# Exact Controllability of a Thermoelastic System with Control in the Thermal Component Only

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**Abstract**

In this work we give a result of exact controllability for a thermoelastic system in which the control term is placed solely in the thermal equation. With such an indirect control input, one is able to control exactly the displacement of the plate, as well as the temperature. This exact controllability occurs in arbitrarily small time. In the case that the moment of inertia parameter for the plate is absent (i.e., $\gamma = 0$ below), then one is provided here with a result of exact controllability for a thermoelastic system which is modelled by the generator of an *analytic* semigroup. The proof here depends upon a multiplier method so as to attain the associated observability inequality. The particular multiplier invoked is of an operator theoretic nature, and has been used previously by the author in deriving stability results for this pde model.

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# 1 Introduction

## 1.1 Statement of the Problem

Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$ with sufficiently smooth boundary $\Gamma$, and $T > 0$. In this work, we will study the exact controllability problem for the following thermoelastic system, with the control function $\overline{u} \in [L^2(0,T;L^2(\Omega))]^2$:

$$\begin{cases} \omega_{tt} - \gamma\Delta\omega_{tt} + \Delta^2\omega + \alpha\Delta\theta = 0 \\ \\ \theta_t - \Delta\theta + \sigma\theta - \alpha\Delta\omega_t = div(\overline{u}) \end{cases} \quad \text{on } (0,T)\times\Omega;$$

$$\omega = \frac{\partial\omega}{\partial\nu} = 0 \quad \text{on } (0,\infty)\times\Gamma; \tag{1}$$

$$\theta = 0 \qquad \text{on } (0,\infty)\times\Gamma;$$

$$\omega(t=0) = \omega_0,\ \omega_t(t=0) = \omega_1,\ \theta(t=0) = \theta_0 \text{ on } \Omega.$$

Here, the coupling parameter $\alpha > 0$; the nonnegative constant $\gamma$ is proportional to the thickness of the plate and assumed to be small with $0 \le \gamma \le M$; the constant $\sigma$ is also nonnegative. There are other physical constants associated with system, but they have been set here to unity for the sake of simplicity. The operator $div$ denotes the divergence of the vector field $\overline{u}(x,y) = [u_1(x,y), u_2(x,y)]$; i.e., $div(\overline{u}) = \dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y}$. As usual, $[\nu_1,\nu_2]$ is the unit normal outward to the boundary. The pde model (1), without the given interior control, is derived in [10], and mathematically describes a Kirchoff plate subjected to a thermal damping. The displacement of the plate is represented by the function $\omega$, and the temperature is denoted by the function $\theta$. The control term $div(\overline{u})$ models a radiative energy flux acting through the volume of the plate.

Defining the space $H^1_{0,\gamma}(\Omega)$ to be

$$H^1_{0,\gamma}(\Omega) = \begin{cases} H^1_0(\Omega) & \text{if } \gamma > 0 \\ \\ L^2(\Omega) & \text{if } \gamma = 0, \end{cases} \tag{2}$$

one can show well–posedness of the uncontrolled thermoelastic system (i.e. $\overline{u} = 0$ in (1)) for initial data $[\omega_0,\omega_1,\theta_0] \in H^2_0(\Omega)\times H^1_{0,\gamma}(\Omega)\times L^2(\Omega)$ (see **Proposition 2.1** below). In general, for given $\overline{u} \in [L^2(0,T;L^2(\Omega)]^2$, the corresponding solution $[\omega,\omega_t,\theta]$ is *a priori* continuous in time into the larger space $\left[D(\mathcal{A}^*_\gamma)\right]'$ (larger with respect to $H^2_0(\Omega)\times H^1_{0,\gamma}(\Omega)\times L^2(\Omega)$), this dual space being defined below in (28). We note here at the outset the dichotomy presented by the parameter $\gamma$: With $\gamma > 0$, the system (1) is *hyperbolic–like*; when $\gamma = 0$, the system is modelled by the generator of an *analytic semigroup*, and so corresponds to parabolic–like dynamics (see [18] and [15]). With the basic space of well–posedness established, we are concerned with the following question of exact controllability on a given time interval $[0, T]$: For data $[\omega_0, \omega_1, \theta_0]$ (initial) and $\left[\omega_0^T, \omega_1^T, \theta_0^T\right]$ (terminal) in $H_0^2(\Omega) \times H_{0,\gamma}^1(\Omega) \times L^2(\Omega)$, is there a suitable control $\overline{u} \in \left[L^2(0, T; L^2(\Omega)\right]^2$ such that the corresponding solution $[\omega, \omega_t, \theta]$ to (1) satisfies at terminal time $T$,

$$[\omega(T), \omega_t(T), \theta(T)] = \left[\omega_0^T, \omega_1^T, \theta_0^T\right] \ ? \tag{3}$$

Controllability properties for this system have been much studied of late, under varying boundary conditions for the displacement, and with different choices of controls. The controllability of the system (1) is initially considered by J. Lagnese in [11], with control being implemented in the boundary conditions for $\omega$ (in this work, free boundary conditions are imposed, instead of the hinged ones in place here). With such a boundary–controlled thermoelastic system, a result of *partial exact controllability* is obtained for $\gamma > 0$ (the hyperbolic case); that is to say, the displacement $\omega$ is exactly controlled, provided the coupling parameter $\alpha$ is small enough. In a more recent work, valid for $\gamma > 0$ (the hyperbolic case), L. de Teresa and E. Zuazua in [6] derive a result of *exact controllability* for the displacement $\omega$ and *approximate controllability* for the temperature $\theta$, in the case that interior control is implemented in the Kirchoff component of (1). In each of these works, given that the control term is acting strictly on the plate component of the dynamics, and that $\gamma > 0$, critical use is made of controllability results for the uncoupled Kirchoff plate so as to eventually treat the system (1) as a perturbation of a Kirchoff plate. Later still, S. Hansen and B. Zhang in [8] study a one–dimensional version of (1) under the influence of a control at one of the boundary conditions for $\omega$. With such a single scalar control in place, they are able to obtain a result of *exact null controllability*; i.e., $[\omega, \omega_t, \theta]$ can be driven to zero at time $T$; this result holds for all $\gamma \geq 0$.

Here, we address the aforementioned question of exact controllability for *both* the displacement and the temperature, and in both the analytic and nonanalytic cases. Our main result in that direction is as follows:

**Theorem 1.1** *For all $\gamma \geq 0$, the system (1) is exactly controllable in arbitrary time $T > 0$. That is to say, for any $T > 0$, and data $[\omega_0, \omega_1, \theta_0], \left[\omega_0^T, \omega_1^T, \theta_0^T\right]$ in the space $H_0^2(\Omega) \times H_{0,\gamma}^1(\Omega) \times L^2(\Omega)$, one can find a control function $\overline{u} \in \left[L^2(0, T; L^2(\Omega)\right]^2$ such that the corresponding solution $[\omega, \omega_t, \theta]$ to (1) satisfies $[\omega(T), \omega_t(T), \theta(T)] = \left[\omega_0^T, \omega_1^T, \theta_0^T\right]$.*

The novelties inherent in this theorem are the following:

*(1)* **Theorem 1.1** states that the displacement of the plate $\omega$ can be controlled exactly by the indirect means of placing the control input term $div\overline{u}$ in the thermal component. Note that since the control term is in the heat equation, the proof of

exact controllability will *not* hinge on perturbation arguments which exploit known controllability results for Kirchoff plates in the case that $\gamma > 0$, or Euler–Bernoulli beams in the case that $\gamma = 0$. The proof of **Theorem 1.1** is necessarily direct.

*(2)* In the case that $\gamma = 0$, it has recently been demonstrated in [15] that the thermoelastic system (1), under all possible boundary conditions for $\omega$, is abstractly modelled by the generator of an analytic semigroup (see (23) and (26) below). Therefore, **Theorem 1.1** constitutes a result of exact controllability for an analytic system in the case that $\gamma = 0$. (It is well–known that exact controllability results for analytic systems are hard to come by. See [5] and [20] for statements of some sufficient conditions for the approximate and exact controllability of analytic systems.) In this respect, our work here complements that recently completed by I. Lasiecka and R. Triggiani in [16], which gives results of exact null controllability for the thermoelastic model (1) in the (analytic) case that $\gamma = 0$, under the influence of either mechanical or thermal control (as we said earlier, the aforementioned paper [8] also recovers the null controllability of a one–dimensional version of (1) for $\gamma = 0$).

The methodology employed in the proving of **Theorem 1.1** is based upon the classical argument of showing the ontoness of the *control* $\rightarrow$ *terminal state* map $\mathcal{L}_T$ (see (25) below for the explicit description of $\mathcal{L}_T$). Establishing the surjectivity for $\mathcal{L}_T$ is in turn tantamount to deriving the following (observability) inequality for some $C_T > 0$:

$$\int_0^T \|\nabla\psi\|^2_{L^2(\Omega)} \geq C_T \,\|[\phi_0, \phi_1, \psi_0]\|^2_{H_0^2(\Omega)\times H^1_{0,\gamma}(\Omega)\times L^2(\Omega)}\,, \tag{4}$$

where $\psi$ is the thermal component of the solution $[\phi, \phi_t, \psi]$ to the following backwards thermoelastic system, adjoint with respect to (1):

$$\begin{cases} \phi_{tt} - \gamma\Delta\phi_{tt} + \Delta^2\phi + \alpha\Delta\psi = 0 \\ \\ \psi_t + \Delta\psi - \sigma\psi - \alpha\Delta\phi_t = 0 \end{cases} \quad \text{on } (0,T)\times\Omega;$$

$$\phi = \frac{\partial\phi}{\partial\nu} = 0 \;\; \text{on } (0,T)\times\Gamma;$$

$$\psi = 0 \qquad \text{on } (0,T)\times\Gamma;$$

$$\phi(T) = \phi_0,\; \phi_t(T) = \phi_1,\; \psi(T) = \psi_0 \text{ on } \Omega.$$

A multiplier technique is invoked here to attain the inequality (4) (see [9] for a treatise of the multiplier method), with the chosen multiplier being of an operator theoretic nature. In fact, the critical multiplier is $A_D^{-1}\theta$, where the operator $A_D$ denotes the Laplacian with Dirichlet boundary conditions (see (9) below). This particular multiplier has also seen service in [1], [3] and [4], works which are concerned with ascertaining stability properties of linear and nonlinear variations of (1).

