Formulation and Mathematical Analysis of a fluid–solid interaction Problem⋆

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ABSTRACT. A general composite model for elastic wave propagation through a region Ω consisting of different elastic subregions is formulated, and analyzed mathematically when Ω consists of an elastic solid region Ω_s and an acoustic fluid region Ω_f. The linear elastodynamic equations complemented with appropriate interface and boundary conditions are used to describe the wave propagation in each subregion. To avoid the traditional integral equation formulations for wave problems in unbounded domains, artificial boundaries are introduced when the region is unbounded and absorbing boundary conditions are imposed on the artificial boundaries to minimize unphysical wave reflections. It is shown that the initial boundary value problem of the mathematical model posses a unique global (in time) quasi–strong solution. Regularity of the quasi–strong solution is also obtained under some reasonable assumptions on the data and on the region.

§0.Introduction. By "a composite medium", we mean that a medium which consists of several distinct parts such that any two adjacent parts may have different material properties. A typical example of such a medium is the one which consists of a solid part and a fluid part. The problems of wave propagation in composite media have long been subjects of both theoretical and practical studies, important applications of such problems are found in inverse scattering, elastoacoustics, geosciences, oceanography. Different composite models were proposed and studied in [2], [3] and [13]. As far as we know most of the existing composite models seem to require one medium is completely submerged in another one, and/or employ some boundary integral formulations, when the medium is unbounded, to handle the difficulty of the unboundedness of the region. (cf. [2] and [3]).

In this paper we specifically consider elastic wave propagation in a composite medium which consists of pure elastic solid subregions and pure acoustic fluid subregions. Furthermore, we confine ourselves to a linear composite model so that linear elastic and acoustic equations hold in the corresponding subregions. In order to model problems from different application areas, we do not impose any restrictions on the geometric configuration of the composite medium. In the unbounded domain case, we truncate the domain to get a bounded (computational) domain and appropriate radiation boundary conditions are imposed on the artificial boundaries to minimize unphysical wave reflections. The advantage

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of using radiation boundary condition approach is that the problem can be solved without using the traditional boundary integral formulation. Hence, many effective numerical methods for partial differential equations can applied easily.

The objective of this paper is to present a mathematical model for wave propagation in a composite medium described above, and to give a complete mathematical analysis for the model from the point of view of existence, uniqueness and regularity of the solution. Unlike previous works in the direction (cf. [2] and [3]), we give a simple and straightforward variational formulation for our model so that numerical methods can be conveniently developed based on the formulation. Indeed, the results of this paper serve as the theoretical foundation for a forthcoming paper [7], where numerical methods and computational algorithmic aspects are developed and analyzed.

The work of this paper is closely related to earlier works by Santos *et al* [14] and by Sheen [16]. In [14] a model for wave propagation through single-phase fluid saturated porous media near a fluid-filled borehole region (single-phase Biot model) is formulated and analyzed. The model was later generalized by Sheen in [16] to the case that the media are saturated by a two-phase immiscible fluid (two-phase Biot model). In these two papers the displacements were used as the primary variables both in the fluid region (borehole) and in the fluid–saturated porous solid region. The model we proposed in this paper is based on a pressure–displacement formulation. In this formulation, because higher order derivatives appear in the interface conditions, it is more delicate to deal with these conditions in the analyses and in the finite element error estimates (cf. [7]).

The organization of the paper is as follows. In §1 we describe a composite model for elastic wave propagation in a composite region by stating its governing partial differential equations, in particular, the external absorbing boundary conditions and the interface conditions between subregions. In §2 we give space notations, definitions of solutions and some useful preliminary results. In §3 we prove existence of quasi–strong solution (see Definition 2.2) by using the method of compactness. Finally, in §4 we establish uniqueness and regularity of quasi–strong solutions.

