A Parallel Nonoverlapping Schwarz Domain Decomposition Method for Elliptic Interface Problems *

Daoqi Yang

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA.
Internet address: dyang@na-net.ornl.gov, http://www.math.wayne.edu/~yang

Abstract

We present a parallel nonoverlapping Schwarz domain decomposition method with interface relaxation for linear elliptic problems, possibly with discontinuities in the solution and its derivatives. This domain decomposition method can be characterized as: the transmission conditions at the interface of subdomains are taken to be Dirichlet at odd iterations and Neumann at even iterations. The convergence analysis of the iterative sequence of subdomain solutions is first established in the energy norm sense at the continuous level. Finite-dimensional discretization schemes, such as finite element approximation, finite element approximation with Lagrange multipliers and hybrid mixed finite element approximation, are then considered and analyzed. Numerical examples are provided to check the performance of this iterative procedure.

Keywords. Domain decomposition, finite element methods, mixed finite element methods, partial differential equations, Lagrange multipliers, parallel computing, transmission problems, interface problems.

AMS Subject Classifications: 65N55, 65N30, 35J25, 65Y05

1 Introduction

In domain decomposition methods, the original problem is divided into subproblems defined on subdomains and possibly on the interface of the subdomains. Overlapping Schwarz methods [11, 24, 38] consist of subdomains with nonzero overlapping, from which efficient preconditioners can be constructed based on additive and multiplicative Schwarz techniques. However, they duplicate the work on the overlapped regions and are sensitive to strong discontinuities in the coefficients. Two major difficulties with the implementation of these methods involve automatic partition of the original domain into overlapping subdomains

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with a specified ratio of subdomain diameter over subdomain overlapping distance, and construction of a coarse space to ensure convergence independence of the number of subdomains and grid size.

Substructuring methods [4, 5, 13, 17, 26] partition the original domain into nonoverlapping regions and reduce the original problem to mainly a global smaller problem defined on the interface of the subdomains. The interface problem involves a Schur complement matrix at the discrete level, and a Steklov-Poincare operator at the continuous level. Substructuring is an efficient, parallel direct method, but requires good and possibly expensive and complicated preconditioners for the iterative interface problem solver [39].

Nonoverlapping Schwarz alternating methods are similar to their overlapping counterpart, but require that transmission conditions on the interface be designed carefully to ensure the convergence. Examples are Funaro, Quarteroni, and Zanolli [19], Marini and Quarteroni [32, 33], Lions [31], Després [14], Douglas, Paes Leme, Roberts, and Wang [15], Kim [23], Quarteroni [34], Le Tallec and Tidriri [25], Benamou and Despres [3], Engquist and Zhao [18], and Yang [40, 41, 42]. In these methods, Robin or alternating Dirichlet-Neumann interface conditions are applied. These methods are especially desirable for the so-called interface or transmission problems that have different characteristics on different regions of the domain. (In Le Tallec and Tidriri [25], the methodology is even used to suppress boundary layers for convection diffusion problems.) However, they often require some relaxation parameter to ensure or accelerate the convergence, due to the lack of a coarse space correction procedure and the fact of not being used as preconditioners for Krylov subspace methods.

In this paper, we present a nonoverlapping Schwarz alternating method for linear second order elliptic problems on general domains. The transmission conditions are taken to be Dirichlet at odd iterations and Neumann at even iterations. This method is in the spirit of the additive overlapping Schwarz method, as opposed to the multiplicative method, in the sense that subdomain problems do not depend on each other at the same iteration level. Thus it can be efficiently implemented on parallel computers. Approximations of this method based on Galerkin method, finite element method with Lagrange multipliers and hybridized mixed finite element method are also studied.

The organization of this paper is as follows. In §2, the domain decomposition method is described for general elliptic problems. In §3, a convergence analysis of the method is carried out in detail at the continuous level. In §4, a finite element approximation is given and analyzed. In §5, a finite element approximation with Lagrange multipliers is considered. Then in §6, a hybridized mixed finite element method with Lagrange multipliers is employed. Finally in §7, numerical examples are provided to check the performance of the method.

2 The Domain Decomposition Method

Let Ω be a smooth bounded two-dimensional domain or a convex polygon with boundary $\partial\Omega$. Assume that Ω is the union of two nonoverlapping subdomains Ω_1 and Ω_2 with interface Γ; that is, $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, $\Omega_1 \cap \Omega_2 = 0$, and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. The subdomains Ω_1 and Ω_2 may be connected or disconnected. When they are disconnected, the decomposition can actually contain more than two computational subdomains.

Consider the following boundary value problem: find $u_1 \in H^1(\Omega_1)$ and $u_2 \in H^1(\Omega_2)$ such that

$$L_k u_k = f \text{ in } \Omega_k, \qquad k = 1, 2, \tag{2.1}$$

$$u_k = g \text{ on } \partial\Omega_k \cap \partial\Omega, \qquad k = 1, 2, \tag{2.2}$$

$$u_1 - u_2 = \mu \text{ on } \Gamma, \tag{2.3}$$

$$\frac{\partial u_1}{\partial \nu_A^1} + \frac{\partial u_2}{\partial \nu_A^2} = \eta \text{ on } \Gamma, \tag{2.4}$$

where for $k = 1, 2$,

$$L_k u = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}^{(k)}(x)\frac{\partial u}{\partial x_j}\Big) + b^{(k)}(x)u, \quad \frac{\partial u}{\partial \nu_A^k} = \sum_{i,j=1}^{2} a_{ij}^{(k)} \frac{\partial u}{\partial x_j}\nu_i^k,$$

$\nu^k = \{\nu_1^k, \nu_2^k\}$ is the outward normal unit vector to $\partial\Omega_k$. We assume that $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$, and that $\mu(x)$ and $\eta(x)$ are given regular functions on Γ. That is, the solution of (2.1)-(2.4) is sought with specified strength of discontinuity, and so is its conormal derivative. We also assume that the coefficients $\{a_{ij}^{(k)}\}$ are symmetric, uniformly positive definite, and bounded in Ω_k, and $b^{(k)} \geq 0$.

Problems of this kind can be encountered in material science, wave propagation, and fluid flow applications [2, 27, 34]. In some applications, the interface conditions (2.3) and (2.4) can also be nonlinear [22], or linear but more complicated [29]. For example, in molecular electrostatics [29], the electrostatic potential has jumps in the form

$$\frac{\partial u_1}{\partial \nu_A^1} + \frac{\epsilon_{out}}{\epsilon_{in}}\frac{\partial u_2}{\partial \nu_A^2} = \eta \text{ on } \Gamma, \tag{2.5}$$

where ϵ_{out} and ϵ_{in} are the uniform dielectric constant outside and inside the molecule, respectively.

Schwarz overlapping and substructuring domain decomposition methods [5, 11, 12, 13, 17, 24, 26, 37, 38, 39] have been analyzed for the special linear case (2.1)-(2.4) in which $\mu = 0$ and $\eta = 0$. It is not clear that any of these methods applies directly to the general case with $\mu \neq 0$ and $\eta \neq 0$. In [27], a finite difference method without domain decomposition was considered for the problem (2.1)-(2.4). Applications to time-dependent interface problems were presented in [28, 21].

We now define formally the following domain decomposition method for problem (2.1)-(2.4): Choose $u_k^0 \in H^1(\Omega_k)$ satisfying $u_k^0|_{\Omega\cap\Omega_k} = g$, $k = 1, 2$. For $n = 0, 1, 2, \cdots$, the sequence $u_k^n \in H^1(\Omega_k)$ with $u_k^n|_{\partial\Omega\cap\partial\Omega_k} = g$ is constructed such that

$$L_1 u_1^{2n+1} = f \text{ in } \Omega_1, \quad u_1^{2n+1} = \alpha u_1^{2n} + (1-\alpha)u_2^{2n} + (1-\alpha)\mu \text{ on } \Gamma; \tag{2.6}$$

$$L_2 u_2^{2n+1} = f \text{ in } \Omega_2, \quad u_2^{2n+1} = \alpha u_1^{2n} + (1-\alpha)u_2^{2n} - \alpha\mu \text{ on } \Gamma; \tag{2.7}$$

$$L_1 u_1^{2n+2} = f \text{ in } \Omega_1, \quad \frac{\partial u_1^{2n+2}}{\partial \nu_A^1} = \beta\frac{\partial u_1^{2n+1}}{\partial \nu_A^1} + (1-\beta)\frac{\partial u_2^{2n+1}}{\partial \nu_A^1} + (1-\beta)\eta \text{ on } \Gamma; \tag{2.8}$$

$$L_2 u_2^{2n+2} = f \text{ in } \Omega_2, \quad \frac{\partial u_2^{2n+2}}{\partial \nu_A^2} = \beta\frac{\partial u_1^{2n+1}}{\partial \nu_A^2} + (1-\beta)\frac{\partial u_2^{2n+1}}{\partial \nu_A^2} - \beta\eta \text{ on } \Gamma; \tag{2.9}$$

where $\alpha, \beta \in (0,1)$ are relaxation parameters that will be determined to ensure (and possibly to accelerate) the convergence of the iterative procedure. Note that this algorithm is very similar to (14)-(15) in [25], where overlapping subdomains are used and subdomain problems are sequentially computed, while our subdomain problems are nonoverlapping and can be solved simultaneously at each iteration level. Indeed, this algorithm is motivated by the ones proposed by Funaro, Quarteroni, and Zanolli [19], Marini and Quarteroni [32, 33], Lions [31], Després [14], Rice, Vavalis, and Yang [36], Quarteroni [34], Benamou and Despres [3], Douglas and Yang [16], and Yang [40, 42, 41], where regular problems with $\mu = 0$ and $\eta = 0$ were considered.