## 2 Abstract Operator Formulation and Analysis

In our proof of controllability (**Theorem 1.1**), the system (1) and its adjoint ((33) below) will be considered as abstract evolution equations in a certain Hilbert space. To develop these operator models, we must introduce the following definitions and notations.

- We define the operator $\mathring{\mathbf{A}}$: $L^2(\Omega) \supset D\left(\mathring{\mathbf{A}}\right) \to L^2(\Omega)$ to be $\mathring{\mathbf{A}} = \Delta^2$, with domain

$$D(\mathring{\mathbf{A}}) = H^4(\Omega) \cap H_0^2(\Omega). \tag{5}$$

- $\mathring{\mathbf{A}}$ is then positive definite, self–adjoint, and consequently from [7] we have the characterizations

$$\begin{aligned} D(\mathring{\mathbf{A}}^{\frac{1}{4}}) &= H_0^1(\Omega); \\ D(\mathring{\mathbf{A}}^{\frac{1}{2}}) &= H_0^2(\Omega); \\ D(\mathring{\mathbf{A}}^{\frac{3}{4}}) &= H^3(\Omega) \cap H_0^2(\Omega). \end{aligned} \tag{6}$$

  In particular, the second characterization in (6) and Green's formula give that for all $\varpi$, $\varpi' \in D(\mathring{\mathbf{A}}^{\frac{1}{2}})$,

$$\left\langle \mathring{\mathbf{A}}\varpi, \varpi' \right\rangle_{\left[D(\mathring{\mathbf{A}}^{\frac{1}{2}})\right]' \times D(\mathring{\mathbf{A}}^{\frac{1}{2}})} = \left(\mathring{\mathbf{A}}^{\frac{1}{2}}\varpi, \mathring{\mathbf{A}}^{\frac{1}{2}}\varpi'\right)_{L^2(\Omega)} = (\Delta\varpi, \Delta\varpi')_{L^2(\Omega)}; \tag{7}$$

  and additionally,

$$\|\varpi\|^2_{D(\mathring{\mathbf{A}}^{\frac{1}{2}})} = \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\varpi\right\|^2_{L^2(\Omega)} = \|\Delta\varpi\|^2_{L^2(\Omega)}. \tag{8}$$

- We define $A_D : L^2(\Omega) \supset D(A_D) \to L^2(\Omega)$ to be $A_D = -\Delta$, with Dirichlet boundary conditions, viz.

$$D(A_D) = H^2(\Omega) \cap H_0^1(\Omega). \tag{9}$$

  $A_D$ is also positive definite, self–adjoint, and by [7]

$$D(A_D^{\frac{1}{2}}) = H_0^1(\Omega). \tag{10}$$

  Moreover, this characterization and Green's Theorem give that for all $\vartheta$, $\vartheta' \in D(A_D^{\frac{1}{2}})$

$$\langle A_D\vartheta, \vartheta' \rangle_{[D(A_D^{\frac{1}{2}})]' \times D(A_D^{\frac{1}{2}})} = \left(A_D^{\frac{1}{2}}\vartheta, A_D^{\frac{1}{2}}\vartheta'\right)_{L^2(\Omega)} = (\nabla\vartheta, \nabla\vartheta')_{[L^2(\Omega)]^2}, \tag{11}$$

and

$$\|\vartheta\|^2_{D(A_D^{\frac{1}{2}})} = \left\|A_D^{\frac{1}{2}}\vartheta\right\|^2_{L^2(\Omega)} = \|\nabla\vartheta\|^2_{[L^2(\Omega)]^2} . \tag{12}$$

- For $\gamma \geq 0$, we define the operator $P_\gamma$ by

$$P_\gamma \equiv \mathbf{I} \; + \gamma A_D, \tag{13}$$

and here consider two cases:

(i) In the case that the parameter $\gamma > 0$, we define a space $H^1_{0,\gamma}(\Omega)$ equivalent to $H^1_0(\Omega)$ with its inner product being defined as

$$(\, \varpi, \, \varpi')_{H^1_{0,\gamma}(\Omega)} \equiv (\varpi, \varpi')_{L^2(\Omega)} + \gamma\,(\nabla\varpi, \nabla\varpi')_{L^2(\Omega)} \quad \forall\, \varpi, \varpi' \in H^1_0(\Omega), \tag{14}$$

and with its dual denoted as $H^{-1}_{0,\gamma}(\Omega)$. The characterization (10), (11) and two extensions by continuity will then yield that

$$P_\gamma \in \mathcal{L}\left(H^1_{0,\gamma}(\Omega), H^{-1}_{0,\gamma}(\Omega)\right), \text{ with} \tag{15}$$

$$\langle P_\gamma\omega_1, \omega_2\rangle_{H^{-1}_{0,\gamma}(\Omega)\times H^1_{\Gamma_0,\gamma}(\Omega)} = (\omega_1, \omega_2)_{H^1_{0,\gamma}(\Omega)} . \tag{16}$$

Furthermore, the obvious $H^1_{0,\gamma}(\Omega)$–ellipticity of $P_\gamma$ and Lax–Milgram give that $P_\gamma$ is boundedly invertible, i.e.

$$P_\gamma^{-1} \in \mathcal{L}\left(H^{-1}_{0,\gamma}(\Omega), H^1_{0,\gamma}(\Omega)\right). \tag{17}$$

In addition, the operator $P_\gamma : L^2(\Omega) \supset D(P_\gamma) \to L^2(\Omega)$, being positive definite and self–adjoint, has its square root $P_\gamma^{\frac{1}{2}}$ as a well–defined operator with $D(P_\gamma^{\frac{1}{2}}) = H^1_{0,\gamma}(\Omega)$ (after using the interpolation theorem in [12], p. 10); it then follows from (14) and (16) that for $\varpi$ and $\varpi' \in H^1_{0,\gamma}(\Omega)$,

$$\left\|P_\gamma^{\frac{1}{2}}\varpi\right\|^2_{L^2(\Omega)} = \|\varpi\|^2_{L^2(\Omega)} + \gamma\,\|\nabla\varpi\|^2_{[L^2(\Omega)]^2} = \|\varpi\|^2_{H^1_{0,\gamma}(\Omega)} ; \tag{18}$$

$$\left(P_\gamma^{\frac{1}{2}}\varpi, P_\gamma^{\frac{1}{2}}\varpi'\right)_{L^2(\Omega)} = (\varpi, \varpi')_{H^1_{0,\gamma}(\Omega)} . \tag{19}$$

(ii) In the case that $\gamma = 0$, then $P_0 = \mathbf{I}$, and we simply set

$$H^1_{0,0}(\Omega) = H^{-1}_{0,0}(\Omega) = L^2(\Omega). \tag{20}$$

- We denote the Hilbert space $\mathbf{H}_\gamma$ to be
$$\mathbf{H}_\gamma \equiv D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times H^1_{0,\gamma}(\Omega) \times L^2(\Omega), \tag{21}$$
with the inner product
$$\begin{aligned}&\left(\begin{bmatrix}\omega_0\\ \omega_1\\ \theta_0\end{bmatrix}, \begin{bmatrix}\widetilde{\omega}_0\\ \widetilde{\omega}_1\\ \widetilde{\theta}_0\end{bmatrix}\right)_{\mathbf{H}_\gamma}\\ &= \left(\mathring{\mathbf{A}}^{\frac{1}{2}}\omega_0, \mathring{\mathbf{A}}^{\frac{1}{2}}\widetilde{\omega}_0\right)_{L^2(\Omega)} + \left(P_\gamma^{\frac{1}{2}}\omega_1, P_\gamma^{\frac{1}{2}}\widetilde{\omega}_1\right)_{L^2(\Omega)} + \left(\theta_0, \widetilde{\theta}_0\right)_{L^2(\Omega)}.\end{aligned} \tag{22}$$
- With the above definitions, we then set $\mathcal{A}_\gamma : \mathbf{H}_\gamma \supset D(\mathcal{A}_\gamma) \to \mathbf{H}_\gamma$ to be
$$\begin{aligned}\mathcal{A}_\gamma &\equiv \begin{pmatrix} 0 & \mathbf{I} & 0 \\ -P_\gamma^{-1}\mathring{\mathbf{A}} & 0 & \alpha P_\gamma^{-1}A_D \\ 0 & -\alpha A_D & -A_D - \sigma\mathbf{I}\end{pmatrix}\\ \text{with } D(\mathcal{A}_\gamma) &= \Big\{[\omega_0, \omega_1, \theta_0] \in D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times D(A_D)\\ &\qquad \text{and such that } \mathring{\mathbf{A}}\omega_0 \in H^{-1}_{0,\gamma}(\Omega)\Big\}.\end{aligned} \tag{23}$$
- We define a (control) operator $\mathcal{B} \in \mathcal{L}\left([L^2(\Omega)]^2, D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times L^2(\Omega) \times \left[D(A_D^{\frac{1}{2}})\right]'\right)$ by having for $\overline{u} \in [L^2(\Omega)]^2$,
$$\mathcal{B}\overline{u} = \begin{bmatrix} 0 \\ 0 \\ div\overline{u}\end{bmatrix}. \tag{24}$$
- Finally, we define the map $\mathcal{L}_T : [L^2(0,T;L^2(\Omega)]^2 \to \mathbf{H}_\gamma$ by having for all $\overline{u} \in [L^2(0,T;L^2(\Omega))]^2$,
$$\mathcal{L}_T\overline{u} \equiv \int_0^T e^{\mathcal{A}_\gamma(T-t)}\mathcal{B}\overline{u}(t)dt. \tag{25}$$

