The evolution of an axi-symmetric Stokes bubble with volumetric change

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Abstract

The evolution of an axi-symmetric Stokes bubble is considered when the bubble volume is changed. Linear stability analysis shows that an expanding spherical bubble is stable, while a shrinking spherical bubble is unstable. Numerical calculations of the time evolution shows that for certain initial condition, such a bubble can form cusps or near cusps or undergo topological change (pinching) before it reduces to zero size, qualitatively similar to earlier exact solutions in

2-D (Tanveer & Vasconcelos, J. FL. Mech. vol 301, p 325-344, 95). Further, for bubbles that pinch, a self similar process is identified.

Key Words: Hydrodynamics Stability (76E), Stokes Flows (76D07)

1 Introduction

Bubble dynamics is important in many applications in engineering [1]. Following the pioneering work by G. I. Taylor, there has been a great deal of research on bubbles and drops in a slow viscous flow [2] [3] [4] [5] [6]. In particular the dynamic break-up of a bubble or a drop through topological changes has been a subject of considerable interest [7]. For two dimensional Stokes flow, exact solutions have been found that for a bubble with infinite viscosity ratio between the exterior and interior fluid, a continual decrease in bubble area generally causes the bubble to form cusps or near cusps or undergo topological changes before its area reduces to zero [8] [9]. A expanding bubble, on the other hand, is found to approach an expanding circle.

A change in bubble area (in 2-D) or volume (in 3-D) can be affected through a number of physical mechanisms. For instance, it is possible for the fluid inside the bubble to dissolve in the ambient fluid without the dissolution process itself significantly affecting the exterior fluid flow. In cases where the fluid inside the bubble sufficiently compressible, a fluctuating pressure field imposed at infinity could alter bubble volume.

The purpose of this paper is to determine if an axi-symmetric bubble would have the same qualitative properties as for a 2-D bubble. Apriori, the differences in the expression for Laplacian and curvature terms in 2-D and axi-symmetric 3-D flows might be expected to have some effects on the phenomena observed in 2-D. Our results suggest that while the quantitative details and the nature of singularity associated with bubble pinching are generally different in 3-D compared to 2-D, the qualitative behavior is quite the same. Further, numerical calculation indicates that pinching is a locally self-similar process, and the similarity variables close to pinching are identified.

We present the equations of the bubble motion in Section 2, and perform a linear stability analysis of an arbitrary contracting or expanding spherical bubble in Section 3. The numerical methods for solving the equations are briefly discussed in Section 4. Numerical results of the evolution of a bubble for different initial conditions are discussed in Section 5. Finally, we summarize our studies in Section 6.

2 Formulation

In the low Reynolds number limit, the flow outside the bubble satisfies the Stokes and continuity equations

$$\mu \nabla^2 \mathbf{u} = \nabla p \qquad \text{and} \qquad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

where $\mathbf{u}$ is the fluid velocity, p is the fluid pressure and μ is the fluid viscosity. We require

$$\mathbf{u} \to 0 \qquad \text{as} \qquad \mathbf{x} \to \infty, \tag{2}$$

The stress on the bubble surface S is balanced by the interfacial tension force as

$$\mathbf{T} \cdot \mathbf{n} = \tau (\nabla \cdot \mathbf{n}) \mathbf{n}, \tag{3}$$

where $\mathbf{T} = -p\mathbf{I} + \mu \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]$, $\mathbf{n}$ is the unit outward normal at the bubble boundary and τ is the surface tension. The interfacial kinematic condition requires

$$\frac{d\mathbf{x}}{dt} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \qquad \text{on } S \tag{4}$$

Finally, the volume of the bubble is required to change at a constant prescribed rate A, i.e.

$$\int_S \mathbf{u} \cdot \mathbf{n} dS = A. \tag{5}$$

We use L, the radius of the initial bubble, as the length scale, $|A|/L^2$ as the velocity scale. The pressure is non-dimensionalized by $\mu|A|/L^3$, and the time scale is chosen as $L/|A|$. Therefore,the equations contain only one non-dimensional parameter,

$$G = \frac{\tau L^2}{\mu|A|}. \tag{6}$$

With this non-dimensionalization, the rate of change of non-dimensional bubble volume is either 1 or -1, corresponding to an expanding or a contracting bubble, respectively. Henceforth, all physical variables are considered non-dimensional.

