Finite Element Methods and Domain Decomposition Algorithms for a Fluid-Solid Interaction Problem⋆

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ABSTRACT. This paper concerns with the finite element Galerkin approximations for a fluid–solid interaction model proposed in [10]. Both Continuous–time and discrete–time approximations are formulated and analyzed. Optimal order *a priori* estimates for the errors in $L^\infty(H^1)$ and $L^\infty(L^2)$ are derived. The main difficulty for the optimal order error estimates is caused by the interface conditions which describe the interaction between a fluid and a solid on their contact surface, and it is overcome by using a boundary duality argument of Douglas and Dupont [5] to handel the terms involving the interface conditions. Finally, several parallelizable domain decomposition algorithms are proposed and analyzed for efficiently solving the finite element systems.

§0. Introduction. The problems of wave propagation in composite media have long been subjects of both theoretical and practical studies, important applications of such problems are found in inverse scattering, elastoacoustics, geosciences, oceanography. Different mathematical and/or numerical composite models were proposed and studied in [3], [4], [10], [16], [15] and [17].

The purpose of this paper is to analyze the finite element Galerkin approximations for a fluid–solid interaction model which was proposed recently by the authors in [10], and to develop some parallelizable domain decomposition algorithms for efficiently solving the finite element systems. In [10] we gave a detailed derivation and the complete mathematical analysis for the model, which will serve as the theoretical foundation for the numerical analysis of this paper. The primary goal of this paper is to establish optimal order *a priori* error estimates in the $L^\infty(H^1)$–norm and in the $L^\infty(L^2)$–norm for the Galerkin approximations to the solution of the model. The main difficulty for obtaining the optimal estimates is caused by the interface conditions which describe the interaction between a fluid and a solid on their contact surface, To overcome the difficulty, the critical idea is to use a boundary duality argument due to Douglas and Dupont [5] to handel the terms involving the interface conditions.

The model and the finite element methods studied in this paper are related to those previously studied by Santos *et al* [16] and by Sheen [17], where the propagation of waves

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through single–phase and two–phase fluid saturated porous media near a fluid–filled borehole region was studied, respectively. Displacements were used as the primary variables in the fluid region (borehole) and in the fluid–saturated porous solid region, and no attempt was made to address the optimality issue of the error estimates for the Galerkin approximations in both [16] and [17].

The domain decomposition methods developed in this paper are based on the idea of using the convex combinations of the interface conditions in place of the original interface conditions to pass the information between subdomains, see [11], [2], [6], [9] and references therein for the expositions and discussions on this approach for problems posed in homogeneous media. So the domain decomposition methods of this paper may be regarded as the generalizations of the methods proposed in those papers to the time–dependent heterogeneous problems.

The organization of this paper is as follows. In §1 we introduce space notations, and state the fluid–solid interaction model and some basic facts about the model. In §2 we formulate the continuous–time Galerkin approximation and establish *a priori* $L^\infty(H^1)$ and $L^\infty(L^2)$ estimates under different assumptions on the the solution and data functions. In §3 two second order (in time) discrete–time Galerkin methods are defined and analyzed. Finally, in §4, several parallelizable non–overlapping domain decomposition algorithms are proposed and analyzed for the problem at the differential level, and these algorithms can be readily adapted for solving the discrete systems of the Galerkin approximations for the fluid–solid interaction problem.

Throughout the paper, unless stated otherwise, C will denote a general positive constant, not necessarily the same in any two places.

§**1. Preliminaries.** We consider the propagation of waves in a composite medium Ω which consists of a fluid part Ω_f and a solid part Ω_f, that is, $\Omega = \Omega_f \cup \Omega_s$. Ω will be identified with a domain in $\mathbb{R}^N$ for $N = 2, 3$, and will be taken to be of unit thickness when $N = 2$. Let $\Gamma = \partial\Omega_f \cap \partial\Omega_s$ denote the interface between two media, and let $\Gamma_f = \partial\Omega_f \setminus \Gamma$ and $\Gamma_s = \partial\Omega_s \setminus \Gamma$. The fluid–solid interaction model we are going to study in this paper is given by

$$\frac{1}{c^2} p_{tt} - \Delta p = g_f, \qquad \text{in } \Omega_f, \tag{2.1.i}$$

$$\rho_s \mathbf{u}_{tt} - \operatorname{div}(\sigma(\mathbf{u})) = \mathbf{g}_s, \qquad \text{in } \Omega_s, \tag{2.1.ii}$$

$$\frac{\partial p}{\partial n_f} - \rho_f \mathbf{u}_{tt} \cdot n_s = 0, \qquad \text{on } \Gamma, \tag{2.1.iii}$$

$$\sigma(\mathbf{u}) n_s - p n_f = 0, \qquad \text{on } \Gamma, \tag{2.1.iv}$$

$$\frac{1}{c} p_t + \frac{\partial p}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{2.1.v}$$

$$\rho_s \mathcal{A}_s \mathbf{u}_t + \sigma(\mathbf{u}) n_s = 0, \qquad \text{on } \Gamma_s, \tag{2.1.vi}$$

$$p(x,0) = p_0(x), \quad p_t(x,0) = p_1(x), \qquad \text{in } \Omega_f, \tag{2.1.vii}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{u}_t(x,0) = \mathbf{u}_1(x), \qquad \text{in } \Omega_s, \tag{2.1.viii}$$

where

$$\sigma(\mathbf{u}) = \lambda_s \operatorname{div} \mathbf{u} I + 2\mu_s \epsilon(\mathbf{u}), \qquad \varepsilon(\mathbf{u}) = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]. \tag{2.1.ix}$$

In the above description, p is the pressure function in Ω_f and $\mathbf{u}$ is the displacement vector in Ω_s. ρ_i $(i = f, s)$ denotes the density of Ω_i, n_i $(i = f, s)$ denotes the unit outward normal to $\partial\Omega_i$. $\lambda_s > 0$ and $\mu_s \geq 0$ are the Lamé constants of Ω_s. Equation (2.1.ix) is the constitutive relation for Ω_s. I stands for the $N \times N$ identity matrix. The boundary conditions in (2.1.v) and (2.1.vi) are the first order absorbing boundary conditions for acoustic and the elastic waves, respectively. These boundary conditions are transparent to waves arriving normally at the boundary (cf. [8], [12]). Finally, equations (2.1.iii) and (2.1.iv) are the interface conditions which describe the interaction between the fluid and the solid. For a detailed derivation of the above model and its analytical analysis, we refer to [10].

The standard space notations are adopted in this paper. For example, $H^k(D)$, $k \geq 0$ integer, denotes the Sobolev spaces over the domain D. When $k = 0$, $H^0(D) = L^2(D)$, and $(\cdot,\cdot)_D$ is used to denote the standard inner product on $L^2(D)$. $\|\cdot\|_{k,D}$ denotes the usual norms on $H^k(D)$. For a Banach space B, $L^q(0,T;B)$ stands for the space of L^q–integrable functions with range in B. $W^{k,q}([0,T];B)$ is the space of functions whose up to kth order derivatives with respect to t are in $L^q(0,T;B)$. $\mathbf{B}$ denotes $(B)^N$, $N = 2, 3$, and a vector in $\mathbf{B}$ is denoted either by $\mathbf{v}$ or by $\underset{\sim}{v}$. In addition, we also introduce the following special space notations:

$$P_f = \bigcap_{k=0}^{1} W^{k,\infty}(0,T;H^{1-k}(\Omega_f)) \cap \left\{ p \in L^2(0,T;L^2(\Omega_f));\ \frac{\partial p}{\partial t} \in L^2(0,T;L^2(\Gamma_f)) \right\},$$

$$Q_f = P_f \cap \bigcap_{k=1}^{2} W^{k,\infty}(0,T;H^{2-k}(\Omega_s)) \cap \left\{ p \in L^2(0,T;L^2(\Omega_f));\ \frac{\partial^2 p}{\partial t^2} \in L^2(0,T;L^2(\Gamma_f)) \right\},$$

$$\mathbf{U}_s = \bigcap_{k=0}^{1} W^{k,\infty}(0,T;\mathbf{H}^{1-k}(\Omega_s)) \cap \left\{ \mathbf{u} \in L^2(0,T;\mathbf{L}^2(\Omega_s));\ \frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T;\mathbf{L}^2(\Gamma_s)) \right\},$$

$$\mathbf{V}_s = \mathbf{U}_s \cap \bigcap_{k=1}^{2} W^{k,\infty}(0,T;\mathbf{H}^{2-k}(\Omega_s)) \cap \left\{ \mathbf{u} \in L^2(0,T;\mathbf{L}^2(\Omega_s));\ \frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2(0,T;\mathbf{L}^2(\Gamma_s)) \right\},$$

$$\widehat{Q}_f = Q_f \cap L^\infty(0,T;H^2(\Omega_f)), \qquad \widehat{\mathbf{V}}_s = V_s \cap L^\infty(0,T;\mathbf{H}^2(\Omega_s)).$$

We shall make the following physical and mathematical assumptions throughout the paper. The same assumptions were made in [10].

Assumption A:

(A1). $\rho_f = \text{constant} > 0, \quad \rho_s = \rho_s(x) \geq \overline{\rho}_s > 0$. λ_f, λ_s, μ_s are all positive constants.

(A2). $\Omega_f \subset \mathbb{R}^N$, $\Omega_s \subset \mathbb{R}^N$ for $N = 2, 3$ are bounded open sets with Lipschitz continuous boundary $\partial\Omega_f$ and $\partial\Omega_s$, respectively.

(A3). $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_s$, $\text{meas}(\Omega_f) \neq \emptyset$, $\text{meas}(\Omega_s) \neq \emptyset$. Assume that $\Gamma \neq \emptyset$. Note that it is acceptable if one of Γ_f and Γ_s is empty.

(A4). Suppose that the initial datum functions satisfy the following compatibility conditions.

Compatibility Condition C:

(C1). $\mathbf{u}_0 \in \mathbf{H}^2(\Omega_s)$ and $\mathbf{u}_1 \in \mathbf{H}^1(\Omega_s)$ are said to be compatible on Γ_s if $\rho_s \mathcal{A}_s \mathbf{u}_1 + \sigma(\mathbf{u}_0) n_s = 0$, on Γ_s.

(C2). $p_0 \in H^1(\Omega_f)$ and $\mathbf{u}_0 \in \mathbf{H}^2(\Omega_s)$ are said to be compatible on Γ if $\sigma(\mathbf{u}_0) n_s - p_0 n_f = 0$, on Γ.

The weak formulation of above problem is defined by seeking $(p, \mathbf{u}) \in P_f \times \mathbf{V}_s$ which satisfy (2.1.vii)–(2.1.viii), and the following two identities

$$
\begin{aligned}
\text{(2.2.i)} \quad & (\frac{1}{c^2} p_{tt}, q)_{\Omega_f} + (\nabla p, \nabla q)_{\Omega_f} + \langle \frac{1}{c} p_t, q \rangle_{\Gamma_f} - \langle \rho_f \mathbf{u}_{tt} \cdot n_s, q \rangle_{\Gamma} \\
& \qquad = (g_f, q)_{\Omega_f}, \quad \forall q \in H^1(\Omega_f), \\
\text{(2.2.ii)} \quad & (\rho_s \mathbf{u}_{tt}, \mathbf{v})_{\Omega_s} + (\sigma(\mathbf{u}), \varepsilon(\mathbf{v}))_{\Omega_s} + \langle \rho_s \mathcal{A}_s \mathbf{u}_t, \mathbf{v} \rangle_{\Gamma_s} + \langle p, \mathbf{v} \cdot n_s \rangle_{\Gamma} \\
& \qquad = (\mathbf{g}_s, v)_{\Omega_s}, \quad \forall \mathbf{v} \in (H^1(\Omega_s))^N.
\end{aligned}
$$

We conclude this section by briefly summarizing some basic facts about the solution of the initial–boundary value problem (2.1). For the precise statements of these results and their proofs, we refer to [10].

Theorem 1.1. *The initial–boundary value problem (2.1) has a unique weak solution $(p, \mathbf{u}) \in P_f \times \mathbf{V}_s$. Moreover, if Ω_f and Ω_s are convex polygon or polyhedron domains, then $(p, \mathbf{u}) \in \widehat{Q}_f \times \widehat{\mathbf{V}}_s$; and both p and $\mathbf{u}$ are smooth in t variable if the source functions and the initial datum functions are smooth in t variable,*

§2. The continuous–time Galerkin approximation. In this section we shall formulate the continuous–time Galerkin approximation to the solution of the initial–boundary value problem (2.1), and derive *a priori* estimates for the errors in $L^\infty(H^1)$ and in $L^\infty(L^2)$ under different assumptions on the approximate starting values and on the smoothness of the solution.

