction-diffusion system, solution, global behavior, existence, blow-up.

## I. Introduction and Main Results

For many years, based on the work of H. Fujita [1] on the problem

$$\begin{cases} u_t - \triangle u = u^p, \quad u \geq 0, \ \text{ in } R^N \times (0, \infty), & (1.1) \\ u|_{t=0} = \varphi \geq 0, \quad \text{on } R^N, & (1.2) \end{cases}$$

great progress has been made on the research of global existence and finite time blow-up of solutions for initial value problems of nonlinear heat equations. The results that have already been obtained shows that, roughly speaking, whether a solution exists globally or blows up at finite time depends at least on three factors: the form of the nonlinear term; the dimension $N$ and the size of the initial value. For instance, for the problem (1.1)–(1.2) we have the following well-known conclusions:

(1) if $0 < p \leq 1$, then every solution with bounded initial value is global;

(2) if $1 < p \leq 1 + 2/N$, then every solution with nontrivial initial value blows up;

(3) if $p > 1 + 2/N$, then solutions with "small" initial values are global while solutions with "large" initial values blow up.

(c.f.[1–3] for the proofs of these conclusions and more details). For more general nonlinear perturbations of the heat equation of the form

$$\begin{cases} u_t - \triangle u = f(u, \partial_x u, \partial_x^2 u), \quad \text{in } R^N \times (0, \infty), & (1.3) \\ u|_{t=0} = \varphi, \quad \text{on } R^N, & (1.4) \end{cases}$$

it has been proved that the solution corresponding to "small" initial function $\varphi$ is global provided $f$ is sufficiently smooth and satisfies

$$|f(u, \xi, \omega)| \leq C(|u| + |\xi| + |\omega|)^p, \quad p > 1 + 2/N, \tag{1.5}$$

0† Project supported by the National Natural Science Foundation of China.

for small $(u, \xi, \omega)$; and when $f$ does not involve $u$, namely, $f(u, \xi, \omega) = f(\xi, \omega)$, then the condition $p > 1 + 2/N$ can be weakened to $p \geq 2$ (c.f. [4–6] for the proof and more details). Recently, the above results have been systematicly generalized to general nonlinear parabolic systems by the author of the present paper in [7–10]. It has been proved that, in especial, for a general nolinear parabolic system of the form

$$u_t - \sum_{|\alpha|=m} A_\alpha \partial_x^\alpha u = F(u, \partial_x u, \cdots, \partial_x^{m-1} u), \tag{1.6}$$

where $u = (u_1, u_2, \cdots, u_n)$, $F = (F_1, F_2, \cdots, F_n)$, $\partial_x u = \{\partial_{x_j} u_k : \ j = 1, 2, \cdots, N, \ k = 1, 2, \cdots, n\}$, $\partial_x^{m-1} u = \{\partial_x^\alpha u_k : \ |\alpha| = m-1, \ k = 1, 2, \cdots, n\}$, and $A_\alpha$'s are $n \times n$ matricies, similar global existence result is valid provided $F$ is continuous (need not be locally Lipchitz continuous) and satisfies

$$|F(w_0, w_1, \cdots, w_{m-1})| \leq C(|w_0|^{p_0} + |w_1|^{p_1} + \cdots + |w_{m-1}|^{p_{m-1}}) \tag{1.7}$$

with $p_j > 1 + (m-j)/(N+j)$ $(j = 0, 1, \cdots, m-1)$ for small $(w_0, w_1, \cdots, w_{m-1})$; see [10] for details.

Restricted to scaler equations, the above result is optimal at least in the case $F(w_0, w_1, \cdots, w_{m-1}) = F(w_0)$ because of the previously mensioned result on the equation (1.1). For systems which involve more than one unknown functions, however, this result is not optimal. This can be seen from the work of M. Escobedo and M. A. Herrero in [11], where they studied nonnegative solutions to the system

$$\begin{cases} u_t - \triangle u = v^p, & (1.8) \\ v_t - \triangle v = u^q, & (1.9) \end{cases}$$

where $p > 0$, $q > 0$, $x \in R^N$ and $t > 0$, and obtained the following conclusions:

(4) if $0 < pq \leq 1$, then every nonnegative solution with bouned initial value is global;

(5) if $pq > 1$ and $N(pq-1)/2(\gamma+1) \leq 1$ $(\gamma = \max(p, q))$, then every nonnegative solution with nontrivial initial value blows up;

(6) if $pq > 1$ and $N(pq-1)/2(\gamma+1) > 1$, then solutions with "small" initial value are global while solutions with "large" initial value blow up.

We are thus lead to the following problem: What is the optimal condition for global existence of small solutions to nonlinear parabolic systems involving more than one unknown functions? This is without doubt a difficult problem. To study this problem, the first step is to find a simple model and get as much as possible information through investigating it as was done for other similar problems. For this purpose, in this paper we study the global behavior of nonnegative solutions to the initial value problem of the following system:

$$\begin{cases} u_t - \triangle u = au^r + bv^p, & (1.10) \\ v_t - \triangle v = cu^q + dv^s, & (1.11) \end{cases}$$

where $a,\ b,\ c,\ d,\ p,\ q,\ r$ and $s$ are positive constants and, without loss of generality, we assume $p \leq q$. We shall make a thorough investigation to the global behavior of solutions to this system. The results show that in the case $r \leq 1,\ s \leq 1$ and $pq \leq 1$, the behavior of this system is similar to a simple combination of the system (1.8)–(1.9) and the two scaler equations

$$u_t - \triangle u = u^r \quad \text{and} \quad v_t - \triangle v = v^s; \tag{1.12}$$

namely, solutions with bounded initial values are always global. However, in the case $r > 1$ or $s > 1$ or $pq > 1$, its behavior is different; that is, even if the conditions

$$r > 1 + \frac{2}{N}, \quad s > 1 + \frac{2}{N} \quad \text{and} \quad \frac{N(pq-1)}{2(\gamma+1)} > 1,$$

which guarantee the global existence of solutions with small initial values for the system(1.8)–(1.9) and the two scaler equations (1.12) are satisfied, solutions of the system (1.10)–(1.11) with small initial values may still blow up unless some additional conditions are satisfied; these additional conditions are the following:

$$p > 1 + \frac{2}{N} \quad \text{or} \quad \frac{2}{N} < p \leq 1 + \frac{2}{N} \quad \text{and} \quad r > \frac{Np}{Np-2}.$$

Let us now introduce our results. In the following the phrase *admissible solution* means a classical solution $(u, v)$ of the problem (1.10) and (1.11) such that $u$ and $v$ are nonnegative and bounded with respect to the space variable. The phrase *admissible* for a function $\varphi$ on $R^N$ means that $\varphi$ is continuous, bounded and nonnegative.

**Theorem 1.1** *Suppose that*

$$r \leq 1, \quad s \leq 1 \quad \textit{and} \quad pq \leq 1. \tag{1.13}$$

*Then every admissible solution of the system (1.10)–(1.11) under the initial conditions*

$$u|_{t=0} = \varphi, \qquad v|_{t=0} = \psi \tag{1.14}$$

*with admissible $\varphi$ and $\psi$ is global. Moreover, for any such $\varphi$ and $\psi$ this problem has a global admissible solution $(u, v)$ possessing the property*

$$0 \leq u(x,t) \leq C(1+t)^{\alpha}, \quad 0 \leq v(x,t) \leq C(1+t)^{1+q\alpha} \tag{1.15}$$

*if $r < 1,\ s < 1$ and $pq < 1$, where $\alpha = \max\left(\frac{1}{1-r}, \frac{s}{q(1-s)}, \frac{1+p}{1-pq}\right)$, or*

$$0 \leq u(x,t) \leq C\mathrm{e}^{\alpha t}, \quad 0 \leq v(x,t) \leq C\mathrm{e}^{q\alpha t} \tag{1.16}$$

*if at least one of $r,\ s$ and $pq$ is equal to 1, where $\alpha$ is a sufficiently large positive constant.*

*Remark*: Under the conditions in the above theorem, every nontrivial admissible solution of the problem (1.10), (1.11) and (1.14) increases to infinitive as $t \to \infty$. In fact, in the case $r < 1,\ s < 1$ and $pq < 1$ we have

$$u(x,t) \geq C_1 t^{\alpha}, \quad v(x,t) \geq C_1 t^{1+q\alpha}, \tag{1.17}$$

where $C_1 > 0$ and $\alpha$ is as in (1.15), and in the case where at least one of $r$, $s$ and $pq$ is equal to 1, the solution increases to infinitive exponentially as $t \to \infty$. See Lemma 3.1 and 3.2 in Section III.

*Remark*: Under the conditions of the above theorem the admissible solution may be not unique, see [11] for the example. This illustrates the phrase "every solution" that we have used in the state of the above theorem and remark. The same remark is also valid for the following three theorems.

**Theorem 1.2** *Suppose that*

$$r > 1 \quad or \quad s > 1 \quad or \quad pq > 1. \tag{1.18}$$

*Then every admissible solution of the problem (1.10), (1.11) and (1.14) with large initial value blows up at finite time. More precisely, we have the following conclusions:*

*(i) If $r > 1$ and $\varphi(x) \geq C(1 + M|x|^2)^{-m}$, where $M$ and $m$ are arbitrary positive numbers, then for sufficiently large $C > 0$ every admissible solution of the problem (1.10)–(1.11) and (1.14) blows up at a time $T^* \leq M^{-1}$.*

*(ii) If $s > 1$ and $\psi \geq C(1+M|x|^2)^{-m}$, where $M$ and $m$ are as above, then similar statement is valid.*

*(iii) If $pq > 1$ and $\varphi \geq C(1 + M|x|^2)^{-m}$, $\psi(x) \geq C(1 + M|x|^2)^{-l}$, where $M$, $m$ and $l$ are arbitrary positive numbers, then the above statement is also valid.*

**Theorem 1.3** *Suppose that (1.18) holds. Suppose furthermore that at least one of the four conditions (a)-(d) below is satisfied:*

*(a)* $r \leq 1 + \frac{2}{N}$;

*(b)* $s \leq 1 + \frac{2}{N}$;

*(c)* $\frac{N(pq-1)}{2(q+1)} \leq 1$;

*(d)* $\frac{N(pq-1)}{2(q+1)} > 1$, $p \leq 1 + \frac{2}{N}$ *and* $r < \frac{Np}{Np-2}$.

*(Remark:* We note that in (d), the condition $\frac{N(pq-1)}{2(q+1)} > 1$ implies $p > \frac{2}{N}$*) Then every admissible solution of the problem (1.10), (1.11) and (1.14) with nontrivial admissible initial value blows up at finite time.*

In the following we use the notation $\|\cdot\|_p$ to denote the norm on the space $L^p(R^N)$. This notation is used throughout the whole paper.