3 Linear stability

A trivial solution to (10-13) is a contracting or expanding spherical bubble. In the spherical polar coordinates (R, θ, ϕ), with the assumption of axi-symmetry, the bubble interface is represented as $R = \mathcal{R}(\theta, t) = R_0(t)$, and the two components of the velocities are

$$\bar{u}_R(R, \theta, t) = \frac{R_0^2 \dot{R}_0}{R^2} \qquad \bar{u}_\theta(R, \theta, t) = 0, \tag{7}$$

and the pressure is

$$\bar{p}(R, \theta, t) = p_\infty(t) = -2\left(2\frac{\dot{R}_0}{R_0} + \frac{G}{R_0}\right) \tag{8}$$

where $R_0(t)$ is determined by

$$\frac{d}{dt}\left(\frac{4\pi}{3}R_0^3(t)\right) = \mathrm{sgn}(A). \tag{9}$$

where $\mathrm{sgn}(A) = 1$ for $A > 0$ and $= -1$ for $A < 0$. However, the results of the linear stability analysis prescribed in this section transcends the restriction (9) on $R_0(t)$ and can be applied to any spherical bubble with specified radius $R_0(t)$. Therefore, the restriction (9) will not be imposed in this section.

We are now in a position to analyze the evolution of small axisymmetric perturbations on a spherical bubble. The analysis is similar in spirit to any linear stability analysis on a spherical bubble, such as [10] except that unlike previous cases, the bubble volume is changed. Since Legendre polynomials, $\{P_k(\cos(\theta))\}_{k=0}^{\infty}$, form a complete set in describing any function of θ, it is enough to consider bubble shape perturbations in the form

$$R = \mathcal{R}(\theta, t) = R_0(t)\left(1 + \epsilon\rho(t)P_k(\cos(\theta))\right) + O(\epsilon^2), \tag{10}$$

where ϵ is a small constant. It is easily seen that the corresponding velocities must take the form,

$$u_R(R, \theta, t) = \bar{u}_R + \epsilon U(R, t)P_k(\cos(\theta)) + O(\epsilon^2) \tag{11}$$

$$u_\theta(R, \theta, t) = \epsilon W(R, t)\sin(\theta)P_k'(\cos(\theta)) + O(\epsilon^2) \tag{12}$$

while pressure is

$$p(R, \theta, t) = \bar{p} + \epsilon q(R, t)P_k(\cos(\theta)) + O(\epsilon^2). \tag{13}$$

For disturbances not affecting the volume, it is clear that the $k = 0$ mode needs to be excluded. So we will assume $k \geq 1$. Substituting (10-13) into the equations of motion (1-4), we obtain the linearized Stokes and continuity equations:

$$-\frac{\partial q}{\partial R}+\frac{\partial^2 U}{\partial R^2}+\frac{2}{R}\frac{\partial U}{\partial R}-\frac{k^2+k+2}{R^2}U-\frac{2k(k+1)}{R^2}W=0, \tag{14}$$

$$\frac{q}{R}+\frac{\partial^2 W}{\partial R^2}+\frac{2}{R}\frac{\partial W}{\partial R}-\frac{k(k+1)}{R^2}W-\frac{2U}{R^2}=0, \tag{15}$$

$$\frac{\partial U}{\partial R}+\frac{2}{R}U+\frac{k(k+1)}{R}W=0. \tag{16}$$

Linearization of normal and tangential components of stress about $R=R_0(t)$ results in the following $O(\epsilon)$ equations:

$$q-2\left(\frac{6\dot{R}_0}{R_0}\rho(t)+\frac{\partial U}{\partial R}\right)+G\frac{\rho(t)}{R_0}(k+2)(k-1)=0 \qquad \text{at } R=R_0(t), \tag{17}$$

$$3\dot{R}_0\frac{\rho(t)}{R_0}-\frac{1}{2}\left\{\frac{U}{R_0}+\frac{1}{k(k+1)}\left(R_0\frac{\partial^2 U}{\partial R^2}+2\frac{\partial U}{\partial R}-\frac{2}{R_0}U\right)\right\}=0 \qquad \text{at } R=R_0(t), \tag{18}$$

while the $O(\epsilon)$ kinematic equation is

$$\dot{\rho}(t)=\frac{U}{R_0}-3\rho\frac{\dot{R}_0}{R_0} \qquad \text{at } R=R_0(t). \tag{19}$$