If we take the initial data $[\omega_0, \omega_1, \theta_0]$ to be in $\mathbf{H}_\gamma$, and control $\overline{u} \in [L^2(0,T;L^2(\Omega)]^2$, then the coupled system (1) becomes formally the operator theoretic model
$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\omega\\ \omega_t\\ \theta\end{bmatrix} &= \mathcal{A}_\gamma\begin{bmatrix}\omega\\ \omega_t\\ \theta\end{bmatrix} + \mathcal{B}\overline{u}\\ \begin{bmatrix}\omega(0)\\ \omega_t(0)\\ \theta(0)\end{bmatrix} &= \begin{bmatrix}\omega_0\\ \omega_1\\ \theta_0\end{bmatrix}.\end{aligned} \tag{26}$$

Similar to what was done in [10] and in [1] for the thermoelastic system (1) with higher order boundary conditions in place, one can show the existence of an associated semigroup $\left\{e^{\mathcal{A}_\gamma t}\right\}_{t\geq 0}$. In particular, we have

**Proposition 2.1** *(well–posedness) Again with the parameter* $\gamma \geq 0$, $\mathcal{A}_\gamma$, *as defined in (23), generates a* $C_0-$*semigroup of contractions* $\left\{e^{\mathcal{A}_\gamma t}\right\}_{t\geq 0}$ *on the energy space* $\boldsymbol{H}_\gamma$.

With these dynamics in hand, the solution $[\omega, \omega_t, \theta]$ to (1) may be written explicitly as

$$\begin{bmatrix} \omega(t) \\ \omega_t(t) \\ \theta(t) \end{bmatrix} = e^{\mathcal{A}_\gamma t} \begin{bmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{bmatrix} + \int_0^t e^{\mathcal{A}_\gamma (t-s)} \mathcal{B}\overline{u}(s) ds; \tag{27}$$

and as we show below that $\mathcal{B} \in \mathcal{L}\left(\left[L^2(0,T;L^2(\Omega))\right]^2, \left[D(\mathcal{A}_\gamma^*)\right]'\right)$ (see **Proposition 2.3** and **Remark 2.4** below), then *a priori* this input to state map gives that

$$[\omega, \omega_t, \theta] \in C\left([0,T]; \left[D(\mathcal{A}_\gamma^*)\right]'\right)^{1}.$$

Given the representation (27) for the solution $[\omega, \omega_t, \theta]$, proving the asserted exact controllability at given time $T > 0$ (**Theorem 1.1**) is then equivalent to the functional analytical principle of showing the surjectivity of the operator $\mathcal{L}_T$, where $\mathcal{L}_T$ is defined in (25) (see [19] and [21]). Note that $\mathcal{L}_T$ is well–defined as an element of $\mathcal{L}\left(\left[L^2(0,T;L^2(\Omega))\right]^2, \left[D\left(\mathcal{A}_\gamma^*\right)\right]'\right)$, as $\mathcal{B}$ is (see **Remark 2.4** below); therefore, the control operator $\mathcal{L}_T$ as a mapping into the state space $\mathbf{H}_\gamma$ makes sense *a priori* only as an unbounded operator with some given domain of definition. However, in what follows below, we show that the map $\mathcal{L}_T$ can be extended to all of $\left[L^2(0,T;L^2(\Omega))\right]^2$. In particular, we have:

**Lemma 2.2** *The operator* $\mathcal{L}_T \in \mathcal{L}\left(\left[L^2(0,T;L^2(\Omega))\right]^2, \boldsymbol{H}_\gamma\right)$.

The proof of this result follows from a chain of propositions.

**Proposition 2.3** *The Hilbert space adjoint* $\mathcal{A}_\gamma^*$ *of* $\mathcal{A}_\gamma$ *is given by*

$$\mathcal{A}_\gamma^* = \begin{pmatrix} 0 & -\boldsymbol{I} & 0 \\ P_\gamma^{-1}\mathring{\boldsymbol{A}} & 0 & -\alpha P_\gamma^{-1} A_D \\ 0 & \alpha A_D & -A_D - \sigma \boldsymbol{I} \end{pmatrix}, \tag{28}$$

$$\textit{with } D(\mathcal{A}_\gamma^*) = D(\mathcal{A}_\gamma).$$

[1]One can work to show that in fact $[\omega, \omega_t, \theta] \in C([0,T]; \mathbf{H}_\gamma)$ for $\gamma \geq 0$.

**Proof:** Define the operator $\mathcal{T}_\gamma : \mathbf{H}_\gamma \to \mathbf{H}_\gamma$ as

$$\mathcal{T}_\gamma = \begin{pmatrix} 0 & -\mathbf{I} & 0 \\ P_\gamma^{-1}\mathring{\mathbf{A}} & 0 & -\alpha P_\gamma^{-1} A_D \\ 0 & \alpha A_D & -A_D - \sigma \mathbf{I} \end{pmatrix}, \qquad \text{with } D(\mathcal{T}_\gamma) = D(\mathcal{A}_\gamma). \tag{29}$$

Then for $[\omega_0, \omega_1, \theta_0]$ and $\left[\widetilde{\omega}_0, \widetilde{\omega}_1, \widetilde{\theta}_0\right] \in D(\mathcal{A}_\gamma)$ we have

$$\begin{aligned}
&\left(\mathcal{A}_\gamma \begin{bmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{bmatrix}, \begin{bmatrix} \widetilde{\omega}_0 \\ \widetilde{\omega}_1 \\ \widetilde{\theta}_0 \end{bmatrix}\right)_{\mathbf{H}_\gamma} \\
&= \left(\mathring{\mathbf{A}}^{\frac{1}{2}}\omega_1, \mathring{\mathbf{A}}^{\frac{1}{2}}\widetilde{\omega}_0\right)_{L^2(\Omega)} - \left(P_\gamma^{\frac{1}{2}} P_\gamma^{-1}\mathring{\mathbf{A}}\omega_0, P_\gamma^{\frac{1}{2}}\widetilde{\omega}_1\right)_{L^2(\Omega)} \\
&+\alpha\left(P_\gamma^{\frac{1}{2}} P_\gamma^{-1} A_D\theta_0, P_\gamma^{\frac{1}{2}}\widetilde{\omega}_1\right)_{L^2(\Omega)} - \alpha\left(A_D\omega_1, \widetilde{\theta}_0\right)_{L^2(\Omega)} \\
&-\left((A_D + \sigma\mathbf{I})\theta_0, \widetilde{\theta}_0\right)_{L^2(\Omega)} \\
&= \left\langle \omega_1, \mathring{\mathbf{A}}\widetilde{\omega}_0\right\rangle_{[D(\mathring{\mathbf{A}}^{\frac{1}{2}})]' \times D(\mathring{\mathbf{A}}^{\frac{1}{2}})} - \left(\mathring{\mathbf{A}}^{\frac{1}{2}}\omega_0, \mathring{\mathbf{A}}^{\frac{1}{2}}\widetilde{\omega}_1\right)_{L^2(\Omega)} \\
&\quad +\alpha\left(\theta_0, A_D\widetilde{\omega}_1\right)_{L^2(\Omega)} - \alpha\left(\omega_1, A_D\widetilde{\theta}_0\right)_{L^2(\Omega)} \\
&\quad -\left(\theta_0, (A_D + \sigma\mathbf{I})\widetilde{\theta}_0\right)_{L^2(\Omega)} \\
&= \left(\mathring{\mathbf{A}}^{\frac{1}{2}}\omega_0, -\mathring{\mathbf{A}}^{\frac{1}{2}}\widetilde{\omega}_1\right)_{L^2(\Omega)} + \left(P_\gamma^{\frac{1}{2}}\omega_1, P_\gamma^{\frac{1}{2}} P_\gamma^{-1}\mathring{\mathbf{A}}\widetilde{\omega}_0\right)_{L^2(\Omega)} \\
&\quad + \left(P_\gamma^{\frac{1}{2}}\omega_1, -\alpha P_\gamma^{\frac{1}{2}} P_\gamma^{-1} A_D\widetilde{\theta}_0\right)_{L^2(\Omega)} + \left(\theta_0, \alpha A_D\widetilde{\omega}_1\right)_{L^2(\Omega)} \\
&\quad + \left(\theta_0, -(A_D + \sigma\mathbf{I})\widetilde{\theta}_0\right)_{L^2(\Omega)} \\
&= \left(\begin{bmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{bmatrix}, \mathcal{T}_\gamma \begin{bmatrix} \widetilde{\omega}_0 \\ \widetilde{\omega}_1 \\ \widetilde{\theta}_0 \end{bmatrix}\right)_{\mathbf{H}_\gamma}.
\end{aligned}$$

$$\text{Thus, } D(\mathcal{T}_\gamma) \subseteq D(\mathcal{A}_\gamma^*) \text{ and } \left.\mathcal{A}_\gamma^*\right|_{D(\mathcal{T}_\gamma)} = \mathcal{T}_\gamma. \tag{30}$$