**Theorem 1.4** *Suppose that*

$$\begin{aligned} &r > 1 + \tfrac{2}{N}, \quad s > 1 + \tfrac{2}{N}, \quad \tfrac{N(pq-1)}{2(q+1)} > 1 \quad \textit{and} \\ &\textit{either} \;\; p > 1 + \tfrac{2}{N} \;\; \textit{or} \;\; p \leq 1 + \tfrac{2}{N} \;\; \textit{and} \;\; r > \tfrac{Np}{Np-2}. \end{aligned} \tag{1.19}$$

*Then there exist $\alpha \geq 1$, $\beta \geq 1$ and $\delta > 0$ such that the problem (1.10), (1.11) and (1.14) has a global admissible solution which decays to zero as $t \to \infty$ provided*

$$\|\varphi\|_\alpha + \|\varphi\|_\infty \leq \delta \quad \textit{and} \quad \|\psi\|_\beta + \|\psi\|_\infty \leq \delta. \tag{1.20}$$

*If, on the other hand,*

$$\varphi(x) \geq c(1 + |x|^2)^{-\mu}, \quad \psi(x) \geq c(1 + |x|^2)^{-\nu}$$

*for some* $0 < \mu < \frac{N}{2\alpha}$, $0 < \nu < \frac{N}{2\beta}$ *and* $c > 0$, *then every admissible solution* $(u, v)$ *of this problem blows up at finite time.*

*Remark*: We note that (1.20) implies that $\varphi(x)$ (resp. $\psi(x)$) decays to zero as $|x| \to \infty$ at a speed faster than $(1 + |x|^2)^{-\frac{N}{2\alpha}}$ (resp. $(1 + |x|^2)^{-\frac{N}{2\beta}}$). Therefore, the second conclusion of the above theorem shows that $\alpha$ and $\beta$ are critical. The explicit expressions of $\alpha$ and $\beta$ are given in Section 4.

Before ending this introduction let us make some agreements and introduce some notation to be used later. In the following sections we always drop the coefficients $a$, $b$, $c$ and $d$ in (1.10) and (1.11) to simplify the notation. We always assume that the initial functions $\varphi$ and $\psi$ are admissible, which will sometimes be restated to make emphasis. We will use the notation $Z_T$ $(0 < T \leq \infty)$ to denote the space of functions $u(x, t)$ on $R^N \times [0, T)$ such that $u \in C^{2,1}(R^N \times (0, T)) \cap C(R^N \times [0, T))$ and is bounded on $R^N \times [0, T_1]$ for arbitrary $0 < T_1 < T$, i.e.,

$$Z_T = C^{2,1}(R^N \times (0, T)) \cap C(R^N \times [0, T)) \cap L^\infty_{loc}([0, T);\, L^\infty(R^N)).$$

Note that a solution $(u, v)$ of the problem (1.10), (1.11) and (1.14) on the domain $R^N \times [0, T)$ is admissible if and only if $u$, $v$ are nonnegative (which is always valid when $\varphi$ and $\psi$ are nonnegative) and belong to $Z_T$. In addition, we usually regard a function $u(x, t)$ in two variables $x$ and $t$ as a function in a single variable $t$ with values taken in some function spaces on $R^N$, and simply write it as $u(t)$. Finally, we denote as usual by $S(t)$ $(0 \leq t < \infty)$ the fundamental-solution operator of the initial value problem of the heat operator, which is, as well-known, the semi-group generated by the Laplacian $\triangle$ when it is restricted to certain function spaces on $R^N$.

The following sections are arranged to present the proofs of Theorem 1.1–1.4.

# II. The Proofs of Theorem 1.1 and 1.2

## II.1 Preparations

The first three theorems are concerned with the following problem:

$$u_t - \triangle u = f(u, v), \quad \text{in } R^N \times (0, T), \tag{2.1}$$

$$v_t - \triangle v = g(u, v), \quad \text{in } R^N \times (0, T), \tag{2.2}$$

$$u|_{t=0} = \varphi, \quad v|_{t=0} = \psi, \quad \text{on } R^N, \tag{2.3}$$

where $0 < T \leq \infty$.

**Theorem 2.1** *Suppose that the functions* $f$ *and* $g$ *are defined, nonnegative and locally Lipchitz continuous on* $[0, \infty) \times [0, \infty)$. *Suppose furthermore that* $f(u, v)$ *is nondecreasing with respect to* $v$ *and* $g(u, v)$ *is nondecreasing with respect to* $u$. *If nonnegative functions* $\overline{u}$, $\overline{v}$, $\underline{u}$, $\underline{v} \in Z_T$ *satisfy*

$$\overline{u}_t - \triangle\overline{u} - f(\overline{u}, \overline{v}) \geq \underline{u}_t - \triangle\underline{u} - f(\underline{u}, \underline{v}), \quad \textit{in } R^N \times (0, T), \tag{2.4}$$

$$\overline{v}_t - \triangle\overline{v} - g(\overline{u}, \overline{v}) \geq \underline{v}_t - \triangle\underline{v} - g(\underline{u}, \underline{v}), \quad \textit{in } R^N \times (0, T), \tag{2.5}$$

$$\overline{u}(x, 0) \geq \underline{u}(x, 0), \quad \overline{v}(x, 0) \geq \underline{v}(x, 0), \quad \textit{for } x \in R^N, \tag{2.6}$$

*then* $\overline{u}(x,t) \geq \underline{u}(x,t)$ *and* $\overline{v}(x,t) \geq \underline{v}(x,t)$ *for all* $(x,t) \in R^N \times [0,T)$.

*Proof.* c.f. [12, Section 28 and 32].

**Theorem 2.2** *Suppose that* $f$ *and* $g$ *are defined, nonnegative and locally Hölder continuous on* $[0,\infty) \times [0,\infty)$ *and there exists constants* $M_1$ *and* $M_2$ *such that* $f(u,v) + M_1 u$ *and* $g(u,v) + M_2 v$ *are monotonically nondecreasing with respect to the two variables* $u$ *and* $v$. *Suppose furthermore that there exists a pair of nonnegative functions* $\overline{u}$, $\overline{v} \in Z_T$ *such that* $(\overline{u}, \overline{v})$ *is an upper solution of the problem (2.1)–(2.3). Then the problem (2.1)–(2.3) has a classical solution* $(u,v)$ *satisfying*

$$0 \leq u(x,t) \leq \overline{u}(x,t), \quad 0 \leq v(x,t) \leq \overline{v}(x,t)$$

*for all* $(x,t) \in R^N \times [0,T)$.

The proof of this theorem is quite similar to that of its correspondence for initial-boundary value problems (c.f. [13]). Thus we omit it here. We note that no Lipchitz continuity condition is required for the validity of the above theorem.

**Theorem 2.3** *Suppose that* $f(u,v)$ *and* $g(u,v)$ *satisfy the conditions stated in Theorem 2.2. Then for any admissible functions* $\varphi$ *and* $\psi$ *on* $R^N$, *there exists a corresponding* $T > 0$ *such that the problem (2.1)–(2.3) has a admissible solution* $(u,v)$ *on* $R^N \times [0,T)$. *Moreover, if we denote by* $T^*$ *the supremum of all such* $T$, *then either* $T^* = \infty$ *or* $T^* < \infty$ *and*

$$\limsup_{t \to T^*-0} \sup_{x \in R^N} (|u(x,t)| + |v(x,t)|) = \infty. \tag{2.7}$$

*Proof.* The existence assertion can be proved by applying Theorem 2.2. Indeed, Let $M = \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}$, and denote by $A$ the larger maximal values of the functions $f(u,v)$ and $g(u,v)$ on the compact set $|u| \leq 2M, |v| \leq 2M$. Then as one may easily verify, the pair $(\overline{u}(x,t), \overline{v}(x,t)) = (M + At, M + At)$ is an upper solution of the problem (2.1)–(2.3) on the domain $R^N \times [0, M/A)$. Hence by Theorem 2.2 we get the assertion. As for the conclusion (2.7), we refer the reader to see [10, Theorem 2.1].

## II.2 The Proof of Theorem 1.1

To prove the first statement in Theorem 1.1, we make separate discussions according to the two different situations $r \leq q$ and $r > q$ respectively. In the first situation we rewrite the problem (1.10), (1.11) and (1.14) as follows:

$$u(t) = S(t)\varphi + \int_0^t S(t-\tau)u(\tau)^r \mathrm{d}\tau + \int_0^t S(t-\tau)v(\tau)^p \mathrm{d}\tau, \tag{2.8}$$

$$v(t) = S(t)\psi + \int_0^t S(t-\tau)u(\tau)^q \mathrm{d}\tau + \int_0^t S(t-\tau)v(\tau)^s \mathrm{d}\tau. \tag{2.9}$$

Now, since $r \leq q,\ p \leq q,\ pq \leq 1$, and, consequently, $p \leq 1,\ r/q \leq 1$, we have, quite similar to the proof of [11, Lemma 3.1],

$$\begin{aligned}
u(t)=&S(t)\varphi + \int_0^t S(t-\tau)u(\tau)^r \mathrm{d}\tau + \int_0^t S(t-\tau)[S(\tau)\psi \\
&+ \int_0^\tau S(\tau-\xi)u(\xi)^q \mathrm{d}\xi + \int_0^\tau S(\tau-\xi)v(\xi)^s \mathrm{d}\xi]^p \mathrm{d}\tau \\
\leq& S(t)\varphi + \int_0^t [S(t-\tau)u(\tau)^q]^{\frac{r}{q}} \mathrm{d}\tau + \int_0^t \{S(t-\tau)[S(\tau)\psi \\
&+ \int_0^\tau S(\tau-\xi)u(\xi)^q \mathrm{d}\xi + \int_0^\tau S(\tau-\xi)v(\xi)^s \mathrm{d}\xi]\}^p \mathrm{d}\tau \\
\leq& S(t)\varphi + t^{1-\frac{r}{q}}[\int_0^t S(t-\tau)u(\tau)^q \mathrm{d}\tau]^{\frac{r}{q}} + t^{1-p}\{tS(t)\psi \\
&+ \int_0^t (t-\xi)S(t-\xi)u(\xi)^q \mathrm{d}\xi + \int_0^t (t-\xi)S(t-\xi)v(\xi)^s \mathrm{d}\xi\}^p \\
\leq& S(t)\varphi + t^{1-\frac{r}{q}} v(t)^{\frac{r}{q}} + t^{1-p}\{tv(t)\}^p.
\end{aligned} \tag{2.10}$$

Applying this result in (1.11) we get

$$v_t - \triangle v \leq \max(1, 3^{q-1})\{[S(t)\varphi]^q + t^{q-r}v^r + t^q v^{pq}\} + v^s.$$

Since $r \leq 1,\ s \leq 1$ and $pq \leq 1$, we have

$$v^r \leq 1+v, \quad v^{pq} \leq 1+v, \quad v^s \leq 1+v.$$

Therefore,

$$v_t - \triangle v \leq C(1+t)^q(1+v),$$

where $C = \max(1, 3^{q-1})(1+\sup\varphi)^q$. From this inequality by comparison we immediately get $v(t) \leq w(t)$ in the existence interval of $v(t)$, where $w(t)$ is the solution of the linear problem

$$w_t - \triangle w = C(1+t)^q(1+w), \quad w(0) = \psi.$$

Hence $v(t)$, and consequently $u(t)$ as well due to (2.10), is bounded on any bounded existence interval, which implies that $(u(t), v(t))$ is global.

In the second situation we have $\max(p, q, r, s) \leq 1$. Therefore,

$$(u+v)_t - \triangle(u+v) = u^r + v^p + u^q + v^s \leq 4 + 2(u+v).$$

Again by comparison we see that $u(t)+v(t)$, and consequently $u(t)$ as well as $v(t)$, is bounded on any bounded existence interval. Thus $(u(t), v(t))$ is still global.