After eliminating W and q among (14-16), we obtain

$$\begin{aligned} & R^2\frac{\partial^4 U}{\partial R^4}+8R\frac{\partial^3 U}{\partial R^3}+(12-2k(k+1))\frac{\partial^2 U}{\partial R^2} \\ - \quad & \frac{4k(k+1)}{R}\frac{\partial U}{\partial R}+k(k+1)(k^2+k-2)\frac{U}{R^2}=0. \end{aligned} \tag{20}$$

The general solutions of (20) are

$$U(R,t)=\frac{a_0(t)}{R^k}+\frac{a_1(t)}{R^{k+2}}+a_2(t)R^{k-1}+a_3(t)R^{k+1} \tag{21}$$

where $a_i(t), i = 0, ..., 3$ are as yet undetermined. Since $k \geq 1$, the relation (2) implies $a_2(t), a_3(t) = 0$. From (15 16), it follows that

$$W(R,t) = \frac{1}{k(k+1)}\left(\frac{ka_1}{R^{k+2}} - \frac{(2-k)a_0}{R^k}\right) \tag{22}$$

$$q(R,t) = \frac{2(2k-1)}{k+1}\frac{a_0}{R^{k+1}} \tag{23}$$

Finally, we use (21-23) in (17, 18) with $a_2 = 0 = a_3$ to eliminate a_0 and a_1 in terms of ρ. Using these resulting expressions in (19), we obtain a first order ODE for $\rho(t)$ which is readily integrated to give

$$\rho(t) = \rho(0)\left(\frac{R_0(0)}{R_0(t)}\right)^{\frac{3k(2k-1)}{(2k^2+1)}} \exp\left\{-G\frac{(2k+1)(k^2-1)}{2(2k^2+1)}\int_0^t \frac{ds}{R_0(s)}\right\} \tag{24}$$

Notice that the surface tension correction factor is 1 for $k = 1$. For any fixed $k > 1$, this correction is small for small G unless $\int_0^t R_0^{-1}(s)ds$ is large. Also, any non-zero G is seen to stabilize disturbances corresponding to sufficiently large k. It is clear from (24) that the spherical solution is unstable when $R_0(t)$ decreases, i.e. for a contracting bubble; on the other hand, it is stable when $R_0(t)$ increases for an expanding bubble. In essence, the sign of A, sgn(A), determines the linear stability. For the case of constant rate of volume change according to (9), $R_0(t)$ appearing in (24) has to be replaced by

$$R_0(t) = \left(1 + \frac{3\text{sgn}(A)}{4\pi}t\right)^{\frac{1}{3}}. \tag{25}$$

4 Numerical Method

The numerical method, which we use to follow the evolution of an axisymmetric bubble, is based on a boundary integral formulation for solving Stokes equations with a moving boundary [11] [2] [12]. The velocities of the bubble surface satisfy the Fredholm integral equation of the second kind,

$$u_j(\mathbf{x}) + P\!\!\!\!\!\!\oint_S u_i(\mathbf{y}) K_{ijk}(\mathbf{x}, \mathbf{y}) n_k(\mathbf{y}) dS(\mathbf{y}) = f_j(\mathbf{x}) \tag{26}$$

where

$$K_{ijk}(\mathbf{x}, \mathbf{y}) = \frac{3}{2\pi} \frac{(y_i - x_i)(y_j - x_j)(y_k - x_k)}{|\mathbf{y} - \mathbf{x}|^5} \tag{27}$$

$$f_j(\mathbf{x}) = \left(\int_S (\nabla \cdot \mathbf{n}(\mathbf{y})) n_i(\mathbf{y}) J_{ij}(\mathbf{x}, \mathbf{y}) dS(\mathbf{y}) \right) G \tag{28}$$

$$J_{ij}(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi} \left(\frac{1}{|\mathbf{y} - \mathbf{x}|} + \frac{(y_i - x_i)(y_j - x_j)}{|\mathbf{y} - \mathbf{x}|^3} \right) \tag{29}$$