§2.1. Formulation of semi–discrete finite element methods. Let $P_{h_1} \subset H^1(\Omega_f)$, $V_{h_2} \subset H^1(\Omega_s)$ be two finite dimensional (finite element) subspaces associated with the triangulation $\mathcal{T}_{h_1}^f$ and $\mathcal{T}_{h_2}^s$ of Ω_f and Ω_s with mesh sizes h_1 and h_2, respectively. Suppose there exist two integers $k \geq 1$ and $r \geq 1$ such that

(i) for any $q \in H^\ell(\Omega_f)$, there exists $q_h \in P_{h_1}$ such that for $0 \leq j \leq \ell$,

$$
\|q - q_h\|_{j,\Omega_f} \leq C h_1^m \|q\|_{\ell,\Omega_f}, \quad m = \min\{\ell - j, k - j\}; \tag{2.3}
$$

(ii) for any $v \in H^\ell(\Omega_f)$, there exists $v_h \in V_{h_2}$ such that for $0 \leq j \leq \ell$

$$
\|v - v_h\|_{j,\Omega_s} \leq C h_2^m \|v\|_{\ell,\Omega_s}, \quad m = \min\{\ell - j, r - j\}. \tag{2.4}
$$

Then the continuous–time finite element method for (2.2) is defined by seeking $(P,\mathbf{U}) \in P_{h_1} \times \mathbf{V}_{h_2}$ such that

$$\begin{aligned}
&\text{(2.5.i)} \quad (\frac{1}{c^2}P_{tt}, q_h)_{\Omega_f} + (\nabla P, \nabla q_h)_{\Omega_f} + \langle \frac{1}{c}P_t, q_h\rangle_{\Gamma_f} - \langle \rho_f \mathbf{U}_{tt}\cdot n_s, q_h\rangle_\Gamma \\
&\qquad\qquad = (g_f, q_h)_{\Omega_f}, \quad \forall q_h \in P_{h_1}, \\
&\text{(2.5.ii)} \quad (\rho_s \mathbf{U}_{tt}, v_h)_{\Omega_s} + (\sigma(\mathbf{U}), \varepsilon(\mathbf{v}_h))_{\Omega_s} + \langle \rho_s \mathcal{A}_s \mathbf{U}_t, \mathbf{v}_h\rangle_{\Gamma_s} + \langle P, \mathbf{v}_h\cdot n_s\rangle_\Gamma \\
&\qquad\qquad = (\mathbf{g}_s, \mathbf{v}_h)_{\Omega_s}, \quad \forall \mathbf{v}_h \in \mathbf{V}_{h_2}, \\
&\text{(2.5.iii)} \quad P(0) = P_0, \quad P_t(0) = P_1, \qquad \text{in } \Omega_f, \\
&\text{(2.5.iv)} \quad \mathbf{U}(0) = \mathbf{U}_0, \quad \mathbf{U}_t(0) = \mathbf{U}_1, \qquad \text{in } \Omega_s,
\end{aligned}$$

where P_0, P_1, $\mathbf{U}_0$ and $\mathbf{U}_1$ are the approximate starting values which will be specified in the next subsection.

Remark. Since (2.5) can be rewritten as a linear system of second order ordinary differential equations (cf. [10]), it is easy to show the well–posedness of (2.5).

§2.2. Optimal H^1 *a priori* error estimate. To estimate the errors $r = p - P$ and $\mathbf{e} = \mathbf{u} - \mathbf{U}$, we use the idea of [19] by comparing the Galerkin approximation with so–called elliptic projection $(\hat{P}, \hat{\mathbf{U}}) \in P_{h_1} \times \mathbf{V}_{h_2}$ of $(p,\mathbf{u})$, defined by

$$(\nabla(p-\hat{P}), \nabla q_h)_{\Omega_f} + (p - \hat{P}, q_h)_{\Omega_f} = 0, \qquad \forall q_h \in P_{h_1}. \tag{2.6}$$

$$(\sigma(\mathbf{u}-\hat{\mathbf{U}}), \varepsilon(\mathbf{v}_h))_{\Omega_s} + (\mathbf{u}-\hat{\mathbf{U}}, \mathbf{v}_h)_{\Omega_s} = 0, \quad \forall v_h \in \mathbf{V}_{h_2}. \tag{2.7}$$

Introduce the notations

$$\begin{aligned}
r &= \hat{p} - P = (p - \hat{P}) + (\hat{P} - P) = \theta + \phi, \\
\mathbf{e} &= \hat{\mathbf{U}} - \mathbf{U} = (\mathbf{u} - \hat{\mathbf{U}}) + (\hat{\mathbf{U}} - \mathbf{U}) = \underset{\sim}{\eta} + \underset{\sim}{\xi}.
\end{aligned}$$

The following estimates of θ and $\underset{\sim}{\eta}$ are well–known. The proof of which can be found in [1] and [19].

Lemma 2.1. *Suppose $p \in W^{j,\infty}(H^k(\Omega_f))$, $\frac{\partial^{j+1}p}{\partial t^{j+1}} \in L^2(H^k(\Omega_f))$, $\mathbf{u} \in W^{j,\infty}(\mathbf{H}^r(\Omega_s))$ and $\frac{\partial^{j+1}\mathbf{u}}{\partial t^{j+1}} \in L^2(\mathbf{H}^r(\Omega_s))$ for $j \ge 0$. Then there exists an h–independent constant $C > 0$ such that*

$$\left\{ \begin{array}{ll} h_1 \| \int_0^t \theta(\tau) d\tau \|_{L^\infty(H^1(\Omega_f))}, & (j=0) \\ h_1 \|\theta\|_{W^{j-1,\infty}(H^1(\Omega_f))}, & (j \ge 1) \end{array} \right\} + \|\theta\|_{W^{j,\infty}(L^2(\Omega_f))} + \left\| \frac{\partial^{j+1}\theta}{\partial t^{j+1}} \right\|_{L^2(L^2(\Omega_f))} \le C h_1^k E_j(p, \Omega_f), \tag{2.8.i}$$

$$\left\{ \begin{array}{ll} h_2 \| \int_0^t \underset{\sim}{\eta}(\tau) d\tau \|_{L^\infty(H^1(\Omega_s))}, & (j=0) \\ h_2 \|\underset{\sim}{\eta}\|_{W^{j-1,\infty}(H^1(\Omega_s))}, & (j \ge 1) \end{array} \right\} + \|\underset{\sim}{\eta}\|_{W^{j,\infty}(L^2(\Omega_s))} + \left\| \frac{\partial^{j+1}\underset{\sim}{\eta}}{\partial t^{j+1}} \right\|_{L^2(L^2(\Omega_s))} \le C h_2^r E_j(\mathbf{u}, \Omega_s), \tag{2.8.ii}$$

where

$$E_j(p,\Omega_f) = \|p\|_{W^{j,\infty}(H^k(\Omega_f))} + \left\| \frac{\partial^{j+1} p}{\partial t^{j+1}} \right\|_{L^2(H^k(\Omega_f))}, \tag{2.9.i}$$

$$E_j(\mathbf{u},\Omega_s) = \|\mathbf{u}\|_{W^{j,\infty}(H^r(\Omega_s))} + \left\| \frac{\partial^{j+1} \mathbf{u}}{\partial t^{j+1}} \right\|_{L^2(H^r(\Omega_s))}. \tag{2.9.ii}$$

To estimate ϕ and $\underset{\sim}{\xi}$, we notice from (2.2), (2.5)–(2.7) that

$$\begin{aligned}
&(2.10.\text{i})\ (\frac{1}{c^2}\hat{P}_{tt}, q_h)_{\Omega_f} + (\nabla \hat{P}, \nabla q_h)_{\Omega_f} + \langle \frac{1}{c}\hat{P}_t, q_h\rangle_{\Gamma_f} - \langle \rho_f \hat{\mathbf{U}}_{tt}\cdot n_s, q_h\rangle_\Gamma = (g_f, q_h)_{\Omega_f} \\
&\qquad - [(\frac{1}{c^2}\theta_{tt}, q_h)_{\Omega_f} + \langle \frac{1}{c}\theta_t, q_h\rangle_{\Gamma_f} - \langle \rho_f \underset{\sim}{\eta}_{tt}\cdot n_s, q_h\rangle_\Gamma + (\theta, q_h)_{\Omega_f}],\ \forall q_h \in P_{h_1}, \\
&(2.10.\text{ii})(\rho_s \mathbf{U}_{tt}, \mathbf{v}_h)_{\Omega_s} + (\sigma(\hat{\mathbf{U}}), \varepsilon(\mathbf{v}_h))_{\Omega_s} + \langle \rho_s \mathcal{A}_s \hat{\mathbf{U}}_t, \mathbf{v}_h\rangle_{\Gamma_s} + \langle \hat{P}, \mathbf{v}_h\cdot n_s\rangle_\Gamma = (\mathbf{g}_s, \mathbf{v}_h)_{\Omega_s} \\
&\qquad - [(\rho_s \underset{\sim}{\eta}_{tt}, \mathbf{v}_h)_{\Omega_s} + \langle \rho_s \mathcal{A}_s \underset{\sim}{\eta}_t, \mathbf{v}_h\rangle_{\Gamma_s} + \langle \theta, \mathbf{v}_h\cdot n_s\rangle_\Gamma + (\underset{\sim}{\eta}, \mathbf{v}_h)_{\Omega_s}],\ \forall \mathbf{v}_h \in \mathbf{V}_{h_2}.
\end{aligned}$$

Subtracting (2.5) from (2.10) gives

$$\begin{aligned}
&(2.11.\text{i})\quad (\frac{1}{c^2}\phi_{tt}, q_h)_{\Omega_f} + (\nabla\phi, \nabla q_h)_{\Omega_f} + \langle \frac{1}{c}\phi_t, q_h\rangle_{\Gamma_f} - \langle \rho_f \underset{\sim}{\xi}_{tt}\cdot n_s, q_h\rangle_\Gamma \\
&\qquad = (\theta - \frac{1}{c^2}\theta_{tt}, q_h)_{\Omega_f} - \langle \frac{1}{c}\theta_t, q_h\rangle_{\Gamma_f} + \langle \rho_f \underset{\sim}{\eta}_{tt}\cdot n_s, q_h\rangle_\Gamma,\ \forall q_h \in P_{h_1}, \\
&(2.11.\text{ii})\quad (\rho_s \underset{\sim}{\xi}_{tt}, \mathbf{v}_h)_{\Omega_s} + (\sigma(\underset{\sim}{\xi}), \varepsilon(\mathbf{v}_h))_{\Omega_s} + \langle \rho_s \mathcal{A}_s \underset{\sim}{\xi}_t, \mathbf{v}_h\rangle_{\Gamma_s} + \langle \phi, \mathbf{v}_h\cdot n_s\rangle_\Gamma \\
&\qquad = (\underset{\sim}{\eta} - \rho_s \underset{\sim}{\eta}_{tt}, \mathbf{v}_h)_{\Omega_s} - \langle \rho_s \mathcal{A}_s \underset{\sim}{\eta}_t, \mathbf{v}_h\rangle_{\Gamma_s} - \langle \theta, \mathbf{v}_h\cdot n_s\rangle_\Gamma,\ \forall \mathbf{v}_h \in \mathbf{V}_{h_2}.
\end{aligned}$$

Now we first differentiate (2.11.ii) with respect to t and then choose the test functions $q_h = \frac{1}{\rho_f}\phi_t$, $\mathbf{v}_h = \underset{\sim}{\xi}_{tt}$ in (2.11) to get

$$\begin{aligned}
&(2.12.\text{i})\quad \frac{1}{2}\frac{d}{dt}\left[\left\| \frac{1}{c\sqrt{\rho_f}}\phi_t \right\|_{0,\Omega_f}^2 + \left\| \frac{1}{\sqrt{\rho_f}}\nabla\phi \right\|_{0,\Omega_f}^2 \right] + \left| \frac{1}{\sqrt{\rho_s c}}\phi_t \right|_{0,\Gamma_f}^2 - \langle \underset{\sim}{\xi}_{tt}\cdot n_s, \phi_t\rangle_\Gamma \\
&\qquad \le \frac{1}{2}\left\| \frac{1}{c\sqrt{\rho_f}}\phi_t \right\|_{0,\Omega_f}^2 + \left\| \frac{1}{c\sqrt{\rho_f}}\theta_{tt} \right\|_{0,\Omega_f}^2 + \left\| \frac{c}{\sqrt{\rho_f}}\theta \right\|_{0,\Omega_f}^2 \\
&\qquad\qquad - [\langle \frac{1}{c\rho_f}\theta_t, \phi_t\rangle_{\Gamma_f} - \langle \underset{\sim}{\eta}_{tt}\cdot n_s, \phi_t\rangle_\Gamma], \\
&(2.12.\text{ii})\quad \frac{1}{2}\frac{d}{dt}\left[\|\sqrt{\rho_s}\underset{\sim}{\xi}_{tt}\|_{0,\Omega_s}^2 + 2\|\sqrt{\mu_s}\varepsilon(\underset{\sim}{\xi}_t)\|_{0,\Omega_s}^2 + \|\sqrt{\lambda_s}\,\mathrm{div}(\underset{\sim}{\xi}_t)\|_{0,\Omega_s}^2 \right] \\
&\qquad\qquad + |\sqrt{\rho_s c_0}\underset{\sim}{\xi}_{tt}|_{0,\Gamma_s}^2 + \langle \phi_t, \underset{\sim}{\xi}_{tt}\cdot n_s\rangle_\Gamma \\
&\qquad \le \frac{1}{2}\|\sqrt{\rho_s}\underset{\sim}{\xi}_{tt}\|_{0,\Omega_s}^2 + \|\sqrt{\rho_s}\underset{\sim}{\eta}_{ttt}\|_{0,\Omega_s}^2 + \left\| \frac{1}{\sqrt{\rho_s}}\underset{\sim}{\eta}_t \right\|_{0,\Omega_s}^2 \\
&\qquad\qquad + [\langle c_1\rho_s \underset{\sim}{\eta}_{tt}, \underset{\sim}{\xi}_{tt}\rangle_{\Gamma_s} - \langle \theta_t, \underset{\sim}{\xi}_{tt}\cdot n_s\rangle_\Gamma].
\end{aligned}$$

The inequalities in (2.12) will be integrated in t in order to get relations to which we can apply Gronwall's Lemma if we can bound the four boundary integrals on the right hand sides of (2.12.i) and (2.12.ii). We could handle two integrals on Γ_f and Γ_s directly by using Schwarz inequality, however this would cause us to loose a factor $h^{\frac{1}{2}}$. In addition, we should not bound two integrals on Γ directly since this will lead to "unclosed" inequalities from which we can not get any estimates. In the following we shall use a boundary duality argument due to Douglas and Dupont (cf. [5], [7]) to bound these four boundary integrals. Note that

$$
\begin{aligned}
(2.13)\ I_1 &\equiv \int_0^\tau \left[\langle \frac{1}{c\rho_f}\theta_t, \phi_t\rangle_{\Gamma_f} - \langle \underset{\sim}{\eta}_{tt}\cdot n_s, \phi_t\rangle_\Gamma\right] dt \\
&= \left[\langle \frac{1}{c\rho_f}\theta_t, \phi\rangle_{\Gamma_f} - \langle \underset{\sim}{\eta}_{tt}\cdot n_s, \phi\rangle_\Gamma\right]\Bigg|_0^\tau - \int_0^\tau \left[\langle \frac{1}{\rho_f}\theta_{tt}, \phi\rangle_{\Gamma_f} - \langle \underset{\sim}{\eta}_{ttt}\cdot n_s, \phi\rangle_\Gamma\right] dt.
\end{aligned}
$$

$$
\begin{aligned}
(2.14)\ I_2 &\equiv \int_0^\tau [\langle c_1\rho_s \underset{\sim}{\eta}_{tt}, \underset{\sim}{\xi}_{tt}\rangle_{\Gamma_s} - \langle \theta_t, \underset{\sim}{\xi}_{tt}\cdot n_s\rangle_\Gamma] dt \\
&= [\langle c_1\rho_s \underset{\sim}{\eta}_{tt}, \underset{\sim}{\xi}_t\rangle_{\Gamma_s} - \langle \theta_t, \underset{\sim}{\xi}_t\cdot n_s\rangle_\Gamma]\Big|_0^\tau - \int_0^\tau [\langle c_1\rho_s \underset{\sim}{\eta}_{ttt}, \underset{\sim}{\xi}_t\rangle_{\Gamma_s} - \langle \theta_{tt}, \underset{\sim}{\xi}_t\cdot n_s\rangle_\Gamma] dt.
\end{aligned}
$$