Note that the scheme (2.6)-(2.9) is a formal idea and its real meaning should be understood in variational forms, which will be given in the next section. The reason is that $\frac{\partial u}{\partial \nu_A^k}|_\Gamma$ is not well defined for $u \in H^1(\Omega_k)$. If $u \in H^1(\Omega_k)$ and $L_k u = f$ in Ω_k, one can prove that $\frac{\partial u}{\partial \nu_A^k} \in H^{-1/2}(\partial\Omega_k)$. But its restriction on the interface $\frac{\partial u}{\partial \nu_A^k}|_\Gamma$ still needs careful interpretation since the space $H^{-1/2}(\partial\Omega_k)$ is not local.

3 Convergence Analysis

We first introduce some notation. Define the Hilbert space by interpolation

$$H_{00}^{\frac{1}{2}}(\Gamma) = [H_0^1(\Gamma), L^2(\Gamma)]_{1/2}.$$

The space $H_{00}^{\frac{1}{2}}(\Gamma)$ is strictly contained in $H_0^{\frac{1}{2}}(\Gamma)$; see Lions and Magenes [30, page 66]. Denote the Hilbert spaces

$$\mathcal{V}_k = \{v \in H^1(\Omega_k), \quad v|_{\partial\Omega\cap\partial\Omega_k} = 0\}, \quad k = 1, 2,$$

and let γ_0 be the trace operator from $H_0^1(\Omega)$ onto $H_{00}^{\frac{1}{2}}(\Gamma)$. For any $\phi \in H_{00}^{\frac{1}{2}}(\Gamma)$, we denote by $R_1\phi$ and $R_2\phi$ the following extensions to Ω_1 and Ω_2, respectively, such that

$$R_k\phi \in \mathcal{V}_k : \quad L_k(R_k\phi) = 0 \text{ in } \Omega_k, \quad \gamma_0(R_k\phi) = \phi \text{ on } \Gamma, \quad k = 1, 2. \tag{3.1}$$

Define the bilinear forms:

$$a_k(u, w) = \sum_{i,j=1}^{2} \int_{\Omega_k} a_{ij}^{(k)} \frac{\partial u}{\partial x_j} \frac{\partial w}{\partial x_i} dx + \int_{\Omega_k} b^{(k)} uw\, dx, \qquad k = 1, 2, \tag{3.2}$$

$$(u, w)_k = \int_{\Omega_k} uw\, dx, \qquad k = 1, 2. \tag{3.3}$$

With all the notation we can write (2.8) and (2.9) in variational form

$$\begin{aligned} a_1(u_1^{2n+2}, w) = \; & (1-\beta)(f, w)_1 + \beta a_1(u_1^{2n+1}, w) + (1-\beta)\langle \eta, \gamma_0 w\rangle \\ & -(1-\beta) a_2(u_2^{2n+1}, R_2\gamma_0 w) + (1-\beta)(f, R_2\gamma_0 w)_2, \quad \forall w \in \mathcal{V}_1, \end{aligned} \tag{3.4}$$

$$\begin{aligned} a_2(u_2^{2n+2}, w) = \; & \beta(f, w)_2 + (1-\beta) a_2(u_2^{2n+1}, w) - \beta\langle \eta, \gamma_0 w\rangle \\ & -\beta a_1(u_1^{2n+1}, R_1\gamma_0 w) + \beta(f, R_1\gamma_0 w)_1, \quad \forall w \in \mathcal{V}_2, \end{aligned} \tag{3.5}$$

where $\langle \cdot, \cdot \rangle$ denote the duality between $H_{00}^{1/2}(\Gamma)$ and its dual space.

To demonstrate the convergence, we introduce the operators T_1 and T_2 as follows:

1) $T_1 : \phi \in H_{00}^{\frac{1}{2}}(\Gamma) \to T_1\phi \in \mathcal{V}_1$ satisfying

$$L_1(T_1\phi) = 0 \text{ in } \Omega_1 \qquad \frac{\partial(T_1\phi)}{\partial \nu_A^1} = -\frac{\partial(R_2\phi)}{\partial \nu_A^2} \text{ on } \Gamma; \tag{3.6}$$

2) $T_2 : \phi \in H_{00}^{\frac{1}{2}}(\Gamma) \to T_2\phi \in \mathcal{V}_2$ satisfying

$$L_2(T_2\phi) = 0 \text{ in } \Omega_2, \qquad \frac{\partial(T_2\phi)}{\partial \nu_A^2} = -\frac{\partial(R_1\phi)}{\partial \nu_A^1} \text{ on } \Gamma, \tag{3.7}$$

which can be easily put in variational form

$$a_1(T_1\phi, w) = -a_2(R_2\phi, R_2\gamma_0 w), \quad \forall w \in \mathcal{V}_1; \tag{3.8}$$

$$a_2(T_2\phi, w) = -a_1(R_1\phi, R_1\gamma_0 w), \quad \forall w \in \mathcal{V}_2. \tag{3.9}$$

For convenience we define the following norms in $\mathcal{V}_1$ and $\mathcal{V}_2$,

$$\|w\|_k^2 = a_k(w, w), \qquad \forall w \in \mathcal{V}_k, \quad k = 1, 2. \tag{3.10}$$

We now begin the convergence analysis of the scheme (2.6)-(2.9) by noting that the error $e_k^n \equiv u_k - u_k^n \in \mathcal{V}_k$ satisfies the following equations:

$$L_1 e_1^{2n+1} = 0 \text{ in } \Omega_1, \qquad e_1^{2n+1} = \alpha e_1^{2n} + (1-\alpha)e_2^{2n} \text{ on } \Gamma; \tag{3.11}$$

$$L_2 e_2^{2n+1} = 0 \text{ in } \Omega_2, \qquad e_2^{2n+1} = \alpha e_1^{2n} + (1-\alpha)e_2^{2n} \text{ on } \Gamma; \tag{3.12}$$

$$L_1 e_1^{2n+2} = 0 \text{ in } \Omega_1, \qquad \frac{\partial e_1^{2n+2}}{\partial \nu_A^1} = \beta\frac{\partial e_1^{2n+1}}{\partial \nu_A^1} + (1-\beta)\frac{\partial e_2^{2n+1}}{\partial \nu_A^1} \text{ on } \Gamma; \tag{3.13}$$

$$L_2 e_2^{2n+2} = 0 \text{ in } \Omega_2, \qquad \frac{\partial e_2^{2n+2}}{\partial \nu_A^2} = \beta\frac{\partial e_1^{2n+1}}{\partial \nu_A^2} + (1-\beta)\frac{\partial e_2^{2n+1}}{\partial \nu_A^2} \text{ on } \Gamma. \tag{3.14}$$

Note that (3.13) and (3.14) have the following variational forms:

$$a_1(e_1^{2n+2}, w) = \beta a_1(e_1^{2n+1}, w) - (1-\beta)a_2(e_2^{2n+1}, R_2\gamma_0 w), \quad \forall w \in \mathcal{V}_1, \tag{3.15}$$

$$a_2(e_2^{2n+2}, w) = (1-\beta)a_2(e_2^{2n+1}, w) - \beta a_1(e_1^{2n+1}, R_1\gamma_0 w), \quad \forall w \in \mathcal{V}_2. \tag{3.16}$$

By (3.11)-(3.12),

$$e_1^{2n+1} = \alpha e_1^{2n} + (1-\alpha)R_1\gamma_0 e_2^{2n}, \tag{3.17}$$

$$e_2^{2n+1} = \alpha R_2\gamma_0 e_1^{2n} + (1-\alpha)e_2^{2n}. \tag{3.18}$$

Combining (3.8) and (3.15), we have

$$e_1^{2n+2} = \beta e_1^{2n+1} + (1-\beta)T_1\gamma_0 e_2^{2n+1}. \tag{3.19}$$

Combining (3.9) and (3.16), we have

$$e_2^{2n+2} = \beta T_2\gamma_0 e_1^{2n+1} + (1-\beta)e_2^{2n+1}. \tag{3.20}$$

Theorem 3.1 *Let*

$$\sigma = \sup_{\phi \in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\|R_1\phi\|_1^2}{\|R_2\phi\|_2^2}, \qquad \tau = \sup_{\phi \in H_{00}^{\frac{1}{2}}(\Gamma)} \frac{\|R_2\phi\|_2^2}{\|R_1\phi\|_1^2}. \tag{3.21}$$

Define

$$\rho_1(\alpha, \sigma, \tau) = \alpha^2 + (1-\alpha)^2\sigma^2 - \frac{2\alpha(1-\alpha)}{\tau}, \tag{3.22}$$

$$\rho_2(\beta, \sigma, \tau) = \beta^2 + (1-\beta)^2\tau^2 - \frac{2\beta(1-\beta)}{\sigma}. \tag{3.23}$$

Then,

$$\|e_1^{2n+2}\|_1^2 \le \rho_1(\alpha, \sigma, \tau)\rho_2(\beta, \sigma, \tau)\|e_1^{2n}\|_1^2, \tag{3.24}$$

$$\|e_2^{2n+2}\|_2^2 \le \rho_1(\alpha, \sigma, \tau)\rho_2(\beta, \sigma, \tau)\|e_2^{2n}\|_2^2. \tag{3.25}$$

Proof: We first look at the Neumann sweep. Note that, by (3.8) and (3.21),

$$\begin{aligned} \|T_1\phi\|_1^2 &= a_1(T_1\phi, T_1\phi) = -a_2(R_2\phi, R_2\gamma_0(T_1\phi)) \\ &\le \|R_2\phi\|_2\|R_2\gamma_0(T_1\phi)\|_2 \le \sqrt{\tau}\|R_2\phi\|_2\|R_1\gamma_0(T_1\phi)\|_1 \\ &= \sqrt{\tau}\|R_2\phi\|_2\|T_1\phi\|_1; \end{aligned} \tag{3.26}$$

that is,

$$\|T_1\phi\|_1 \le \sqrt{\tau}\|R_2\phi\|_2, \ \ \forall \phi \in H_{00}^{1/2}(\Gamma). \tag{3.27}$$