$\mathbf{J}$ and $\mathbf{K}$ are the Green's functions for the velocity and stress fields, respectively. P indicates the principal-value integral. Equation (5) implies that

$$\int_S u_i(\mathbf{y}) n_i(\mathbf{y}) dS(\mathbf{y}) = \operatorname{sgn}(A). \tag{30}$$

The eigenvalue problem for the homogeneous equation associated with (26) has an eigenvalue one with multiplicity one. For (26) to have a solution, f_j must satisfy the Fredholm alternative condition. Note that the corresponding adjoint problem is

$$w_j(\mathbf{x}) + n_k(\mathbf{x}) P\!\!\!\!\!\!\oint_S w_i(\mathbf{y}) K_{jik}(\mathbf{y}, \mathbf{x}) dS(\mathbf{y}) = 0. \tag{31}$$

It is easy to check that $w_j(\mathbf{x}) = n_j(\mathbf{x})$ is the eigenfunction of (31) by using the identity

$$P\!\!\!\!\!\!\oint_s K_{ijk}(\mathbf{x}, \mathbf{y}) n_j(\mathbf{x}) dS(\mathbf{x}) = -\delta_{ik}. \tag{32}$$

Also, it can be shown that

$$\int_S f_i(\mathbf{y}) n_i(\mathbf{y}) dS(\mathbf{y}) = 0 \tag{33}$$

by using

$$\int_S J_{ji}(\mathbf{x},\mathbf{y})n_i(\mathbf{x})dS(\mathbf{x}) = 0. \tag{34}$$

Thus f_j indeed satisfies the Fredholm alternative condition. The arbitrary constant in the solution of (26) corresponds to the freedom of choice of bubble volume. By demanding (30), a unique solution to (26) can be obtained. In fact (26) and (30) can be combined into one equation as

$$\begin{aligned} & u_j(\mathbf{x}) + PV\!\!\int_S u_i(\mathbf{y})K_{ijk}(\mathbf{x},\mathbf{y})n_k(\mathbf{y})dS(\mathbf{y}) \\ = \;& f_j(\mathbf{x}) + n_j(\mathbf{x})\left(\int_S u_i(\mathbf{y})n_i(\mathbf{y})dS(\mathbf{y}) - \mathrm{sgn}(A)\right). \end{aligned} \tag{35}$$

It is clear that (26, 30) itself implies (35). We now show that (35) implies (26, 30). We multiply (35) by $n_j(\mathbf{x})$ and integrate over S, the left-hand side of (35) vanishes by using (32), and the first term on the right vanishes due to (34). Consequently, we obtain equation (30). From (35, 30), clearly (26) follows. The advantage in having equations in the form (35) is that the resulting integral equation has a unique solution for u_j.

When the motion of flow is assumed to be axisymmetric, the surface integrals can be reduced to line integrals by performing the azimuthal integration analytically. The resulting complete elliptic integrals of the first and second kind are computed by the recursive formulae [13]. The numerical techniques of solving the system of equations is similar to those discussed in [14] [4] [12] [15]. The bubble surface is approximated by a set of boundary nodes along the contour in the (r, z) meridional plane. All other values on the surface are obtained through the interpolation of a quintic spline [16]. To solve (35), we apply the equation at the collocation point where the integration is performed using a six-point Gaussian quadrature. Rather than using a LU decomposition or Gaussian elimination, we find it more efficient to solve the resulting linear system by an iteration procedure known as GMRES [17]. To accelerate the convergence of the iterations, the initial guess is improved by a fourth-order extrapolation from the previous time steps. The bubble interface is advanced by solving (4). The time integration here is carried out with a fourth-order Adams-Moulton predictor-corrector. The starting values are obtained through a standard fourth-order Runge-Kutta method. All computation were performed with a 64 bit arithmetic.