Using (2.13), (2.14), and the trace theorem $|v|_{\frac{1}{2},\partial D} \le C\|v\|_{1,D}$ we get

$$
\begin{aligned}
(2.15)\qquad |I_1|+|I_2| &\le \delta[\|\phi(\tau)\|_{1,\Omega_f}^2 + \|\underset{\sim}{\xi}_t(\tau)\|_{1,\Omega_s}^2] \\
&+C\int_0^\tau [\|\phi(t)\|_{1,\Omega_f}^2 + \|\underset{\sim}{\xi}_t(t)\|_{1,\Omega_s}^2] dt \\
&+C\int_0^\tau [|\theta_{tt}(t)|_{-\frac{1}{2},\Gamma_f}^2 + |\theta_{tt}(t)|_{-\frac{1}{2},\Gamma}^2 + |\underset{\sim}{\eta}_{ttt}(t)|_{-\frac{1}{2},\Gamma_s}^2 + |\underset{\sim}{\eta}_{ttt}(t)|_{-\frac{1}{2},\Gamma}^2] dt \\
&+C[|\theta_t(\tau)|_{-\frac{1}{2},\Gamma_f}^2 + |\theta_t(\tau)|_{-\frac{1}{2},\Gamma}^2 + |\underset{\sim}{\eta}_{tt}(\tau)|_{-\frac{1}{2},\Gamma_s}^2 + |\underset{\sim}{\eta}_{tt}(\tau)|_{-\frac{1}{2},\Gamma}^2] \\
&+C[|\theta_t(0)|_{-\frac{1}{2},\Gamma_f}^2 + |\theta_t(0)|_{-\frac{1}{2},\Gamma}^2 + |\underset{\sim}{\eta}_{tt}(0)|_{-\frac{1}{2},\Gamma_s}^2 + |\underset{\sim}{\eta}_{tt}(0)|_{-\frac{1}{2},\Gamma}^2] \\
&+C[\|\phi(0)\|_{1,\Omega_f}^2 + \|\underset{\sim}{\xi}_t(0)\|_{1,\Omega_s}^2].
\end{aligned}
$$

Finally, using the relation

$$v^2(\tau) = v^2(0) + 2\int_0^\tau v(t)v_t(t)\ dt$$

we have

$$(2.16)\qquad \|\phi(\tau)\|_{0,\Omega_f}^2 \le \|\phi(0)\|_{0,\Omega_f}^2 + \int_0^\tau \left[\|\phi(t)\|_{0,\Omega_f}^2 + \|\phi_t(t)\|_{0,\Omega_f}^2\right] dt.$$

$$(2.17)\qquad \|\underset{\sim}{\xi}_t(\tau)\|_{0,\Omega_s}^2 \le \|\underset{\sim}{\xi}_t(0)\|_{0,\Omega_f}^2 + \int_0^\tau [\|\underset{\sim}{\xi}_t(t)\|_{0,\Omega_s}^2 + \|\underset{\sim}{\xi}_{tt}(t)\|_{0,\Omega_s}^2] dt.$$

Combing the above estimates we get the following lemma.

Lemma 2.2. *There exists a constant $C > 0$ such that*

$$
\begin{aligned}
(2.18)\ & \|\phi_t\|^2_{L^\infty(L^2(\Omega_f))} + \|\phi\|^2_{L^\infty(H^1(\Omega_f))} + \|\phi_t\|^2_{L^2(L^2(\Gamma_f))} + \|\underset{\sim}{\xi}_{tt}\|^2_{L^\infty(L^2(\Omega_s))} \\
& \qquad + \|\underset{\sim}{\xi}_t\|^2_{L^\infty(H^1(\Omega_s))} + \|\underset{\sim}{\xi}_{tt}\|^2_{L^2(L^2(\Gamma_s))} \\
& \le C[\|\theta\|^2_{L^2(L^2(\Omega_f))} + \|\theta_{tt}\|^2_{L^2(L^2(\Omega_f))} + \|\underset{\sim}{\eta}_t\|^2_{L^2(L^2(\Omega_s))} + \|\underset{\sim}{\eta}_{ttt}\|^2_{L^2(L^2(\Omega_s))} \\
& + \|\theta_t\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega_f))} + \|\theta_{tt}\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_f))} + \|\underset{\sim}{\eta}_{tt}\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega_s))} \\
& + \|\underset{\sim}{\eta}_{ttt}\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_s))} + \|\phi_t(0)\|^2_{0,\Omega_f} + \|\phi(0)\|^2_{1,\Omega_f} + \|\underset{\sim}{\xi}_{tt}(0)\|^2_{0,\Omega_s} + \|\underset{\sim}{\xi}_t(0)\|^2_{1,\Omega_s}].
\end{aligned}
$$

Proof. Adding (2.12.i) and (2.12.ii), integrating the sum and applying (2.15) we get that there exists some constant $C > 0$ such that for any $\tau \in (0, T)$,

$$
\begin{aligned}
(2.19)\quad & \|\phi_t(\tau)\|^2_{0,\Omega_f} + \|\nabla\phi(\tau)\|^2_{0,\Omega_f} + \int_0^\tau |\phi_t(t)|^2_{0,\Gamma_f}\,dt + \|\underset{\sim}{\xi}_{tt}(\tau)\|^2_{0,\Omega_s} \\
& \qquad + \|\varepsilon(\underset{\sim}{\xi}_t(\tau))\|^2_{0,\Omega_s} + \|\operatorname{div}(\underset{\sim}{\xi}_t(\tau))\|^2_{0,\Omega_s} + \int_0^\tau |\underset{\sim}{\xi}_{tt}(t)|^2_{0,\Gamma_s}\,dt \\
& \le \delta[\|\phi(\tau)\|^2_{1,\Omega_f} + \|\underset{\sim}{\xi}_t(\tau)\|^2_{1,\Omega_s}] + C\int_0^\tau \Big[\|\phi_t(t)\|^2_{0,\Omega_f} + \|\phi(t)\|^2_{1,\Omega_f} \\
& + \|\underset{\sim}{\xi}_{tt}(t)\|^2_{0,\Omega_s} + \|\underset{\sim}{\xi}_t(t)\|^2_{1,\Omega_s}\Big]\,dt + C\{\|\theta\|^2_{L^2(L^2(\Omega_f))} + \|\theta_{tt}\|^2_{L^2(L^2(\Omega_f))} \\
& + \|\underset{\sim}{\eta}_t\|^2_{L^2(L^2(\Omega_s))} + \|\underset{\sim}{\eta}_{ttt}\|^2_{L^2(L^2(\Omega_s))} + \|\theta_t\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega_f))} \\
& + \|\theta_{tt}\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_f))} + \|\underset{\sim}{\eta}_{tt}\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega_s))} + \|\underset{\sim}{\eta}_{ttt}\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_s))} \\
& + \|\phi_t(0)\|^2_{0,\Omega_f} + \|\phi(0)\|^2_{1,\Omega_f} + \|\underset{\sim}{\xi}_{tt}(0)\|^2_{0,\Omega_s} + \|\underset{\sim}{\xi}_t(0)\|^2_{1,\Omega_s}\}.
\end{aligned}
$$

Recall that Korn's inequality ([13])

$$
(2.20)\qquad \|\varepsilon(\mathbf{v})\|^2_{0,\Omega_s} + \|\mathbf{v}\|^2_{0,\Omega_s} \ge C_0\|\mathbf{v}\|^2_{1,\Omega_s}, \quad \forall \mathbf{v} \in (H^1(\Omega_s))^N.
$$

Take $\delta = \min\{\frac{1}{2}, \frac{C_0}{2}\}$, add (2.16) and (2.17) to (2.19), the lemma then follows from the resulting relation and Gronwall's lemma.

To bound the negative half norms appeared in (2.18), we need the following lemma of Douglas and Dupont ([5]).

Lemma 2.3. *Suppose $E_j(p, \Omega_f)$ and $E_j(\mathbf{u}, \Omega_s)$ are bounded for some integer $j \ge 0$. Then there exists a constant $C > 0$ such that*

$$
(2.21)\qquad \|\theta\|_{W^{j,\infty}(H^{-\frac{1}{2}}(\partial\Omega_f))} + \left\|\frac{\partial^{j+1}\theta}{\partial t^{j+1}}\right\|_{L^2(H^{-\frac{1}{2}}(\partial\Omega_f))} \le Ch_1^k E_j(p, \Omega_f).
$$

$$
(2.22)\qquad \|\underset{\sim}{\eta}\|_{W^{j,\infty}(H^{-\frac{1}{2}}(\partial\Omega_s))} + \left\|\frac{\partial^{j+1}\underset{\sim}{\eta}}{\partial t^{j+1}}\right\|_{L^2(H^{-\frac{1}{2}}(\partial\Omega_s))} \le Ch_2^r E_j(\mathbf{u}, \Omega_s).
$$

From Lemmas 2.1–2.3 we get the following theorem.

Theorem 2.1. *Suppose $E_1(p,\Omega_f)$ and $E_2(\mathbf{u},\Omega_s)$ are bounded. Then there exists a constant $C>0$ such that*

$$(2.23)\quad \|p-P\|_{W^{1,\infty}(L^2(\Omega_f))}+\|\mathbf{u}-\mathbf{U}\|_{W^{2,\infty}(L^2(\Omega_s))}\le C\left[h_1^kE_1(p,\Omega_f)+h_2^kE_2(\mathbf{u},\Omega_s)\right].$$

$$(2.24)\quad \|p-P\|_{L^\infty(H^1(\Omega_f))}+\|\mathbf{u}-\mathbf{U}\|_{W^{1,\infty}(H^1(\Omega_s))}\le C\left[h_1^{k-1}E_1(p,\Omega_f)+h_2^{k-1}E_2(\mathbf{u},\Omega_s)\right].$$

$$(2.25)\quad \|(p-P)_t\|_{L^2(L^2(\Gamma_f))}+\|(\mathbf{u}-\mathbf{U})_{tt}\|_{L^2(L^2(\Gamma_s))}$$
$$\le\begin{cases} C\left[h_1^kE_1(p,\Omega_f)+h_2^kE_2(\mathbf{u},\Omega_s)\right]^{\frac12}\left[h_1^{k-1}E_1(p,\Omega_f)+h_2^{k-1}E_2(\mathbf{u},\Omega_s)\right]^{\frac12}, & N=2,\\ C\left[h_1^kE_1(p,\Omega_f)+h_2^kE_2(\mathbf{u},\Omega_s)\right]^{\frac14}\left[h_1^{k-1}E_1(p,\Omega_f)+h_2^{k-1}E_2(\mathbf{u},\Omega_s)\right]^{\frac34}, & N=3.\end{cases}$$

Remark. The estimate (2.23) and (2.25) hold under the assumption

$$(2.26.\text{i})\quad \|(\hat P-P)_t(0)\|_{0,\Omega_f}+\|(\hat P-P)(0)\|_{1,\Omega_f}+\|(\hat{\mathbf{U}}-U)(0)\|_{1,\Omega_s}$$
$$+\|(\hat{\mathbf{U}}-U)_t(0)\|_{1,\Omega_s}+\|(\hat{\mathbf{U}}-U)_{tt}(0)\|_{0,\Omega_s}\le C(h_1^k+h_2^r),$$

but (2.24) holds under the weaker assumption

$$(2.26.\text{ii})\quad \|(\hat P-P)_t(0)\|_{0,\Omega_f}+\|(\hat P-P)(0)\|_{1,\Omega_f}+\|(\hat{\mathbf{U}}-U)(0)\|_{1,\Omega_s}$$
$$+\|(\hat{\mathbf{U}}-U)_t(0)\|_{1,\Omega_s}+\|(\hat{\mathbf{U}}-U)_{tt}(0)\|_{0,\Omega_s}\le C(h_1^{k-1}+h_2^{r-1}).$$

Proof. The inequalities (2.23) and (2.24) follow from Lemmas 2.1–2.3, (2.16)–(2.17), and the triangle inequality; and (2.25) can be obtained by using Lemmas 2.2 and 2.3 and the following fact

$$|v|_{0,\partial D}\le\begin{cases}\|v\|_{0,D}^{\frac12}\|v\|_{0,D}^{\frac12}, & \text{if } N=2,\\ \|v\|_{0,D}^{\frac14}\|v\|_{0,D}^{\frac34}, & \text{if } N=3.\end{cases}$$

§**2.3. Optimal L^2 *a priori* error estimate.** In the previous subsection we derive optimal L^2-estimate for $p-P$, $(p-P)_t$, $\mathbf{u}-\mathbf{U}$, $(\mathbf{u}-\mathbf{U})_t$ and $(\mathbf{u}-\mathbf{U})_{tt}$ under the assumption (2.26.i), which is a super approximation requirement for the approximate starting values (see [7] and [15] for a discussion and for the strategies to construct such approximate starting values). On the other hand, the $L^\infty(H^1)$ estimate (2.24) is obtained under the assumption (2.26.ii), which is satisfied when L^2–projections of the initial data are used as the approximate starting values (cf. [1]).