¿From (3.27) and (3.21) we have

$$\begin{aligned} \|T_1\gamma_0 e_2^{2n+1}\|_1 &\le \sqrt{\tau}\|R_2\gamma_0 e_2^{2n+1}\|_2 = \sqrt{\tau}\|R_2\gamma_0 e_1^{2n+1}\|_2 \\ &\le \tau\|R_1\gamma_0 e_1^{2n+1}\|_1 = \tau\|e_1^{2n+1}\|_1. \end{aligned} \tag{3.28}$$

Also by (3.8),

$$\begin{aligned} a_1(T_1\gamma_0 e_2^{2n+1}, e_1^{2n+1}) &= -a_2(e_2^{2n+1}, R_2\gamma_0 e_1^{2n+1}) = -a_2(e_2^{2n+1}, R_2\gamma_0 e_2^{2n+1}) \\ &= -a_2(e_2^{2n+1}, e_2^{2n+1}) = -\|e_2^{2n+1}\|_2^2. \end{aligned} \tag{3.29}$$

¿From (3.19), (3.28), and (3.29), we see that

$$\begin{aligned} \|e_1^{2n+2}\|_1^2 &= \beta^2\|e_1^{2n+1}\|_1^2 + 2\beta(1-\beta)a_1(e_1^{2n+1}, T_1\gamma_0 e_2^{2n+1}) + (1-\beta)^2\|T_1\gamma_0 e_2^{2n+1}\|_1^2 \\ &\le \beta^2\|e_1^{2n+1}\|_1^2 - 2\beta(1-\beta)\|e_2^{2n+1}\|_2^2 + (1-\beta)^2\tau^2\|e_1^{2n+1}\|_1^2 \\ &\le \left[\beta^2 - \frac{2\beta(1-\beta)}{\sigma} + (1-\beta)^2\tau^2\right]\|e_1^{2n+1}\|_1^2. \end{aligned} \tag{3.30}$$

Similarly to (3.27), (3.28), and (3.29), we can prove that

$$\|T_2\phi\|_2 \le \sqrt{\sigma}\|R_1\phi\|_1, \ \ \forall \phi \in H_{00}^{1/2}(\Gamma), \tag{3.31}$$

$$\|T_2\gamma_0 e_1^{2n+1}\|_2 \le \sqrt{\sigma}\|e_1^{2n+1}\|_1, \tag{3.32}$$

$$a_2(T_2\gamma_0 e_1^{2n+1}, e_2^{2n+1}) = -\|e_1^{2n+1}\|_1^2. \tag{3.33}$$

On the other hand, from (3.33),

$$\begin{aligned}\|e_2^{2n+1}\|_2^2 &= \|R_2\gamma_0 e_1^{2n+1}\|_2^2 \le \tau \|e_1^{2n+1}\|_1^2\\ &= -\tau a_2(T_2\gamma_0 e_1^{2n+1}, e_2^{2n+1}) \le \tau \|T_2\gamma_0 e_1^{2n+1}\|_2 \|e_2^{2n+1}\|_2,\end{aligned}$$

which gives

$$\|e_2^{2n+1}\|_2 \le \tau \|T_2\gamma_0 e_1^{2n+1}\|_2. \tag{3.34}$$

Combining (3.20), (3.31), (3.32), (3.33), and (3.34) we derive that

$$\begin{aligned}\|e_2^{2n+2}\|_2^2 &= \beta^2 \|T_2\gamma_0 e_1^{2n+1}\|_2^2 + 2\beta(1-\beta) a_2(T_2\gamma_0 e_1^{2n+1}, e_2^{2n+1}) + (1-\beta)^2 \|e_2^{2n+1}\|_2^2\\ &\le \beta^2 \|T_2\gamma_0 e_1^{2n+1}\|_2^2 - 2\beta(1-\beta)\|e_1^{2n+1}\|_1^2 + (1-\beta)^2\tau^2 \|T_2\gamma_0 e_2^{2n+1}\|_2^2\\ &\le \left[\beta^2 - \frac{2\beta(1-\beta)}{\sigma} + (1-\beta)^2\tau^2\right] \|T_2\gamma_0 e_2^{2n+1}\|_2^2.\end{aligned} \tag{3.35}$$

Next, consider the Dirichlet sweep. Notice that

$$a_1(e_1^{2n}, v|_{\Omega_1}) + a_2(e_2^{2n}, v|_{\Omega_2}) = 0, \quad \forall v \in H_0^1(\Omega), \tag{3.36}$$

which implies

$$a_1(e_1^{2n}, e_1^{2n}) + a_2(e_2^{2n}, R_2\gamma_0 e_1^{2n}) = 0. \tag{3.37}$$

Thus,

$$\|e_1^{2n}\|_1^2 \le \|e_2^{2n}\|_2 \|R_2\gamma_0 e_1^{2n}\|_2 \le \sqrt{\tau}\|e_2^{2n}\|_2 \|e_1^{2n}\|_1;$$

that is,

$$\|e_1^{2n}\|_1 \le \sqrt{\tau}\|e_2^{2n}\|_2. \tag{3.38}$$

Similarly,

$$\|e_2^{2n}\|_2 \le \sqrt{\sigma}\|e_1^{2n}\|_1. \tag{3.39}$$

¿From (3.17), (3.36), (3.38), and (3.39) we see that

$$\begin{aligned}&a_1(e_1^{2n+1}, e_1^{2n+1})\\ &\quad = \alpha^2 a_1(e_1^{2n}, e_1^{2n}) + 2\alpha(1-\alpha) a_1(e_1^{2n}, R_1\gamma_0 e_2^{2n}) + (1-\alpha)^2 a_1(R_1\gamma_0 e_2^{2n}, R_1\gamma_0 e_2^{2n})\\ &\quad = \alpha^2 a_1(e_1^{2n}, e_1^{2n}) - 2\alpha(1-\alpha) a_2(e_2^{2n}, e_2^{2n}) + (1-\alpha)^2 a_1(R_1\gamma_0 e_2^{2n}, R_1\gamma_0 e_2^{2n})\\ &\quad \le \alpha^2 \|e_1^{2n}\|_1^2 - \frac{2\alpha(1-\alpha)}{\tau}\|e_1^{2n}\|_1^2 + (1-\alpha)^2\sigma^2 \|e_1^{2n}\|_1^2\\ &\quad = \left[\alpha^2 - \frac{2\alpha(1-\alpha)}{\tau} + (1-\alpha)^2\sigma^2\right] \|e_1^{2n}\|_1^2.\end{aligned} \tag{3.40}$$

¿From (3.18), (3.9), (3.21), and (3.31), we obtain

$$
\begin{aligned}
\|T_2\gamma_0 e_2^{2n+1}\|_2^2 &= \|\alpha T_2\gamma_0(R_2\gamma_0 e_1^{2n}) + (1-\alpha)T_2\gamma_0 e_2^{2n}\|_2^2 \\
&= \|\alpha e_2^{2n} + (1-\alpha)T_2\gamma_0 e_2^{2n}\|_2^2 \\
&= \alpha^2 a_2(e_2^{2n}, e_2^{2n}) + 2\alpha(1-\alpha)a_2(e_2^{2n}, T_2\gamma_0 e_2^{2n}) + (1-\alpha)^2 a_2(T_2\gamma_0 e_2^{2n}, T_2\gamma_0 e_2^{2n}) \\
&\le \alpha^2 a_2(e_2^{2n}, e_2^{2n}) - 2\alpha(1-\alpha)a_1(R_1\gamma_0 e_2^{2n}, R_1\gamma_0 e_2^{2n}) + (1-\alpha)^2\sigma^2\|e_2^{2n}\|_2^2 \\
&= \left[\alpha^2 - \frac{2\alpha(1-\alpha)}{\tau} + (1-\alpha)^2\sigma^2\right]\|e_2^{2n}\|_2^2,
\end{aligned}
\tag{3.41}
$$

where we have used the fact that $T_2\gamma_0 e_1^{2n} = e_2^{2n}$. Inequalities (3.30) and (3.40) lead to (3.24), and (3.35) and (3.41) lead to (3.25). The proof of the theorem is now complete. ∎

Corollary 3.1 *The iterative scheme converges in the energy norm if we choose α and β such that* $\max\{0, 1 - \frac{2(\tau+1)}{\tau\sigma^2+\tau+2}\} < \alpha < 1$, *and* $\max\{0, 1 - \frac{2(\sigma+1)}{\tau^2\sigma+\sigma+2}\} < \beta < 1$. *The optimal relaxation parameters are* $\alpha = \frac{\sigma^2\tau+1}{\tau\sigma^2+\tau+2}$ *and* $\beta = \frac{\sigma\tau^2+1}{\tau^2\sigma+\sigma+2}$.

Corollary 3.2 *Let $\sigma = \tau = 1$, which is true when Ω_1 is the reflection of Ω_2 across Γ and the coefficients $a_{ij}^{(k)}$ and $b^{(k)}$, $k = 1, 2$, are symmetric with respect to the interface Γ. If $\alpha = \beta = \frac{1}{2}$, then the iterative method (2.6)-(2.9) is convergent in the energy norm in two iterations no matter what the initial guesses are.*

This agrees with the results of Rice, Vavalis, and the author in [36], where the explicit solution of the Helmholz operator on a rectangular domain is used to obtain optimal values for the relaxation parameters.

It should be noted that, in general, the constants σ and τ are not easy to obtain. Nevertheless, when applying the finite difference or the finite element version of the iterative method, the discrete counterparts of σ and τ can be approximated. The evaluation of these constants can help one design algorithms with approximately optimal convergent rates [33].