To prove the second statement in Theorem 1.1, we note that if $r < 1,\ s < 1$ and $pq < 1$ then the pair of functions $(\overline{u}, \overline{v})$ defined by

$$\overline{u}(x,t) = C(1+t)^\alpha, \quad \overline{v}(x,t) = C^{q'}(1+t)^{1+q\alpha},$$

where $\alpha = \max\left(\frac{1}{1-r}, \frac{s}{q(1-s)}, \frac{1+p}{1-pq}\right)$ and $q < q' < \frac{1}{p}$, is an upper solution of the problem (1.10), (1.11) and (1.14) provided $C$ is sufficiently large. If at least one of $r,\ s$ and $pq$ takes the value 1 then $(\overline{u}, \overline{v})$ defined by

$$\overline{u}(x,t) = C\mathrm{e}^{\alpha t}, \quad \overline{v}(x,t) = C^q \mathrm{e}^{q\alpha t},$$

with $C$ and $\alpha$ sufficiently large, is still an upper solution of that problem. Therefore by Theorem 2.2 we get the conclusion. This finishes the proof of Theorem 1.1.

## II.3 The Proof of Theorem 1.2

First we assume that $r > 1$ and $\varphi(x) \geq C(1 + M|x|^2)^{-m}$. Without loss of generality we may assume $m > \max\left(\frac{2}{r-1}, \frac{N}{2} - \frac{3}{4}\right)$. Let

$$\underline{u}(x.t) = C[1 + M(|x|^2 - t)]^{-m},$$

and let $T \in (0, \infty]$ be the number such that $[0, T)$ is the maximal existence interval of the solution $(u, v)$ of the problem (1.10), (1.11) and (1.14). We claim that $\underline{u}$ is a lower solution of the problem

$$w_t - \triangle w = w^r, \quad \text{in } R^N \times (0, \min(M^{-1}, T)), \tag{2.11}$$

$$w(x, 0) = C(1 + M|x|^2)^{-m}, \quad x \in R^N. \tag{2.12}$$

To see this we employ the abbreviate notation $A = 1 + M(|x|^2 - t)$ and write

$$\begin{aligned} \underline{u}_t &= mMCA^{-m-1}, \quad \forall (x,t) \in R^N \times (0, \min(M^{-1}, T)); \\ \triangle \underline{u} &= -2mMNCA^{-m-1} + 4m(m+1)M^2 C|x|^2 A^{-m-2} \\ &\geq mMC(4m+4-2N)A^{-m-1} - 4m(m+1)MCA^{-m-2}, \quad \forall (x,t) \in R^N \times (0, \min(M^{-1}, T)). \end{aligned}$$

Consequently,

$$\underline{u}_t - \triangle \underline{u} - \underline{u}^r \leq -mMC(4m+3-2N)A^{-m-1} + 4m(m+1)MCA^{-m-2} - C^r A^{-mr},$$
$$\forall (x,t) \in R^N \times (0, \min(M^{-1}, T)).$$

Since $4m+3-2N>0$ and $m+1<m+2<mr$, by applying Young's inequality we get a constant $C_0>0$ depending only on $m$, $N$ and $r$ such that

$$A^{-m-2} \leq \frac{4m+3-2N}{4(m+1)} A^{-m-1} + C_0 A^{-mr}, \quad \forall (x,t) \in R^N \times (0, \min(M^{-1}, T)).$$

Therefore,

$$\underline{u}_t - \triangle \underline{u} - \underline{u}^r \leq -C[C^{r-1} - 4m(m+1)MC_0]A^{-mr}, \quad \forall (x,t) \in R^N \times (0, \min(M^{-1}, T)),$$

which implies that if $C^{r-1} \geq 4m(m+1)MC_0$ then $\underline{u}$ is a lower solution of the problem (2.11)–(2.12) and hence our claim is true. Now, since $u$ is obviously an upper solution of (2.11)–(2.12) and the assumption $r>1$ implies that the function $w \to w^r$ is locally Lipchitz continuous on $[0, \infty)$, we conclude by comparison that $u(x,t) \geq \underline{u}(x,t)$ on $R^N \times [0, \min(M^{-1}, T))$, which proves the conclusion *(i)* because $\lim_{t \to M^{-1}-0} \underline{u}(0,t) = \infty$.

The proof of conclusion *(ii)* is similar.

Next we assume $pq > 1$, $p \geq 1$ and

$$\varphi(x) \geq C(1 + M|x|^2)^{-m}, \quad \psi(x) \geq C(1 + M|x|^2)^{-l}. \tag{2.13}$$

Again, without loss of generality we may assume

$$m > \max\left[\frac{2(p+1)}{pq-1},\ \frac{2p(q+1)}{q(pq-1)},\ \frac{N}{2}-\frac{3}{4},\ \left(\frac{N}{2}+\frac{5}{4}\right)/q\right] \tag{2.14}$$

and

$$\max\left(\frac{m+2}{p},\ \frac{N}{2}-\frac{3}{4}\right) < l < mq-2. \tag{2.15}$$

If this fails we first choose a number $m' > \max\left(m, \frac{l+2}{q}\right)$ such that (2.14) holds with $m$ replaced by $m'$ and then a number $l' \geq l$ such that (2.15) holds with $l$ replaced by $l'$ and $m$ replaced by $m'$, and substitute $m$ and $l$ with $m'$ and $l'$ respectively. Let

$$\underline{u}(x,t) = \overline{C}A^{-m}, \quad \underline{v}(x,t) = \overline{C}^{q'}[1+M(|x|^2-t)]^{-l},$$

where $q' = 1$ if $p > 1$ and $1 < q' < q$ if $p = 1$. A similar discussion as we have made above shows that $(\underline{u}, \underline{v})$ is a lower solution (defined in obvious way) of the problem

$$\begin{cases} u_t - \triangle u = v^p, & \text{in } R^N \times (0, \min(M^{-1}, T)), \\ v_t - \triangle v = u^q, & \text{in } R^N \times (0, \min(M^{-1}, T)), \\ u(x,0) = \overline{C}(1+M|x|^2)^{-m}, & v(x,0) = \overline{C}^{q'}(1+M|x|^2)^{-l}, \ x \in R^N, \end{cases}$$

provided $\overline{C}$ is sufficiently large. Since the solution $(u, v)$ of the problem (1.10), (1.11) and (1.14) is an upper solution of the above problem provided $C \geq \max(\overline{C}, \overline{C}^{q'})$, by Theorem 2.1 we conclude that $u(x,t) \geq \underline{u}(x,t)$ and $v(x,t) \geq \underline{v}(x,t)$ on $R^N \times [0, \min(M^{-1}, T))$. Therefore, the conclusion *(iii)* is valid in the case $p \geq 1$.

Finally, we assume that $pq > 1$, $p < 1$ and (3.6) holds and, moreover,

$$m > \max\left[\frac{4(1+pq)}{pq(pq-1)},\ \frac{N}{2}-\frac{3}{4},\ \left(\frac{N}{2}+\frac{5}{4}\right)/pq\right], \tag{2.16}$$

$$\max\left(m+2,\ \frac{N}{2}-\frac{3}{4}\right)/p < l < (mpq-2)/p. \tag{2.17}$$

From (2.9) we have

$$v(t) \geq S(t)\psi + \int_0^t S(t-\tau)u(\tau)^q \mathrm{d}\tau.$$

Consequently,

$$\begin{aligned} v(t)^p &\geq 2^{p-1}[S(t)\psi]^p + 2^{p-1}\left[\textstyle\int_0^t S(t-\tau)u(\tau)^q \mathrm{d}\tau\right]^p \\ &\geq 2^{p-1}S(t)\psi^p + 2^{p-1}t^{p-1}\textstyle\int_0^t S(t-\tau)u(\tau)^{pq}\mathrm{d}\tau \\ &\geq 2^{p-1}S(t)\psi^p + 2^{p-1}M^{1-p}\textstyle\int_0^t S(t-\tau)u(\tau)^{pq}\mathrm{d}\tau, \end{aligned} \tag{2.18}$$

for $0 < t < \min(M^{-1}, T)$. Now let $w(t) = M^{p-1}S(t)\psi^p + \int_0^t S(t-\tau)u(\tau)^{pq}\mathrm{d}\tau$. Then

$$u_t - \triangle u \geq v^p \geq 2^{p-1}M^{1-p}w, \quad \text{for } 0 < t < \min(M^{-1}, T).$$

Thus $(u, w)$ is an upper solution of the problem

$$\begin{cases} u_t - \triangle u = 2^{p-1} M^{1-p} w, & \text{in } R^N \times (0, \min(M^{-1}, T)), \\ w_t - \triangle w = u^{pq}, & \text{in } R^N \times (0, \min(M^{-1}, T)), \\ u(0) = \varphi, \ \ w(0) = M^{p-1}\psi^p. \end{cases}$$

On the other hand, as one can verify, the pair of functions $(\underline{u}, \underline{w})$ defined by

$$\underline{u}(x,t) = \overline{C} A^{-m}, \quad \underline{w}(x,t) = \overline{C}^h A^{-lp}$$

is a lower solution of the above problem under the assumptions (2.13), (2.16) and (3.17) provided $1 < h < pq$, $\overline{C}$ is sufficiently large and $C \geq \max(\overline{C}, \ M^{\frac{1}{p}-1}\overline{C}^{\frac{h}{p}})$. Therefore, by applying Theorem 2.1 once more we conclude that $u(x,t) \geq \underline{u}(x,t)$ and $w(x,t) \geq \underline{w}(x,t)$ on $R^N \times [0, \min(M^{-1}, T))$. Note that $w(x,t) \geq \underline{w}(x,t)$ and (2.18) combined together implies that

$$v(x,t)^p \geq 2^{p-1} M^{1-p} \underline{w}(x,t).$$

Hence the conclusion *(iii)* is still valid in the case $p < 1$. The proof of Theorem 1.2 is finished.

# III. The Proof of Theorem 1.3

## III.1 Preliminary Lemmas

The proof of Theorem 1.3 is based on a series of preliminary lemmas which have their own importance. For instance, the conclusions in the remark following Theorem 1.1 are direct corollaries of the first two lemmas.