In this subsection we shall derive optimal L^2–estimates for $p-P$, $\mathbf{u}-\mathbf{U}$ and $(\mathbf{u}-\mathbf{U})_t$ under weaker requirements on the approximate starting values and on the smoothness of the solution. The main idea is to employ a modified energy method of Baker [1] for the p–equation and to use the standard energy method for the $\mathbf{u}$–equation.

First, we notice that (2.11.i) can be rewritten as

$$(2.27)\qquad -(\frac{1}{c^2}\phi_t,q_{ht})_{\Omega_f}+(\nabla\phi,\nabla q_h)_{\Omega_f}-\langle\frac1c\phi,q_{ht}\rangle_{\Gamma_f}+\langle\rho_f\underset{\sim}{\xi}_t\cdot n_s,q_{ht}\rangle_\Gamma$$
$$=(\theta,q_h)_{\Omega_f}+(\frac{1}{c^2}\theta_t,q_{ht})_{\Omega_f}-\frac{d}{dt}\Big[(\frac{1}{c^2}r_t,q_h)_{\Omega_f}+\langle\frac1c r,q_h\rangle_{\Gamma_f}$$
$$-\langle\rho_f\mathbf{e}_t\cdot n_s,q_h\rangle_\Gamma\Big]+\langle\frac1c\theta,q_{ht}\rangle_{\Gamma_f}-\langle\rho_f\underset{\sim}{\eta}_t\cdot n_s,q_{ht}\rangle_\Gamma.$$

Take $q_h(t) = \int_t^\tau \phi(\lambda)d\lambda$ in (2.27) and $\mathbf{v}_h(t) = \rho_f \underset{\sim}{\xi}_t(t)$ in (2.11.ii). Clearly, $q_h(\tau) = 0$ and $q_h'(t) = -\phi(t)$. Hence,

$$\begin{aligned}(2.28)\qquad &\frac{1}{2}\frac{d}{dt}\left\|\frac{1}{c}\phi\right\|_{0,\Omega_f}^2 - \frac{1}{2}\frac{d}{dt}\left\|\nabla\int_t^\tau \phi(\lambda)d\lambda\right\|_{0,\Omega_f}^2 + \left|\frac{1}{\sqrt{c}}\phi\right|_{0,\Gamma_f}^2 - \langle \rho_f \underset{\sim}{\xi}_t\cdot n_s, \phi\rangle_\Gamma \\ &= (\theta, \int_t^\tau \phi(\lambda)d\lambda)_{\Omega_f} - (\frac{1}{c^2}\theta_t, \phi)_{\Omega_f} - \langle\frac{1}{c}\theta, \phi\rangle_{\Gamma_f} + \langle \rho_f \underset{\sim}{\eta}_t\cdot n_s, \phi\rangle_\Gamma \\ &\quad + \frac{d}{dt}\left[(\frac{1}{c^2}r_t, q_h)_{\Omega_f} + \langle\frac{1}{c}r, q_h\rangle_{\Gamma_f} - \langle\rho_f \mathbf{e}_t\cdot n_s, q_h\rangle_\Gamma\right].\end{aligned}$$

$$\begin{aligned}(2.29)\quad &\frac{1}{2}\frac{d}{dt}\left[\left\|\sqrt{\rho_s\rho_f}\underset{\sim}{\xi}_t\right\|_{0,\Omega_s}^2 + 2\left\|\sqrt{\mu_s\rho_f}\varepsilon(\underset{\sim}{\xi})\right\|_{0,\Omega_s}^2 + \left\|\sqrt{\lambda_s\rho_f}\,\mathrm{div}(\underset{\sim}{\xi})\right\|_{0,\Omega_s}^2\right] \\ &\quad + \left|\sqrt{c_0\rho_s\rho_f}\underset{\sim}{\xi}_t\right|_{0,\Gamma_s}^2 + \langle\phi, \rho_f\underset{\sim}{\xi}_t\cdot n_s\rangle_\Gamma \\ &\le C\left[\|\underset{\sim}{\xi}_t\|_{0,\Omega_s}^2 + \|\underset{\sim}{\eta}_{tt}\|_{0,\Omega_s}^2 + \|\underset{\sim}{\eta}\|_{0,\Omega_s}^2\right] - \langle\rho_s\mathcal{A}_s\underset{\sim}{\eta}_t, \rho_f\underset{\sim}{\xi}_t\rangle_{\Gamma_s} - \langle\theta, \rho_f\underset{\sim}{\xi}_t\cdot n_s\rangle_\Gamma.\end{aligned}$$

Adding (2.29) to (2.28) and integrating over $(0,\tau)$ we get

$$\begin{aligned}(2.30)\qquad &\|\phi(\tau)\|_{0,\Omega_f}^2 + \int_0^\tau |\phi(t)|_{0,\Gamma_f}^2 dt + \|\underset{\sim}{\xi}_t(\tau)\|_{0,\Omega_s}^2 + \|\varepsilon(\underset{\sim}{\xi}(\tau))\|_{0,\Omega_s}^2 \\ &\quad + \|\mathrm{div}(\underset{\sim}{\xi}(\tau))\|_{0,\Omega_s}^2 + \int_0^\tau |\underset{\sim}{\xi}_t(t)|_{0,\Gamma_s}^2 dt + \left\|\nabla\int_0^\tau\phi(\lambda)d\lambda\right\|_{0,\Omega_f}^2 \\ &\le C\int_0^\tau\Bigg[\|\phi(t)\|_{0,\Omega_f}^2 + \left\|\int_t^\tau \phi(\lambda)d\lambda\right\|_{0,\Omega_f}^2 + \|\theta_t(t)\|_{0,\Omega_f}^2 \\ &\quad + \|\theta(t)\|_{0,\Omega_f}^2 + \|\xi_t(t)\|_{0,\Omega_s}^2 + \|\underset{\sim}{\eta}_{tt}(t)\|_{0,\Omega_s}^2 + \|\underset{\sim}{\eta}\|_{0,\Omega_s}^2\Bigg] dt \\ &\quad - 2\int_0^\tau \Bigg[\langle\frac{1}{c}\theta(t), \phi(t)\rangle_{\Gamma_f} - \langle\rho_f\underset{\sim}{\eta}_t(t)\cdot n_s, \phi(t)\rangle_\Gamma \\ &\quad + \langle\rho_s\mathcal{A}_s\underset{\sim}{\eta}_t(t), \rho_f\underset{\sim}{\xi}_t(t)\rangle_{\Gamma_s} + \langle\theta(t), \rho_f\underset{\sim}{\xi}_t(t)\cdot n_s\rangle_\Gamma\Bigg] dt \\ &\quad + 2\Bigg[(\frac{1}{c^2}r_t(0), \int_0^\tau\phi(\lambda)d\lambda)_{\Omega_f} + \langle\frac{1}{c}r(0), \int_0^\tau\phi(\lambda)d\lambda\rangle_{\Gamma_f} \\ &\quad - \langle\rho_f\mathbf{e}_t(0)\cdot n_s, \int_0^\tau\phi(\lambda)d\lambda\rangle_\Gamma\Bigg] + C\Bigg[\|\phi(0)\|_{0,\Omega_f}^2 + \|\underset{\sim}{\xi}_t(0)\|_{0,\Omega_s}^2 \\ &\quad + \|\varepsilon(\underset{\sim}{\xi}(0))\|_{0,\Omega_s}^2 + \|\mathrm{div}(\underset{\sim}{\xi}(0))\|_{0,\Omega_s}^2\Bigg].\end{aligned}$$

Since

$$\begin{aligned}(2.31)\quad &\left|\langle \frac{1}{c} r(0), \int_0^\tau \phi(\lambda)d\lambda\rangle_{\Gamma_f} - \langle \rho_f \mathbf{e}_t(0)\cdot n_s, \int_0^\tau \phi(\lambda)d\lambda\rangle_\Gamma\right| \\ &\qquad \le \frac{\delta}{2}\left\|\int_0^\tau \phi(\lambda)d\lambda\right\|^2_{1,\Omega_f} + C\left[|r(0)|^2_{-\frac{1}{2},\Gamma_f} + |\mathbf{e}_t(0)|^2_{-\frac{1}{2},\Gamma}\right].\end{aligned}$$

$$(2.32)\quad \left|(\frac{1}{c^2} r_t(0), \int_0^\tau \phi(\lambda)d\lambda)_{\Omega_f}\right| \le \frac{1}{2}\left\|\int_0^\tau \phi(\lambda)d\lambda\right\|^2_{0,\Omega_f} + C\|r_t(0)\|^2_{0,\Omega_f}.$$

$$\begin{aligned}(2.33)\quad &\left|\int_0^\tau \left[\langle \frac{1}{c}\theta(t), \phi(t)\rangle_{\Gamma_f} - \langle \rho_f \underset{\sim}{\eta}_t(t), \phi(t)\rangle_\Gamma + \langle \rho_s \mathcal{A}_s \underset{\sim}{\eta}_t(t), \rho_f \underset{\sim}{\xi}_t(t)\rangle_{\Gamma_s}\right.\right. \\ &\qquad\qquad \left.\left. + \langle \theta(t), \rho_f \underset{\sim}{\xi}_t(t)\cdot n_s\rangle_\Gamma\right] dt\right| \\ &= \left|\langle \frac{1}{c}\theta(0), \int_0^\tau \phi(\lambda)d\lambda\rangle_{\Gamma_f} - \langle \rho_f \underset{\sim}{\eta}_t(0), \int_0^\tau \phi(\lambda)d\lambda\rangle_\Gamma\right. \\ &\quad + \langle \rho_s \mathcal{A}_s \underset{\sim}{\eta}_t(t), \rho_f \underset{\sim}{\xi}(t)\rangle_{\Gamma_s}\big|_0^\tau + \langle \theta(t), \rho_f \underset{\sim}{\xi}(t)\cdot n_s\rangle_\Gamma\big|_0^\tau \\ &\quad + \int_0^\tau \left[\langle \frac{1}{c}\theta_t(t), \int_t^\tau \phi(\lambda)d\lambda\rangle_{\Gamma_f} - \langle \rho_s \mathcal{A}_s \underset{\sim}{\eta}_{tt}(t), \rho_f \underset{\sim}{\xi}(t)\rangle_{\Gamma_s}\right] dt \\ &\quad \left.- \int_0^\tau \left[\langle \rho_f \underset{\sim}{\eta}_{tt}(t), \int_0^\tau \phi(\lambda)d\lambda\rangle_\Gamma + \langle \theta_t(t), \underset{\sim}{\xi}(t)\cdot n_s\rangle_\Gamma\right] dt\right| \\ &\le \frac{\delta}{2}\left[\left\|\int_0^\tau \phi(\lambda)d\lambda\right\|^2_{1,\Omega_f} + \|\underset{\sim}{\xi}(\tau)\|^2_{1,\Omega_s}\right] \\ &\quad + C\int_0^\tau \left[\left\|\int_0^t \phi(\lambda)d\lambda\right\|^2_{1,\Omega_f} + \|\underset{\sim}{\xi}(t)\|^2_{1,\Omega_s}\right] dt \\ &\quad + C\left[\|\theta_t\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_f))} + \|\underset{\sim}{\eta}_{tt}\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_s))} + \|\theta\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega_f))}\right. \\ &\quad \left. + \|\underset{\sim}{\eta}_t\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega_s))} + \|\underset{\sim}{\xi}(0)\|^2_{1,\Omega_s}\right].\end{aligned}$$

$$(2.34)\quad \|\underset{\sim}{\xi}(\tau)\|^2_{0,\Omega_s} \le \|\underset{\sim}{\xi}(0)\|^2_{0,\Omega_s} + \int_0^\tau \left[\|\underset{\sim}{\xi}(t)\|^2_{0,\Omega_s} + \|\underset{\sim}{\xi}_t(t)\|^2_{0,\Omega_s}\right] dt.$$

$$(2.35)\quad \left\|\int_0^\tau \phi(\lambda)d\lambda\right\|^2_{0,\Omega_f} \le \tau\int_0^\tau \|\phi(t)\|^2_{0,\Omega_f}.$$

Take $\delta = \min\{\frac{1}{2}, \frac{C_0}{2}\}$ and add (2.34) and (2.35) to (2.30), and using (2.31)–(2.33) and

Korn's inequality (2.20) we get (let $q(t) = \int_0^t \phi(\lambda)d\lambda$)

$$
\begin{aligned}
(2.36) \qquad & \|\phi(\tau)\|^2_{0,\Omega_f} + \|q(\tau)\|^2_{1,\Omega_f} + \int_0^\tau |\phi(t)|^2_{0,\Gamma_f} dt + \|\underset{\sim}{\xi}_t(\tau)\|^2_{0,\Omega_s} \\
& \qquad + \|\underset{\sim}{\xi}(\tau)\|^2_{1,\Omega_s} + \int_0^\tau |\underset{\sim}{\xi}_t(t)|^2_{0,\Gamma_s} dt \\
& \leq C \Bigg\{ \int_0^\tau \Big[\|\phi(t)\|^2_{0,\Omega_f} dt + \|q(t)\|^2_{1,\Omega_f} + \|\underset{\sim}{\xi}_t(t)\|^2_{0,\Omega_s} + \|\underset{\sim}{\xi}(t)\|^2_{1,\Omega_s} \Big] dt \\
& \quad + \|\theta\|^2_{L^2(L^2(\Omega_f))} + \|\theta_t\|^2_{L^2(L^2(\Omega_f))} + \|\underset{\sim}{\eta}\|^2_{L^2(L^2(\Omega_s))} + \|\underset{\sim}{\eta}_{tt}\|^2_{L^2(L^2(\Omega_s))} \\
& \quad + \|\theta\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega_f))} + \|\theta_t\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_f))} + \|\underset{\sim}{\eta}_t\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega_s))} \\
& \quad + \|\underset{\sim}{\eta}_{tt}\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_s))} + |r(0)|^2_{-\frac{1}{2},\Omega_f} + |\mathbf{e}_t(0)|^2_{-\frac{1}{2},\Gamma} \\
& \quad + \|\phi(0)\|^2_{0,\Omega_f} + \|\underset{\sim}{\xi}_t(0)\|^2_{0,\Omega_s} + \|\underset{\sim}{\xi}(0)\|^2_{1,\Omega_s} + \|r_t(0)\|^2_{0,\Omega_f} \Bigg\}.
\end{aligned}
$$