4 Finite Element Approximation

For an easy presentation we assume that $g = 0$ in (2.2). To avoid technical difficulty, we also assume that $\mu = 0$ in (2.3); that is, the solution is assumed to be continuous across the interface. Let $T_h = \{T\}$ be a regular triangulation of Ω with no elements crossing the interface Γ; h is the grid size. Define a conforming or nonconforming finite element space [33] [20]

$$
W^h = \{w \in C(\bar{\Omega}) : \ w|_T \in P_r(T) \ \text{ for } \ \forall T \in T_h, w|_{\partial\Omega} = 0\},
\tag{4.1}
$$

where $P_r(T)$ denotes the space of polynomials of degree $\le r$ on T. The finite element approximation of the problem (2.1)-(2.4) reads: Find $u^h \in W^h$ such that

$$
a(u^h, w) = (f, w)_1 + (f, w)_2 + \langle \eta, \gamma_0 w\rangle, \qquad \forall w \in W^h,
\tag{4.2}
$$

where $a(u^h, w) = a(u^h|_{\Omega_1}, w|_{\Omega_1}) + a(u^h|_{\Omega_2}, w|_{\Omega_2})$. By standard finite element theory,

$$\|u - u^h\|_{H^1(\Omega)} \leq Ch^r \|u\|_{H^{r+1}(\Omega)}. \tag{4.3}$$

Now, we define finite element spaces over the subdomains:

$$W_k^h = \{w \in H^1(\Omega_k) : \ w|_T \in P_r(T) \text{ for } \forall T \in T_h, \ w|_{\partial\Omega \cap \partial\Omega_k} = 0\}, \tag{4.4}$$

$$M_k^h = \{w \in W_k^h : w|_{\partial\Omega_k} = 0\}. \tag{4.5}$$

Let Σ_h denote the subdivision of Γ induced by T_h. Set

$$\Phi^h = \{\phi \in C(\Gamma) : \ \phi|_I \in P_r(I) \text{ for } \forall I \in \Sigma_h, \ \phi|_{\partial\Gamma} = 0\}. \tag{4.6}$$

For $k = 1, 2$, we define the discrete extension operators $R_k^h : \Phi^h \to W_k^h$ by

$$R_k^h\phi \in W_k^h : \ a_k(R_k^h\phi, w) = 0, \quad \forall w \in M_k^h; \quad R_k^h\phi = \phi \text{ on } \Gamma. \tag{4.7}$$

Let U_k^n be the finite element approximation of u_k^n. We define the following discrete version of our domain decomposition method: Choose $U_k^0 \in W_k^h, \ k = 1, 2$. For $n = 0, 1, 2, \cdots$, find $U_k^{2n+1}, U_k^{2n+2} \in W_k^h$ such that

$$a_1(U_1^{2n+1}, w) = (f, w)_1, \quad \forall w \in M_1^h, \quad U_1^{2n+1} = \alpha U_1^{2n} + (1-\alpha)U_2^{2n} \text{ on } \Gamma; \tag{4.8}$$

$$a_2(U_2^{2n+1}, w) = (f, w)_2, \quad \forall w \in M_2^h, \quad U_2^{2n+1} = \alpha U_1^{2n} + (1-\alpha)U_2^{2n} \text{ on } \Gamma; \tag{4.9}$$

$$\begin{aligned} a_1(U_1^{2n+2}, w) &= (1-\beta)(f, w)_1 + (1-\beta)(f, R_2^h\gamma_0 w)_2 + (1-\beta)\langle \eta, \gamma_0 w\rangle \\ &\quad + \beta a_1(U_1^{2n+1}, w) - (1-\beta)a_2(U_2^{2n+1}, R_2^h\gamma_0 w), \quad \forall w \in W_1^h; \end{aligned} \tag{4.10}$$

$$\begin{aligned} a_2(U_2^{2n+2}, w) &= \beta(f, R_1^h\gamma_0 w)_1 + \beta(f, w)_2 - \beta\langle \eta, \gamma_0 w\rangle \\ &\quad - \beta a_1(U_1^{2n+1}, R_1^h\gamma_0 w) + (1-\beta)a_2(U_2^{2n+1}, w), \quad \forall w \in W_2^h. \end{aligned} \tag{4.11}$$

In order to carry out the convergence analysis, we define the errors for $n = 0, 1, 2, \cdots$, by

$$E_k^n = u^h|_{\Omega_k} - U_k^n, \quad k = 1, 2.$$

Then, combining (4.8)-(4.11) and (4.2) we have

$$a_1(E_1^{2n+1}, w) = 0, \quad \forall w \in M_1^h, \quad E_1^{2n+1} = \alpha E_1^{2n} + (1-\alpha)E_2^{2n} \text{ on } \Gamma; \tag{4.12}$$

$$a_2(E_2^{2n+1}, w) = 0, \quad \forall w \in M_2^h, \quad E_2^{2n+1} = \alpha E_1^{2n} + (1-\alpha)E_2^{2n} \text{ on } \Gamma; \tag{4.13}$$

$$a_1(E_1^{2n+2}, w) = \beta a_1(E_1^{2n+1}, w) - (1-\beta)a_2(E_2^{2n+1}, R_2^h\gamma_0 w), \quad \forall w \in W_1^h; \tag{4.14}$$

$$a_2(E_2^{2n+2}, w) = -\beta a_1(E_1^{2n+1}, R_1^h\gamma_0 w) + (1-\beta)a_2(E_2^{2n+1}, w), \quad \forall w \in W_2^h. \tag{4.15}$$

Now, define the discrete counterparts of the operators T_1 and T_2 in an analogous way to (3.8) and (3.9):

$$\begin{aligned} &T_1^h : \phi \in \Phi^h \to T_1^h\phi \in W_1^h \text{ such that} \\ &a_1(T_1^h\phi, w) = -a_2(R_2^h\phi, R_2^h\gamma_0 w), \quad \forall w \in W_1^h, \end{aligned} \tag{4.16}$$

$$\begin{aligned} &T_2^h : \phi \in \Phi^h \to T_2^h\phi \in W_2^h \text{ such that} \\ &a_2(T_2^h\phi, w) = -a_1(R_1^h\phi, R_1^h\gamma_0 w), \quad \forall w \in W_2^h, \end{aligned} \tag{4.17}$$

Then, combining (4.12)-(4.15) and (4.16)-(4.17), we obtain the following error equations

$$E_1^{2n+1} = \alpha E_1^{2n} + (1-\alpha) R_1^h \gamma_0 E_2^{2n}; \tag{4.18}$$

$$E_2^{2n+1} = \alpha R_2^h \gamma_0 E_1^{2n} + (1-\alpha) E_2^{n}; \tag{4.19}$$

$$a_1(E_1^{2n+2}, w) = \beta a_1(E_1^{2n+1}, w) + (1-\beta) a_1(T_1^h \gamma_0 E_2^{2n+1}, w), \quad \forall w \in W_1^h; \tag{4.20}$$

$$a_2(E_2^{2n+2}, w) = \beta a_2(T_2^h \gamma_0 E_1^{2n+1}, w) + (1-\beta) a_2(E_2^{2n+1}, w), \quad \forall w \in W_2^h. \tag{4.21}$$

It can be shown that there exist finite positive numbers $\bar{\sigma}$ and $\bar{\tau}$ that do not depend on the finite element grid size h such that

$$\sup_{\phi \in \Phi^h} \frac{\|R_1^h \phi\|_1^2}{\|R_2^h \phi\|_2^2} \le \bar{\sigma}, \qquad \sup_{\phi \in \Phi^h} \frac{\|R_2^h \phi\|_2^2}{\|R_1^h \phi\|_1^2} \le \bar{\tau}; \tag{4.22}$$

(see [4, Lemma 5.1] and [33, Lemma 4.1]). In analogy with the proofs of Theorem 3.1 and Corollary 3.1, from (4.18)-(4.21) we can easily establish the following results.

Theorem 4.1 *For the finite element version (4.8)-(4.11) of our domain decomposition method,*

$$\|E_1^{2n+2}\|_1^2 + \|E_2^{2n+2}\|_2^2 \le \rho_1(\alpha, \bar{\sigma}, \bar{\tau}) \rho_2(\beta, \bar{\sigma}, \bar{\tau}) \left(\|E_1^{2n}\|_1^2 + \|E_2^{2n}\|_2^2 \right), \tag{4.23}$$

where ρ_1 and ρ_2 are defined by (3.22) and (3.23).

Corollary 4.1 *The domain decomposition method (4.8)-(4.11) is convergent in the energy norm if* $\max\{0, 1 - \frac{2(\bar{\tau}+1)}{\bar{\tau}\bar{\sigma}^2 + \bar{\tau} + 2}\} < \alpha < 1$ *and* $\max\{0, 1 - \frac{2(\bar{\sigma}+1)}{\bar{\tau}^2 \sigma + \bar{\sigma} + 2}\} < \beta < 1$. *In particular, if* $\alpha = \beta = \frac{1}{2}$ *and* $\bar{\sigma} = \bar{\tau} = 1$ *then the procedure converges to the finite element solution u^h, defined in (4.2), in two iterations.*

Corollary 4.2 *The rate of convergence of the finite element domain decomposition method (4.8)-(4.11) is independent of the grid size.*

5 Finite Element Approximation with Lagrange Multipliers

In this section, we will introduce Lagrange multipliers in place of the conormal derivatives at the interface. As one may expect, this will give rise to a simpler algorithm and have the capability of dealing with non-matching grids.