On the other hand, one can explicitly compute the inverse of $\mathcal{A}_\gamma \in \mathcal{L}(\mathbf{H}_\gamma, D(\mathcal{A}_\gamma))$ as

$$\mathcal{A}_\gamma^{-1} = \begin{pmatrix} -\alpha^2 \mathring{\mathbf{A}}^{-1} A_D (A_D + \sigma \mathbf{I})^{-1} A_D & -\mathring{\mathbf{A}}^{-1} P_\gamma & -\alpha \mathring{\mathbf{A}}^{-1} A_D (A_D + \sigma \mathbf{I})^{-1} \\ \mathbf{I} & 0 & 0 \\ -\alpha (A_D + \sigma \mathbf{I})^{-1} A_D & 0 & -(A_D + \sigma \mathbf{I})^{-1} \end{pmatrix}; \tag{31}$$

in turn, its Hilbert space adjoint $\left(\mathcal{A}_\gamma^*\right)^{-1} \in \mathcal{L}(\mathbf{H}_\gamma, D(\mathcal{A}_\gamma^*))$ can be computed as

$$\left(\mathcal{A}_\gamma^*\right)^{-1} = \begin{pmatrix} -\alpha^2 \mathring{\mathbf{A}}^{-1} A_D (A_D + \sigma \mathbf{I})^{-1} A_D & \mathring{\mathbf{A}}^{-1} P_\gamma & -\alpha \mathring{\mathbf{A}}^{-1} A_D (A_D + \sigma \mathbf{I})^{-1} \\ -\mathbf{I} & 0 & 0 \\ -\alpha (A_D + \sigma \mathbf{I})^{-1} A_D & 0 & -(A_D + \sigma \mathbf{I})^{-1} \end{pmatrix}.$$

Thus for $[\omega_0, \omega_1, \theta_0] \in \mathbf{H}_\gamma$, we have that

$$\begin{pmatrix} -\alpha^2 \mathring{\mathbf{A}}^{-1} A_D (A_D + \sigma \mathbf{I})^{-1} A_D & \mathring{\mathbf{A}}^{-1} P_\gamma & -\alpha \mathring{\mathbf{A}}^{-1} A_D (A_D + \sigma \mathbf{I})^{-1} \\ -\mathbf{I} & 0 & 0 \\ -\alpha (A_D + \sigma \mathbf{I})^{-1} A_D & 0 & -(A_D + \sigma \mathbf{I})^{-1} \end{pmatrix} \begin{bmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{bmatrix} =$$

$$\begin{bmatrix} -\alpha^2 \mathring{\mathbf{A}}^{-1} A_D (A_D + \sigma \mathbf{I})^{-1} A_D \omega_0 + \mathring{\mathbf{A}}^{-1} P_\gamma \omega_1 - \alpha \mathring{\mathbf{A}}^{-1} A_D (A_D + \sigma \mathbf{I})^{-1} \theta_0 \\ -\omega_0 \\ -\alpha (A_D + \sigma \mathbf{I})^{-1} A_D \omega_0 - (A_D + \sigma \mathbf{I})^{-1} \theta_0 \end{bmatrix},$$

and so

$$D(\mathcal{A}_\gamma^*) \subseteq D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \times D(A_D).$$

In addition,

$$-\alpha^2 A_D (A_D + \sigma \mathbf{I})^{-1} A_D \omega_0 + P_\gamma \omega_1 - \alpha A_D (A_D + \sigma \mathbf{I})^{-1} \theta_0 \in H_{0,\gamma}^{-1}(\Omega).$$

From these two containments, the definition of $D(\mathcal{T}_\gamma)$ in (29), and (30), we then deduce that the adjoint $\mathcal{A}_\gamma^*$ is as given in (28). ∎

**Remark 2.4** *Since div* $\in \mathcal{L}\left(\left[L^2(\Omega)\right]^2, H^{-1}(\Omega)\right)$, *we have from (24) and (28) that* $\mathcal{B} \in \mathcal{L}\left(\left[L^2(\Omega)\right]^2, \left[D(\mathcal{A}_\gamma^*)\right]'\right)$.

Using the form of $\mathcal{A}_\gamma^*$ given in **Proposition 2.3**, and its associated semigroup $\left\{e^{\mathcal{A}_\gamma^* t}\right\}_{t \geq 0}$, we quickly have the following:

**Corollary 2.5** *For terminal data* $[\phi_0, \phi_1, \psi_0] \in \boldsymbol{H}_\gamma$*, the function* $[\phi, \phi_t, \psi] \in C([0,T];\boldsymbol{H}_\gamma)$*, defined by*

$$\begin{bmatrix} \phi(t) \\ \phi_t(t) \\ \psi(t) \end{bmatrix} \equiv e^{\mathcal{A}_\gamma^*(T-t)} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix}, \tag{32}$$

*is a weak solution of the following system (adjoint with respect to (1)):*

$$\begin{cases} \phi_{tt} - \gamma\Delta\phi_{tt} + \Delta^2\phi + \alpha\Delta\psi = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } (0,T)\times\Omega; \\ \psi_t + \Delta\psi - \sigma\psi - \alpha\Delta\phi_t = 0 \end{cases}$$

$$\phi = \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } (0,T)\times\Gamma; \tag{33}$$

$$\psi = 0 \qquad \text{on } (0,T)\times\Gamma;$$

$$\phi(T) = \phi_0,\ \phi_t(T) = \phi_1,\ \psi(T) = \psi_0 \quad \text{on } \Omega.$$

**Remark 2.6** *Note that for data* $[\phi_0, \phi_1, \psi_0] \in D(\mathcal{A}_\gamma)$*, the system (33) may be written abstractly as (see (28) and (32))*

$$\begin{cases} P_\gamma\phi_{tt} = -\mathring{\boldsymbol{A}}\phi + \alpha A_D\psi \quad \text{in } H_{0,\gamma}^{-1}(\Omega); \\ \psi_t = -\alpha A_D\phi_t + (A_D + \sigma\boldsymbol{I})\psi \quad \text{in } L^2(\Omega); \\ [\phi(T), \phi(T), \psi(T)] = [\phi_0, \phi_1, \psi_0]. \end{cases} \tag{34}$$

Concerning this adjoint system, we have the following additional regularity and energy relation:

**Proposition 2.7** *The component* $\psi$ *of the solution* $[\phi, \phi_t, \psi]$ *to the backward system (33) satisfies* $\psi \in L^2\left(0,\infty; D(A_D^{\frac{1}{2}})\right)$*. Indeed, we have the following relation valid for all data* $[\phi_0, \phi_1, \psi_0] \in \boldsymbol{H}_\gamma$*:*

$$\frac{1}{2}\left[\|[\phi(0), \phi_t(0), \psi(0)]\|_{\boldsymbol{H}_\gamma}^2 - \|[\phi_0, \phi_1, \psi_0]\|_{\boldsymbol{H}_\gamma}^2\right]$$

$$= -\int_0^T \left\|A_D^{\frac{1}{2}}\psi(t)\right\|_{L^2(\Omega)}^2 dt - \sigma\int_0^T \|\psi(t)\|_{L^2(\Omega)}^2 dt. \tag{35}$$

**Proof:** Taking $[\phi_0, \phi_1, \psi_0] \in D(\mathcal{A}_\gamma^*)$, we have, via **Corollary 2.5**,

$$\frac{1}{2}\frac{d}{dt}\left\| \begin{bmatrix} \phi(t) \\ \phi_t(t) \\ \psi(t) \end{bmatrix} \right\|_{\mathbf{H}_\gamma}^2 dt = -\left( \mathcal{A}_\gamma^* \begin{bmatrix} \phi(t) \\ \phi_t(t) \\ \psi(t) \end{bmatrix}, \begin{bmatrix} \phi(t) \\ \phi_t(t) \\ \psi(t) \end{bmatrix} \right)_{\mathbf{H}_\gamma}$$

$$\begin{aligned} = \quad & \left(\mathring{\mathbf{A}}^{\frac{1}{2}}\phi_t(t), \mathring{\mathbf{A}}^{\frac{1}{2}}\phi(t)\right)_{L^2(\Omega)} - \left(\mathring{\mathbf{A}}^{\frac{1}{2}}\phi(t), \mathring{\mathbf{A}}^{\frac{1}{2}}\phi_t(t)\right)_{L^2(\Omega)} \\ & +\alpha\left(A_D\psi(t), \phi_t(t)\right)_{L^2(\Omega)} - \alpha\left(A_D\phi_t(t), \psi(t)\right)_{L^2(\Omega)} \\ & +\left(A_D^{\frac{1}{2}}\psi(t), A_D^{\frac{1}{2}}\psi(t)\right)_{L^2(\Omega)} + \sigma\left\|\psi(t)\right\|_{L^2(\Omega)}^2 . \end{aligned}$$