§1. Formulation of the composite model. The composite medium Ω will be identified with a domain in $\mathbb{R}^N$ for $N=2,3$, and will be taken to be of unit thickness when $N=2$. Suppose that Ω consists of different elastic subregions Ω_i, for $i=1,2,...,M$. We assume that each Ω_i is isotropic, thus the following linear elastic wave equations hold in Ω_i:

$$\rho_i \frac{\partial^2 \mathbf{u}_i}{\partial t^2} = \operatorname{div} \sigma(\mathbf{u}_i) + \mathbf{g}_i, \tag{1.1.i}$$

$$\sigma(\mathbf{u}_i) = \lambda_i \operatorname{div} \mathbf{u}_i I + 2\mu_i \varepsilon(\mathbf{u}_i), \tag{1.1.ii}$$

$$\varepsilon(\mathbf{u}_i) = \frac{1}{2}[\nabla \mathbf{u}_i + (\nabla \mathbf{u}_i)^T], \tag{1.1.iii}$$

where $\mathbf{u}_i$ denotes the displacement vector in Ω_i and $(\nabla \mathbf{u}_i)^T$ is the transpose of $\nabla \mathbf{u}_i$. ρ_i is the density of Ω_i, $\lambda_i > 0$ and $\mu_i \geq 0$ are its Lamé constants, I is the $N \times N$ identity matrix, $\mathbf{g}_i$ is the source.

To describe the boundary conditions, two cases are considered here. First, if the domain Ω is bounded and computationally feasible, some usual boundary condition is imposed on the boundary $\partial\Omega$. Secondly, if Ω is either unbounded or so large that it is not feasible to compute a solution in the entire domain, we truncate Ω into a computationally feasible finite domain by introducing an artificial (usually flat) boundary, and absorbing boundary conditions are used on the artificial boundary to minimize unphysical reflections caused by the artificial boundary. we will abuse the notation by letting the truncated region be still denoted by Ω and its boundary by $\partial\Omega$.

Let

$$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j, \quad \Gamma = \cup\Gamma_{ij}, \quad \Gamma_i = \partial\Omega \cap \partial\Omega_i, \quad i,j = 1,2,\cdots,M.$$

So if the exterior boundary portion Γ_i is a part of the original boundary, we have

$$\mathbf{u}_i \cdot n_i = 0, \qquad \text{on } \Gamma_i. \tag{1.2}$$

If Γ_i is a piece of artificial boundary, we impose

$$\rho_i \mathcal{A}_i \frac{\partial \mathbf{u}_i}{\partial t} + \sigma(\mathbf{u}_i) \cdot n_i = 0, \qquad \text{on } \Gamma_i, \tag{1.3}$$

where $\mathcal{A}_i$ is a $N \times N$ symmetric positive definite matrix and its entries only depend on n_i, ρ_i, λ_i and μ_i. For instance, when $N = 2$ it has the following form:

$$\mathcal{A}_i = \begin{pmatrix} n_{i,1} & n_{i,2} \\ n_{i,2} & -n_{i,1} \end{pmatrix} \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix} \begin{pmatrix} n_{i,1} & n_{i,2} \\ n_{i,2} & -n_{i,1} \end{pmatrix}, \tag{1.4.i}$$

and

$$\alpha_i = \sqrt{\frac{\lambda_i + 2\mu_i}{\rho_i}}, \qquad \beta_i = \sqrt{\frac{\mu_i}{\rho_i}}, \tag{1.4.ii}$$

where $n_i = (n_{i,1}, n_{i,2})$ is the outward normal to Γ_i. The equation (1.3) is the first–order absorbing boundary condition for elastic wave, the derivation of which can be found in [5], [6], [8] and [12].