We assume again that $g = 0$ in (2.2). Then the variational formulation of the scheme (2.6)-(2.9) can be written

$$\begin{cases} a_1(u_1^{2n+1}, w) - \left\langle \dfrac{\partial u_1^{2n+1}}{\partial \nu_A^1}, \gamma_0 w \right\rangle = (f, w)_1, & \forall w \in \mathcal{V}_1, \\ u_1^{2n+1} = \alpha u_1^{2n} + (1-\alpha) u_2^{2n} + (1-\alpha)\mu \text{ on } \Gamma; \end{cases} \tag{5.1}$$

$$\begin{cases} a_2(u_2^{2n+1}, w) - \left\langle \dfrac{\partial u_2^{2n+1}}{\partial \nu_A^2}, \gamma_0 w \right\rangle = (f, w)_2, & \forall w \in \mathcal{V}_2, \\ u_2^{2n+1} = \alpha u_1^{2n} + (1-\alpha) u_2^{2n} - \alpha\mu \text{ on } \Gamma; \end{cases} \tag{5.2}$$

$$(5.3)\qquad \begin{aligned} &a_1(u_1^{2n+2}, w) \\ &= \beta\Big\langle \frac{\partial u_1^{2n+1}}{\partial \nu_A^1}, \gamma_0 w\Big\rangle + (1-\beta)\Big\langle \frac{\partial u_2^{2n+1}}{\partial \nu_A^1} + \eta, \gamma_0 w\Big\rangle + (f,w)_1,\ \forall w \in \mathcal{V}_1;\end{aligned}$$

$$(5.4)\qquad \begin{aligned} &a_2(u_2^{2n+2}, w) \\ &= \beta\Big\langle \frac{\partial u_1^{2n+1}}{\partial \nu_A^2} - \eta, \gamma_0 w\Big\rangle + (1-\beta)\Big\langle \frac{\partial u_2^{2n+1}}{\partial \nu_A^2}, \gamma_0 w\Big\rangle + (f,w)_2,\ \forall w \in \mathcal{V}_2;\end{aligned}$$

for $k = 1, 2$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product over the interface Γ.

Replacing the conormal derivatives $\dfrac{\partial u_k^n}{\partial \nu_A^k}$ by the Lagrange multipliers λ_k^n, (5.1)-(5.4) become

$$(5.5)\qquad \begin{cases} a_1(u_1^{2n+1}, w) - \langle \lambda_1^{2n+1}, \gamma_0 w\rangle = (f,w)_1, & \forall w \in \mathcal{V}_1, \\ u_1^{2n+1} = \alpha u_1^{2n} + (1-\alpha) u_2^{2n} + (1-\alpha)\mu \text{ on } \Gamma; \end{cases}$$

$$(5.6)\qquad \begin{cases} a_2(u_2^{2N+1}, w) - \langle \lambda_2^{2n+1}, \gamma_0 w\rangle = (f,w)_2, & \forall w \in \mathcal{V}_2, \\ u_2^{2n+1} = \alpha u_1^{2n} + (1-\alpha) u_2^{2n} - \alpha\mu \text{ on } \Gamma; \end{cases}$$

$$(5.7)\qquad a_1(u_1^{2n+2}, w) = \langle \beta\lambda_1^{2n+1} - (1-\beta)\lambda_2^{2n+1} + (1-\beta)\eta, \gamma_0 w\rangle + (f,w)_1,\ \ \forall w \in \mathcal{V}_1;$$

$$(5.8)\qquad a_2(u_2^{2n+2}, w) = \langle -\beta\lambda_1^{2n+1} + (1-\beta)\lambda_2^{2n+1} - \beta\eta, \gamma_0 w\rangle + (f,w)_2,\ \ \forall w \in \mathcal{V}_2.$$

The procedure (5.5)-(5.8) is the domain decomposition method with Lagrange multipliers at the continuous level, a variant of (2.6)-(2.9). We now formulate its finite element version. Let $T_h = \{T\}$ be a regular triangulation of Ω with no elements crossing the interface Γ. As in the last section, we define finite element spaces

$$(5.9)\qquad W_k^h = \{w \in H^1(\Omega_k) :\ \ w|_T \in P_r(T)\ \ \forall T \in T_h, w|_{\partial\Omega\cap\partial\Omega_k} = 0\},\ \ k = 1, 2,$$

where $P_r(T)$ denotes the space of polynomials of degree $\leq r$ on T. Let Z^h be the space of the restrictions on the interface of the functions in W_k^h. Note that there are two copies of such space assigned on Γ, one from Ω_1 and the other from Ω_2. We denote them by Z_1^h and Z_2^h, respectively. Let $\{U_k^n, \Lambda_k^n\} \in W_k^h \times Z_k^h$ denote the finite element approximation of $\{u_k^n, \lambda_k^n\}$. Then, the finite element domain decomposition method with Lagrange multipliers is constructed as follows:

$$(5.10)\qquad \begin{cases} a_1(U_1^{2n+1}, w) - \langle \Lambda_1^{2n+1}, \gamma_0 w\rangle = (f,w)_1, & \forall w \in W_1^h, \\ U_1^{2n+1} = \alpha U_1^{2n} + (1-\alpha) U_2^{2n} + (1-\alpha)\mu \text{ on } \Gamma; \end{cases}$$

$$(5.11)\qquad \begin{cases} a_2(U_2^{2n+1}, w) - \langle \Lambda_2^{2n+1}, \gamma_0 w\rangle = (f,w)_2, & \forall w \in W_2^h, \\ U_2^{2n+1} = \alpha U_1^{2n} + (1-\alpha) U_2^{2n} - \alpha\mu \text{ on } \Gamma; \end{cases}$$

$$(5.12)\qquad a_1(U_1^{2n+2}, w) = \langle \beta\Lambda_1^{2n+1} - (1-\beta)\Lambda_2^{2n+1} + (1-\beta)\eta, \gamma_0 w\rangle + (f,w)_1,\ \ \forall w \in W_1^h;$$

$$(5.13)\qquad a_2(U_2^{2n+2}, w) = \langle -\beta\Lambda_1^{2n+1} + (1-\beta)\Lambda_2^{2n+1} - \beta\eta, \gamma_0 w\rangle + (f,w)_2,\ \ \forall w \in W_2^h.$$

Note that the finite element approximation with Lagrange multipliers on the interface of the problem (2.1)-(2.4) reads: find $\{u_k^h, \lambda_k^h\} \in W_k^h \times Z_k^h$ such that

$$(5.14)\qquad a_k(u_k^h, w) - \langle \lambda_k^h, \gamma_0 w\rangle = (f,w)_k, \qquad \forall w \in W_k^h,\ \ k = 1, 2,$$

$$(5.15)\qquad u_1^h - u_2^h = \mu \text{ on } \Gamma,$$

$$(5.16)\qquad \lambda_1^h + \lambda_2^h = \eta \text{ on } \Gamma.$$

Let

$$E_k^n = u_k^h - U_k^n, \quad r_k^n = \lambda_k^h - \Lambda_k^n, \quad k = 1, 2.$$

Then, combining (5.10)-(5.13) and (5.14)-(5.16) leads to the error equations

$$a_1(E_1^{2n+1}, w) - \langle r_1^{2n+1}, \gamma_0 w\rangle = 0, \quad \forall w \in W_1^h, \quad E_1^{2n+1} = \alpha E_1^{2n} + (1-\alpha)E_2^{2n} \text{ on } \Gamma; \tag{5.17}$$

$$a_2(E_2^{2n+1}, w) - \langle r_2^{2n+1}, \gamma_0 w\rangle = 0, \quad \forall w \in W_2^h, \quad E_2^{2n+1} = \alpha E_1^{2n} + (1-\alpha)E_2^{2n} \text{ on } \Gamma; \tag{5.18}$$

$$a_1(E_1^{2n+2}, w) = \langle \beta r_1^{2n+1} - (1-\beta) r_2^{2n+1}, \gamma_0 w\rangle, \quad \forall w \in W_1^h; \tag{5.19}$$

$$a_2(E_2^{2n+2}, w) = \langle -\beta r_1^{2n+1} + (1-\beta) r_2^{2n+1}, \gamma_0 w\rangle, \quad \forall w \in W_2^h. \tag{5.20}$$

In order to analyze the error equations (5.17)-(5.20), we need to introduce some notation. Let $\Phi^h = \{w|_\Gamma : \quad w \in W_k^h, \ k = 1, 2\}$. Define the extension operators $R_k : \phi \in \Phi^h \to \{R_k^1\phi, R_k^2\phi\} \in W_k^h \times Z_k^h$ by

$$a_k(R_k^1\phi, w) - \langle R_k^2\phi, w\rangle = 0, \quad \forall w \in W_k^h, \quad R_k^1\phi = \phi \text{ on } \Gamma, \tag{5.21}$$

and the linear operators T_1^h and T_2^h by

1) $T_1^h : \phi \in \Phi^h \to T_{21}^h\phi \in W_1^h$,

$$a_1(T_{21}^h\phi, w) = -a_2(R_2^1\phi, R_2^1\gamma_0 w), \quad \forall w \in W_1^h. \tag{5.22}$$

2) $T_2^h : \phi \in \Phi^h \to T_{12}^h\phi \in W_1^h$,

$$a_2(T_{12}^h\phi, w) = -a_1(R_1^1\phi, R_1^1\gamma_0 w), \quad \forall w \in W_2^h. \tag{5.23}$$

We now begin to derive error reduction estimates. From (5.17)-(5.18) and (5.21), we have

$$E_1^{2n+1} = \alpha E_1^{2n} + (1-\alpha) R_1^1\gamma_0 E_2^{2n}, \tag{5.24}$$

$$E_2^{2n+1} = \alpha R_2^1\gamma_0 E_1^{2n} + (1-\alpha) E_2^{2n}. \tag{5.25}$$

¿From (5.19)-(5.20) and (5.22)-(5.23), we have

$$E_1^{2n+2} = \beta E_1^{2n+1} + (1-\beta) T_1^h\gamma_0 E_2^{2n+1}, \tag{5.26}$$

$$E_2^{2n+2} = \beta T_2^h\gamma_0 E_1^{2n+1} + (1-\beta) E_2^{2n+1}. \tag{5.27}$$

Define $\bar\sigma$ and $\bar\tau$ such that

$$\sup_{\phi\in\Phi^h} \frac{\|R_1^1\phi\|_1^2}{\|R_2^1\phi\|_2^2} \le \bar\sigma, \qquad \sup_{\phi\in\Phi^h} \frac{\|R_2^1\phi\|_2^2}{\|R_1^1\phi\|_1^2} \le \bar\tau. \tag{5.28}$$

In analogy with the proofs of Theorem 3.1 and Corollary 3.1, from (5.24)-(5.27) we can easily establish the following results.