Integrating both sides of this equation from 0 to $T$ and using the terminal condition of (33) gives the relation (35), at least for smooth $[\phi_0, \phi_1, \psi_0]$. The fact that $\psi \in L^2\left(0, \infty; D(A_D^{\frac{1}{2}})\right)$, with continuous dependence on the data, now comes from the contraction of the semigroup $\left\{e^{\mathcal{A}_\gamma^* t}\right\}_{t\geq 0}$. A density argument then concludes the proof. ■

**Proposition 2.8** *The operator $\mathcal{B}^* e^{\mathcal{A}_\gamma^*(T-\cdot)} \in \mathcal{L}\left(\boldsymbol{H}_\gamma, [L^2(0,T;L^2(\Omega))]^2\right)$, and for every $[\phi_0, \phi_1, \psi_0] \in \boldsymbol{H}_\gamma$*

$$\mathcal{B}^* e^{\mathcal{A}_\gamma^*(T-\cdot)} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} = -\nabla\psi, \tag{36}$$

*where $\psi$ is the thermal component of the solution $[\phi, \phi_t, \psi]$ to the adjoint system (33).*

**Proof:** Using the definition of $\mathcal{B}$ in (24) and **Remark 2.4**, we compute its adjoint $\mathcal{B}^* \in \mathcal{L}\left(D(\mathcal{A}_\gamma^*), [L^2(0,T;L^2(\Omega))]^2\right)$: For every $\overline{u} \in [L^2(0,T;L^2(\Omega))]^2$ and $[\phi_0, \phi_1, \psi_0] \in D(\mathcal{A}_\gamma^*)$,

$$\begin{aligned} & \left\langle \mathcal{B}\overline{u}, \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} \right\rangle_{[D(\mathcal{A}_\gamma^*)]' \times D(\mathcal{A}_\gamma^*)} \\ = \quad & \left( \mathcal{A}_\gamma^{-1}\mathcal{B}\overline{u}(t), \mathcal{A}_\gamma^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} \right)_{\mathbf{H}_\gamma} \\ = \quad & \left( \begin{bmatrix} -\alpha\mathring{\mathbf{A}}^{-1}A_D(A_D+\sigma\mathbf{I})^{-1}div\overline{u} \\ 0 \\ -(A_D+\sigma\mathbf{I})^{-1}div\overline{u} \end{bmatrix}, \begin{bmatrix} -\phi_1 \\ P_\gamma^{-1}\mathring{\mathbf{A}}\phi_0 - \alpha P_\gamma^{-1}A_D\psi_0 \\ \alpha A_D\phi_1 - (A_D+\sigma\mathbf{I})\psi_0 \end{bmatrix} \right)_{\mathbf{H}_\gamma} \end{aligned}$$

(after using (31) and (28))

$$= \alpha \left\langle A_D(A_D+\sigma\mathbf{I})^{-1}div\overline{u}, \phi_1 \right\rangle_{[D(A_D^{\frac{1}{2}})]'\times D(A_D^{\frac{1}{2}})}$$

$$-\alpha \left((A_D+\sigma\mathbf{I})^{-1}div\overline{u}, A_D\phi_1\right)_{L^2(\Omega)} + \langle div\overline{u}, \psi_0\rangle_{[D(A_D^{\frac{1}{2}})]'\times D(A_D^{\frac{1}{2}})}$$

$$= \langle div\overline{u}, \psi_0\rangle_{[D(A_D^{\frac{1}{2}})]'\times D(A_D^{\frac{1}{2}})}$$

$$= (\overline{u}, -\nabla\psi_0)_{[L^2(\Omega)]^2} = \left(\overline{u}, \mathcal{B}^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix}\right)_{\mathbf{H}_\gamma}. \tag{37}$$

This form of the adjoint $\mathcal{B}^*$, the fact that $e^{\mathcal{A}_\gamma^*(\cdot)} \in \mathcal{L}\left(D(\mathcal{A}_\gamma^*), C([0,T]; D(\mathcal{A}_\gamma^*)\right)$, **Proposition 2.7**, and a density argument give the result. ■

*Proof of Lemma 2.2*: From a computation along the lines of that undertaken for (37), it can be shown that the adjoint $\mathcal{L}_T^* \in \mathcal{L}\left(\left[D(\mathcal{A}_\gamma^*)\right]', [L^2(0,T;L^2(\Omega))]^2\right)$ has the classical form

$$\mathcal{L}_T^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} (\cdot) = \mathcal{B}^* e^{\mathcal{A}_\gamma^*(T-\cdot)} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} \tag{38}$$

(see [21])). **Proposition 2.8** and the use of duality then provide the asserted **Lemma 2.2**.

# 3 Proof of Theorem 1.1

As mentioned above, showing the exact controllability for given $T > 0$ is equivalent to showing the ontoness of the operator $\mathcal{L}_T \in \mathcal{L}\left([L^2(0,T;L^2(\Omega))]^2, \mathbf{H}_\gamma\right)$ (see (25) and **Lemma 2.2**). In turn, by the classical functional analysis, the surjectivity of $\mathcal{L}_T$ is equivalent to showing that there exists a constant $C_T > 0$ such that the following injectivity condition holds for all $[\phi_0, \phi_1, \psi_0] \in \mathbf{H}_\gamma$:

$$\left\| \mathcal{L}_T^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} \right\|_{[L^2(0,T;L^2(\Omega))]^2} \geq C_T \left\| \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} \right\|_{\mathbf{H}_\gamma}. \tag{39}$$

It is this inequality which we will proceed to verify. Note that in "pde form", the inequality (39) becomes (see (38), **Proposition 2.8** and (11))

$$\int_0^T \left\| A_D^{\frac{1}{2}}\psi \right\|_{L^2(\Omega)}^2 dt \geq C_T \left\| \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} \right\|_{\mathbf{H}_\gamma}^2, \tag{40}$$

where $\psi$ is the (thermal) component of the solution $[\phi, \phi_t, \psi]$ to the backwards system (33).

Recall, that using the semigroup generated by $\mathcal{A}_\gamma^*$, $[\phi, \phi_t, \psi]$ has the explicit form given in (32). Accordingly, one can show, in a fashion similar to that in employed in [1], [3] and [10], that for data $[\phi_0, \phi_1, \psi_0] \in D\left(\left(\mathcal{A}_\gamma^*\right)^2\right)$, the corresponding solution $[\phi, \phi_t, \psi]$ to (33), besides residing in $C\left([0,T]; D\left(\left(\mathcal{A}_\gamma^*\right)^2\right)\right)$, enjoys the following regularity:

$$\begin{aligned} \phi &\in C([0,T]; D(\mathbf{\mathring{A}})); \\ \phi_t &\in C([0,T]; H^3(\Omega)); \\ \psi &\in C([0,T]; H^3(\Omega)). \end{aligned} \tag{41}$$

In view of this regularity for solutions corresponding to smooth initial data, a density argument will therefore allow the assumption throughout that $[\phi, \phi_t, \psi]$ has the regularity needed to justify the computations performed below.

Also, we will have frequent need throughout of the following Green's Theorem which is derived in [10] for functions $\varpi$ and $\varpi'$ "smooth enough":

$$\begin{aligned} \int_\Omega (\Delta^2 \varpi)\varpi' d\Omega = a\left(\varpi, \varpi'\right) + \int_\Gamma \left[\frac{\partial \Delta \varpi}{\partial \nu} + (1-\mu)\frac{\partial B_2 \varpi}{\partial \tau}\right] \varpi' d\Gamma \\ - \int_\Gamma \left[\Delta \varpi + (1-\mu) B_1 \varpi\right] \frac{\partial \varpi'}{\partial \nu} d\Gamma, \end{aligned} \tag{42}$$

where the bilinear form $a(\cdot,\cdot)$ is defined by

$$a\left(\varpi, \varpi'\right) \equiv \int_\Omega \left[\varpi_{xx}\varpi'_{xx} + \varpi_{yy}\varpi'_{yy} + \mu\left(\varpi_{xx}\varpi'_{yy} + \varpi_{yy}\varpi'_{xx}\right) + 2(1-\mu)\varpi_{xy}\varpi'_{xy}\right] d\Omega. \tag{43}$$

Here, $\mu \in (0, \frac{1}{2})$ is Poisson's ratio, and the boundary operators $B_i$ are given by

$$\begin{aligned} B_1 \varpi &\equiv 2\nu_1\nu_2 \frac{\partial^2 \varpi}{\partial x \partial y} - \nu_1^2 \frac{\partial^2 \varpi}{\partial y^2} - \nu_2^2 \frac{\partial^2 \varpi}{\partial x^2}; \\ B_2 \varpi &\equiv (\nu_1^2 - \nu_2^2)\frac{\partial^2 \varpi}{\partial x \partial y} + \nu_1\nu_2\left(\frac{\partial^2 \varpi}{\partial y^2} - \frac{\partial^2 \varpi}{\partial x^2}\right). \end{aligned} \tag{44}$$

**Step 1** *(Proof of a requisite trace result)***.** We first derive a trace regularity result for the adjoint system (33) which does not follow from the standard Sobolev trace theory, and which is critical in obtaining the estimate (40). This result is analogous to that proved in [1] and [3]. We note that related trace regularity results for Euler–Bernoulli plates were proved in [17], and for Kirchoff plates in [13].