Finally, (2.36) and Gronwall's inequality yield

Lemma 2.4. *There is a constant $C > 0$ such that*

$$
\begin{aligned}
(2.37) \quad & \|\phi\|^2_{L^\infty(L^2(\Omega_f))} + \left\| \int_0^t \phi(\lambda)d\lambda \right\|^2_{L^\infty(H^1(\Omega_f))} + \|\underset{\sim}{\xi}_t\|_{L^\infty(L^2(\Omega_s))} + \|\underset{\sim}{\xi}\|^2_{L^\infty(H^1(\Omega_s))} \\
& \qquad + \|\phi\|^2_{L^2(L^2(\Gamma_s))} + \|\underset{\sim}{\xi}_t\|^2_{L^2(L^2(\Omega_s))} \\
& \leq C \Bigg\{ \|\theta\|^2_{L^2(L^2(\Omega_f))} + \|\theta_t\|^2_{L^2(L^2(\Omega_f))} + \|\underset{\sim}{\eta}\|^2_{L^2(L^2(\Omega_s))} + \|\underset{\sim}{\eta}_{tt}\|^2_{L^2(L^2(\Omega_s))} \\
& \quad + \|\theta\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega_f))} + \|\theta_t\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_f))} + \|\underset{\sim}{\eta}_t\|^2_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega_s))} \\
& \quad + \|\underset{\sim}{\eta}_{tt}\|^2_{L^2(H^{-\frac{1}{2}}(\partial\Omega_s))} + |r(0)|^2_{-\frac{1}{2},\Gamma_f} + |\mathbf{e}_t(0)|^2_{-\frac{1}{2},\Gamma} \\
& \quad + \|\phi(0)\|^2_{0,\Omega_f} + \|\underset{\sim}{\xi}_t(0)\|^2_{0,\Omega_s} + \|\underset{\sim}{\xi}(0)\|^2_{1,\Omega_s} + \|r_t(0)\|^2_{0,\Omega_f} \Bigg\}.
\end{aligned}
$$

From Lemmas 2.1, 2.3 and 2.4 and triangle inequality we get the following theorem.

Theorem 2.2. *Suppose $E_0(p,\Omega_f)$ and $E_1(\mathbf{u},\Omega_s)$ are bounded. Then there exists a constant $C > 0$ such that*

$$(2.40)\ \|p - P\|_{L^\infty(L^2(\Omega_f))} + \|\mathbf{u} - \mathbf{U}\|_{W^{1,\infty}(L^2(\Omega_s))} \leq C \left[h_1^k E_0(p,\Omega_f) + h_2^k E_1(\mathbf{u},\Omega_s) \right].$$

$$
\begin{aligned}
(2.41)\ & \left\| \int_0^t (p - P)(\lambda)d\lambda \right\|_{L^\infty(H^1(\Omega_f))} + \|\mathbf{u} - \mathbf{U}\|_{L^\infty(H^1(\Omega_s))} \\
& \qquad \leq C \left[h_1^{k-1} E_0(p,\Omega_f) + h_2^{k-1} E_1(\mathbf{u},\Omega_s) \right].
\end{aligned}
$$

$$
\begin{aligned}
(2.42)\ & \|(p - P)\|_{L^2(L^2(\Gamma_f))} + \|(\mathbf{u} - \mathbf{U})_t\|_{L^2(L^2(\Gamma_s))} \\
& \leq \begin{cases} C \left[h_1^k E_0(p,\Omega_f) + h_2^k E_1(\mathbf{u},\Omega_s) \right]^{\frac{1}{2}} \left[h_1^{k-1} E_0(p,\Omega_f) + h_2^{k-1} E_1(\mathbf{u},\Omega_s) \right]^{\frac{1}{2}}, & N = 2, \\ C \left[h_1^k E_0(p,\Omega_f) + h_2^k E_1(\mathbf{u},\Omega_s) \right]^{\frac{1}{4}} \left[h_1^{k-1} E_0(p,\Omega_f) + h_2^{k-1} E_1(\mathbf{u},\Omega_s) \right]^{\frac{3}{4}}, & N = 3. \end{cases}
\end{aligned}
$$

Remark. Estimate (2.40) holds under the following assumption

$$\|(\hat{P}-P)_t(0)\|_{0,\Omega_f}+\|(\hat{P}-P)(0)\|_{0,\Omega_f}+\|(\hat{\mathbf{U}}-\mathbf{U})_t(0)\|_{0,\Omega_s} + \|(\hat{\mathbf{U}}-\mathbf{U})(0)\|_{1,\Omega_s}\le C(h_1^k+h_2^r). \tag{2.43.i}$$

But estimate (2.41) holds when

$$\|(\hat{P}-P)_t(0)\|_{0,\Omega_f}+\|(\hat{P}-P)(0)\|_{0,\Omega_f}+\|(\hat{\mathbf{U}}-\mathbf{U})_t(0)\|_{0,\Omega_s} + \|(\hat{\mathbf{U}}-\mathbf{U})(0)\|_{1,\Omega_s}\le C(h_1^{k-1}+h_2^{r-1}). \tag{2.43.ii}$$

§3. The discrete–time Galerkin approximation. In this section we shall introduce fully–discrete finite element methods for the initial–boundary value problem (2.1) by discretizing the system of ordinary partial differential system (2.5) using the finite difference method. We shall derive error estimates analogous to Theorems 2.1 and 2.2, and show that the time–stepping schemes are of second order accuracy.

§3.1. Formulation of fully–discrete finite element methods. Let J be a positive integer. Let $\Delta t=\frac{T}{J}$, $t_n=n\Delta t$, and

$$\mathbf{u}^n=\mathbf{u}(t_n),\quad p^n=p(t_n),\quad \mathbf{U}^n=\mathbf{U}(t_n),\quad P^n=P(t_n).$$

We also let

$$\begin{aligned}
&P^{n+\frac12}=\frac{P^n+P^{n+1}}{2}, && \partial_f P^n=\frac{P^{n+1}-P^n}{\Delta t},\\
&\partial_b P^n=\frac{P^n-P^{n-1}}{\Delta t}, && \partial_c P^n=\frac{P^{n+1}-P^{n-1}}{2\Delta t},\\
&\overline{\partial}_c P^n=\frac{P^{n+1}+P^{n-1}}{2\Delta t}, && \partial^2 P^n=\frac{P^{n+1}-2P^n+P^{n-1}}{\Delta t^2},\\
&P^{n,\gamma}=\gamma P^{n-1}+(1-2\gamma)P^n+\gamma P^{n+1}.
\end{aligned}$$

It is easy to check the following identities

$$\partial_c P^n=\frac12[\partial_f P^n+\partial_f P^{n-1}]=\frac{P^{n+\frac12}-P^{n-\frac12}}{\Delta t}, \tag{3.1.i}$$

$$\partial^2 P^n=\partial_f(\partial_b P^n)=\partial_b(\partial_f P^n). \tag{3.1.ii}$$

The discrete–time finite element method is defined by seeking a sequence $\{(P^n,\mathbf{U}^n)\}_{n=0}^J$ in $P_{h_1}\times\mathbf{V}_{h_2}$ such that for for $n=1,2,\cdots,J-1$,

$$(\frac{1}{c^2}\partial^2 P^n,q^h)_{\Omega_f}+(\nabla P^{n,\frac14},\nabla q^h)_{\Omega_f}+\langle\frac1c\partial_c P^n,q^h\rangle_{\Gamma_f} - \langle\rho_f\partial_c^2\mathbf{U}^n\cdot n_s,q^h\rangle_\Gamma=(g_f^{n,\frac14},q^h)_{\Omega_f},\quad \forall q^h\in P_{h_1}, \tag{3.2.i}$$

$$(\rho_s\partial^2\mathbf{U}^n,\mathbf{v}^h)_{\Omega_s}+(\sigma(\mathbf{U}^{n,\frac14}),\varepsilon(\mathbf{v}^h))_{\Omega_s}+\langle\rho_s\mathcal{A}_s\partial_c\mathbf{U}^n,\mathbf{v}^h\rangle_{\Gamma_s} + \langle P^n,\mathbf{v}^h\cdot n_s\rangle_\Gamma=(\mathbf{g}_s^{n,\frac14},\mathbf{v}^h)_{\Omega_s},\quad \forall \mathbf{v}^h\in\mathbf{V}_{h_2}, \tag{3.2.ii}$$

$$P^0,P^1\in P_{h_1},\qquad \mathbf{U}^0,\mathbf{U}^1\in\mathbf{V}_{h_2}. \tag{3.2.iii}$$

Remark. $P^0,P^1,\mathbf{U}^0$ and $\mathbf{U}^1$ are some approximations to the initial values $p_0,p_1,\mathbf{u}_0$ and $\mathbf{u}_1$, they will be specified later in the next subsection.

§**3.2. Optimal H^1 *a priori* error estimate.** Introduce error functions

$$r^n = p^n - P^n = (p^n - \hat{P}^n) + (\hat{P}^n - P^n) = \theta^n + \phi^n,$$

$$\mathbf{e}^n = \mathbf{u}^n - \mathbf{U}^n = (\mathbf{u}^n - \hat{\mathbf{U}}^n) + (\hat{\mathbf{U}}^n - \mathbf{U}^n) = \underset{\sim}{\eta}^n + \underset{\sim}{\xi}^n.$$

Since the error derivation for the discrete–time approximation is analogous to that of the continuous–time approximation, in the following we shall only highlight the steps which are worth noting. First note that $(p^n, \mathbf{u}^n)$ satisfies

$$\text{(3.3.i)} \quad (\frac{1}{c^2}\partial^2 p^n, q^h)_{\Omega_f} + (\nabla p^{n,\frac14}, \nabla q^h)_{\Omega_f} + \langle \frac{1}{c}\partial_c p^n, q^h\rangle_{\Gamma_f} - \langle \rho_f \partial_c^2 \mathbf{u}^n \cdot n_s, q^h\rangle_\Gamma$$
$$= (g_f^{n,\frac14} + \alpha_n, q^h)_{\Omega_f} + \langle \beta_n, q^h\rangle_{\Gamma_f} + \langle \underset{\sim}{\delta}_n \cdot n_s, q^h\rangle_\Gamma, \quad \forall q^h \in P_{h_1},$$

$$\text{(3.3.ii)} \quad (\rho_s \partial^2 \mathbf{u}^n, \mathbf{v}^h)_{\Omega_s} + (\sigma(\mathbf{u}^{n,\frac14}), \varepsilon(\mathbf{v}^h))_{\Omega_s} + \langle \rho_s \mathcal{A}_s \partial_c \mathbf{u}^n, \mathbf{v}^h\rangle_{\Gamma_s} + \langle p^n, \mathbf{v}^h \cdot n_s\rangle_\Gamma$$
$$= (\mathbf{g}_s^{n,\frac14} + \underset{\sim}{\pi}_n, \mathbf{v}^h)_{\Omega_s} + \langle \underset{\sim}{\lambda}_n, \mathbf{v}^h\rangle_{\Gamma_s} + \langle w_n, \mathbf{v}^h \cdot n_s\rangle_\Gamma, \quad \forall \mathbf{v}^h \in \mathbf{V}_{h_2}.$$

It follows from Taylor's formula that

$$\|\alpha_n\|^2_{0,\Omega_f} \le C\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} \|p_{tttt}(t)\|^2_{0,\Omega_f}\, dt,$$

$$\|\underset{\sim}{\pi}_n\|^2_{0,\Omega_s} \le C\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} \|\mathbf{u}_{tttt}(t)\|^2_{0,\Omega_s}\, dt,$$

$$|\beta_n|^2_{-\frac12,\Gamma_f} \le C\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} |p_{ttt}(t)|^2_{-\frac12,\partial\Omega_f}\, dt,$$

$$|\underset{\sim}{\delta}_n|^2_{-\frac12,\Gamma} \le C\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} |\mathbf{u}_{tttt}(t)|^2_{-\frac12,\partial\Omega_s}\, dt,$$

$$|\underset{\sim}{\lambda}_n|^2_{-\frac12,\Gamma_s} \le C\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} |\mathbf{u}_{ttt}(t)|^2_{-\frac12,\partial\Omega_s}\, dt,$$

$$|w_n|^2_{-\frac12,\Gamma} \le C\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} |p_{ttt}(t)|^2_{-\frac12,\partial\Omega_f}\, dt.$$

From (3.3) and the definition of $\hat{P}$ and $\hat{\mathbf{U}}$ we get

$$\text{(3.4.i)} \quad (\frac{1}{c^2}\partial^2 \phi^n, q^h)_{\Omega_f} + (\nabla \phi^{n,\frac14}, \nabla q^h)_{\Omega_f} + \langle \frac{1}{c}\partial_c \phi^n, q^h\rangle_{\Gamma_f} - \langle \rho_f \partial_c^2 \underset{\sim}{\xi}^n \cdot n_s, q^h\rangle_\Gamma$$
$$= (\alpha_n - \frac{1}{c^2}\partial^2\theta^n + \theta^{n,\frac14}, q^h)_{\Omega_f} + \langle \beta_n - \frac{1}{c}\partial_c\theta^n, q_h\rangle_{\Gamma_f}$$
$$+ \langle (\underset{\sim}{\delta}_n - \rho_f \partial_c^2 \underset{\sim}{\eta}^n) \cdot n_s, q_h\rangle_\Gamma, \quad \forall q_h \in P_{h_1},$$

$$\text{(3.4.ii)} \quad (\rho_s \partial^2 \underset{\sim}{\xi}^n, \mathbf{v}_h)_{\Omega_s} + (\sigma(\underset{\sim}{\xi}^{n,\frac14}), \varepsilon(\mathbf{v}_h))_{\Omega_s} + \langle \rho_s \mathcal{A}_s \partial_c \underset{\sim}{\xi}^n, \mathbf{v}^h\rangle_{\Gamma_s}$$
$$+ \langle \phi^n, \mathbf{v}^h \cdot n_s\rangle_\Gamma$$
$$= (\underset{\sim}{\pi}_n - \rho_s \partial^2 \underset{\sim}{\eta}^n + \underset{\sim}{\eta}^{n,\frac14}, \mathbf{v}_h)_{\Omega_s} + \langle \underset{\sim}{\lambda}_n - \rho_s \mathcal{A}_s \partial_c \underset{\sim}{\eta}^n, \mathbf{v}^h\rangle_{\Gamma_s}$$
$$+ \langle w_n - \theta^n, \mathbf{v}^h n_s\rangle_\Gamma, \quad \forall \mathbf{v}^h \in \mathbf{V}_{h_2}.$$