5 Numerical Studies

First of all, we check the codes by comparing the numerically computed solutions against those from the linear stability analysis in Section 2. We consider the initial condition of the form

$$r(\theta) = R(\theta)\sin(\theta) \qquad z(\theta) = R(\theta)\cos(\theta) \tag{36}$$

where

$$R(\theta) = 1 + \epsilon\rho(0)P_k(\cos(\theta)) + \epsilon^2 d \tag{37}$$

with d chosen to ensure that the computed initial volume of the bubble is exactly $\frac{4}{3}\pi$. Also we set $\rho(0) = 1$ without loss of generality. The difference between the linearized stability solutions, with $\rho(t)$ given by (24) and the numerically computed ones is found to be $O(\epsilon^2)$ as $\epsilon \to 0$. As expected from linear stability analysis, when $\mathrm{sgn}(A) = 1$, the bubble is found to always approach an expanding sphere. This is found to be true even when ϵ is large. However, the situation is different for $\mathrm{sgn}(A) = -1$. In particular, for $G = 0.02$ and $k = 2$, the shape of bubbles is plotted in Figure 1 for $\epsilon = -0.3$, and in Figure 2 for $\epsilon = 0.3$. Clearly, the difference between the numerically computed solutions and the linear stability ones become visible at $t = 1$ and slowly increase. We notice that nonlinear effects slow down the tendency of the bubble to become concave on parts of the interface. We have plotted the the linear stability solutions upto $t = 3$, while the numerically computed ones upto time when the bubbles are close to pinching. Notice that at later time, the corresponding physical shape is similar to a dimpled pancake in Figure 1 and a peanut in Figure 2. The computation corresponding to Figure 1 and 2 has to be eventually stopped for loss of the resolution when the bubble comes close to pinching. For a boundary integral method, the accuracy of calculation deteriorates when two parts of the interface approach each other. This is due to the inaccurate calculation of integrals when the distance between neighboring collocation points on one part of the interface is comparable to the distance between two collocation points on different parts of the interface. It is necessary to maintain a large value of the ratio λ of the smallest distance between the two parts of the bubble surface to the distance between the pinching point and its neighbor on the same side of the interface. Computation has to be stopped when λ fails to be sufficiently large, as is the case when pinching is approached.

In order to determine if bubbles in Figure 1 and 2 continue the trend toward pinching, it is convenient to start with initial bubble shapes that resemble $t = 4$ profiles in Figures 1 and 2. This allows us to choose boundary parameterization that concentrates points at the interface location where pinching occurs.

In particular, for the case of a dimpled pancake we choose

$$r(\nu) = (b - a)\sin(\nu) + c\ \sin(3\nu) \qquad z(\nu) = (b + a)\cos(\nu) + c\ \cos(3\nu) \quad (38)$$

where $a = -d$, $b = 0.4d$, $c = 0.1d$, with the constant d chosen to ensure that the initial volume of the bubble is $\frac{4}{3}\pi$. Further, we use a stretching transformation

$$\nu = \alpha - 0.5\gamma\ \sin(2\alpha), \qquad (39)$$

where $\gamma = 0.8$ is chosen for this set of initial conditions. For equally spacing α, the collocation points cluster near $r = 0$ initially as seen from (38, 39), and it is found that these remain reasonably clustered near pinching as well. With the parameterization (39), it is possible to calculate accurately the interface evolution much closer to pinching than for calculation shown in Figure 1.

In Figure 3, we illustrate the evolution of a bubble subjected to the initial condition (38) for $G = 0$. As shown in Figure 3, the bubble surface comes close to touching far before the bubble volume becomes zero. The overall error tolerance for solving the integral equation (35) is set to 10^{-10}, and the typical iteration number in GMRES is 9 and it increases to as much as 30 as the bubble surface comes close to pinching. The accuracy of the calculation is examined in two ways: studying its resolution by doubling the number of points and by monitoring the volume of the bubble. The spatial convergence is demonstrated by comparing computations for $N = 33, 65, 129$ with those for $N = 257$ for $\Delta t = 0.0025$. Defining the difference of solutions corresponding to $N = 257$ with that obtained for another N as the error, we plot this at at $r = 0$ on a negative $\log_{10}$ vertical scale in Figure 4. For early time when $t < 2.5$, the accuracy improves from $N = 33$ to $N = 65$ by about one digit, and shows no significant improvement for $N = 65$ to $N = 129$. This is due to the fact that the interface is well represented by 65 collocation points far before pinching, and this is also confirmed by checking the spectrum of the interface representation. However, as time approaches about 3.4, close to the estimated pinching time, higher resolution clearly shows better accuracy as shown in the close-up window in Figure 4. The difference between the computed volume and the exact expression appears to be a less sensitive measure of error. Also the spatial resolution study presented in Figure 4 is found to remain unchanged if Δt is further halved.