**Lemma 3.1** *Suppose that $\varphi$ is a nontrivial admissible function on $R^N$. If a nonnegative function $u \in Z_\infty$ satisfies*

$$u_t - \triangle u \geq u^r, \quad \text{in } R^N \times (0, \infty), \tag{3.1}$$

$$u(0) = \varphi, \quad \text{on } R^N, \tag{3.2}$$

*then the following statements hold:*

*(i) If $0 < r < 1$ then*

$$u(t) \geq (1-r)^{\frac{1}{1-r}} t^{\frac{1}{1-r}}, \qquad \forall t \geq 0; \tag{3.3}$$

*(ii) If $r = 1$ then*

$$u(t) \geq \mathrm{e}^t S(t)\varphi, \quad \forall t \geq 0; \tag{3.4}$$

*(iii) If $r > 1$ then*

$$\|S(t)u(t)\|_\infty \leq (r-1)^{-\frac{1}{r-1}} t^{-\frac{1}{r-1}}, \quad \forall t > 0. \tag{3.5}$$

*Proof.* The estimate (3.5) can be proved in the same way as in the proof of [11, Lemma 4.1 and 4.3] (or alternatively, as in the proof of [1, Lemma 2.1] and [11, Lemma 4.3]), so we omit its proof here. To prove the estimate (3.3) we first assume $\varphi(x) > 0$ for all $x \in R^N$. Then one can deduce in the same way as in the proof of [11, Lemma 4.1] and obtain

$$u(t) \geq A_k[S(t)\varphi^{r^k}]t^{\frac{1-r^k}{1-r}}, \quad \forall t \geq 0, \;\; \forall k \in Z_+, \tag{3.6}$$

where

$$A_k = \prod_{j=1}^{k-1} \left( \frac{1-r}{1-r^{j+1}} \right)^{r^{k-j-1}}, \qquad k = 1, 2, \cdots.$$

Note that

$$A_k \geq (1-r)^{\frac{1-r^{k-1}}{1-r}} \geq (1-r)^{\frac{1}{1-r}}, \qquad k = 1, 2, \cdots.$$

Thus by replacing $A_k$ with $(1-r)^{\frac{1}{1-r}}$ in (3.6) and then letting $k \to \infty$, we get (3.3). For general $\varphi(x) \geq 0$, $\varphi \neq 0$, since (3.1) and (3.2) implies $u(x,t) > 0$ for all $x \in R^N$ and $t > 0$, we can use (3.3) to the function $u_\varepsilon(t) = u(t+\varepsilon)$ $(\varepsilon > 0)$ first and then let $\varepsilon \to 0^+$. The estimate (3.4) can be obtained by direct comparison. *Q. E. D.*

**Lemma 3.2** *Suppose that $\varphi$ and $\psi$ are admissible and $(\varphi, \psi) \neq (0,0)$. If nonnegative functions $u, v \in Z_\infty$ satisfy*

$$u_t - \triangle u \geq v^p, \quad \text{in } R^N \times (0, \infty), \tag{3.7}$$

$$v_t - \triangle v \geq u^q, \quad \text{in } R^N \times (0, \infty), \tag{3.8}$$

$$u(0) = \varphi, \;\; v(0) = \psi, \quad \text{on } R^N, \tag{3.9}$$

*then there exists a function $C(p,q)$ defined and positive on $(0,\infty) \times (0,\infty)$ such that the following statements hold:*

*(i) If $p > 0$, $q > 0$ and $pq < 1$ then*

$$u(t) \geq C(p,q)t^{\frac{1+p}{1-pq}}, \quad \forall t \geq 0, \tag{3.10}$$

$$v(t) \geq C(q,p)t^{\frac{1+q}{1-pq}}, \quad \forall t \geq 0; \tag{3.11}$$

*(ii) If $0 < p \leq q$ and $pq = 1$ then*

$$u(t) \geq C(p,q)\mathrm{e}^{\frac{1}{2p}t^p} S(t)\varphi, \quad \forall t \geq 1, \tag{3.12}$$

$$v(t)^p \geq 2^{p-1}S(t)\psi^p + C(p,q)\mathrm{e}^{\frac{1}{2p}t^p} S(t)\varphi, \quad \forall t \geq 1; \tag{3.13}$$

*(iii) If $p > 0$, $q > 0$ and $pq > 1$ then*

$$||S(t)u(t)||_\infty \leq C(p,q)t^{-\frac{1+p}{pq-1}}, \qquad if \;\; q \geq 1, \tag{3.14}$$

$$||S(t)u(t)^q||_\infty \leq C(p,q)t^{-\frac{q(1+p)}{pq-1}}, \qquad if \;\; q < 1, \tag{3.15}$$

$$||S(t)v(t)||_\infty \leq C(q,p)t^{-\frac{1+q}{pq-1}}, \qquad if \;\; p \geq 1, \tag{3.16}$$

$$||S(t)v(t)^p||_\infty \leq C(q,p)t^{-\frac{p(1+q)}{pq-1}}, \qquad if \;\; p < 1. \tag{3.17}$$

*Proof.* The estimates (3.16) and (3.17) have already been proved by [11, Lemma 4.3], and the estimates (3.14) and (3.15) can be proved with the same method. Thus we only prove (3.10)–(3.13).

We first assume $pq < 1$. Then by deducing in the same way as in the proof of [11, Lemma 4.1 and 4.3] we get

$$u(t) \geq \begin{cases} B_k(p,q)[S(t)\varphi]^{(pq)^k} t^{\frac{(1+p)[1-(pq)^k]}{1-pq}}, & \text{if } q \geq 1, \\ B_k(p,q)[S(t)\varphi^q]^{\frac{(pq)^k}{q}} t^{\frac{(1+p)[1-(pq)^k]}{1-pq}}, & \text{if } q < 1, \end{cases} \tag{3.18}$$

$$v(t) \geq \begin{cases} B_k(q,p)[S(t)\psi]^{(pq)^k} t^{\frac{(1+q)[1-(pq)^k]}{1-pq}}, & \text{if } p \geq 1, \\ B_k(q,p)[S(t)\psi^p]^{\frac{(pq)^k}{p}} t^{\frac{(1+q)[1-(pq)^k]}{1-pq}}, & \text{if } p < 1, \end{cases} \tag{3.19}$$

where

$$\begin{aligned} B_k(p,q) =& (1+p)^{-\frac{1-(pq)^k}{1-pq}} \prod_{j=1}^{k-1} \left(\frac{1-pq}{1-(pq)^{j+1}}\right)^{(pq)^{k-j-1}} \\ &\cdot \{q(1+p)[1+pq+\cdots+(pq)^{j-1}]+1\}^{-\frac{(pq)^{k-j}}{q}}, \quad k = 1, 2, \cdots. \end{aligned}$$

Since

$$\begin{aligned} \ln B_k(p,q) =& -\frac{1-(pq)^k}{1-pq} \ln(1+p) + \sum_{j=1}^{k-1} (pq)^{k-j-1} \{\ln(1-pq) - \ln[1-(pq)^{j+1}]\} \\ &- \frac{1}{q} \sum_{j=1}^{k-1} (pq)^{k-j} \ln\{q(1+p)[1+pq+\cdots+(pq)^{j-1}]+1\} \\ \geq& -\frac{1}{1-pq} \ln(1+p) + \frac{1}{1-pq} \ln(1-pq) - \frac{1}{q(1-pq)} \ln\left(\frac{q(1+p)}{1-pq}+1\right), \end{aligned}$$

we have

$$B_k(p,q) \geq C(p,q) \overset{def}{=} (1-pq)^{\frac{1+q}{q(1-pq)}} / (1+p)^{\frac{1}{1-pq}} (1+q)^{\frac{1}{q(1-pq)}}, \quad k = 1, 2, \cdots.$$

Thus by replacing $B_k(p,q), B_k(q,p)$ with $C(p,q), C(q,p)$ respectively and then letting $k \to \infty$, we obtain (3.10) and (3.11)

Next we assume $pq = 1$. In the case $p = q = 1$, we get by direct comparison the following:

$$\begin{cases} u(t) \geq (\cosh t) S(t)\varphi + (\sinh t) S(t)\psi, \\ v(t) \geq (\sinh t) S(t)\varphi + (\cosh t) S(t)\psi, \end{cases}$$

and consequently,

$$u(t) \geq \frac{1}{2} e^t S(t)\varphi, \quad v(t) \geq \frac{1}{2} e^t S(t)\psi,$$

which implies (3.12) and (3.13). In the case $p < 1 < q$, we define

$$w(t) = \int_0^t S(t-\tau) u(\tau) d\tau. \tag{3.20}$$

Since

$$v(t) \geq S(t)\psi + \int_0^t S(t-\tau) u(\tau)^q d\tau, \tag{3.21}$$

we have

$$\begin{aligned} v(t)^p &\geq 2^{p-1}S(t)\psi^p + 2^{p-1}t^{p-1}\int_0^t S(t-\tau)u(\tau)^{pq}\,\mathrm{d}\tau \\ &= 2^{p-1}S(t)\psi^p + 2^{p-1}t^{p-1}w(t) \geq \frac{1}{2}(1+t)^{p-1}w(t). \end{aligned} \tag{3.22}$$

Taking this into (3.7) we get

$$u_t - \triangle u \geq \frac{1}{2}(1+t)^{p-1}w(t),$$

which combined with $w_t - \triangle w = u \geq \frac{1}{2}(1+t)^{p-1}u$ implies

$$(u+w)_t - \triangle(u+w) \geq \frac{1}{2}(1+t)^{p-1}(u+w).$$

Therefore,

$$u(t) + w(t) \geq \mathrm{e}^{\frac{1}{2p}(1+t)^p}S(t)\varphi \geq \mathrm{e}^{\frac{1}{2p}t^p}S(t)\varphi.$$

From this we obtain

$$\begin{aligned} (u\mathrm{e}^t)_t - \triangle(u\mathrm{e}^t) &= (u_t - \triangle u + u)\mathrm{e}^t \\ &\geq \tfrac{1}{2}(1+t)^{p-1}\mathrm{e}^t(u+w) \geq \tfrac{1}{2}(1+t)^{p-1}\mathrm{e}^{t+\frac{1}{2p}t^p}S(t)\varphi, \end{aligned}$$

and consequently,

$$u(t)\mathrm{e}^t \geq \frac{1}{2}\left(\int_0^t (1+\tau)^{p-1}\mathrm{e}^{\tau+\frac{1}{2p}\tau^p}\mathrm{d}\tau\right)S(t)\varphi \geq C(p,q)\mathrm{e}^{t+\frac{1}{2p}t^p}S(t)\varphi, \quad \forall t \geq 1,$$

or equivalently,

$$u(t) \geq C(p,q)\mathrm{e}^{\frac{1}{2p}t^p}S(t)\varphi, \quad \forall t \geq 1.$$

That is, (3.12) is valid. To prove (3.13) we first apply (3.12) in $w_t - \triangle w = u$ and get

$$w(t) \geq C(p,q)\left(\int_0^t \mathrm{e}^{\frac{1}{2p}\tau^p}\mathrm{d}\tau\right) \geq C'(p,q)t^{1-p}\mathrm{e}^{\frac{1}{2p}t^p}S(t)\varphi,$$

and then apply this result in (3.22). *Q. E. D.*

**Lemma 3.3** *Suppose that $\varphi$ is a admissible function and satisfies*

$$\varphi(x) \geq c\mathrm{e}^{-\alpha|x|^2}, \qquad \forall x \in R^N, \tag{3.23}$$

*for some positive constants $c$ and $\alpha$. If $u \in Z_\infty$ satisfies (3.1) and (3.2), then the following holds for all $t \geq 0$:*

$$u(x,t) \geq \begin{cases} C(1+t)^{-\frac{N}{2}r+1}\exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } 0 < r < 1+\frac{2}{N}, \\ C(1+t)^{-\frac{N}{2}}\ln(2+t)\exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } r = 1+\frac{2}{N}, \\ C(1+t)^{-\frac{N}{2}}\exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } r > 1+\frac{2}{N}, \end{cases} \tag{3.24}$$

*where $C$ and $\alpha'$ are positive constants depending only on $r$, $N$, $c$ and $\alpha$.*

The proof of this lemma is similar to that of [11, Lemma 4.4, (4.18)]. Thus we omit it here.