Apply the operator ∂_c to both sides of (3.4.ii), take $q^h = \partial_c \phi^n$ in (3.4.i) and $\mathbf{v}_h = \rho_f \partial_c(\partial_c \underset{\sim}{\xi}^n)$, add the resulting equations to get

$$
\begin{aligned}
(3.5)\quad & \frac{1}{2\Delta t}\left[\left\|\frac{1}{c}\partial_f\phi^n\right\|^2_{0,\Omega_f} - \left\|\frac{1}{c}\partial_f\phi^{n-1}\right\|^2_{0,\Omega_f} + \|\nabla\phi^{n+\frac12}\|^2_{0,\Omega_f} - \|\nabla\phi^{n-\frac12}\|^2_{0,\Omega_f}\right] \\
&+ \left|\frac{1}{\sqrt{c}}\partial_c\phi^n\right|^2_{0,\Gamma_f} + \frac{1}{2\Delta t}\Big[\|\sqrt{\rho_s\rho_f}\partial_f\partial_c\underset{\sim}{\xi}^n\|^2_{0,\Omega_s} - \|\sqrt{\rho_s\rho_f}\partial_f\partial_c\underset{\sim}{\xi}^{n-1}\|^2_{0,\Omega_s} \\
&+ 2\|\sqrt{\mu_s\rho_f}\varepsilon(\partial_c\underset{\sim}{\xi}^{n+\frac12})\|^2_{0,\Omega_s} - 2\|\sqrt{\mu_s\rho_s}\varepsilon(\partial_c\xi^{n-\frac12})\|^2_{0,\Omega_s} \\
&+ \|\sqrt{\lambda_s\rho_f}\mathrm{div}(\partial_c\underset{\sim}{\xi}^{n+\frac12})\|^2_{0,\Omega_s} - \|\sqrt{\lambda_s\rho_f}\mathrm{div}(\partial_c\underset{\sim}{\xi}^{n-\frac12})\|^2_{0,\Omega_s}\Big] \\
&\quad + c_0|\sqrt{\rho_s\rho_f}\partial_c(\partial_c\underset{\sim}{\xi}^n)|^n_{0,\Gamma_s} \\
\le\ & \|\partial_c\phi^n\|^2_{0,\Omega_f} + \|\alpha_n\|^2_{0,\Omega_f} + \|\frac{1}{c}\partial^2\theta^n\|^2_{0,\Omega_f} + \|\theta^{n,\frac14}\|_{0,\Omega_f} \\
&+ \|\sqrt{\rho_f}\partial_c(\partial_c\underset{\sim}{\xi}^n)\|^2_{0,\Omega_s} + \|\sqrt{\rho_f}\partial_c\underset{\sim}{\pi}_n\|^2_{0,\Omega_s} + \|\sqrt{\rho_f\rho_s}\partial^2(\partial_c\underset{\sim}{\eta}^n)\|^2_{0,\Omega_s} \\
&+ \|\sqrt{\rho_f}\partial_c\underset{\sim}{\eta}^{n,\frac14}\|^2_{0,\Omega_s} + \langle\beta_n - \frac{1}{c}\partial_c\theta^n, \partial_c\phi^n\rangle_{\Gamma_f} + \langle(\underset{\sim}{\delta}_n - \rho_f\partial^2\underset{\sim}{\eta}^n)\cdot n_s, \partial_c\phi^n\rangle_\Gamma \\
&+ \langle\partial_c(\underset{\sim}{\lambda}_n - \rho_s\mathcal{A}_s\partial_c\underset{\sim}{\eta}^n), \rho_f\partial_c(\partial_c\underset{\sim}{\xi}^n)\rangle_{\Gamma_s} + \langle\partial_c(w_n - \theta^n), \rho_f\partial_c(\partial_c\underset{\sim}{\xi}^n)\cdot n_s\rangle_\Gamma.
\end{aligned}
$$

Applying $\sum_{n=1}^{\ell}$ ($\ell \le J-1$) to (3.5) yields

$$
\begin{aligned}
(3.6)\quad & \|\partial_f\phi^\ell\|^2_{0,\Omega_f} + \|\nabla\phi^{\ell+\frac12}\|^2_{0,\Omega_f} + \|\partial_f(\partial_c\underset{\sim}{\xi}^\ell)\|^2_{0,\Omega_f} + \|\varepsilon(\partial_c\xi^{\ell+\frac12})\|^2_{0,\Omega_s} \\
&\quad + \|\mathrm{div}(\partial_c\underset{\sim}{\xi}^{\ell+\frac12})\|^2_{0,\Omega_s} + \Delta t\sum_{n=1}^{\ell}\left[|\partial_c\phi^n|^2_{0,\Gamma_f} + |\partial_c(\partial_c\underset{\sim}{\xi}^n)|^2_{0,\Gamma_s}\right] \\
\le\ & C\Bigg\{\sum_{n=1}^{\ell}\left[\|\phi^{n+\frac12} - \phi^{n-\frac12}\|^2_{0,\Omega_f} + \|\partial_c\underset{\sim}{\xi}^{n+\frac12} - \partial_c\underset{\sim}{\xi}^{n-\frac12}\|^2_{0,\Omega_s}\right] \\
&+ \Delta t\sum_{n=1}^{\ell}\Big[\|\alpha_n\|^2_{0,\Omega_f} + \|\partial^2\theta^n\|^2_{0,\Omega_f} + \|\theta^{n,\frac14}\|^2_{0,\Omega_f} + \|\partial_c\underset{\sim}{\pi}_n\|^2_{0,\Omega_s} \\
&+ \|\partial^2(\partial_c\underset{\sim}{\eta}^n)\|^2_{0,\Omega_s} + \|\partial_c\underset{\sim}{\eta}^{n,\frac14}\|^2_{0,\Omega_s}\Big] + \Delta t\sum_{n=1}^{\ell}\Big[\langle\beta_n - \frac{1}{c}\partial_c\theta^n, \partial_c\phi^n\rangle_{\Gamma_f} \\
&+ \langle(\underset{\sim}{\delta}_n - \rho_f\partial_c^2\underset{\sim}{\eta}^n)\cdot n_s, \partial_c\phi^n\rangle_\Gamma + \langle\partial_c(w_n - \theta^n), \rho_f\partial_c(\partial_c\underset{\sim}{\xi}^n)\cdot n_s\rangle_\Gamma \\
&+ \langle\partial_c(\underset{\sim}{\lambda}_n - \rho_s\mathcal{A}_s\partial_c\underset{\sim}{\eta}^n), \rho_f\partial_c(\partial_c\underset{\sim}{\xi}^n)\rangle_{\Gamma_s}\Big] + \Big[\|\partial_f\phi^0\|^2_{0,\Omega_f} + \|\nabla\phi^{\frac12}\|^2_{0,\Omega_f} \\
&+ \|\partial_f(\partial_c\underset{\sim}{\xi}^0)\|^2_{0,\Omega_s} + \|\varepsilon(\partial_c\underset{\sim}{\xi}^{\frac12})\|^2_{0,\Omega_s} + \|\mathrm{div}(\partial_c\underset{\sim}{\xi}^{\frac12})\|^2_{0,\Omega_s}\Big]\Bigg\}.
\end{aligned}
$$

For the boundary integral terms using discrete integration by parts we get

$$
\begin{aligned}
J_\ell =&\Delta t \sum_{n=1}^{\ell} \Big[\langle \beta_n - \frac{1}{c}\partial_c\theta^n, \partial_c\phi^n\rangle_{\Gamma_f} + \langle (\underset{\sim}{\delta}_n - \rho_f\partial_c^2\underset{\sim}{\eta}^n)\cdot n_s, \partial_c\phi^n\rangle_{\Gamma} \\
&+ \langle \partial_c(\underset{\sim}{\lambda}_n - \rho_s\mathcal{A}_s\partial_c\underset{\sim}{\eta}^n), \rho_f\partial_c(\partial_c\underset{\sim}{\xi}^n)\rangle_{\Gamma_s} + \langle \partial_c(w_n-\theta^n), \rho_f\partial_c(\partial_c\underset{\sim}{\xi}^n)\cdot n_s\rangle_{\Gamma} \Big] \\
=&-\frac{\Delta t}{2}\sum_{n=1}^{\ell} \Big[\langle \partial_f(\beta_n - \frac{1}{c}\partial_c\theta^n), \phi^{n+\frac12}\rangle_{\Gamma_f} + \langle \partial_f\partial_c(\underset{\sim}{\lambda}_n - \rho_s\mathcal{A}_s\partial_c\underset{\sim}{\eta}^n), \rho_f\partial_c\underset{\sim}{\xi}^{n+\frac12}\rangle_{\Gamma_s} \\
&+\langle \partial_f(\underset{\sim}{\delta}_n - \rho_f\partial_c^2\underset{\sim}{\eta}^n)\cdot n_s, \phi^{n+\frac12}\rangle_{\Gamma} + \langle \partial_f(\partial_c(w_n-\theta^n)), \rho_f\partial_c\underset{\sim}{\xi}^{n+\frac12}\cdot n_s\rangle_{\Gamma} \Big] \\
&+\frac{\Delta t}{2}\Big[\langle \beta_{\ell+1} - \frac{1}{c}\partial_c\theta^{\ell+1}, \phi^{\ell+\frac12}\rangle_{\Gamma_f} - \langle \beta_1 - \frac{1}{c}\partial_c\theta^1, \phi^{\frac12}\rangle_{\Gamma_f} \\
&+ \langle (\underset{\sim}{\delta}_{\ell+1} - \rho_f\partial_c^2\underset{\sim}{\eta}^{\ell+1})\cdot n_s, \phi^{\ell+\frac12}\rangle_{\Gamma} - \langle (\underset{\sim}{\delta}_1 - \rho_f\partial_c^2\underset{\sim}{\eta}^1)\cdot n_s, \phi^{\frac12}\rangle_{\Gamma} \\
&+ \langle \partial_c(\underset{\sim}{\lambda}_{\ell+1} - \rho_f\partial_c\underset{\sim}{\eta}^{\ell+1}), \rho_f\partial_c\underset{\sim}{\xi}^{\ell+\frac12}\rangle_{\Gamma_s} - \langle \partial_c(\underset{\sim}{\lambda}_1 - \rho_f\partial_c\underset{\sim}{\eta}^1), \rho_f\partial_c\underset{\sim}{\xi}^{\frac12}\rangle_{\Gamma_s} \\
&+\langle \partial_c(w_{\ell+1}-\theta^{\ell+1}), \rho_f\partial_c\underset{\sim}{\xi}^{\ell+\frac12}\cdot n_s\rangle_{\Gamma} - \langle \partial_c(w_1-\theta^1), \rho_f\partial_c\underset{\sim}{\xi}^{\frac12}\cdot n_s\rangle_{\Gamma} \Big].
\end{aligned}
\tag{3.7}
$$

Hence,

$$
\begin{aligned}
|J_\ell| \le& C\Delta t \sum_{n=1}^{\ell} \Big[\|\phi^{n+\frac12}\|_{1,\Omega_f}^2 + \|\partial_c\underset{\sim}{\xi}^{n+\frac12}\|_{1,\Omega_s}^2 + |\partial_f\partial_c\theta^n|_{-\frac12,\partial\Omega_f}^2 \\
&+ |\partial_f\partial_c^2\underset{\sim}{\eta}^n|_{-\frac12,\partial\Omega_s}^2 + |\partial_f\beta_n|_{-\frac12,\Gamma_f}^2 + |\partial_f\underset{\sim}{\delta}_n|_{-\frac12,\Gamma}^2 + |\partial_f\partial_c\underset{\sim}{\lambda}_n|_{-\frac12,\Gamma}^2 \\
&\quad +|\partial_f\partial_c w_n|_{-\frac12,\Gamma}^2 \Big] \\
&+ \Delta t \Big[\|\phi^{\ell+\frac12}\|_{1,\Omega_f}^2 + \|\partial_c\underset{\sim}{\xi}^{\ell+\frac12}\|_{1,\Omega_s}^2 \Big] + C\Delta t \Big[\|\partial_c\theta\|_{\tilde{L}^\infty(H^{-\frac12}(\partial\Omega_f))}^2 \\
&+ \|\partial_c^2\underset{\sim}{\eta}\|_{\tilde{L}^\infty(H^{-\frac12}(\partial\Omega_s))}^2 + \|\beta\|_{\tilde{L}^\infty(H^{-\frac12}(\Gamma_f))}^2 + \|\underset{\sim}{\delta}\|_{\tilde{L}^\infty(H^{-\frac12}(\Gamma))}^2 \\
&+\|\underset{\sim}{\lambda}\|_{\tilde{L}^\infty(H^{-\frac12}(\partial\Omega_s))}^2 + \|w\|_{\tilde{L}^\infty(H^{-\frac12}(\Gamma))}^2 + \|\phi^{\frac12}\|_{1,\Omega_f}^2 + \|\partial_c\underset{\sim}{\xi}^{\frac12}\|_{1,\Omega_s}^2 \Big].
\end{aligned}
\tag{3.8}
$$

Using the inequality

$$
\begin{aligned}
(\phi^{\ell+\frac12})^2 =& (\phi^{\frac12})^2 + \sum_{n=1}^{\ell}(\phi^{n+\frac12}+\phi^{n-\frac12})(\phi^{n+\frac12}-\phi^{n-\frac12}) \\
&\le \frac{3}{2}(\phi^{\frac12})^2 - \frac{1}{2}(\phi^{k+1})^2 + \sum_{n=1}^{k}\Big[2(\phi^{n+\frac12})^2 + (\phi^{n+\frac12}-\phi^{n-\frac12})^2\Big],
\end{aligned}
\tag{3.9}
$$

we obtain

$$(3.10)\|\phi^{\ell+\frac{1}{2}}\|_{0,\Omega_f}^2 \le \|\phi^{\frac{1}{2}}\|_{0,\Omega_f}^2 + \sum_{n=1}^{\ell}\left[\frac{4}{3}\|\phi^{n+\frac{1}{2}}\|_{0,\Omega_f}^2 + \frac{2}{3}\|\phi^{n+\frac{1}{2}} - \phi^{n-\frac{1}{2}}\|_{0,\Omega_f}^2\right].$$

$$(3.11)\|\partial_c \underset{\sim}{\xi}^{\ell+\frac{1}{2}}\|_{0,\Omega_s}^2 \le \|\partial_c \underset{\sim}{\xi}^{\frac{1}{2}}\|_{0,\Omega_s}^2 + \sum_{n=1}^{\ell}\left[\frac{4}{3}\|\partial_c \underset{\sim}{\xi}^{n+\frac{1}{2}}\|_{0,\Omega_s}^2 + \frac{2}{3}\|\partial_c(\underset{\sim}{\xi}^{n+\frac{1}{2}} - \underset{\sim}{\xi}^{n-\frac{1}{2}})\|_{0,\Omega_s}^2\right].$$

Finally, add (3.10) and (3.11) to (3.6), then use (3.8), Lemmas 2.1 and 2.3, Korn's inequality, Gronwall's lemma and the triangle inequality, we have the following theorem.