We now turn to investigating a possible self similar process describing pinching. For this initial condition, we assume a self-similar relationship near pinching:

$$z(\alpha,t) \sim (t_c - t)^p \, F\left(\frac{r(\alpha,t)}{(t_c - t)^q}\right). \tag{40}$$

with $q > 0$. To obtain p, q and t_c, we fit the numerically calculated $z(0,t)$ and $z_{\alpha\alpha}(0,t)$ to the following forms,

$$z(0,t) \sim (t_c - t)^p F(0); \tag{41}$$

$$z_{\alpha\alpha}(0,t) \sim (t_c - t)^{p-2q} F''(0) r_\alpha^2(0,t_c). \tag{42}$$

Numerically, $r_\alpha(0,t_c)$ is found to be nonzero. The procedure to compute p, q and t_c is as following. First we apply a nonlinear least square fit to (41) over M (> 3) consecutive points in time to determine p, t_c and $F(0)$. Once p is determined, q is determined from using (42) over two adjacent time steps. The largest time t_f used for the purpose of this fit depends on the estimated

N	t_f	t_c	p	q	$F(0)$	deviation
33	3.25	3.4245	0.9684	0.5119	0.1692	9,2$\times 10^{-8}$
65	3.35	3.4271	0.9837	0.5041	0.1714	1.7 $\times 10^{-7}$
129	3.40	3.4277	0.9913	0.5004	0.1732	3.8 $\times 10^{-6}$
257	3.415	3.4283	0.9966	0.4997	0.1746	5.4 $\times 10^{-6}$

Table 1: The estimation of p, q, t_c for (38) and different N using 30 time steps

accuracy of the solutions. As measured by comparing with higher resolution, the accuracy of the solution close to pinching depends on the ratio λ of the neck distance over the local spacing. We determine t_f by requiring that both $\lambda > 2$ and the spatial accuracy $< 10^{-4}$. Either one of these criteria fails for the first time beyond $t = t_f$. We define "deviation" in Tables 1 and 2 as the least square error, corresponding to the estimated p, t_c and $F(0)$. In Table 1, p, q , t_c, $F(0)$, t_f and deviation are shown for different N, where $M = 30$, i.e. 30 pairs of consecutive values $(t_i, z(0, t_i))$ are used to compute the least square with $t_i = t_f - (i-1) \times 0.0025, i = 1...30$. Similarly in Table 2, the first 15 pairs of values from the computation in Table 1 is employed. The convergence of the computed p, q, t_c and $F(0)$ is easily observed from both tables and the results indicate consistency between $M = 15$ and 30. Upto three digit precision, we determine $p = 1$, $q = 1/2$, $t_c = 3.428 \pm 10^{-3}$, and $F(0) = 0.174 \pm 10^{-3}$.

In order to check the consistency of the form-fit, both computed neck distance of the bubble as well as the the right-hand side of (41), denoted as the "fitted" in Figure 5, are plotted as functions of the time. Both curves agree with each other very well when $t > 3$ (see close-up window). To check if indeed the similarity hypothesis (40) for t close to t_c is valid, $z(\alpha, t)/(t_c - t)$

N	t_f	t_c	p	q	$F(0)$	deviation
33	3.25	3.4251	0.9712	0.5133	0.1695	8.6×10^{-8}
65	3.35	3.4274	0.9863	0.5054	0.1719	1.6×10^{-7}
129	3.40	3.4279	0.9940	0.5015	0.1742	6.5×10^{-6}
257	3.415	3.4283	0.9972	0.5000	0.1749	4.3×10^{-6}

Table 2: The estimation of p, q, t_c for (38) and different N using 15 time steps

is plotted as a function of $r(\alpha,t)/(t_c-t)^{\frac{1}{2}}$ at $t = 2.7+0.1\times i$ for $i = 1, ..., 7$ in Figure 6. The upper curves correspond to later times. Note that $t = 3.3, 3.4$ curves are almost on top of each other as is expected if (40) is valid for t approaching t_c.