**Lemma 3.1** *The component $\phi$ of the solution $[\phi, \phi_t, \psi]$ of (33) satisfies $\Delta\phi|_\Gamma \in L^2(0,T; L^2(\Gamma))$, with the accompanying estimate*

$$\int_0^T \|\Delta\phi\|^2_{L^2(\Gamma)}\, dt \leq C\left(\int_0^T \left[\left\|\mathring{\boldsymbol{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} + \left\|P_\gamma^{\frac{1}{2}}\phi_t\right\|^2_{L^2(\Omega)} + \left\|A_D^{\frac{1}{2}}\psi\right\|^2_{L^2(\Omega)}\right] dt \right.$$
$$\left. + \|[\phi_0, \phi_1, \psi_0]\|^2_{\boldsymbol{H}_\gamma}\right), \qquad (45)$$

*where $C$ does not depend on the parameter $\gamma$.*

**Proof:** We start by multiplying the first equation of (33) by the quantity $h \cdot \nabla\phi$, where $h(x,y) \equiv [h_1(x,y), h_2(x,y)]$ is a $\left[C^2(\overline{\Omega})\right]^2$ vector field such that $h|_\Gamma = [\nu_1, \nu_2]$ on $\Gamma$, and subsequently integrate from 0 to $T$ so as to obtain the equation

$$\int_0^T \left(\phi_{tt} - \gamma\Delta\phi_{tt} + \Delta^2\phi + \alpha\Delta\psi, h \cdot \nabla\phi\right)_{L^2(\Omega)} dt = 0. \qquad (46)$$

We now estimate the left hand side.

(i) To start off,

$$\int_0^T (\phi_{tt}, h \cdot \nabla\phi)_{L^2(\Omega)}\, dt = (\phi_t, h \cdot \nabla\phi)_{L^2(\Omega)}\Big|_0^T - \int_0^T (\phi_t, h \cdot \nabla\phi_t)_{L^2(\Omega)}\, dt$$

$$= (\phi_t, h \cdot \nabla\phi)_{L^2(\Omega)}\Big|_0^T - \frac{1}{2}\int_0^T \int_\Omega div\left(\phi_t^2 h\right) dt d\Omega$$
$$+ \frac{1}{2}\int_0^T \int_\Omega \phi_t^2\left[h_{1x} + h_{2y}\right] dt d\Omega$$
$$= (\phi_t, h \cdot \nabla\phi)_{L^2(\Omega)}\Big|_0^T + \frac{1}{2}\int_0^T \int_\Omega \phi_t^2\left[h_{1x} + h_{2y}\right] dt d\Omega, \qquad (47)$$

after making use of the divergence theorem and the fact that $\phi_t = 0$ on $\Gamma$.

(ii) Next,

$$\int_0^T (-\Delta\phi_{tt}, h \cdot \nabla\phi)_{L^2(\Omega)}\, dt = \left(\nabla\phi_t, \nabla\left(h \cdot \nabla\phi\right)\right)_{[L^2(\Omega)]^2}\Big|_0^T$$
$$- \int_0^T \left(\nabla\phi_t, \nabla\left(h \cdot \nabla\phi_t\right)\right)_{[L^2(\Omega)]^2} dt$$

(after using in part the fact that $h \cdot \nabla\phi = \dfrac{\partial\phi}{\partial\nu} = 0$ on $\Gamma$)

$$
\begin{aligned}
= \quad & \left(\nabla\phi_t, \nabla\left(h\cdot\nabla\phi\right)\right)_{[L^2(\Omega)]^2}\Big|_0^T - \frac{1}{2}\int_0^T\int_\Omega div\left(\left|\nabla\phi_t\right|^2 h\right) dtd\Omega \\
& -\int_0^T\int_\Omega\left[\frac{\phi_{tx}^2 h_{1x}}{2}+\frac{\phi_{ty}^2 h_{2y}}{2}\right] dtd\Omega - \int_0^T\int_\Omega\left[\phi_{tx}\phi_{ty}h_{2x}+\phi_{tx}\phi_{ty}h_{1y}\right] dtd\Omega \\
& +\int_0^T\int_\Omega\left[\frac{\phi_{tx}^2 h_{2y}}{2}+\frac{\phi_{ty}^2 h_{1x}}{2}\right] dtd\Omega
\end{aligned}
$$

$$
\begin{aligned}
& = \left(\nabla\phi_t, \nabla(h\cdot\nabla\phi)\right)_{[L^2(\Omega)]^2}\Big|_0^T \\
& +\int_0^T\int_\Omega\left[\frac{\phi_{tx}^2 h_{2y}}{2}+\frac{\phi_{ty}^2 h_{1x}}{2}-\frac{\phi_{tx}^2 h_{1x}}{2}-\frac{\phi_{ty}^2 h_{2y}}{2}\right] dtd\Omega \\
& -\int_0^T\int_\Omega\left[\phi_{tx}\phi_{ty}h_{2x}+\phi_{tx}\phi_{ty}h_{1y}\right] dtd\Omega, \qquad (48)
\end{aligned}
$$

after again using the divergence theorem and the fact that $\int_\Omega div\left(\left|\nabla\phi_t\right|^2 h\right) d\Omega = \int_\Gamma \left|\nabla\phi_t\right|^2 d\Gamma = 0$ (as $\phi_t(t)\in H_0^2(\Omega)$).

(iii) To handle the fourth order term in the expression (46), we use the Green's Theorem (42) and the fact that $h\cdot\nabla\phi = 0$ on $\Gamma$ to obtain

$$
\begin{aligned}
\int_0^T \left(\Delta^2\phi, h\cdot\nabla\phi\right)_{L^2(\Omega)} dt = \int_0^T a\left(\phi, h\cdot\nabla\phi\right) dt \\
-\int_0^T\int_\Gamma\left(\Delta\phi+(1-\mu)B_1\phi\right)\frac{\partial^2\phi}{\partial\nu^2} d\Gamma dt. \qquad (49)
\end{aligned}
$$

We note at this point that we can rewrite the first term on the right hand side of (49) as

$$
\begin{aligned}
\int_0^T a\left(\phi, h\cdot\nabla\phi\right) dt \quad = \quad & \frac{1}{2}\int_0^T\int_\Omega h\cdot\nabla\left[\phi_{xx}^2+\phi_{yy}^2+2\mu\phi_{xx}\phi_{yy}+2(1-\mu)\phi_{xy}^2\right] dtd\Omega \\
& + \mathcal{O}\left(\int_0^T\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|_{L^2(\Omega)}^2 dt\right), \qquad (50)
\end{aligned}
$$

where $\mathcal{O}\left(\int_0^T\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|_{L^2(\Omega)}^2 dt\right)$ denotes a series of terms which can be majorized by $\int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \phi \right\|^2_{L^2(\Omega)} dt$. We consequently have by the divergence theorem that

$$\begin{aligned}
& \int_0^T a\left(\phi, h \cdot \nabla \phi\right) dt \\
= & \ \frac{1}{2} \int_0^T \int_\Omega h \cdot \nabla \left[ \phi_{xx}^2 + \phi_{yy}^2 + 2\mu \phi_{xx} \phi_{yy} + 2(1-\mu) \phi_{xy}^2 \right] dt d\Omega \\
& + \mathcal{O}\left( \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \phi \right\|^2_{L^2(\Omega)} dt \right) \\
= & \ \frac{1}{2} \int_0^T \int_\Omega div \left\{ h \left[ \phi_{xx}^2 + \phi_{yy}^2 + 2\mu \phi_{xx} \phi_{yy} + 2(1-\mu) \phi_{xy}^2 \right] \right\} \\
& + \mathcal{O}\left( \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \phi \right\|^2_{L^2(\Omega)} dt \right) \\
= & \ \frac{1}{2} \int_0^T \int_\Gamma \left[ \phi_{xx}^2 + \phi_{yy}^2 + 2\mu \phi_{xx} \phi_{yy} + 2(1-\mu) \phi_{xy}^2 \right] dt d\Gamma \\
& + \mathcal{O}\left( \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \phi \right\|^2_{L^2(\Omega)} dt \right) \\
= & \ \frac{1}{2} \int_0^T \int_\Gamma (\Delta \phi)^2 dt + \mathcal{O}\left( \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \phi \right\|^2_{L^2(\Omega)} dt \right), \qquad (51)
\end{aligned}$$

where in the last step above, we have used the fact (as reasoned in [10], Ch. 4) that $\phi|_\Gamma = \left. \frac{\partial \phi}{\partial \nu} \right|_\Gamma = 0$ implies that $\phi_{xx}^2 + \phi_{yy}^2 + 2\mu \phi_{xx} \phi_{yy} + 2(1-\mu) \phi_{xy}^2 = (\Delta \phi)^2$ on $\Gamma$.