Theorem 5.1 *For the finite element domain decomposition method with Lagrange multipliers (5.10)-(5.13),*

$$\|E_1^{2n+2}\|_1^2 + \|E_2^{2n+2}\|_2^2 \le \rho_1(\alpha, \bar\sigma, \bar\tau)\rho_2(\beta, \bar\sigma, \bar\tau)\left(\|E_1^{2n}\|_1^2 + \|E_2^{2n}\|_2^2\right), \tag{5.29}$$

where ρ_1 and ρ_2 are defined by (3.22) and (3.23).

Corollary 5.1 *The domain decomposition method (5.10)-(5.13) is convergent in the energy norm if* $\max\{0, 1 - \frac{2(\bar\tau+1)}{\bar\tau\bar\sigma^2+\bar\tau+2}\} < \alpha < 1$ *and* $\max\{0, 1 - \frac{2(\bar\sigma+1)}{\bar\tau^2\sigma+\bar\sigma+2}\} < \beta < 1$.

6 Hybrid Mixed Finite Element Approximation

In this section, we will introduce Lagrange multipliers in place of the solution values at the interface. Let flux $q_k = -A_k \nabla u_k$, where $A_k = (a_{ij}^{(k)})$ is a 2×2 matrix. The mixed differential form is as follows:

$$A_k^{-1} q_k + \nabla u_k = 0 \text{ in } \Omega_k, \tag{6.1}$$

$$\text{div } q_k + b^{(k)} u_k = f \text{ in } \Omega_k, \tag{6.2}$$

$$u_k = 0 \text{ on } \partial\Omega_k \cap \partial\Omega, \tag{6.3}$$

$$u_1 - u_2 = \mu \text{ on } \Gamma, \tag{6.4}$$

$$q_1 \cdot \nu_1 + q_2 \cdot \nu_2 = \eta \text{ on } \Gamma, \tag{6.5}$$

Again, we assumed that $g = 0$ in (2.2).

Define subdomain spaces: $V_k = H(\text{div}, \Omega_k)$, $W_k = L^2(\Omega_k)$, $k = 1, 2$. Then a formal iterative algorithm in the mixed differential case would be as follows: select the initial guess $\{q_k^0, u_k^0\} \in V_k \times W_k$, $k = 1, 2$, and compute $\{q_k^{2n+1}, u_k^{2n+1}\} \in V_k \times W_k$ and $\{q_k^{2n+2}, u_k^{2n+2}\} \in V_k \times W_k$ satisfying:

$$\left\{ \begin{array}{l} (A_1^{-1} q_1^{2n+1}, v)_1 - (u_1^{2n+1}, \text{div } v)_1 = \\ \qquad -\langle \alpha u_1^{2n} + (1-\alpha) u_2^{2n} + (1-\alpha)\mu, v \cdot \nu_1 \rangle, \ \forall v \in V_1, \\ (\text{div } q_1^{2n+1}, w)_1 + (b^{(1)} u_1^{2n+1}, w)_1 = (f, w)_1, \quad \forall w \in W_1; \end{array} \right. \tag{6.6}$$

$$\left\{ \begin{array}{l} (A_2^{-1} q_2^{2n+1}, v)_2 - (u_2^{2n+1}, \text{div } v)_2 = \\ \qquad -\langle \alpha u_1^{2n} + (1-\alpha) u_2^{2n} - \alpha\mu, v \cdot \nu_2 \rangle, \ \forall v \in V_2, \\ (\text{div } q_2^{2n+1}, w)_2 + (b^{(2)} u_2^{2n+1}, w)_2 = (f, w)_2, \quad \forall w \in W_2; \end{array} \right. \tag{6.7}$$

and

$$\left\{ \begin{array}{l} (A_1^{-1} q_1^{2n+2}, v)_1 - (u_1^{2n+2}, \text{div } v)_1 + \langle u_1^{2n+2}, v \cdot \nu_1 \rangle = 0, \quad \forall v \in V_1, \\ (\text{div } q_1^{2n+2}, w)_1 + (b^{(1)} u_1^{2n+2}, w)_1 = (f, w)_1, \quad \forall w \in W_1, \\ q_1^{2n+2} \cdot \nu_1 = \beta q_1^{2n+1} \cdot \nu_1 + (1-\beta) q_2^{2n+1} \cdot \nu_1 + (1-\beta)\eta \quad \text{on} \quad \Gamma; \end{array} \right. \tag{6.8}$$

$$\left\{ \begin{array}{l} (A_2^{-1} q_2^{2n+2}, v)_2 - (u_2^{2n+2}, \text{div } v)_2 + \langle u_2^{2n+2}, v \cdot \nu_2 \rangle = 0, \quad \forall v \in V_2, \\ (\text{div } q_2^{2n+2}, w)_2 + (b^{(2)} u_2^{2n+2}, w)_2 = (f, w)_2, \quad \forall w \in W_2, \\ q_2^{2n+2} \cdot \nu_2 = \beta q_1^{2n+1} \cdot \nu_2 + (1-\beta) q_2^{2n+1} \cdot \nu_2 - \beta\eta \quad \text{on} \quad \Gamma. \end{array} \right. \tag{6.9}$$

Since the restriction of an L^2 function on the interface Γ is not clear, we will introduce Lagrange multipliers [1] for u_k on the interface Γ and limit the discussion below to the discrete approximate problem.

Let T_h be a subdivision of the domain Ω with no elements crossing the interface Γ, and let $V_k^h \times W_k^h \subset H(\text{div}, \Omega_k) \times L^2(\Omega_k)$ be mixed finite element spaces [6, 7, 8, 9, 10, 35]. We introduce Lagrange multipliers [1] at the interface Γ in place of the solution values, instead of solution derivatives. Assume that the normal component $q_k \cdot \nu_k$ on the interface Γ is a polynomial of some fixed degree τ , where for simplicity we assume τ constant on the edges of the elements. Let Y_k^h be the space of all such polynomials from Ω_k. Then, the hybridized mixed finite element method is given by seeking $\{q_k^h, u_k^h, \lambda_k^h\} \in V_k^h \times W_k^h \times Y_k^h$ such that

$$(A^{-1} q_k^h, v)_k - (u_k^h, \text{div } v)_k + \langle \lambda_k^h, v \cdot \nu_k \rangle_\Gamma = 0, \quad \forall v \in V_k, \ k = 1, 2, \tag{6.10}$$

$$(6.11) \qquad (\text{div } q_k^h, w)_k + (b^{(k)}u_k^h, w)_k = (f, w)_k, \quad \forall w \in W_k,\ k = 1,2,$$

$$(6.12) \qquad \lambda_1^h - \lambda_2^h = \mu \quad \text{on} \quad \Gamma,$$

$$(6.13) \qquad q_1^h \cdot \nu_1 + q_2^h \cdot \nu_2 = \eta \quad \text{on} \quad \Gamma.$$

We are now in a position to formulate our hybrid mixed finite element domain decomposition method. Choose the initial guess $\{Q_k^0, U_k^0, \Lambda_k^0\} \in V_k^h \times W_k^h \times Y_k^h$ arbitrarily. For $n = 0, 1, 2, \cdots$, we construct the functions $\{Q_k^{2n+1}, U_k^{2n+1}\} \in V_k^h \times W_k^h$ and $\{Q_k^{2n+2}, U_k^{2n+2}, \Lambda_k^{2n+2}\} \in V_k^h \times W_k^h \times Y_k^h$ satisfying:

$$(6.14) \qquad \begin{cases} (A_1^{-1}Q_1^{2n+1}, v)_1 - (U_1^{2n+1}, \text{div } v)_1 = \\ \qquad -\langle \alpha\Lambda_1^{2n} + (1-\alpha)\Lambda_2^{2n} + (1-\alpha)\mu, v\cdot\nu_1\rangle,\ \forall v \in V_1, \\ (\text{div } Q_1^{2n+1}, w)_1 + (b^{(1)}U_1^{2n+1}, w)_1 = (f, w)_1,\ \forall w \in W_1; \end{cases}$$

$$(6.15) \qquad \begin{cases} (A_2^{-1}Q_2^{2n+1}, v)_2 - (U_2^{2n+1}, \text{div } v)_2 = \\ \qquad -\langle \alpha\Lambda_1^{2n} + (1-\alpha)\Lambda_2^{2n} - \alpha\mu, v\cdot\nu_2\rangle,\ \forall v \in V_2, \\ (\text{div } Q_2^{2n+1}, w)_2 + (b^{(2)}U_2^{2n+1}, w)_2 = (f, w)_2, \quad \forall w \in W_2; \end{cases}$$

and

$$(6.16) \qquad \begin{cases} (A_1^{-1}Q_1^{2n+2}, v)_1 - (U_1^{2n+2}, \text{div } v)_1 + \langle \Lambda_1^{2n+2}, v\cdot\nu_1\rangle = 0, \quad \forall v \in V_1, \\ (\text{div } Q_1^{2n+2}, w)_1 + (b^{(1)}U_1^{2n+2}, w)_1 = (f, w)_1, \quad \forall w \in W_1, \\ Q_1^{2n+2}\cdot\nu_1 = \beta Q_1^{2n+1}\cdot\nu_1 + (1-\beta)Q_2^{2n+1}\cdot\nu_1 + (1-\beta)\eta \text{ on } \Gamma; \end{cases}$$

$$(6.17) \qquad \begin{cases} (A_2^{-1}Q_2^{2n+2}, v)_2 - (U_2^{2n+2}, \text{div } v)_2 + \langle \Lambda_2^{2n+2}, v\cdot\nu_2\rangle = 0, \quad \forall v \in V_2, \\ (\text{div } Q_2^{2n+2}, w)_2 + (b^{(2)}U_2^{2n+2}, w)_2 = (f, w)_2, \quad \forall w \in W_2, \\ Q_2^{2n+2}\cdot\nu_2 = \beta Q_1^{2n+1}\cdot\nu_2 + (1-\beta)Q_2^{2n+1}\cdot\nu_2 - \beta\eta \text{ on } \Gamma. \end{cases}$$

Analysis of the convergence of the iterates $\{Q_k^n, U_k^n, \Lambda_k^n\}$ of (6.14)-(6.17) towards the mixed finite element solution $\{q_k^h, u_k^h, \lambda_k^h\}$ of (6.10)-(6.13) requires a rather lengthy and delicate argument and might be considered later.