Adding surface-tension has no significant effect on the pinching behavior for this case since curvature is small near the pinching point. As shown in Figure 7, the bubble pinches when $G = 0.1$. A similar study demonstrates that $p = 1$, $q = \frac{1}{2}$, $t_c = 3.911 \pm 10^{-3}$, and $F(0) = 0.247 \pm 10^{-3}$ for this case. Quantities as in Figures 5 and 6 are plotted in Figures 8 and 9 for this case. In Figure 9, The curves from the bottom to the top corresponds to the time $t = 3.5 + i \times 0.1$ for $i = 1, ..., 4$.

The results in the case of an initial dimpled pancake is not surprising since the fluid on top and bottom of the bubble are "unaware" of each other as they are drawn together by the global condition of volume reduction. This is similar to the 2-D case [8] [9]. The pinching is consistent with a local quadratic behavior

$$z \sim a_0 r^2 + b_0(t_c - t) \tag{43}$$

which implies

$$\frac{z}{t_c - t} \sim a_0 \left(\frac{r}{(t_c - t)^{\frac{1}{2}}} \right)^2 + b_0 \tag{44}$$

This is consistent with (41) with $p = 1$ and $q = \frac{1}{2}$. Confirmation of this result suggests the reliability of the form-fit and indirectly the accuracy of the numerics, even close to pinching.

We turn now to studying another set of initial condition.

$$r(\nu) = (b + a)\sin(\nu) - c\,\sin(3\nu) \qquad z(\nu) = (a - b)\cos(\nu) + c\,\cos(3\nu) \tag{45}$$

where $a = -d$, $b = 0.5d$, $c = 0.2d$, and d is a constant such that the initial volume of the bubble is $\frac{4}{3}\pi$. This initial bubble, unlike the previous case, has the shape of a peanut and qualitatively resemble the $t = 4$ curve in Figure 2. The evolution of a bubble (only half of the profile plotted) for $G = 0$ is shown in Figure 10. Here the bubble surface develops a cusp, within numerical precision. Such cusps have been observed earlier for exact 2-D solutions in [8]. In that case a small nonzero surface tension smoothes out the cusp locally, but the shapes are near cusps. This is expected for the axi-symmetric case as well for small G, though numerically calculation becomes difficult. Not surprisingly, when $G = 0.5$, the cusp formation with initial condition (45) is smoothed out over a large scale. The bubble then tends to pinch at the origin as seen in Figure 11. In order to better resolve the pinching, we cluster the collocation points around $z = 0$ initially by choosing uniformly spaced points in α and choosing

$$\nu = \alpha + 0.5\gamma\,\sin(2\alpha), \tag{46}$$

with $\gamma = 0.8$.

N	t_f	t_c	p	q	$F(0)$
129	4.050	4.1495	0.9828	0.3988	0.3175
257	4.060	4.1508	0.9921	0.4000	0.3234

Table 3: The estimation of p, q, t_c for (45) and different N

Before proceeding with discussion of the pinching behavior, we remind the reader once again about the essential difference between the two initial conditions (38) and (45). Although both initial bubble shapes look similar in the meridional plane with r and z interchanged, their actual 3-D shapes are significantly different as discussed above (one is pancake shaped and the other resembles a peanut). From a computational point view, the second case provides more numerical difficulties due to the presence of term $1/r$ in the curvature, which is sensitive to round-off error as r approaches 0. As a result, we can not carry out the computation in the second case as close to the pinching time as that in the first case. However, we calculated as far as we could and it appears to be sufficient to identify a self similar pinch process in this case as well.

As in the previous case, the self similar form (40) is assumed, except that r and z are interchanged. In order to keep errors acceptable ($< 10^{-4}$), it is found that we need $\lambda > 5$. Since there are not enough available time steps near pinching where asymptotic behavior (40) is valid, we abandon the least square fit approach in favor of a direct fit to (41).