Before stating the next lemma we remind the reader to recall that $p$ and $q$ in (3.7)-(3.8) are assumed to satisfy $p \le q$.

**Lemma 3.4** *Suppose that $pq > 1$, $\varphi$ and $\psi$ are admissible and satisfy*

$$\varphi(x) \ge ce^{-\alpha|x|^2}, \quad \psi(x) \ge ce^{-\alpha|x|^2}, \quad \forall x \in R^N \tag{3.25}$$

*for some positive constants $c$ and $\alpha$. If $u, v \in Z_\infty$ satisfy (3.7)–(3.9), then the following estimates hold for all $t \ge 0$:*

$$u(x,t) \ge \begin{cases} C(1+t)^{-\frac{N}{2}p+1} \exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } 0 < p < 1 + \frac{2}{N}, \\ C(1+t)^{-\frac{N}{2}} \ln(2+t) \exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } p = 1 + \frac{2}{N}, \\ C(1+t)^{-\frac{N}{2}} \exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } p > 1 + \frac{2}{N}; \end{cases} \tag{3.26}$$

$$v(x,t) \ge \begin{cases} C(1+t)^{-\frac{N}{2}pq+q+1} \exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } \frac{N(pq-1)}{2(q+1)} < 1, \\ C(1+t)^{-\frac{N}{2}} \ln(2+t) \exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } \frac{N(pq-1)}{2(q+1)} = 1, \\ C(1+t)^{-\frac{N}{2}} \exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } \frac{N(pq-1)}{2(q+1)} > 1, \end{cases} \tag{3.27}$$

*where $\alpha'$ and $C$ are positive constants depending only on $p$, $q$, $c$, $\alpha$ and $N$.*

The proof of this Lemma is similar to that of [11, Lemma 4.4, (4.19)] and hence is omitted.

## III.2 The Proof of Theorem 1.3

We are now ready to prove Theorem 1.3. We use the method of reducing to contradiction. So we suppose that a global solution exists under the conditions of this theorem. Since $(\varphi, \psi) \neq (0,0)$ implies that for any $t_0 > 0$ there exist constants $c > 0$ and $\alpha > 0$ such that

$$u(x,t_0) \ge ce^{-\alpha|x|^2}, \quad v(x,t_0) \ge ce^{-\alpha|x|^2}, \quad \forall x \in R^N$$

(c.f. [11, Lemma 2.4]), by shifting the origin of the time variable if necessary we may assume that (3.25) is valid. Now, we first assume $r > 1$. Then from Lemma 3.1 we get

$$\|S(t)u(t)\|_\infty \le Ct^{-\frac{1}{r-1}}, \quad \forall t > 0. \tag{3.28}$$

In the following we use the same notation $C$ to denote various different positive constants. If the condition *(a)* is satisfied, namely, $1 < r \le 1 + 2/N$, then by Lemma 3.3 we have

$$u(x,t) \ge \begin{cases} C(1+t)^{-\frac{N}{2}r+1} \exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } 1 < r < 1 + \frac{2}{N}, \\ C(1+t)^{-\frac{N}{2}} \ln(2+t) \exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } r = 1 + \frac{2}{N}, \end{cases}$$

which implies

$$S(t)u(t)|_{x=0} \geq \begin{cases} C(1+t)^{-\frac{N}{2}r+1}, & \text{if } 1 < r < 1 + \frac{2}{N}, \\ C(1+t)^{-\frac{N}{2}} \ln(2+t), & \text{if } r = 1 + \frac{2}{N}. \end{cases} \tag{3.29}$$

One can easily verify that (3.29) is contrary to (3.28). If the condition *(b)* is satisfied, then in the case $1 < s \leq 1 + 2/N$ a similar deduction as we have just made with $u$ replaced by $v$ leads to absurdity, and in the case $0 < s \leq 1$ we have, by Lemma 3.1,

$$v(t) \geq \begin{cases} Ct^{\frac{1}{1-s}}, & \text{if } 0 < s < 1, \\ \mathrm{e}^t S(t)\psi, & \text{if } s = 1, \end{cases}$$

which implies

$$u(t) \geq \int_0^t S(t-\tau)v(\tau)^p \mathrm{d}\tau \geq \begin{cases} Ct^{1+\frac{p}{1-s}}, & \text{if } 0 < s < 1, \\ C(\mathrm{e}^{pt} - 1)S(t)\psi^p, & \text{if } s = 1 \text{ and } p < 1, \\ C(\mathrm{e}^{pt} - 1)[S(t)\psi]^p, & \text{if } s = 1 \text{ and } p \geq 1, \end{cases}$$

and consequently,

$$S(t)u(t) \geq \begin{cases} Ct^{1+\frac{p}{1-s}}, & \text{if } 0 < s < 1, \\ C(\mathrm{e}^{pt} - 1)S(t)^2\psi^p, & \text{if } s = 1, \ p < 1, \\ C(\mathrm{e}^{pt} - 1)[S(t)^2\psi]^p, & \text{if } s = 1, \ p \geq 1. \end{cases} \tag{3.30}$$

(3.30) is contrary to (3.28) because at best $S(t)^2\psi^p$ and $S(t)^2\psi$ decay to zero at polynomial speed as $t \to \infty$. If the condition *(c)* is satisfied, then in the case $pq > 1$ and $\frac{N(pq-1)}{2(q+1)} \leq 1$, a similar deduction as in the proof of [11, Lemma 4.4] shows the contradiction, and in the case $pq \leq 1$ we apply Lemma 3.2 and get

$$u(t) \geq \begin{cases} Ct^{\frac{1+p}{1-pq}}, & \text{if } pq < 1, \\ C\mathrm{e}^{\frac{1}{2p}t^p} S(t)\varphi, & \text{if } pq = 1, \end{cases}$$

which is also contrary to (3.28). If the condition *(d)* is satisfied, then by Lemma 3.4 we have

$$u(x,t) \geq \begin{cases} C(1+t)^{-\frac{N}{2}p+1} \exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } 0 < p < 1 + \frac{2}{N}, \\ C(1+t)^{-\frac{N}{2}} \ln(2+t) \exp\left(-\frac{\alpha'|x|^2}{1+t}\right), & \text{if } p = 1 + \frac{2}{N}, \end{cases}$$

which implies

$$S(t)u(t)|_{x=0} \geq \begin{cases} C(1+t)^{-\frac{N}{2}p+1}, & \text{if } 0 < p < 1 + \frac{2}{N}, \\ C(1+t)^{-\frac{N}{2}} \ln(2+t), & \text{if } p = 1 + \frac{2}{N}. \end{cases}$$

Again, we get a contradiction with (3.28).

The situations $s > 1$ and $pq > 1$ can be discussed similarly. We left it to the reader.

## IV. The Proof of Theorem 1.4

We will use the extension method to prove Theorem 1.4. That is, we will prove that under the condition (1.19) the solution $(u(t), v(t))$ corresponding to small initial datum is bounded on any bounded existence interval and thus its maximal existence interval is $[0, \infty)$.

Let us first impose an additional condition on $p$, $q$, $r$ and $s$. This condition is

$$r \geq \frac{p(q+1)}{p+1}, \qquad s \geq \frac{q(p+1)}{q+1}. \tag{4.1}$$

We now prove

**Theorem 4.1** *Suppose that in addition to (1.19), the condition (4.1) is also satisfied. Then there exists $\delta > 0$ such that for any admissible functions $\varphi \in L^{\frac{N(pq-1)}{2(p+1)}}(R^N)$ and $\psi \in L^{\frac{N(pq-1)}{2(q+1)}}(R^N)$ satisfying*

$$\|\varphi\|_{\frac{N(pq-1)}{2(p+1)}} + \|\varphi\|_\infty \leq \delta, \quad \|\psi\|_{\frac{N(pq-1)}{2(q+1)}} + \|\psi\|_\infty \leq \delta, \tag{4.2}$$

*the admissible solution $(u, v)$ of the problem (1.10), (1.11) and (1.14) is global. Moreover, $u$ and $v$ satisfy the following decay estimates:*

$$\|u(t)\|_\alpha \leq \begin{cases} C\delta(1+t)^{-(\frac{p+1}{pq-1}-\frac{N}{2\alpha})}, & \text{if } \frac{N(pq-1)}{2(p+1)} \leq \alpha < \alpha^* \text{ or} \\ \qquad \frac{N(pq-1)}{2(p+1)} \leq \alpha \leq \infty & \text{and } p(q+1)^2 < q(p+1)(pq-1), \\ C\delta(1+t)^{-\frac{p(q+1)}{q(p+1)}} \ln(2+t), & \text{if } \alpha = \alpha^* \text{and } p(q+1)^2 \geq q(p+1)(pq-1), \\ C\delta(1+t)^{-\frac{p(q+1)}{q(p+1)}}, & \text{if } \alpha > \alpha^* \text{ and } p(q+1)^2 \geq q(p+1)(pq-1), \end{cases} \tag{4.3}$$

$$\|v(t)\|_\beta \leq \begin{cases} C\delta(1+t)^{-(\frac{q+1}{pq-1}-\frac{N}{2\alpha})}, & \text{if } \frac{N(pq-1)}{2(q+1)} \leq \beta < \beta^* \text{ or} \\ \qquad \frac{N(pq-1)}{2(q+1)} \leq \beta \leq \infty & \text{and } p > 1 + \frac{2}{q}, \\ C\delta(1+t)^{-1} \ln(2+t), & \text{if } \beta = \beta^* \text{ and } p \leq 1 + \frac{2}{q}, \\ C\delta(1+t)^{-1}, & \text{if } \beta > \beta^* \text{ and } p \leq 1 + \frac{2}{q}, \end{cases} \tag{4.4}$$

*where*

$$\alpha^* = \begin{cases} \infty, & \text{if } p(q+1)^2 \leq q(p+1)(pq-1), \\ \frac{Nq(p+1)(pq-1)}{2[p(q+1)^2 - q(p+1)(pq-1)]}, & \text{if } p(q+1)^2 > q(p+1)(pq-1); \end{cases}$$

$$\beta^* = \begin{cases} \infty, & \text{if } p \geq 1 + \frac{2}{q}, \\ \frac{N(pq-1)}{2(q+2-pq)}, & \text{if } p < 1 + \frac{2}{q}. \end{cases}$$

*Remark.* We can also get decay estimates for the first order derivatives of $u$ and $v$ with respect to the space variable $x$. However, to shorten the discussion we shall not do so. The reader is refered to [10] for the method.