7 Numerical Examples

In this section, we present some numerical experiments for the iterative procedure (2.6)-(2.9) using a nine point finite difference discretization. In all of our test, we let the domain $\Omega = (0,1)\times(0,1)$ and the interface Γ be at the line $x = 0.5$, which divides the domain into $\Omega_1 = (0, 0.5)\times(0,1)$ and $\Omega_2 = (0.5, 1)\times(0,1)$. The relaxation parameters α and β are taken to be 0.5. Zero initial guess is used.

Example 7.1 *Let*

$$-\frac{\partial}{\partial x}(e^x\frac{\partial u_1}{\partial x}) - \frac{\partial}{\partial y}(e^y\frac{\partial u_1}{\partial y}) + \frac{1}{1+x+y}u_1 = f_1, \qquad in\ \Omega_1,$$
$$-\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^2 u_2}{\partial y^2} = f_2, \qquad in\ \Omega_2,$$
$$u_k = g_k,\ on\ \partial\Omega_k \cap \partial\Omega, k = 1,2.$$

The functions f_1, f_2, g_1, g_2, μ, and η are chosen such that the exact solution is

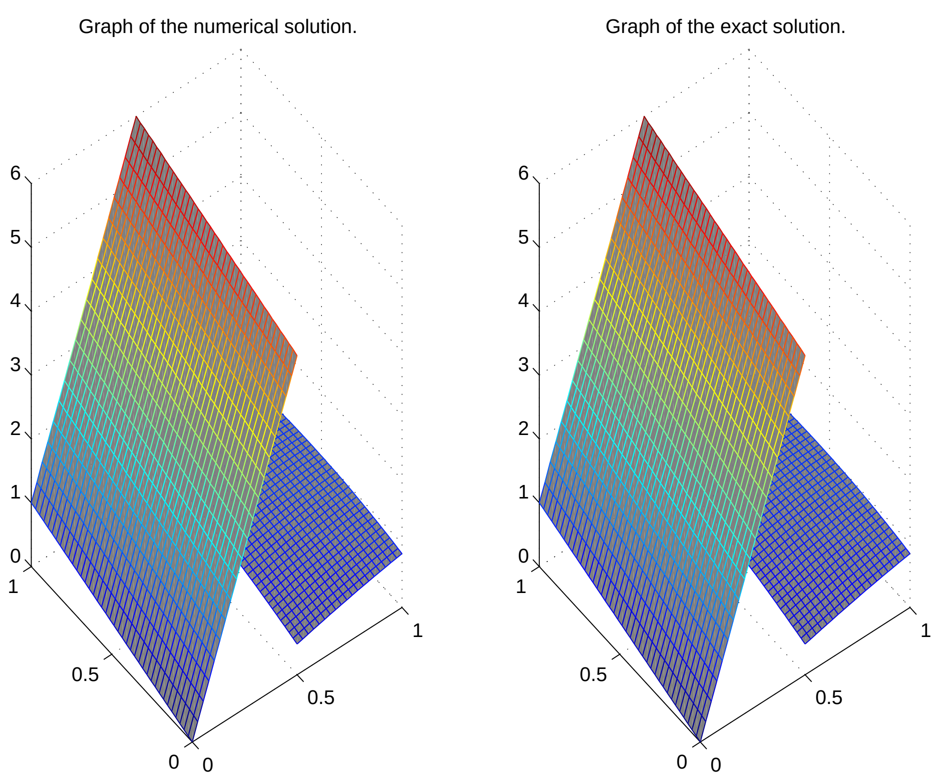
$$u_1(x,y) = 10x + y,\ in\ \Omega_1, \quad u_2(x,y) = sin(x+y),\ in\ \Omega_2.$$

Table 7.1 shows the results for the iterative and true errors in the maximum norm on the whole domain Ω. Figure 7.1 gives the plots of the true solution and the numerical solution (with grid size $\frac{1}{40} \times \frac{1}{40}$), which shows a good resolution of the sharp interface for the approximate solution.

Table 7.1: Iterative and true errors for Example 7.1 with interface at $x = 0.5$. The errors are shown in the L^∞-norm.

	Grid size $\frac{1}{40} \times \frac{1}{40}$		Grid size $\frac{1}{80} \times \frac{1}{80}$	
Iteration	Iterative error	True error	Iterative error	True error
1	9.17E-1	7.50E-1	9.61E-1	8.92E-1
2	9.39E-2	7.45E-2	1.10E-1	9.58E-2
3	9.27E-3	6.44E-3	1.22E-2	1.06E-2
4	9.24E-4	6.40E-4	1.35E-3	1.13E-3
5	9.18E-5	3.12E-4	1.50E-4	2.07E-4
6	9.10E-6	3.97E-4	1.67E-5	1.45E-4
7	9.07E-7	2.96E-4	1.85E-6	1.44E-4

Figure 7.1: Plots of the approximate and exact solutions for Example 7.1. A sharp resolution of the interface is observed.



Now we consider a more difficult problem with convection on one side of the interface and general variable coefficients.

Example 7.2 *Let*

$$-\nabla \cdot \left(\begin{bmatrix} e^x & x \\ x & e^y \end{bmatrix} \nabla u_1 \right) + \left[\frac{1}{1+x+y}, \frac{1}{1+x+y} \right] \nabla u_1 + \frac{u_1}{1+x+y} = f_1(x,y), \qquad in\ \Omega_1,$$
$$-\nabla \cdot \left(\begin{bmatrix} x+1 & \sin(xy) \\ \sin(xy) & y+1 \end{bmatrix} \nabla u_2 \right) + (2+\sin(x)+\cos(y))u_2 = f_2(x,y), \qquad in\ \Omega_2,$$
$$u_k = g_k, \qquad on\ \partial\Omega_k \cap \partial\Omega, k = 1, 2.$$

The functions f_1, f_2, g_1, g_2, μ, and η are chosen such that the exact solution is

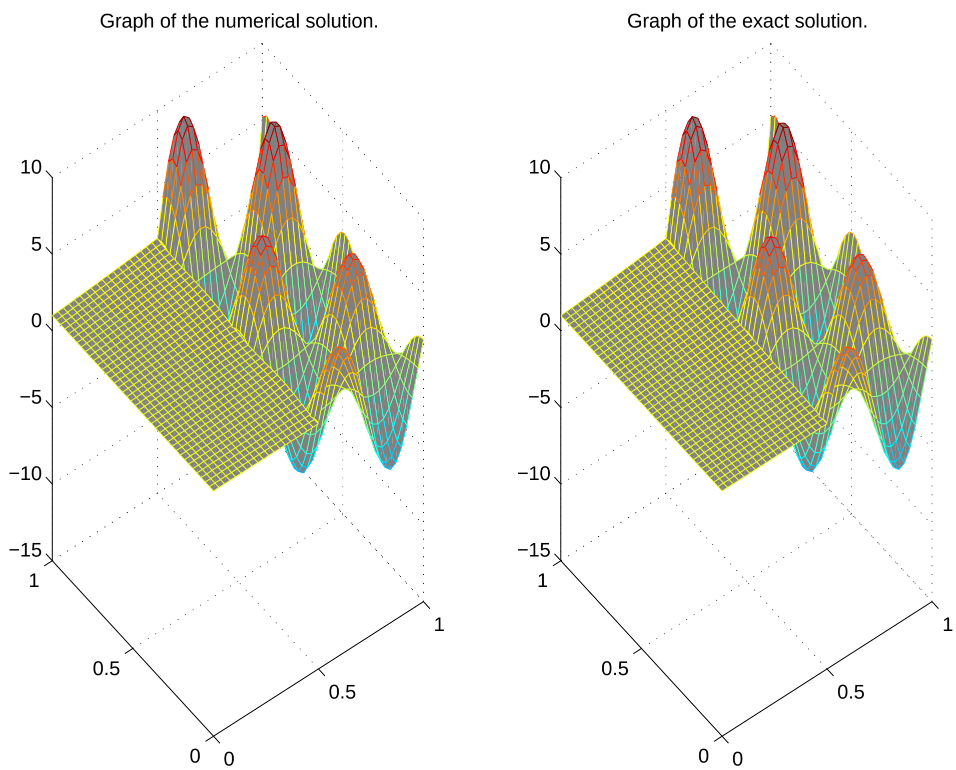
$$u_1(x,y) = e^{xy}, \ in\ \Omega_1, \quad u_2(x,y) = 5e^{xy}sin(13x)cos(13y), \ in\ \Omega_2.$$

Table 7.2 shows the results for the iterative and true errors in the maximum norm on the whole domain Ω. Figure 7.2 gives the plots of the true solution and the numerical solution (with grid size $\frac{1}{40} \times \frac{1}{40}$), which shows a good resolution of the sharp interface for the approximate solution.