To handle the second term on the right hand side of (49), we note that $B_1 \phi = 0$ on $\Gamma$, which implies that

$$\Delta \phi = \Delta \phi + (1-\mu) B_1 \phi = \frac{\partial^2 \phi}{\partial \nu^2} \text{ on } \Gamma . \qquad (52)$$

The insertion of (51) into (49), followed by the consideration of (52) then yields that

$$\begin{aligned}
\int_0^T \left( \Delta^2 \phi, h \cdot \nabla \phi \right)_{L^2(\Omega)} dt = & -\frac{1}{2} \int_0^T \| \Delta \phi \|^2_{L^2(\Gamma)} dt \\
& + \mathcal{O}\left( \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \phi \right\|^2_{L^2(\Omega)} dt \right). \qquad (53)
\end{aligned}$$

(iv) To handle the last term on the left hand side of equation (46), we use the classical Green's theorem and the fact that $h \cdot \nabla \phi = 0$ on $\Gamma$ to obtain

$$\int_0^T (\Delta\psi, h \cdot \nabla\phi)_{L^2(\Omega)} \, dt = -\int_0^T (\nabla\psi, \nabla (h \cdot \nabla\phi))_{[L^2(\Omega)]^2} \, dt. \tag{54}$$

To finish the proof, we rewrite (46) by collecting the relations given above in (47), (48), (53) and (54) to obtain the relation

$$\frac{1}{2}\int_0^T \|\Delta\phi\|^2_{L^2(\Gamma)} \, dt = -\int_0^T (\nabla\psi, \nabla (h \cdot \nabla\phi))_{[L^2(\Omega)]^2} \, dt + \mathcal{O}\left(\int_0^T \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} dt\right)$$

$$+ \mathcal{O}\left(\int_0^T \left\|P_\gamma^{\frac{1}{2}}\phi_t\right\|^2_{L^2(\Omega)} dt\right) + (\phi_t, h \cdot \nabla\phi)_{L^2(\Omega)}\Big|_0^T + \gamma \, (\nabla\phi_t, \nabla (h \cdot \nabla\phi))_{[L^2(\Omega)]^2}\Big|_0^T.$$

A majorization of this quantity, which in part uses the contraction of the semigroup $\left\{e^{\mathcal{A}_\gamma^* t}\right\}_{t\geq 0}$, gives the desired inequality (45). ■

**Step 2** *(Proof of the Inequality (40))*

We multiply the first equation in (33) by $A_D^{-1}\psi$ and integrate in time and space to obtain

$$\int_0^T \left(\phi_{tt} - \gamma\Delta\phi_{tt} + \Delta^2\phi + \alpha\Delta\psi, A_D^{-1}\psi\right)_{L^2(\Omega)} dt = 0, \tag{55}$$

and proceed to estimate this quantity.

**(A.1)** *Dealing with* $\int_0^T \left(\phi_{tt} - \gamma\Delta\phi_{tt}, A_D^{-1}\psi\right)_{L^2(\Omega)} dt$ : Using an integration by parts and the second differential equation of (34) yields

$$\begin{aligned}
&\int_0^T \left(\phi_{tt} - \gamma\Delta\phi_{tt}, A_D^{-1}\psi\right)_{L^2(\Omega)} dt \\
&= \left(\phi_t, A_D^{-1}\psi\right)_{L^2(\Omega)}\Big|_0^T + \gamma \left(\nabla\phi_t, \nabla A_D^{-1}\psi\right)_{[L^2(\Omega)]^2}\Big|_0^T \\
&\quad - \int_0^T \left[\left(\phi_t, A_D^{-1}\psi_t\right)_{L^2(\Omega)} + \gamma \left(\nabla\phi_t, \nabla A_D^{-1}\psi_t\right)_{[L^2(\Omega)]^2}\right] dt \\
&= \alpha \int_0^T \left[\|\phi_t\|^2_{L^2(\Omega)} + \gamma \|\nabla\phi_t\|^2_{L^2(\Omega)}\right] dt \\
&\quad - \int_0^T \left[\left(\phi_t, \left(\mathbf{I} + \sigma A_D^{-1}\right)\psi\right)_{L^2(\Omega)} + \gamma \left(\nabla\phi_t, \nabla \left(\mathbf{I} + \sigma A_D^{-1}\right)\psi\right)_{[L^2(\Omega)]^2}\right] dt \\
&\quad + \left(\phi_t, A_D^{-1}\psi\right)_{L^2(\Omega)}\Big|_0^T + \gamma \left(\nabla\phi_t, \nabla A_D^{-1}\psi\right)_{[L^2(\Omega)]^2}\Big|_0^T.
\end{aligned} \tag{56}$$

In regards to the last two terms of this relation, we have by Green's Theorem and the fact that $\phi_t \in H_0^1(\Omega)$ for $\gamma > 0$, that for any $t \in [0,T]$

$$\gamma\left(\nabla\phi_t(t), \nabla A_D^{-1}\psi(t)\right)_{[L^2(\Omega)]^2} = \gamma(\phi_t(t), \psi(t))_{L^2(\Omega)}$$

$$\leq \gamma C_\epsilon \|\phi_t\|^2_{C([0,T];L^2(\Omega))} + \gamma\frac{\epsilon}{8}\|\psi(t)\|^2_{L^2(\Omega)}. \tag{57}$$

Moreover for $T \geq t \geq 0$,

$$\begin{aligned} \left(\phi_t(t), A_D^{-1}\psi(t)\right)_{L^2(\Omega)} &\leq \frac{\epsilon}{8}\|\phi_t(t)\|^2_{L^2(\Omega)} + C_\epsilon\left\|A_D^{-1}\psi(t)\right\|^2_{L^2(\Omega)} \\ &\leq \frac{\epsilon}{8}\|\phi_t(t)\|^2_{L^2(\Omega)} + C_\epsilon\|\psi(t)\|^2_{[D(A_D^{\frac{1}{2}})]'} \\ &\leq \frac{\epsilon}{8}\|\phi_t(t)\|^2_{L^2(\Omega)} + C_\epsilon\|\psi\|^2_{C([0,T];H^{-1}(\Omega))}, \end{aligned} \tag{58}$$

after using (10).

Combining (57) and (58), using the contraction of the semigroup $\left\{e^{\mathcal{A}_\gamma^* t}\right\}_{t\geq 0}$ and (56), we then have the estimate

$$\begin{aligned} &\left|\int_0^T\left(\phi_{tt} - \gamma\Delta\phi_{tt}, A_D^{-1}\psi\right)_{L^2(\Omega)} dt - \alpha\int_0^T\left[\|\phi_t\|^2_{L^2(\Omega)} + \gamma\|\nabla\phi_t\|^2_{[L^2(\Omega)]^2}\right]dt\right| \\ \leq\ & C\int_0^T\left[\|\phi_t\|_{L^2(\Omega)}\left\|A_D^{\frac{1}{2}}\psi\right\|_{L^2(\Omega)} + \gamma\|\nabla\phi_t\|_{[L^2(\Omega)]^2}\left\|A_D^{\frac{1}{2}}\psi\right\|_{L^2(\Omega)}\right]dt \\ &+\frac{\epsilon}{2}\|\phi_0,\phi_1,\psi_0\|^2_{\mathbf{H}_\gamma} + C_\epsilon\left(\gamma\|\phi_t\|^2_{C([0,T];L^2(\Omega))} + \|\psi\|^2_{C([0,T];H^{-1}(\Omega))}\right) \\ \leq\ & \epsilon\left(\int_0^T\left\|P_\gamma^{\frac{1}{2}}\phi_t\right\|^2_{L^2(\Omega)}dt + \frac{1}{2}\|\phi_0,\phi_1,\psi_0\|^2_{\mathbf{H}_\gamma}\right) + C\left(\int_0^T\left\|A_D^{\frac{1}{2}}\psi\right\|^2_{L^2(\Omega)}dt\right. \\ &\left.+\gamma\|\phi_t\|^2_{C([0,T];L^2(\Omega))} + \|\psi\|^2_{C([0,T];H^{-1}(\Omega))}\right). \end{aligned} \tag{59}$$

where the constant $C$ does not depend on $\gamma$, $0 \leq \gamma \leq M$.

**(A. 2)** *Dealing with* $\int_0^T\left(\Delta^2\omega, A_D^{-1}\theta\right)dt$ : Another application of Green's theorem in (42) and the fact that $A_D^{-1}\psi\big|_\Gamma = 0$ give

$$\int_0^T\left(\Delta^2\phi, A_D^{-1}\psi\right)_{L^2(\Omega)}dt = \int_0^T a\left(\phi, A_D^{-1}\psi\right)dt - \int_0^T\left(\Delta\phi, \frac{\partial A_D^{-1}\psi}{\partial\nu}\right)_{L^2(\Gamma)}dt. \tag{60}$$

Estimating the right hand side of (60) yields, after the use of trace theory, elliptic regularity and the mean inequality,

$$\begin{aligned}
&\left| \int_0^T \left(\Delta^2\phi, A_D^{-1}\psi\right)_{L^2(\Omega)} dt \right| \\
\leq\ & C_0 \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}}\phi \right\|_{L^2(\Omega)} \left\| A_D^{\frac{1}{2}}\psi \right\|_{L^2(\Omega)} dt \\
&+\frac{\epsilon}{2C} \int_0^T \|\Delta\phi\|_{L^2(\Gamma)}^2\, dt + C_\epsilon \int_0^T \left\| A_D^{\frac{1}{2}}\psi \right\|_{L^2(\Omega)}^2 dt
\end{aligned}$$

(where the inverted $C$ is the same constant present in (45))

$$\begin{aligned}
\leq\ & C_0 \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}}\phi \right\|_{L^2(\Omega)} \left\| A_D^{\frac{1}{2}}\psi \right\|_{L^2(\Omega)} dt \\
&+\frac{\epsilon}{2} \left( \int_0^T \left[ \left\| \mathring{\mathbf{A}}^{\frac{1}{2}}\phi \right\|_{L^2(\Omega)}^2 + \left\| P_\gamma^{\frac{1}{2}}\phi_t \right\|_{L^2(\Omega)}^2 \right] dt + \|[\phi_0,\phi_1,\psi_0]\|_{\mathbf{H}_\gamma}^2 \right) \\
&+C_\epsilon \int_0^T \left\| A_D^{\frac{1}{2}}\psi \right\|_{L^2(\Omega)}^2 dt
\end{aligned}$$

(by **Lemma 3.1**)

$$\begin{aligned}
\leq\ & \epsilon \left( \int_0^T \left[ \left\| \mathring{\mathbf{A}}^{\frac{1}{2}}\phi \right\|_{L^2(\Omega)}^2 + \left\| P_\gamma^{\frac{1}{2}}\phi_t \right\|_{L^2(\Omega)}^2 \right] dt + \frac{1}{2} \|[\phi_0,\phi_1,\psi_0]\|_{\mathbf{H}_\gamma}^2 \right) \\
&+C_\epsilon \int_0^T \left\| A_D^{\frac{1}{2}}\psi \right\|_{L^2(\Omega)}^2 dt\,, \qquad (61)
\end{aligned}$$

after the use of the mean inequality.