Theorem 3.1. *Suppose $E_1(p,\Omega_f)$ and $E_2(\mathbf{u},\Omega_s)$ are bounded. Then there exists an h–independent constant $C>0$ such that*

$$(3.12)\quad \begin{aligned} &\|\partial_f(p-P)\|_{\tilde{L}^\infty(L^2(\Omega_f))} + \|p-P\|_{\tilde{L}^\infty(L^2(\Omega_f))} + \|\partial_f\partial_c(\mathbf{u}-\mathbf{U})\|_{\tilde{L}^\infty(L^2(\Omega_s))} \\ &\qquad + \|\partial_c(\mathbf{u}-\mathbf{U})\|_{\tilde{L}^\infty(L^2(\Omega_s))} \le C\left[h_1^k + h_2^r + (\Delta t)^2\right]. \end{aligned}$$

$$(3.13)\quad \|p-P\|_{\tilde{L}^\infty(H^1(\Omega_f))} + \|\partial_c(\mathbf{u}-\mathbf{U})\|_{\tilde{L}^2(H^1(\Omega_s))} \le C\left[h_1^{k-1} + h_2^{r-1} + (\Delta t)^2\right].$$

$$(3.14)\quad \begin{aligned} &\|\partial_c(p-P)\|_{\tilde{L}^2(L^2(\Gamma_f))} + \|\partial_c^2(\mathbf{u}-\mathbf{U})\|_{\tilde{L}^2(L^2(\Gamma_s))} \\ &\le \begin{cases} C\left[h_1^k + h_2^r + (\Delta t)^2\right]^{\frac{1}{2}}\left[h_1^{k-1} + h_2^{r-1} + (\Delta t)^2\right]^{\frac{1}{2}}, & N=2, \\ C\left[h_1^k + h_2^r + (\Delta t)^2\right]^{\frac{1}{4}}\left[h_1^{k-1} + h_2^{r-1} + (\Delta t)^2\right]^{\frac{3}{4}}, & N=3, \end{cases} \end{aligned}$$

where

$$\|f\|_{\tilde{L}^\infty(X)} = \max_{0\le\ell<J}\|f^{\ell+\frac{1}{2}}\|_X, \qquad \|f\|_{\tilde{L}^2(X)} = \Delta t\left[\sum_{\ell=0}^{J-1}\|f^\ell\|_X^2\right]^{\frac{1}{2}}.$$

§3.3. Optimal L^2 *a priori* error estimate. We consider a modified version of (3.2), which is obtained by replacing P^n by $P^{n,\frac{1}{4}}$ in (3.2.ii). By repeating the above derivation and that of Theorem 2.2, we can show the following theorem.

Theorem 3.2. *Suppose $E_0(p,\Omega_f)$ and $E_1(\mathbf{u},\Omega_s)$ are bounded. Then there exists an h–independent constant $C>0$ such that*

$$(3.15)\quad \begin{aligned} &\|p-P\|_{\tilde{L}^\infty(L^2(\Omega_f))} + \left\|\Delta t\sum_{n=1}^{\ell}(p^n - P^n)\right\|_{\tilde{L}^\infty(L^2(\Omega_f))} + \|\mathbf{u}-\mathbf{U}\|_{\tilde{L}^\infty(L^2(\Omega_s))} \\ &\qquad + \|\partial_f(\mathbf{u}-\mathbf{U})\|_{\tilde{L}^\infty(L^2(\Omega_s))} \le C\left[h_1^k + h_2^r + (\Delta t)^2\right]. \end{aligned}$$

$$(3.16)\quad \left\|\Delta t\sum_{n=0}^{\ell}(p-P)\right\|_{\tilde{L}^\infty(H^1(\Omega_f))} + \|\mathbf{u}-\mathbf{U}\|_{\tilde{L}^\infty(H^1(\Omega_s))} \le C\left[h_1^{k-1} + h_2^{r-1} + (\Delta t)^2\right].$$

$$(3.17)\quad \begin{aligned} &\|(p-P)\|_{\tilde{L}^2(L^2(\Gamma_f))} + \|\partial_c(\mathbf{u}-\mathbf{U})\|_{\tilde{L}^2(L^2(\Gamma_s))} \\ &\le \begin{cases} C\left[h_1^k + h_2^r + (\Delta t)^2\right]^{\frac{1}{2}}\left[h_1^{k-1} + h_2^{r-1} + (\Delta t)^2\right]^{\frac{1}{2}}, & N=2, \\ C\left[h_1^k + h_2^r + (\Delta t)^2\right]^{\frac{1}{4}}\left[h_1^{k-1} + h_2^{r-1} + (\Delta t)^2\right]^{\frac{3}{4}}, & N=3, \end{cases} \end{aligned}$$

where $(P, \mathbf{U})$ *is defined by*

$$\text{(3.18.i)} \qquad (\frac{1}{c^2}\partial^2 P^n, q^h)_{\Omega_f} + (\nabla P^{n,\frac14}, \nabla q^h)_{\Omega_f} + \langle \frac{1}{c}\partial_c P^n, q^h\rangle_{\Gamma_f} - \langle \rho_f \partial_c^2 \mathbf{U}^n \cdot n_s, q^h\rangle_\Gamma = (g_f^{n,\frac14}, q^h)_{\Omega_f}, \quad \forall q^h \in P_{h_1},$$

$$\text{(3.18.ii)} \qquad (\rho_s \partial^2 \mathbf{U}^n, \mathbf{v}^h)_{\Omega_s} + (\sigma(\mathbf{U}^{n,\frac14}), \varepsilon(\mathbf{v}^h))_{\Omega_s} + \langle \rho_s \mathcal{A}_s \partial_c \mathbf{U}^n, \mathbf{v}^h\rangle_{\Gamma_s} + \langle P^{n,\frac14}, \mathbf{v}^h \cdot n_s\rangle_\Gamma = (\mathbf{g}_s^{n,\frac14}, \mathbf{v}^h)_{\Omega_s}, \quad \forall \mathbf{v}^h \in \mathbf{V}_{h_2}.$$

Proof. We use the same error function notations as in the previous subsection. Define

$$f^0 = 0, \qquad f^n = \Delta t \sum_{m=0}^{n-1} \phi^{m+\frac12}.$$

Notice that

$$\partial_c f^n = \frac12(\phi^{n+\frac12} + \phi^{n-\frac12}),$$

$$\phi^{n,\frac14} = \frac14(\phi^{n+1} + 2\phi^n + \phi^{n-1}) = \frac12(\phi^{n+\frac12} + \phi^{n-\frac12}),$$

$$\text{(3.19.i)} \qquad \sum_{n=1}^{\ell}(\partial_c f^n, g^n) = -\sum_{n=1}^{\ell}(f^n, \partial_c g^n) + \frac{1}{2\Delta t}\left[(f^{\ell+1}, g^\ell) + (f^\ell, g^{\ell+1}) - (f^1, g^0) - (f^0, g^1)\right],$$

$$\text{(3.19.ii)} \qquad \sum_{n=1}^{\ell}(\partial_c f^n, g^n) = -\sum_{n=1}^{\ell}(f^{n+\frac12}, \partial_f g^n) + \frac{1}{2\Delta t}\left[(f^{\ell+\frac12}, g^{\ell+1}) - (f^{\frac12}, g^1)\right].$$

Taking $q^h = f^n$ and $\mathbf{v}^h = \rho_f \partial_c \underset{\sim}{\xi}^n$, applying $\sum_{n=1}^{\ell}$ to the error equations corresponding to (3.18), using (3.19.ii) on the term $(\nabla \phi^{n,\frac14}, \nabla q^h)_{\Omega_f}$ and (3.19.i) on the term $\langle \rho_f \partial_c^2 \underset{\sim}{\xi}^n \cdot n_s, q^h\rangle_\Gamma$ yield the desired estimates.

Remark. One may consider the following family of discrete schemes

$$\text{(3.20.i)} \qquad (\frac{1}{c^2}\partial^2 P^n, q^h)_{\Omega_f} + (\nabla P^{n,\gamma}, \nabla q^h)_{\Omega_f} + \langle \frac{1}{c}\partial_c P^n, q^h\rangle_{\Gamma_f} - \langle \rho_f \partial_c^2 \mathbf{U}^n \cdot n_s, q^h\rangle_\Gamma = (g_f^{n,\gamma}, q^h)_{\Omega_f}, \quad \forall q^h \in P_{h_1},$$

$$\text{(3.20.ii)} \qquad (\rho_s \partial^2 \mathbf{U}^n, \mathbf{v}^h)_{\Omega_s} + (\sigma(\mathbf{U}^{n,\gamma}), \varepsilon(\mathbf{v}^h))_{\Omega_s} + \langle \rho_s \mathcal{A}_s \partial_c \mathbf{U}^n, \mathbf{v}^h\rangle_{\Gamma_s} + \langle P^{n,\gamma}, \mathbf{v}^h \cdot n_s\rangle_\Gamma = (\mathbf{g}_s^{n,\gamma}, \mathbf{v}^h)_{\Omega_s}, \quad \forall \mathbf{v}^h \in \mathbf{V}_{h_2}.$$

In particular, for $\gamma = \frac12$. It can be shown that these schemes are absolutely stable if $\gamma \geq \frac14$.

§4. Non–overlapping domain decomposition methods. In this section we shall present a couple of parallelizable non–overlapping domain decomposition iterative algorithms for efficiently solving the finite element systems (3.2), (3.18) and (3.20). Considering the heterogeneous nature of the fluid–solid interaction problem, it is very nature to use non–overlapping domain decomposition method to solve the problem. Indeed, the non–overlapping domain decomposition approach is a natural and effective way for solving heterogeneous and/or interface problems arising from many scientific applications. We refer to [14] and reference therein for more discussions in this direction.

For simplicity we shall only describe and analyze our domain decomposition algorithms at the differential level in this section. Following the ideas of [2], [6] and [9], it is not very hard but rather technical and tedious to construct and analyze the discrete analogues of the differential domain decomposition algorithms to be introduced in the following. Those analyses along with the computation test results will be reported elsewhere in a forthcoming paper. Another point which is worth mentioning is that the domain decomposition algorithms of this paper can be used for solving the discrete systems of the fluid–solid interaction problem (2.1) which arise from using other discretization methods such as finite difference and spectral methods, even hybrid methods of using different discretization methods in different media (subdomains).

§4.1. Algorithms. Recall the interface conditions on the fluid–solid contact surface are

$$\frac{\partial p}{\partial n_f} = \rho_f \mathbf{u}_{tt} \cdot n_s, \qquad p n_f = \sigma(\mathbf{u}) n_s, \qquad \text{on } \Gamma. \tag{4.1}$$

Rewrite the second equation in (4.1) as

$$-p_t = \sigma(\mathbf{u}_t) n_s \cdot n_s, \qquad 0 = \sigma(\mathbf{u}_t) n_s \cdot \tau_s, \qquad \text{on } \Gamma, \tag{4.2}$$

where τ_s denotes the unit tangential vector on $\partial\Omega_s$. The equivalence of (4.1.ii) and (4.2) holds if the initial conditions satisfy the compatibility conditions (C1) and (C2) (cf. §1).

Lemma 4.1. *The interface conditions in (4.1) are equivalent to*

$$\frac{\partial p}{\partial n_f} + \alpha p_t = \rho_f \mathbf{u}_{tt} \cdot n_s - \alpha \sigma(\mathbf{u}_t) n_s \cdot n_s, \quad \textit{on } \Gamma, \tag{4.3.i}$$

$$\rho_f \mathbf{u}_{tt} + \beta \sigma(\mathbf{u}_t) n_s = \frac{\partial p}{\partial n_f} n_s - \beta p_t n_s, \qquad \textit{on } \Gamma, \tag{4.3.ii}$$

$$\sigma(\mathbf{u}_t) n_s \tau_s = 0, \qquad \textit{on } \Gamma, \tag{4.3.iii}$$

for any pair of constants α and β such that $\alpha + \beta \neq 0$.

Proof. Trivial.

Based on the above new form of the interface conditions we propose the following iterative algorithms, one of which resembles to Jacobi type iteration and the other resembles to Gauss–Seidel type iteration.