Table 7.2: Iterative and true errors for Example 7.2 with interface at $x = 0.5$. The errors are shown in the L^∞-norm.

	Grid size $\frac{1}{40} \times \frac{1}{40}$		Grid size $\frac{1}{80} \times \frac{1}{80}$	
Iteration	Iterative error	True error	Iterative error	True error
1	4.79E-1	4.48E-1	5.13E-1	5.02E-1
2	9.24E-3	5.84E-2	1.44E-2	2.55E-2
3	3.85E-4	6.21E-2	7.81E-4	3.00E-2
4	1.64E-5	6.21E-2	4.04E-5	3.02E-2
5	6.83E-7	6.21E-2	2.04E-6	3.02E-2
6	2.80E-8	6.21E-2	1.00E-7	3.02E-2

Figure 7.2:[1] Plots of the approximate and exact solutions for Example 7.2.
A sharp resolution of the interface is observed.



Numerical experiments show that our iterative method is insensitive to variable coefficients even with big jumps across the interface. Although the relaxation parameters α and β can be chosen in some optimal fashion, our method converges pretty fast with $\alpha = \beta = 1/2$ even for very complicated problems. Also, sharp interfaces of the true solution can be captured fairly easily and accurately. Our future work will try to solve two and three dimensional application problems with more complex geometry and interface.

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[1]Note to the reader: A color version of this figure can be seen more clearly at http://www.math.wayne.edu/~yang/images/interface.gif

References

[1] D. N. Arnold and F. Brezzi, Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates, RAIRO Modél. Math. Anal. Numér. 19(1985) 7-32.

[2] G. R. Baker, D. I. Meiron, and S. A. Orszag, Generalized vortex methods for free-surface flow problems, J. Fluid Mech., 123(1982) 477-501.

[3] J.-D. Benamou and B. Despres, A domain decomposition method for the Helmholtz equation and related optimal control problems, Technical Report 2791, INRIA, France, Feb 1996.

[4] P. Bjorstad and O. Widlund, Iterative methods for the solution of elliptic problems on regions partitioned into substructures, SIAM J. Numer. Anal. 23(1986) 1097-1120.

[5] J. H. Bramble, J. E. Pasciak and A. H. Schatz, The construction of preconditioners for elliptic problems by substructuring IV, Math. Comp. 53(1989) 1-24.

[6] F. Brezzi, J. Douglas, Jr., R. Durán, and M. Fortin, Mixed finite elements for second order elliptic problems in three variables, Numer. Math. 51(1987) 237-250.

[7] F. Brezzi, J. Douglas, Jr., M. Fortin, and L. D. Marini, Efficient rectangular mixed finite elements in two and three space variables, RAIRO Modél. Math. Anal. Numér. 21(1987) 581-604.

[8] F. Brezzi, J. Douglas, Jr., and L. D. Marini, Variable degree mixed methods for second order elliptic problems, Matemática Aplicada e Computacional 4(1985) 19-34.

[9] F. Brezzi, J. Douglas, Jr., and L. D. Marini, Two families of mixed finite elements for second order elliptic problems, Numer. Math. 47(1985) 217-235.

[10] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.

[11] T. Chan and T. P. Mathew, Domain decomposition, Acta Numerica 1994, pp. 61-143.

[12] T. Chan and J. Zou, A convergence theory of multilevel additive Schwarz methods on unstructured meshes, to appear.

[13] L. C. Cowsar, J. Mandel, and M. F. Wheeler, Balancing domain decomposition for mixed finite elements, Math. Comp. 64(1995) 989-1015.

[14] B. Després, Domain decomposition method and the Helmholz problem, in: G. Cohen, L. Halpern and P. Joly, Eds., Mathematical and Numerical Aspects of Wave Propagation Phenomena, SIAM, Philadelphia, PA, 1991, pp. 44-52.

[15] J. Douglas, Jr., P. J. Paes Leme, J. E. Roberts and J. Wang, A parallel iterative procedure applicable to the approximate solution of second order partial differential equations by mixed finite element methods, Numer. Math. 65(1993) 95-108.

[16] J. Douglas, Jr. and D. Q. Yang, Numerical experiments of a nonoverlapping domain decomposition method for partial differential equations, in: D. Griffiths and G. A. Watson, Eds., Numerical Analysis: A. R. Mitchell 75th Birthday Volume, World Scientific Publishing Co., Singpore, 1996, pp. 85-97.

[17] M. Dryja, B. F. Smith, and O. Widlund, Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions, SIAM J. Numer. Anal. 31(1994) 1662-1694.

[18] B. Engquist and H. K. Zhao, Analysis of generalized Schwarz alternating procedure for domain decomposition, Technical Report 96-28, Department of Mathematics, UCLA, August 1996.

[19] D. Funaro, A. Quarteroni and P. Zanolli, An iterative procedure with interface relaxation for domain decomposition methods, SIAM J. Numer. Anal. 25(1988) 1213-1236.

[20] J. S. Gu and X. C. Hu, On an essential estimate in the analysis of domain decomposition methods, J. Comput. Math. 12(1994) 132-137

[21] T. Y. Hou, Z. L. Li, S. Osher, and H. K. Zhao, A hybrid method for moving interface problems with application to the Hele-Shaw flow, to appear.

[22] C. E. Kenig, Monotonicity formulas and regularity of two-phase free boundary problems.

[23] S. Kim, A parallelizable iterative procedure for the Helmholz problem, Appl. Numer. Math. 14(1994) 435-449.

[24] P. Le Tallec, Domain decomposition methods in computational mechanics, Computational Mechanics Advances 1(1994) 121-220.

[25] P. Le Tallec and M. D. Tidriri, Convergence analysis of domain decomposition algorithms with full overlapping for advection-diffusion problems, Technical Report 2435, INRIA, France, October 1994.

[26] P. Le Tallec, Y. H. De Roeck and M. Vidrascu, Domain decomposition methods for large linearly elliptic three-dimensional problems, J. Comput. Appl. Math. 34(1991) 93-117.

[27] R. J. LeVeque and Z. L. Li, The immersed interface method for elliptic equations with discontinuous coefficients and singular sources, SIAM J. Numer. Anal. 31(1994) 1019-1044.

[28] R. J. LeVeque and Z. L. Li, Immersed interface methods for Stokes flow with elastic boundaries or surface tension, SIAM J. Sci. Comput. 18(1997) 709-735.

[29] J. Liang and S. Subramaniam, Numerical computing of molecular electrostatics through boundary integral equations, IMA preprint 1440, University of Minnesota, November 1996.

[30] J. L. Lions and E. Magenes, Nonhomogeneous Boundary Value Problems and Applications I, Springer-Verlag, Berlin, 1972.

[31] P. L. Lions, On the Schwarz alternating method III: a variant for nonoverlapping subdomains, in: T. F. Chan, R. Glowinski, J. Periaux and O. B. Widlund, Eds., Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM, Philadelphia, PA, 1990, pp. 202-223.

[32] L. D. Marini and A. Quarteroni, An iterative procedure for domain decomposition methods: A finite element approach, in: R. Glowinski, G. H. Golub, G. A. Meurant and J. Periaux, Eds., Domain decomposition methods for partial differential equations, SIAM, Philadelphia, PA, 1988, pp. 129-143.

[33] L. D. Marini and A. Quarteroni, A relaxation procedure for domain decomposition methods using finite elements, Numer. Math. 55(1989) 575-598.

[34] A. Quarteroni, Domain decomposition methods for wave propagation problems, in: D. E. Keyes, Y. Saad, and D. G. Truhlar, Eds., Domain-Based Parallelism and Problem Decomposition Methods in Computational Science and Engineering, SIAM, Philadelphia, PA, 1995, pp. 21-38.

[35] P.–A. Raviart and J.–M. Thomas, A mixed finite element method for second order elliptic problems, in: I. Galligani and E. Magenes, Eds., Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics 606, Springer-Verlag, Berlin and New York, 1977, pp. 292-315.

[36] J. R. Rice, E. A. Vavalis and D. Q. Yang, Convergence analysis of a nonoverlapping domain decomposition method for elliptic PDEs, CSD-TR-93-048, Dept. of Computer Sciences, Purdue University, West Lafayette, IN 47907, July 1993.

[37] B. F. Smith, P. E. Bjorstad, and W. D. Gropp, Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations, Cambridge University Press, Cambridge, 1996.

[38] J. Xu, Iterative methods by space decomposition and subspace correction, SIAM Rev. 34(1992) 581-613.

[39] J. Xu and J. Zou, Nonoverlapping domain decomposition methods, to appear.

[40] D. Q. Yang, A parallel iterative nonoverlapping domain decomposition procedure for elliptic problems, IMA J. Numer. Anal. 16(1996) 75-91.

[41] D. Q. Yang, A parallel nonoverlapping Schwarz domain decomposition algorithm for elliptic partial differential equations, Proceedings of the Eighth SIAM Conference on Parallel Processing for Scientific Computing (Minneapolis, March 14-17, 1997), Proceedings in Applied Mathematics 94, Eds., M. Heath, V. Torczon, G. Astfalk, P. Bjorstad, A. H. Karp, C. H. Koebel, V. Kumar, R. E. Lucas, L. T. Watson, and D. E. Womble, SIAM, Philadelphia, PA, 1997.

[42] D. Q. Yang, A parallel iterative domain decomposition algorithm for elliptic problems, J. Comput. Math., to appear.