**(A.3)** *Dealing with* $\int_0^T \left(\alpha\Delta\psi, A_D^{-1}\psi\right)_{L^2(\Omega)} dt$ : Easily we have

$$\int_0^T \left(\alpha\Delta\psi, A_D^{-1}\psi\right)_{L^2(\Omega)} dt = -\alpha \int_0^T \left(A_D\psi, A_D^{-1}\psi\right)_{L^2(\Omega)} dt = -\alpha \int_0^T \|\psi\|_{L^2(\Omega)}^2\, dt. \qquad (62)$$

**(A.4)** *Combining (55), (59), (61) and (62) thus results in the following*: For $\epsilon > 0$ small enough there exists a constant $C > 0$ (independent of $\gamma$) such that the solution

$[\omega, \omega_t, \theta]$ of (1) satisfies

$$(\alpha - 2\epsilon) \int_0^T \left[ \|\phi_t\|^2_{L^2(\Omega)} + \gamma \|\nabla \phi_t\|^2_{[L^2(\Omega)]^2} \right] dt$$

$$\leq C \left( \int_0^T \left\| A_D^{\frac{1}{2}} \psi \right\|^2_{L^2(\Omega)} dt + \gamma \|\phi_t\|^2_{C([0,T];L^2(\Omega))} + \|\psi\|^2_{C([0,T];H^{-1}(\Omega))} \right)$$

$$\epsilon \left( \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \phi \right\|^2_{L^2(\Omega)} dt + \|[\phi_0, \phi_1, \psi_0]\|^2_{\mathbf{H}_\gamma} \right), \tag{63}$$

where the noncrucial dependence of $C$ upon $\epsilon$ has not been noted.

**(A.5)** *The Conclusion of the Proof of Theorem 1.1*: To majorize the norm of the component $\phi$, we multiply the first equation of (34) by $\phi$, integrate from 0 to $T$ and integrate by parts to thereby obtain the relation

$$\left( P_\gamma^{\frac{1}{2}} \phi_t, P_\gamma^{\frac{1}{2}} \phi \right)_{L^2(\Omega)} \Big|_0^T - \int_0^T \left\| P_\gamma^{\frac{1}{2}} \phi_t \right\|^2_{L^2(\Omega)} dt$$

$$= - \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \phi \right\|^2_{L^2(\Omega)} dt + \alpha \int_0^T \left( A_D^{\frac{1}{2}} \psi, A_D^{\frac{1}{2}} \phi \right)_{L^2(\Omega)} dt. \tag{64}$$

Concerning the first term in this relation, we have for $\forall\, t \in [0, T]$

$$\begin{aligned} \left( P_\gamma^{\frac{1}{2}} \phi_t(t), P_\gamma^{\frac{1}{2}} \phi(t) \right)_{L^2(\Omega)} &\leq \frac{\epsilon}{2} \left\| P_\gamma^{\frac{1}{2}} \phi_t(t) \right\|^2_{L^2(\Omega)} + C_\epsilon \left\| P_\gamma^{\frac{1}{2}} \phi(t) \right\|^2_{L^2(\Omega)} \\ &\leq \frac{\epsilon}{2} \|[\phi_0, \phi_1, \psi_0]\|^2_{\mathbf{H}_\gamma} + C_\epsilon \left\| P_\gamma^{\frac{1}{2}} \phi(t) \right\|^2_{L^2(\Omega)}. \end{aligned} \tag{65}$$

Combining (64) and (65), we eventually arrive at the following estimate for $\epsilon > 0$ small enough:

$$(1 - \epsilon) \int_0^T \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \phi \right\|^2_{L^2(\Omega)} dt$$

$$\leq \int_0^T \left\| P_\gamma^{\frac{1}{2}} \phi_t \right\|^2_{L^2(\Omega)} dt + C \left( \int_0^T \left\| A_D^{\frac{1}{2}} \psi \right\|^2_{L^2(\Omega)} dt \right.$$

$$\left. + \left\| P_\gamma^{\frac{1}{2}} \phi \right\|^2_{C([0,T];L^2(\Omega))} \right) + \epsilon \|[\phi_0, \phi_1, \psi_0]\|^2_{\mathbf{H}_\gamma}, \tag{66}$$

where the noncrucial dependence of $C$ upon $\epsilon$ has not been noted.

Thus, if $\epsilon$ is small enough, we then have, upon combining (63) and (66), the existence of a constant $C$ such that

$$\int_0^T \left[ \left\| \mathring{\mathbf{A}}^{\frac{1}{2}} \phi \right\|^2_{L^2(\Omega)} + \|\phi_t\|^2_{L^2(\Omega)} + \gamma \|\nabla \phi_t\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)} \right] dt$$

$$\leq \frac{2\epsilon + \alpha\epsilon - 3\epsilon^2}{(\alpha - 2\epsilon)(1-\epsilon)} \|[\phi_0, \phi_1, \psi_0]\|^2_{\mathbf{H}_\gamma} + C \left( \int_0^T \left\| A_D^{\frac{1}{2}} \psi \right\|^2_{L^2(\Omega)} dt \right.$$

$$\left. + \left\| P_\gamma^{\frac{1}{2}} \phi \right\|^2_{C([0,T];L^2(\Omega))} + \gamma \|\phi_t\|^2_{C([0,T];L^2(\Omega))} + \|\psi\|^2_{C([0,T];H^{-1}(\Omega))} \right). \quad (67)$$

From here, we apply the relation (35) and its inherent property that $\|[\phi(t), \phi_t(t), \psi(t)]\|^2_{\mathbf{H}_\gamma} \geq \|[\phi(0), \phi_t(0), \psi(0)]\|^2_{\mathbf{H}_\gamma}$ $\forall\, t \in [0,T]$ (recall that $[\phi, \phi_t, \psi]$ solves the *backward* problem (33)) to obtain

$$T \left[ \|[\phi_0, \phi_1, \psi_0]\|^2_{\mathbf{H}_\gamma} - 2\int_0^T \left\| A_D^{\frac{1}{2}} \psi(t) \right\|^2_{L^2(\Omega)} dt - 2\sigma \int_0^T \|\psi(t)\|^2_{L^2(\Omega)} dt \right]$$

$$= T \|[\phi(0), \phi_t(0), \psi(0)]\|^2_{\mathbf{H}_\gamma} \leq \int_0^T \|[\phi(t), \phi_t(t), \psi(t)]\|^2_{\mathbf{H}_\gamma} dt$$

$$\leq \frac{2\epsilon + \alpha\epsilon - 3\epsilon^2}{(\alpha - 2\epsilon)(1-\epsilon)} \|[\phi_0, \phi_1, \psi_0]\|^2_{\mathbf{H}_\gamma} + C \left( \int_0^T \left\| A_D^{\frac{1}{2}} \psi \right\|^2_{L^2(\Omega)} dt \right.$$

$$\left. + \left\| P_\gamma^{\frac{1}{2}} \phi \right\|^2_{C([0,T];L^2(\Omega))} + \gamma \|\phi_t\|^2_{C([0,T];L^2(\Omega))} + \|\psi\|^2_{C([0,T];H^{-1}(\Omega))} \right). \quad (68)$$

Taking $\epsilon > 0$ small enough in (68), we then have the following preliminary inequality valid for all $\gamma \geq 0$ and $T > 0$ :

$$\|[\phi_0, \phi_1, \psi_0]\|^2_{\mathbf{H}_\gamma} \leq C_T \left( \int_0^T \left\| A_D^{\frac{1}{2}} \psi \right\|^2_{L^2(\Omega)} dt \right.$$

$$\left. + \left\| P_\gamma^{\frac{1}{2}} \phi \right\|^2_{C([0,T];L^2(\Omega))} + \gamma \|\phi_t\|^2_{C([0,T];L^2(\Omega))} + \|\psi\|^2_{C([0,T];H^{-1}(\Omega))} \right). \quad (69)$$

The inequality (40) is finally obtained after invoking the following proposition which can be derived through a (by now) classical compactness/uniqueness argument (see e.g., [2] and [14].

**Proposition 3.2** *The inequality (69) implies the existence of a constant $C_T$ such that the corresponding solution $[\phi, \phi_t, \psi]$ of (33) satisfies*

$$\left\| P_\gamma^{\frac{1}{2}} \phi \right\|^2_{C([0,T];L^2(\Omega))} + \gamma \|\phi_t\|^2_{C([0,T];L^2(\Omega))} + \|\psi\|^2_{C([0,T];H^{-1}(\Omega))}$$

$$\leq C_T \int_0^T \left\| A_D^{\frac{1}{2}} \psi \right\|^2_{L^2(\Omega)} dt. \quad (70)$$

These last two inequalities complete the proof of **Theorem 1.1**.

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