*Proof.* First we note that a similar discussion as in the proof of [10, Theorem 2.2] shows that for any $p > 0$, $q > 0$, $r > 0$ and $s > 0$ and any admissible $\varphi \in L^{\frac{N(pq-1)}{2(p+1)}}(R^N)$ and $\psi \in L^{\frac{N(pq-1)}{2(q+1)}}(R^N)$, there exists $\varepsilon > 0$ such that (1.10), (1.11) and (1.14) has a classical solution $(u,v)$ on $R^N \times [0,\varepsilon]$ with $u$ belonging to $C([0,\varepsilon]; L^{\frac{N(pq-1)}{2(p+1)}}(R^N) \cap L^\infty(R^N))$ and $v$ belonging to $C([0,\varepsilon]; L^{\frac{N(pq-1)}{2(q+1)}}(R^N) \cap L^\infty(R^N))$. Let $[0,T^*)$ be the maximal interval such that $(u,v)$ exists on $R^N \times [0,T^*)$ and for any $T \in (0,T^*)$,

$$u \in C([0,T]; L^{\frac{N(pq-1)}{2(p+1)}}(R^N) \cap L^\infty(R^N)), \qquad v \in C([0,T]; L^{\frac{N(pq-1)}{2(q+1)}}(R^N) \cap L^\infty(R^N)).$$

Then as well-known, we have either $T^* = \infty$ or $T^* < \infty$ and

$$\limsup_{t \to T^*-0} \left( \|u(t)\|_{\frac{N(pq-1)}{2(p+1)}} + \|u(t)\|_\infty + \|v(t)\|_{\frac{N(pq-1)}{2(q+1)}} + \|v(t)\|_\infty \right) = \infty.$$

We now prove that there exists $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0]$ and any $T \in (0,\infty)$, if $(u,v)$ exists on $R^N \times [0,T)$ and

$$u \in C([0,T_1]; L^{\frac{N(pq-1)}{2(p+1)}}(R^N) \cap L^\infty(R^N)), \qquad v \in C([0,T_1]; L^{\frac{N(pq-1)}{2(q+1)}}(R^N) \cap L^\infty(R^N)),$$

for any $T_1 \in (0,T)$, then the functions

$$f(t) = \|u(t)\|_{\frac{N(pq-1)}{2(p+1)}} + (1+t)^{\frac{1}{q}} \|u(t)\|_{\frac{Nq(pq-1)}{2(q+1)}} + \mu(t)\|u(t)\|_\infty$$

and

$$g(t) = \|v(t)\|_{\frac{N(pq-1)}{2(q+1)}} + (1+t)^{\frac{q+1}{q(p+1)}} \|v(t)\|_{\frac{Nq(p+1)(pq-1)}{2(q+1)^2}} + \nu(t)\|v(t)\|_\infty$$

are bounded on $[0,T)$, where

$$\mu(t) = \begin{cases} (1+t)^{\frac{p+1}{pq-1}}, & \text{if } \ p(q+1)^2 < q(p+1)(pq-1), \\ (1+t)^{\frac{p+1}{pq-1}}[\ln(2+t)]^{-1}, & \text{if } \ p(q+1)^2 = q(p+1)(pq-1), \\ (1+t)^{\frac{p(q+1)}{q(p+1)}}, & \text{if } \ p(q+1)^2 > q(p+1)(pq-1), \end{cases}$$

$$\nu(t) = \begin{cases} (1+t)^{\frac{q+1}{pq-1}}, & \text{if } \ p > 1 + \frac{2}{q}, \\ (1+t)[\ln(2+t)]^{-1}, & \text{if } \ p = 1 + \frac{2}{q}, \\ 1+t, & \text{if } \ p < 1 + \frac{2}{q}. \end{cases}$$

If this statement is proved, then $T^* = \infty$ and $(u,v)$ is global.

First, take $\alpha \geq 1$ such that

$$\frac{Nq(pq-1)}{2(pq^2+1)} < \alpha < \frac{Nq(p+1)(pq-1)}{2p(q+1)^2} \ \left( \leq \frac{N(pq-1)}{2(p+1)} \right).$$

Obviously, such $\alpha$ satisfies

$$\frac{N(pq-1)}{2(p+1)} < \alpha \cdot \frac{p(q+1)}{p+1} < \frac{Nq(pq-1)}{2(q+1)},$$

$$\frac{p(q+1)}{pq-1}-1<\frac{N}{2\alpha}<1+\frac{q+1}{q(pq-1)}<1+\frac{p+1}{pq-1},$$
$$\frac{N(pq-1)}{2(q+1)}<\alpha p<\frac{Nq(p+1)(pq-1)}{2(q+1)^2}.$$

Therefore, from (2.8) we get

$$\begin{aligned}\|u(t)\|_{\frac{N(pq-1)}{2(p+1)}}\leq&\|\varphi\|_{\frac{N(pq-1)}{2(p+1)}}+\int_0^t(t-\tau)^{-\frac{N}{2}\left(\frac{1}{\alpha}-\frac{2(p+1)}{N(pq-1)}\right)}\left(\|u(\tau)\|_{\alpha r}^r+\|v(\tau)\|_{\alpha p}^p\right)\mathrm{d}\tau\\
\leq&\delta+\int_0^t(t-\tau)^{-\left(\frac{N}{2\alpha}-\frac{p+1}{pq-1}\right)}\{\|u(\tau)\|_{\frac{N(pq-1)}{2(p+1)}}^{\frac{Nq}{2\alpha}-\frac{p(q+1)^2}{(p+1)(pq-1)}}\|u(\tau)\|_{\frac{Nq(pq-1)}{2(q+1)}}^{\frac{pq(q+1)}{pq-1}-\frac{Nq}{2\alpha}}\\
&\cdot\|u(\tau)\|_\infty^{r-\frac{p(q+1)}{p+1}}+\|v(\tau)\|_{\frac{N(pq-1)}{2(q+1)}}^{\frac{Nq(p+1)}{2\alpha(q+1)}-\frac{p(q+1)}{pq-1}}\|v(\tau)\|_{\frac{Nq(p+1)(pq-1)}{2(q+1)^2}}^{\frac{pq(p+1)}{pq-1}-\frac{Nq(p+1)}{2\alpha(q+1)}}\}\mathrm{d}\tau\qquad(4.5)\\
\leq&\delta+\left([\sup_{0\leq\tau\leq t}f(\tau)]^r+[\sup_{0\leq\tau\leq t}g(\tau)]^p\right)\\
&\cdot\int_0^t(t-\tau)^{-\left(\frac{N}{2\alpha}-\frac{p+1}{pq-1}\right)}(1+\tau)^{-\left(\frac{p(q+1)}{pq-1}-\frac{N}{2\alpha}\right)}\mathrm{d}\tau.\end{aligned}$$

Note that the term $\mu(\tau)^{-\left(r-\frac{p(q+1)}{p+1}\right)}$ has been omitted because it is not greater than 1. Now, as one can easily verify, for any $a<1$ and $b<1$ there exists a corresponding constant $C=C(a,b)>0$ such that

$$\int_0^t(t-\tau)^{-a}(1+\tau)^{-b}\mathrm{d}\tau\leq C(1+t)^{1-a-b}\qquad(4.6)$$

(c.f. [10, lemma 4.1]). Hence we have

$$\int_0^t(t-\tau)^{-\left(\frac{N}{2\alpha}-\frac{p+1}{pq-1}\right)}(1+\tau)^{-\left(\frac{p(q+1)}{pq-1}-\frac{N}{2\alpha}\right)}\mathrm{d}\tau\leq C.$$

Taking this into (4.5) we obtain

$$\|u(t)\|_{\frac{N(pq-1)}{2(p+1)}}\leq\delta+C\left([\sup_{0\leq\tau\leq t}f(\tau)]^r+[\sup_{0\leq\tau\leq t}g(\tau)]^p\right),\quad 0<t<T.\qquad(4.7)$$

Similarly,

$$\begin{aligned}\|u(t)\|_{\frac{Nq(pq-1)}{2(q+1)}}\leq&C(1+t)^{-\frac{N}{2}\left(\frac{2(p+1)}{N(pq-1)}-\frac{2(q+1)}{Nq(pq-1)}\right)}\left(\|\varphi\|_{\frac{N(pq-1)}{2(p+1)}}+\|\varphi\|_{\frac{Nq(pq-1)}{2(q+1)}}\right)\\
&+\int_0^t(t-\tau)^{\frac{N}{2}\left(\frac{1}{\alpha}-\frac{2(q+1)}{Nq(pq-1)}\right)}\left(\|u(\tau)\|_{\alpha r}^r+\|v(\tau)\|_{\alpha p}^p\right)\mathrm{d}\tau\\
\leq&C\delta(1+t)^{-\frac{1}{q}}+C\int_0^t(t-\tau)^{-\left(\frac{N}{2\alpha}-\frac{q+1}{q(pq-1)}\right)}(1+\tau)^{-\left(\frac{p(q+1)}{pq-1}-\frac{N}{2\alpha}\right)}\mathrm{d}\tau\\
&\cdot\left([\sup_{0\leq\tau\leq t}f(\tau)]^r+[\sup_{0\leq\tau\leq t}g(\tau)]^p\right)\\
\leq&C\delta(1+t)^{-\frac{1}{q}}+C(1+t)^{1-\left(\frac{N}{2\alpha}-\frac{q+1}{q(pq-1)}\right)-\left(\frac{p(q+1)}{pq-1}-\frac{N}{2\alpha}\right)}\\
&\cdot\left([\sup_{0\leq\tau\leq t}f(\tau)]^r+[\sup_{0\leq\tau\leq 0}g(\tau)]^p\right)\\
\leq&C\delta(1+t)^{-\frac{1}{q}}+C(1+t)^{-\frac{1}{q}}\left([\sup_{0\leq\tau\leq t}f(\tau)]^r+[\sup_{0\leq\tau\leq t}g(\tau)]^p\right),\quad 0<t<T,\end{aligned}$$

namely,

$$(1+t)^{\frac{1}{q}}\|u(t)\|_{\frac{Nq(pq-1)}{2(q+1)}} \le C\delta + C\left([\sup_{0\le\tau\le t} f(\tau)]^r + [\sup_{0\le\tau\le t} g(\tau)]^p\right), \quad 0<t<T. \tag{4.8}$$

Next we take $\beta \ge 1$ such that

$$\frac{N}{2}\Big/\left(1+\frac{(q+1)^2}{q(p+1)(pq-1)}\right) < \beta < \frac{N(pq-1)}{2(q+1)}.$$

Evidently, such $\beta$ satisfies

$$\frac{N(pq-1)}{2(q+1)} < \beta\cdot\frac{q(p+1)}{q+1} < \frac{Nq(p+1)(pq-1)}{2(q+1)^2},$$

$$\frac{N(pq-1)}{2(p+1)} < \beta q < \frac{Nq(pq-1)}{2(q+1)},$$

$$\frac{q(p+1)}{pq-1} - 1 < \frac{N}{2\beta} < 1 + \frac{(q+1)^2}{q(p+1)(pq-1)} < 1 + \frac{q+1}{pq-1}.$$