Specifically, we enforce (41) at three adjacent time steps ending at t_f, and solve the nonlinear equations through Newton iteration to obtain p, t_c and $F(0)$. The computation of q is similar to the previous case. In Table 3, a estimate of p, q, and t_c are given. The difference between the two solutions

$N = 129$ and $N = 257$ agree up to 10^{-4} at $t = 4.06$, and the volume of the bubble is accurate to 10^{-6} for $N = 129$ and 10^{-8} for $N = 257$ respectively.

Based on the data in Table 3, we estimate that $p = 1$, $q = 2/5$, $t_c = 4.15 \pm 10^{-2}$ and $F(0) = 0.32 \pm 10^{-2}$. In Figure 12 and Figure 13, graphs similar to Figures 5 and 6 are shown for this initial condition (45) . In Figure 13, the bottom curve corresponds to $t = 3.9$ while all other curves overlap corresponding to $t = 4.0 + (i-1) \times 0.01$ for $i = 1, ..., 7$. We note different similarity exponent in this case compared to the previous case.

An analytic similarity solution describing this pinching appears to be difficult as the far-field behavior must come into play since the fluid velocity and pressure at any part of the interface is globally dependent on values at other parts of the interface. For instance, it is known within the context of 2-D Stokes flow that the interface between a zero viscosity fluid displacing a viscous fluid can be stable for an expanding inviscid bubble [8] or unstable for a contracting viscous blob [18]. This observation suggests that the far-field plays a very important role on the similarity process; and this makes it difficult to obtain a locally self-similar analytic solution.

6 Discussions

We have studied the evolution of an axi-symmetric Stokes bubble whose volume changes. A linear stability analysis suggests that an expanding bubble is stable while a contracting bubble is unstable to arbitrary axi-symmetric disturbances. Numerical computation for bubbles with a constant rate of change of bubble volume suggests the same behavior for large deformation, though the growth of the instability is weakened by nonlinear effects. Numerical computation also suggests that a certain class of disturbances cause a contracting bubble to undergo change in topology as different parts of the interface touch each other. The pinching appears to be a self-similar process, though the exponents depend on whether the initial shape is pancake or peanut shaped. Our calculations ignore the fluid inside the bubble which can affect the phenomenon presented here. Such inclusion of fluid motion inside must also account for the precise mechanism by which the volume of the interior fluid is decreasing.

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List of Figure Captions

1. Comparison between the numerically computed solutions (the solid curves) and the linear stability results (the dotted ones) for $\epsilon = -0.3$. The solid curves correspond to $t = i \times 0.5, i = 0, ..., 8$ and $t = 4.15$, while the dotted ones are for $t = i \times 0.5, i = 1, ..., 6$.

2. The same as Figure 1 except that $\epsilon = 0.3$ and the most inside curve is at $t = 4.29$.

3. The evolution of a bubble surface for initial condition (38) with $G = 0$ at $t = (i - 1)$ for $i = 1...4$, and $t = 3 + i \times 0.1$ for $i = 1, ..., 4$.

4. "Error" in the neck-width, defined as the difference between computation for $N = 257$ and $\Delta t = 0.0025$ with any other N.

5. The neck distance vs. time for initial condition (38) with $G = 0$.

6. Local behavior of $F(y)$ near $y = 0$ for initial condition (38) for $G = 0$.

7. The evolution of a bubble surface for initial condition (38) for $G = 0.1$ at $t = (i - 1)$ for $i = 1...4$, and $t = 3.4 + i \times 0.1$ for $i = 1, ..., 5$.

8. The neck distance vs. time for initial condition (38) for $G = 0.1$.

9. Local behavior of $F(y)$ near $y = 0$ for initial condition (38) with $G = 0.1$.

10. The evolution of a bubble surface for initial condition (45) with $G = 0$ at $t = (i - 1)$ for $i = 1...4$.

11. The evolution of a bubble surface for initial condition (45) with $G = 0.5$ at $t = (i-1)$ for $i = 1...5$, and $t = 4.06$.

12. The neck distance vs. time for initial condition (45) with $G = 0.5$.

13. Local behavior of $F(y)$ near $y = 0$ for initial condition (45) with $G = 0.5$.