Algorithm 1: (Jacobi type iteration)

Step 1 $\forall p^0 \in P_f, \quad \forall \mathbf{u}^0 \in \mathbf{V}_s$.
Step 2 Generate $\{(p^n, \mathbf{u}^n)\}_{n\geq 1}$ iteratively by solving

$$\frac{1}{c^2} p_{tt}^n - \Delta p^n = g_f, \qquad \text{in } \Omega_f, \tag{4.4.i}$$

$$\frac{1}{c} p_t^n + \frac{\partial p^n}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{4.4.ii}$$

$$\frac{\partial p^n}{\partial n_f} + \alpha p_t^n = \rho_f \mathbf{u}_{tt}^{n-1} \cdot n_s - \alpha \sigma(\mathbf{u}_t^{n-1}) n_s \cdot n_s, \quad \text{on } \Gamma; \tag{4.4.iii}$$

$$\rho_s \mathbf{u}_{tt}^n - \operatorname{div} \sigma(\mathbf{u}^n) = \mathbf{g}_s, \qquad \text{in } \Omega_s, \tag{4.4.iv}$$

$$\rho_s \mathcal{A}_s \mathbf{u}_t^n + \sigma(\mathbf{u}^n) n_s = 0, \qquad \text{on } \Gamma_s, \tag{4.4.v}$$

$$\rho_f \mathbf{u}_{tt}^n + \beta \sigma(\mathbf{u}_t^n) n_s = \frac{\partial p^{n-1}}{\partial n_f} n_s - \beta p_t^{n-1} n_s, \qquad \text{on } \Gamma, \tag{4.4.vi}$$

$$\sigma(\mathbf{u}_t^n) n_s \cdot \tau_s = 0, \qquad \text{on } \Gamma. \tag{4.4.vii}$$

Algorithm 2: (Gauss-Seidel type iteration)

Step 1 $\forall \mathbf{u}^0 \in \mathbf{V}_s$.
Step 2 Generate $\{p^n\}_{n\geq 0}$ and $\{\mathbf{u}^n\}_{n\geq 1}$ iteratively by solving

$$\frac{1}{c^2} p_{tt}^n - \Delta p^n = g_f, \qquad \text{in } \Omega_f, \tag{4.5.i}$$

$$\frac{1}{c} p_t^n + \frac{\partial p^n}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{4.5.ii}$$

$$\frac{\partial p^n}{\partial n_f} + \alpha p_t^n = \rho_f \mathbf{u}_{tt}^n \cdot n_s - \alpha \sigma(\mathbf{u}_t^n) n_s \cdot n_s, \quad \text{on } \Gamma; \tag{4.5.iii}$$

$$\rho_s \mathbf{u}_{tt}^{n+1} - \operatorname{div} \sigma(\mathbf{u}^{n+1}) = \mathbf{g}_s, \qquad \text{in } \Omega_s, \tag{4.5.iv}$$

$$\rho_s \mathcal{A}_s \mathbf{u}_t^{n+1} + \sigma(\mathbf{u}^{n+1}) n_s = 0, \qquad \text{on } \Gamma_s, \tag{4.5.v}$$

$$\rho_f \mathbf{u}_{tt}^{n+1} + \beta \sigma(\mathbf{u}_t^{n+1}) n_s = \frac{\partial p^n}{\partial n_f} n_s - \beta p_t^n n_s, \quad \text{on } \Gamma, \tag{4.5.vi}$$

$$\sigma(\mathbf{u}_t^{n+1}) n_s \cdot \tau_s = 0, \qquad \text{on } \Gamma. \tag{4.5.vii}$$

Remark. Appropriate initial conditions must be provided in the above algorithms. We omit these conditions for notation brevity, and assume no ambiguity will be caused by the omission.

§**4.1. Convergence Analysis.** In this subsection we shall establish the utility of Algorithms 1 and 2 by proving their convergence. Because the convergence proof for Algorithm 2 is almost same as the proof of Algorithm 1, we only give the proof for Algorithm 1 in the following.

Introduce the error functions at the nth iteration

$$r^n = p - p^n, \quad \mathbf{e}^n = \mathbf{u} - \mathbf{u}^n.$$

It is easy to check that $(r^n, \mathbf{e}^n)$ satisfies the error equations

$$\frac{1}{c^2} r^n_{tt} - \Delta r^n = 0, \qquad \text{in } \Omega_f, \tag{4.6.i}$$

$$\frac{1}{c} r^n_t + \frac{\partial r^n}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{4.6.ii}$$

$$\frac{\partial r^n}{\partial n_f} + \alpha r^n_t = \rho_f \mathbf{e}^{n-1}_{tt} \cdot n_s - \alpha \sigma(\mathbf{e}^{n-1}_t) n_s \cdot n_s, \quad \text{on } \Gamma; \tag{4.6.iii}$$

$$\rho_s \mathbf{e}^n_{tt} - \operatorname{div} \sigma(\mathbf{e}^n) = 0, \qquad \text{in } \Omega_s, \tag{4.6.iv}$$

$$\rho_s \mathcal{A}_s \mathbf{e}^n_t + \sigma(\mathbf{e}^n) n_s = 0, \qquad \text{on } \Gamma_s, \tag{4.6.v}$$

$$\rho_s \mathbf{e}^n_{tt} + \beta \sigma(\mathbf{e}^n_t) n_s = \frac{\partial r^{n-1}}{\partial n_f} n_s - \beta r^{n-1}_t n_s, \qquad \text{on } \Gamma, \tag{4.6.vi}$$

$$\sigma(\mathbf{e}^n_t) n_s \cdot \tau_s = 0, \qquad \text{on } \Gamma. \tag{4.6.vii}$$

Define the "pseudo–energy"

$$E_n = E(\{r^n, \mathbf{e}^n\}) = \left\| \frac{\partial r^n}{\partial n_f} + \alpha r^n \right\|^2_{L^2(L^2(\Gamma))} + \left\| \rho_f \mathbf{e}^n_{tt} + \beta \sigma(\mathbf{e}^n) n_s \right\|^2_{L^2(L^2(\Gamma))}. \tag{4.7}$$

Lemma 4.2. *There holds the following inequality*

$$E_{n+1}(\tau) \le E_n(\tau) - R_n(\tau), \tag{4.8}$$

where

$$R_n(\tau) = 4 \int_0^\tau \int_\Gamma \left[\alpha \frac{\partial r^n}{\partial n_f} r^n_t + \beta \sigma(\mathbf{e}^n_t) n_s \cdot \mathbf{e}^n_{tt} \right] dx dt.$$

Proof.

$$
\begin{aligned}
E_{n+1} &= \left\| \frac{\partial r^{n+1}}{\partial n_f} + \alpha r_t^{n+1} \right\|^2_{L^2(L^2(\Gamma))} + \left\| \rho_f \mathbf{e}_{tt}^{n+1} + \beta\sigma(\mathbf{e}_t^{n+1}) n_s \right\|^2_{L^2(L^2(\Gamma))} \\
&= \|\rho_f \mathbf{e}_{tt}^n \cdot n_s - \beta\sigma(\mathbf{e}_t^n) n_s \cdot n_s\|^2_{L^2(L^2(\Gamma))} + \left\| \frac{\partial r^n}{\partial n_f} n_s - \alpha r_t^n n_s \right\|^2_{L^2(L^2(\Gamma))} \\
&\leq ||\rho_f \mathbf{e}_{tt}^n + \beta\sigma(\mathbf{e}_t^n) n_s||^2_{L^2(L^2(\Gamma))} + \left\| \frac{\partial r^n}{\partial n_f} + \alpha r_t^n \right\|^2_{L^2(L^2(\Gamma))} \\
&\qquad - 2\beta \int_0^\tau \int_\Gamma [\sigma(\mathbf{e}_t^n) n_s \cdot n_s \rho_f \mathbf{e}_{tt}^n \cdot n_s + \rho_f \mathbf{e}_{tt}^n \cdot \sigma(\mathbf{e}_t^n) n_s]\, dx d_t \\
&\qquad + 2\alpha \int_0^\tau \int_\Gamma \left[\frac{\partial r^n}{\partial n_f} r_t^n n_s \cdot n_s + \frac{\partial r^n}{\partial n_f} r_t^n \right] dxdt \\
&= E_n - 4 \int_0^\tau \int_\Gamma \left[\alpha \frac{\partial r^n}{\partial n_f} r_t^n + \beta \rho_s \sigma(\mathbf{e}_t^n) n_s \cdot \mathbf{e}_{tt}^n \right] dxdt \\
&= E_n - R_n.
\end{aligned}
$$

Here we have used the fact that

$$
\sigma(\mathbf{e}_t^n) n_s \cdot n_s \mathbf{e}_{tt}^n \cdot n_s = \sigma(\mathbf{e}_t^n) n_s \cdot \mathbf{e}_{tt}^n, \quad \text{since} \quad \sigma(\mathbf{e}_t^n) n_s \cdot \tau_s = 0.
$$

The proof is completed.

To find a lower bound for $R_n(\tau)$, we test (4.6.i) against r_t^n to get

$$
\frac{1}{2}\frac{d}{dt} \left\| \frac{1}{c} r_t^n \right\|^2_{0,\Omega_f} + \frac{1}{2}\frac{d}{dt} \|\nabla r^n\|^2_{0,\Omega_f} + \left| \frac{1}{\sqrt{c}} r_t^n \right|^2_{0,\Gamma_f} = \int_\Gamma \frac{\partial r^n}{\partial n_f} r_t^n dx,
$$

which implies that

$$
\int_0^\tau \int_\Gamma \frac{\partial r^n}{\partial n_f} r_t^n dxdt = \frac{1}{2} \left\| \frac{1}{c} r_t^n(\tau) \right\|^2_{0,\Omega_f} + \frac{1}{2} \|\nabla r^n(\tau)\|^2_{0,\Omega_f} + \left\| \frac{1}{\sqrt{c}} r_t^n \right\|_{L^2(L^2(\Gamma_f))}. \tag{4.9}
$$

Here we have implicitly assumed that $r^n(0) = r_t^n(0) = 0$.

Differentiating (4.6.iv) with respect to t and testing it against $\mathbf{e}_{tt}^n$ give us

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} \|\sqrt{\rho_s} \mathbf{e}_{tt}^n\|^2_{0,\Omega_s} + \frac{d}{dt} \|\sqrt{\mu_s} \varepsilon(\mathbf{e}_t^n)\|^2_{0,\Omega_s} + \frac{1}{2}\frac{d}{dt} \left\| \sqrt{\lambda_s}\, \mathrm{div}(\mathbf{e}_t^n) \right\|^2_{0,\Omega_s} \\
+ c_0 |\sqrt{\rho_s} \mathbf{e}_t^n|^2_{0,\Gamma_s} \leq \int_\Gamma \sigma(\mathbf{e}_t^n) \cdot n_s \mathbf{e}_{tt}^n dx,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\int_0^\tau \int_\Gamma \rho_s \sigma(\mathbf{e}_t^n) n_s \cdot \mathbf{e}_{tt}^n dxdt \geq& \frac{1}{2} \|\sqrt{\rho_s} \mathbf{e}_{tt}^n(\tau)\|^2_{0,\Omega_s} + \|\sqrt{\mu_s} \varepsilon(\mathbf{e}_t^n(\tau))\|^2_{0,\Omega_s} \\
&+ \frac{1}{2} \left\| \sqrt{\lambda_s}\, \mathrm{div}(\mathbf{e}_t^n(\tau)) \right\|^2_{0,\Omega_s} + c_0 ||\sqrt{\rho_s} \mathbf{e}_t^n||^2_{L^2(L^2(\Gamma_s))} \\
&- \frac{1}{2} \|\sqrt{\rho_s} \mathbf{e}_{tt}^n(0)\|^2_{0,\Omega_s}.
\end{aligned} \tag{4.10}
$$

Finally, suppose $\mathbf{e}^n(0) = \mathbf{e}_t^n(0) = 0$, it follows from (4.6.i) that

$$\|\sqrt{\rho_s}\mathbf{e}_{tt}^n(0)\|_{0,\Omega_s} = \left\| \frac{1}{\sqrt{\rho_s}} \operatorname{div}(\mathbf{e}^n(0)) \right\|_{0,\Omega_s} = 0.$$

Combining (4.9) and (4.10) we get the following lemma.

Lemma 4.3. *$R_n(\tau)$ satisfies the following inequality*

$$\begin{aligned} R_n(\tau) \geq 2\alpha & \left[\left\| \frac{1}{c} r_t^n(\tau) \right\|_{0,\Omega_f}^2 + \|\nabla r^n(\tau)\|_{0,\Omega_f}^2 + 2 \left\| \frac{1}{\sqrt{c}} r_t^n \right\|_{L^2(L^2(\Gamma_f))}^2 \right] \\ & + 2\beta \left[\|\sqrt{\rho_s}\mathbf{e}_{tt}^n(\tau)\|_{0,\Omega_s}^2 + \|\sqrt{\mu_s}\varepsilon(\mathbf{e}_t^n(\tau))\|_{0,\Omega_s}^2 \right. \\ & \left. + \|\sqrt{\lambda_s} div(\mathbf{e}_t^n(\tau)\|_{0,\Omega_s}^2 + 2c_0 \, \|\sqrt{\rho_s}\mathbf{e}_t^n\|_{L^2(L^2(\Gamma_s))}^2 \right]. \end{aligned}$$

We are now ready to state our convergence theorem.

Theorem. *For $\alpha > 0$ and $\beta > 0$ we have*

(i) *$p^n \to p$ strongly in P_f,*
(ii) *$\mathbf{u}^n \to \mathbf{u}$ strongly in V_s.*

Proof. The conclusion of the theorem is just an immediate consequence of the combination of Lemmas 4.2 and 4.3.

Remark. If we choose $\alpha = 0$, $\beta = \infty$, Algorithms 1 and 2 become N–N alternating type algorithms. This is possible because the Neumann data $\frac{\partial p}{\partial n_f}$ and $\sigma(\mathbf{u})n_s$ are not related to each other directly on the interface. In addition, at the end of each N–N iteration one can add the following relaxation step to speed up the convergence

$$\begin{aligned} p^n &:= \mu p^n + (1-\mu)p^{n-1}, \\ \mathbf{u}_{tt}^n &:= \mu \mathbf{u}_{tt}^n + (1-\mu)\mathbf{u}_{tt}^{n-1}, \end{aligned}$$

where μ is any constant satisfying $0 < \mu < 1$.

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