Therefore, from (2.9) we get

$$\begin{aligned}
\|v(t)\|_{\frac{N(pq-1)}{2(q+1)}} \le& \|\psi\|_{\frac{N(pq-1)}{2(q+1)}} + \textstyle\int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{1}{\beta}-\frac{2(q+1)}{N(pq-1)}\right)}\{\|u(\tau)\|_{\beta q}^q \\
&+ \|v(\tau)\|_{\frac{\beta q(p+1)}{q+1}}^{\frac{q(p+1)}{q+1}} \|v(\tau)\|_\infty^{s-\frac{q(p+1)}{q+1}}\}\mathrm{d}\tau \\
\le& \delta + \textstyle\int_0^t (t-\tau)^{-\left(\frac{N}{2\beta}-\frac{q+1}{pq-1}\right)}(1+\tau)^{-\left(\frac{q(p+1)}{pq-1}-\frac{N}{2\beta}\right)}\mathrm{d}\tau \\
&\cdot\left([\sup_{0\le\tau\le t} f(\tau)]^q + [\sup_{0\le\tau\le t} g(\tau)]^s\right) \\
\le& \delta + C\left([\sup_{0\le\tau\le t} f(\tau)]^q + [\sup_{0\le\tau\le t} g(\tau)]^s\right), \quad 0<t<T.
\end{aligned} \tag{4.9}$$

Similarly,

$$\begin{aligned}
\|v(t)\|_{\frac{Nq(p+1)(pq-1)}{2(q+1)^2}} \le& C(1+t)^{-\frac{N}{2}\left(\frac{2(q+1)}{N(pq-1)}-\frac{2(q+1)^2}{Nq(p+1)(pq-1)}\right)}\left(\|\psi\|_{\frac{N(pq-1)}{2(q+1)}} + \|\psi\|_{\frac{Nq(p+1)(pq-1)}{2(q+1)^2}}\right) \\
&+ C\textstyle\int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{1}{\beta}-\frac{2(q+1)^2}{Nq(p+1)(pq-1)}\right)}\left(\|u(\tau)\|_{\beta q}^q + \|v(\tau)\|_{\frac{\beta q(p+1)}{q+1}}^{\frac{q(p+1)}{q+1}}\|v(\tau)\|_\infty^{s-\frac{q(p+1)}{q+1}}\right)\mathrm{d}\tau \\
\le& C\delta(1+t)^{-\frac{q+1}{q(p+1)}} + C\textstyle\int_0^t (t-\tau)^{-\left(\frac{N}{2\beta}-\frac{(q+1)^2}{q(p+1)(pq-1)}\right)}(1+\tau)^{-\left(\frac{q(p+1)}{pq-1}-\frac{N}{2\beta}\right)}\mathrm{d}\tau \\
&\cdot\left([\sup_{0\le\tau\le t} f(\tau)]^q + [\sup_{0\le\tau\le t} g(\tau)]^s\right) \\
\le& C\delta(1+t)^{-\frac{q+1}{q(p+1)}} + C(1+t)^{-\frac{q+1}{q(p+1)}}\left([\sup_{0\le\tau\le t} f(\tau)]^q + [\sup_{0\le\tau\le t} g(\tau)]^s\right), \quad 0<t<T,
\end{aligned}$$

namely,

$$(1+t)^{\frac{q+1}{q(p+1)}}\|v(t)\|_{\frac{Nq(p+1)(pq-1)}{2(q+1)^2}} \le C\delta + C\left([\sup_{0\le\tau\le t} f(\tau)]^q + [\sup_{0\le\tau\le t} g(\tau)]^s\right), \quad 0<t<T. \tag{4.10}$$

To estimate $\|u(t)\|_\infty$, we should make a little variation to the method used above. Let us write

$$\begin{aligned}
\|u(t)\|_\infty \leq & C(1+t)^{-\frac{p+1}{pq-1}}\left(\|\varphi\|_{\frac{N(pq-1)}{2(p+1)}}+\|\varphi\|_\infty\right)\\
& +C\int_0^t(1+t+\tau)^{-\frac{p(q+1)^2}{q(p+1)(pq-1)}}\{(\|u(\tau)\|_{\frac{Nqr(p+1)(pq-1)}{2p(q+1)^2}}+\|u(\tau)\|_\infty)^r\\
& +(\|v(\tau)\|_{\frac{Nq(p+1)(pq-1)}{2(q+1)^2}}+\|v(\tau)\|_\infty)^p\}\mathrm{d}\tau\\
\leq & C\delta(1+t)^{-\frac{p+1}{pq-1}}+C\int_0^t(1+t-\tau)^{-\frac{p(q+1)^2}{q(p+1)(pq-1)}}\{\|u(\tau)\|^{\frac{p(q+1)}{p+1}}_{\frac{Nq(pq-1)}{2(q+1)}}\\
& \cdot\|u(\tau)\|_\infty^{r-\frac{p(q+1)}{p+1}}+\|u(\tau)\|_\infty^r+\|v(\tau)\|^p_{\frac{Nq(p+1)(pq-1)}{2(q+1)^2}}+\|v(\tau)\|_\infty^p\}\mathrm{d}\tau\\
\leq & C\delta(1+t)^{-\frac{p+1}{pq-1}}+C\left([\sup_{0\leq\tau\leq t}f(\tau)]^r+[\sup_{0\leq\tau\leq t}g(\tau)]^p\right)\\
& \cdot\int_0^t(1+t-\tau)^{-\frac{p(q+1)^2}{q(p+1)(pq-1)}}(1+\tau)^{-\frac{p(q+1)}{q(p+1)}}\mathrm{d}\tau
\end{aligned}\tag{4.11}$$

We have

$$\int_0^t(1+t-\tau)^{-a}(1+\tau)^{-b}\mathrm{d}\tau\leq\begin{cases}C(1+t)^{-b}, & \text{if } \ a>1,\ \ a\geq b,\\ C(1+t)^{-b}\ln(2+t), & \text{if } \ a=1,\ b\leq 1,\\ C(1+t)^{1-a-b}, & \text{if } \ a<1,\ \ b<1.\end{cases}$$

(c.f. [10, Lemma 4.1]). Therefore, (4.11) implies

$$\begin{aligned}
\|u(t)\|_\infty \leq & C\delta(1+t)^{-\frac{p+1}{pq-1}}+\left([\sup_{0\leq\tau\leq t}f(\tau)]^r+[\sup_{0\leq\tau\leq t}g(\tau)]^p\right)\\
& \cdot\begin{cases}C(1+t)^{-\frac{p(q+1)}{q(p+1)}}, & \text{if } \ p(q+1)^2>q(p+1)(pq-1)\\ C(1+t)^{-\frac{p(q+1)}{q(p+1)}}\ln(2+t), & \text{if } \ p(q+1)^2=q(p+1)(pq-1)\\ C(1+t)^{-\frac{p+1}{pq-1}}, & \text{if } \ p(q+1)^2<q(p+1)(pq-1)\end{cases}\\
\leq & C\mu(t)^{-1}\{\delta+[\sup_{0\leq\tau\leq t}f(\tau)]^r+[\sup_{0\leq\tau\leq t}g(\tau)]^p\},\quad 0<t<T,
\end{aligned}$$

namely,

$$\mu(t)\|u(t)\|_\infty\leq C\delta+C\left([\sup_{0\leq\tau\leq t}f(\tau)]^r+[\sup_{0\leq\tau\leq t}g(\tau)]^p\right),\quad 0<t<T.\tag{4.12}$$

To estimate $\|v(t)\|_\infty$, we take $\gamma\geq 1$ such that

$$\max\left(\frac{N}{2},\frac{N(pq-1)}{2q(p+1)}\right)<\gamma<\frac{N(pq-1)}{2(q+1)},\quad \text{if } \ p>1+\frac{2}{q};$$

$$\gamma=\frac{N(pq-1)}{2(q+1)},\quad \text{if } \ p\leq 1+\frac{2}{q}.$$

Then we have

$$\begin{aligned}\|v(t)\|_\infty \le & C(1+t)^{-\frac{q+1}{pq-1}}\left(\|\psi\|_{\frac{N(pq-1)}{2(q+1)}}+\|\psi\|_\infty\right)+C\int_0^t(1+t-\tau)^{-\frac{N}{2\gamma}}\\ & \cdot\{(\|u(\tau)\|_{\gamma q}+\|u(\tau)\|_\infty)^q+(\|v(\tau)\|_{\gamma s}+\|v(\tau)\|_\infty)^s\}\mathrm{d}\tau\\ \le & C\delta(1+t)^{-\frac{q+1}{pq-1}}+C\left([\sup_{0\le\tau\le t}f(\tau)]^q+[\sup_{0\le\tau\le t}g(\tau)]^s\right)\\ & \cdot\int_0^t(1+t-\tau)^{-\frac{N}{2\gamma}}(1+\tau)^{-\left(\frac{p+1}{pq-1}-\frac{N}{2\gamma}\right)}\mathrm{d}\tau\\ \le & C\nu(t)^{-1}\left(\delta+[\sup_{0\le\tau\le t}f(\tau)]^q+[\sup_{0\le\tau\le t}g(\tau)]^s\right),\quad 0<t<T,\end{aligned}$$

namely,

$$\nu(t)\|v(t)\|_\infty\le C\delta+C\left([\sup_{0\le\tau\le t}f(\tau)]^q+[\sup_{0\le\tau\le t}g(\tau)]^s\right),\quad 0<t<T. \tag{4.13}$$

From(4.7)-(4.10), (4.12) and (4.13) we finally get

$$f(t)\le C\delta+C\left([\sup_{0\le\tau\le t}f(\tau)]^r+[\sup_{0\le\tau\le t}g(\tau)]^p\right), \tag{4.14}$$

$$g(t)\le C\delta+C\left([\sup_{0\le\tau\le t}f(\tau)]^q+[\sup_{0\le\tau\le t}g(\tau)]^s\right), \tag{4.15}$$

for all $t\in[0,T)$, where $C$ is a positive constant depending only on $p,\ q,\ r,\ s,\ N$ and the choice of $\alpha,\ \beta$ and $\gamma$. We now demonstrate from the above estimates that there exists $\delta_0>0$ such that $f(t)$ and $g(t)$ are bounded from above when $0<\delta\le\delta_0$. To do so we write $h(t)=f(t)+g(t)$ and prove that such assertion is valid for $h(t)$. We first assume that $p>1$. Then by the hypothesis we see that $p,\ q,\ r$ and $s$ are all greater than one. Summing (4.14) and (4.15) up we obtain

$$h(t)\le 2C\delta+C\left([\sup_{0\le\tau\le t}h(\tau)]^r+[\sup_{0\le\tau\le t}h(\tau)]^p+[\sup_{0\le\tau\le t}h(\tau)]^q+[\sup_{0\le\tau\le t}h(\tau)]^s\right),\quad \forall t\in[0,T). \tag{4.16}$$

This shows that the curve $\xi=\xi(t)\equiv\sup_{0\le\tau\le t}h(\tau)$ $(0\le t<T)$ lies in the region

$$J_\delta=\{\xi\ge0:\ C(\xi^r+\xi^p+\xi^q+\xi^s)+2C\delta-\xi\ge0\},$$

where $C$ is the same constant as appearing in (4.16). Without loss of generality we may assume that $C\ge2$, for else we may substitute $C$ with 2. Since $r>1$, $p>1$, $q>1$ and $s>1$, there exists $\delta_0>0$ such that for $0<\delta\le\delta_0$, $J_\delta$ is disconnected, consisting of two disjoint intervals, with the left interval of the form $[0,A(\delta)]$ ($A(\delta)$ =the smaller positive root of the function $F_\delta(\xi)=C(\xi^r+\xi^p+\xi^q+\xi^q+\xi^s)+2C\delta-\xi$) and the right interval of the form $[B(\delta),\infty)$ ($B(\delta)$ =the larger positive root of the function $F_\delta(\xi)$). We note that $A(\delta)\ge2C\delta$ because $F_\delta(\xi)\ge0$ for all $0\le\xi\le2C\delta$. Since the curve $\xi=\xi(t)$ $(0\le t<T)$ is continuous (which is guaranteed by the continuity of $(u(t),v(t))$) and $\xi(0)\le4\delta\le2C\delta\le A(\delta)$, we conclude that $\xi(t)\le A(\delta)$ for all $0\le t<T$. This proves the assertion in the case $p>1$. Next we assume that $p\le1$. By repeatedly using (4.15) $n$ times, we obtain from (4.14),

$$f(t)\le C\left(\delta+\sum_{k=0}^{n-1}\delta^{ps^k}\right)+C\left([\sup_{0\le\tau\le t}f(\tau)]^r+\sum_{k=0}^{n-1}[\sup_{0\le\tau\le t}f(\tau)]^{pqs^k}+[\sup_{0\le\tau\le t}g(\tau)]^{ps^n}\right). \tag{4.17}$$

Since $pq > 1$ and $s > 1$, we see that $pqs^k > 1$ for all $k = 0, 1, \cdots, n-1$ and $ps^n > 1$ when $n$ is sufficiently large. Similarly, by using (4.14) we obtain from (4.15),

$$g(t) \le C(\delta + \delta^q) + \left( [\sup_{0\le\tau\le t} f(\tau)]^{qr} + [\sup_{0\le\tau\le t} g(\tau)]^{pq} + [\sup_{0\le\tau\le t} g(\tau)]^{s} \right). \tag{4.18}$$

We have $qr > 1$, $pq > 1$ and $s > 1$. Now by summing (4.17) and (4.18) up and making a similar deduction as we have made in the case $p > 1$, we get still the desired assertion in the present case.

In summary, we have proved that there exists $\delta_0 > 0$ such that for $0 < \delta \le \delta_0$,

$$f(t) \le h(t) \le A(\delta), \quad g(t) \le h(t) \le A(\delta), \quad \forall t \in [0, T), \tag{4.19}$$

where $A(\delta)$ is a positive constant independent of $T$. Hence the statement is proved. Consequently, if $0 < \delta \le \delta_0$ and $\varphi$ and $\psi$ satisfy (4.2), then the solution $(u, v)$ of the problem (1.10), (1.11) and (1.14) is global. Moreover, since $A(\delta)$ is independent of $T$, we see that (4.19) is valid for all $t \in [0, \infty)$. From this conclusion one can easily obtain (4.3) and (4.4) by deducing similarly as we have done in getting (4.5)–(4.13). The proof of Theorem 4.1 is finished. *Q. E. D.*

We note that the condition (4.2) implies that $\varphi(x)$ (resp. $\psi(x)$) decays to zero as $|x| \to \infty$ at speed not slower than $(1+|x|)^{-\frac{2(p+1)}{pq-1}}$ (resp. $(1+|x|)^{-\frac{2(q+1)}{pq-1}}$). The following theorem shows that $\frac{2(p+1)}{pq-1}$ and $\frac{2(q+1)}{pq-1}$ are exactly the critical values.

**Theorem 4.2** *Suppose that*

$$\text{either} \qquad \varphi(x) \ge c(1+|x|^2)^{-\alpha} \qquad \text{or} \qquad \psi(x) \ge c(1+|x|^2)^{-\beta}, \tag{4.20}$$

*where $c > 0$, $0 < \alpha < \frac{p+1}{pq-1}$ and $0 < \beta < \frac{q+1}{pq-1}$. Then the admissible solution $(u, v)$ of the problem (1.10), (1.11) and (1.14) blows up at finite time.*

*Proof.* Suppose that the first inequality in (4.20) holds. If $(u, v)$ is global, then we have

$$\begin{aligned} u(t) &\ge S(t)\varphi \\ &\ge c(4\pi t)^{-\frac{N}{2}} \int_{R^N} (1+|x-\xi|^2)^{-\alpha} \mathrm{e}^{-\frac{|\xi|^2}{4t}} \mathrm{d}\xi \\ &\ge C \int_{R^N} (1+|x|^2+4t|\xi|^2)^{-\alpha} \mathrm{e}^{-|\xi|^2} \mathrm{d}\xi \\ &\ge C(1+4t+|x|^2)^{-\alpha} \int_{R^N} (1+|\xi|^2)^{-\alpha} \mathrm{e}^{-|\xi|^2} \mathrm{d}\xi \\ &\ge C(1+4t+|x|^2)^{-\alpha}, \qquad \forall (x,t) \in R^N \times [0,\infty), \end{aligned}$$

and consequently,

$$\begin{aligned} S(t)u(t) &\ge C(4\pi t)^{-\frac{N}{2}} \int_{R^N} (1+4t+|x-\xi|^2)^{-\alpha} \mathrm{e}^{-\frac{|\xi|^2}{4t}} \mathrm{d}\xi \\ &\ge C \int_{R^N} (1+4t+|x|^2+4t|\xi|^2)^{-\alpha} \mathrm{e}^{-|\xi|^2} \mathrm{d}\xi \\ &\ge C(1+4t+|x|^2)^{-\alpha} \int_{R^N} (1+|\xi|^2)^{-\alpha} \mathrm{e}^{-|\xi|^2} \mathrm{d}\xi \\ &\ge C(1+4t+|x|^2)^{-\alpha}, \qquad \forall (x,t) \in R^N \times [0,\infty). \end{aligned}$$

Since $-\alpha > -\frac{p+1}{pq-1}$, the last result is contrary to (3.14). Similarly, if the second inequality in (4.20) holds, then we shall get a contradiction to (3.16) (if $p \geq 1$) or (3.17) (if $p < 1$). *Q. E. D.*

To finish the proof of Theorem 1.4, there still remains the case

$$s < \frac{q(p+1)}{q+1} \quad \text{or} \quad r < \frac{p(q+1)}{p+1}$$

to be considered. Let us now tackle this case. If $p > 1+\frac{2}{N}$, then $\min(p,q,r,s) > 1+\frac{2}{N}$, and we can apply [10, Theorem 2.1] to get the desired conclusion. So we now suppose that $p \leq 1+\frac{2}{N}$. Then the total condition that remains is the following:

$$\begin{aligned} &r > 1+\tfrac{2}{N}, \;\; s > 1+\tfrac{2}{N}, \;\; \tfrac{N(pq-1)}{2(q+1)} > 1, \;\; p \leq 1+\tfrac{2}{N}, \;\; r > \tfrac{Np}{Np-2} \\ &\text{and either} \quad r < \tfrac{p(q+1)}{p+1} \quad \text{or} \quad s < \tfrac{q(p+1)}{q+1}. \end{aligned} \tag{4.21}$$

Let $q_0 = \frac{N+2}{Np-2}$, i.e., the number such that $\frac{N(pq_0-1)}{2(q_0+1)} = 1$. Since $\frac{N(pq-1)}{2(q+1)} > 1$, $s > 1+\frac{2}{N} = \frac{q_0(p+1)}{q_0+1}$ and $r > \frac{Np}{Np-2} = \frac{p(q_0+1)}{p+1}$, we can find another number $q' \in (q_0, q)$ such that

$$\frac{N(pq'-1)}{2(q'+1)} > 1, \quad r \geq \frac{p(q'+1)}{p+1} \quad \text{and} \quad s \geq \frac{q'(p+1)}{q'+1}. \tag{4.22}$$

Now rewrite the problem (1.10), (1.11) and (1.14) as follows:

$$u_t - \triangle u = u^r + v^p, \quad \text{in } R^N \times (0,\infty), \tag{4.23}$$

$$v_t - \triangle v = u^{q'} \cdot u^{q-q'} + v^s, \quad \text{in } R^N \times (0,\infty), \tag{4.24}$$

$$u|_{t=0} = \varphi, \quad v|_{t=0} = \psi, \quad \text{on } R^N. \tag{4.25}$$

By using the same method as in the proof of Theorem 4.1, we can derive from (4.23)–(4.25) the following

**Theorem 4.3** *Suppose that the condition (4.21) is satisfied. Let $q' \in (q_0, q)$ be the maximal number such that (4.22) is valid. Then there exists $\delta > 0$ such that for any admissible functions $\varphi \in L^{\frac{N(pq'-1)}{2(p+1)}}(R^N)$ and $\psi \in L^{\frac{N(pq'-1)}{2(q'+1)}}(R^N)$ satisfying*

$$\|\varphi\|_{\frac{N(pq'-1)}{2(p+1)}} + \|\varphi\|_\infty \leq \delta, \quad \|\psi\|_{\frac{N(pq'-1)}{2(q'+1)}} + \|\psi\|_\infty \leq \delta, \tag{4.26}$$

*the admissible solution $(u,v)$ of the problem (1.10), (1.11) and (1.14) is global. Moreover, the decay estimates (4.3) and (4.4) are both valid with $q$ replaced by $q'$.*

Indeed, since $q - q' > 0$, the term $u^{q-q'}$ in (4.22) can always be substituted with $\|u\|_\infty^{q-q'}$ when we make estimates for $v$. Obviously, it is needless to write out the details.

Note that the maximumness of $q'$ implies that at least one of the two equalities $r = \frac{p(q'+1)}{p+1}$ and $s = \frac{q'(p+1)}{q'+1}$ is valid. Therefore, by using the same method as in the proof of Theorem 4.2 and applying Lemma 3.1 instead of Lemma 3.2 we get

**Theorem 4.4** *Let $q'$ be as above. Suppose that either*

$$r = \frac{p(q'+1)}{p+1} \quad \textit{and} \quad \varphi(x) \geq c(1+|x|^2)^{-\alpha}$$

*or*

$$s = \frac{q'(p+1)}{q'+1} \quad \textit{and} \quad \psi(x) \geq c(1+|x|^2)^{-\beta},$$

*where* $c > 0$, $0 < \alpha < \frac{p+1}{pq'-1}$ *and* $0 < \beta < \frac{q'+1}{pq'-1}$. *Then the admissible solution* $(u, v)$ *of the problem (1.10), (1.11) and (1.14) blows up at finite time.*

The proof is left to the reader.

By now, we have completed the proof of Theorem 1.4.

*Acknowledgement*—The author is glad to express his sincere thanks to the referee for his valuable suggestion on modifying this paper.

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# GLOBAL BEHAVIOR OF SOLUTIONS TO A REACTION-DIFFUSION SYSTEM

S. B. Cui(Cui Shangbin)

Department of Mathematics, Lanzhou University,
Lanzhou, Gansu 730000, People's Republic of China

**Abstract**

In this paper the global behavior of solutions to the semilinear reaction diffusion system

$$\begin{cases} u_t - \triangle u = u^r + v^p, & \text{in } R^N \times (0, \infty) \\ v_t - \triangle v = u^q + v^s, & \text{in } R^N \times (0, \infty) \end{cases}$$

is studied, where $p$, $q$, $r$ and $s$ are positive constants. It is proved that under certain conditions the solution is global and when those conditions are not satisfied the solution blows up at finite time.

**Key words and phrases**: Reaction-diffusion system, solution, global behavior, existence, blow-up.

**AMS 1991 subject classification number**: 35K45.