The conditions on the interfaces of two media are complicated. On a solid–solid interface, the interface conditions should impose the vanishing of the relative movements (welded boundary condition) and the continuity of the normal stresses, that is,

$$\mathbf{u}_i - \mathbf{u}_j = 0, \quad \text{on } \Gamma_{ij}, \tag{1.5.i}$$

$$\sigma(\mathbf{u}_i)n_i + \sigma(\mathbf{u}_j)n_j = 0, \quad \text{on } \Gamma_{ij}. \tag{1.5.ii}$$

On a solid–fluid interface Γ_{sf}, the interface conditions should require the vanishing of only normal components of the relative movements (slip boundary condition) and the continuity of the normal stresses. Thus (1.5) has to be modified as

$$\mathbf{u}_s \cdot n_s + \mathbf{u}_f \cdot n_f = 0, \quad \text{on } \Gamma_{sf}, \tag{1.6.i}$$

$$\sigma(\mathbf{u}_s)n_s + \sigma(\mathbf{u}_f)n_f = 0, \quad \text{on } \Gamma_{sf}, \tag{1.6.ii}$$

where the sub–indices s and f stand for solid and fluid media respectively. We have assumed here that the fluid is an elastic inviscid fluid.

Finally, to close the system (1.1)–(1.6) the following initial conditions must be given:

$$\mathbf{u}_i(x,0) = \mathbf{u}_{i,0}(x), \qquad \frac{\partial \mathbf{u}_i}{\partial t}(x,0) = \mathbf{u}_{i,1}(x). \tag{1.7}$$

The system (1.1)–(1.7) is a general model for elastic wave propagation in a fluid–solid composite medium. The displacements are used as unknowns in both elastic media and acoustic media. In practice, it is more convenient to use pressure field in acoustic media. In the rest of this section, we will modify the model above by replacing the displacement variable $\mathbf{u}_f$ by a pressure variable p defined by

$$p = -\lambda_f \operatorname{div} \mathbf{u}_f. \tag{1.8}$$

Without loss of generality we assume that the domain Ω of interest consists of an elastic solid part Ω_s where $\mu_s \geq \mu_0 > 0$, and an acoustic fluid part Ω_f where $\mu_f = 0$.

First, taking divergence of (1.1.i) and substituting (1.1.ii) into (1.1.i), we have

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \Delta p + g_f, \quad c = (\lambda_f/\rho_f)^{1/2}, \quad g_f = -\operatorname{div}(\mathbf{g}_f). \tag{1.9}$$

Equation (1.9) is the standard acoustic wave equation in inviscid fluids.

To simplify the absorbing boundary condition on the exterior boundary Γ_f, we note that $\mu_f = 0$ implies $\sigma(\mathbf{u}_f) = -pI$, where p is defined by (1.8). Now rewrite boundary condition (1.3) as

$$\rho_f \mathcal{A}_f \frac{\partial \mathbf{u}_f}{\partial t} - p n_f = 0, \qquad \text{on } \Gamma_f. \tag{1.10}$$

Differentiating the above equation with respect to t, and using (1.1.i), we get

$$0 = \mathcal{A}_f \operatorname{div}\sigma(\mathbf{u}_f) - \frac{\partial p}{\partial t} n_f = -\mathcal{A}_f \operatorname{div}(pI) - \frac{\partial p}{\partial t} n_f = -\mathcal{A}_f \nabla p - \frac{\partial p}{\partial t} n_f, \tag{1.11}$$

where we have assumed that the source $\mathbf{g}_f$ vanishes on Γ_f. Since $\mu_f = 0$, the matrix $\mathcal{A}_f$ defined by (1.4) can be simplified as

$$\mathcal{A}_f = \alpha_f \begin{pmatrix} n_{f1}n_{f1} & n_{f1}n_{f2} \\ n_{f2}n_{f1} & n_{f2}n_{f2} \end{pmatrix} = \alpha_f (n_f n_f^t),$$

where $n_f^t = (n_{f1}, n_{f2})$ denotes the transpose of n_f on Γ_f. It follows that

$$\mathcal{A}_f \nabla p = \alpha_f (n_f n_f^t) \nabla p = \alpha_f n_f (n_f^t \nabla p) = \alpha_f \frac{\partial p}{\partial n_f} n_f. \tag{1.12}$$

Substituting (1.12) into (1.11) yields

$$\frac{1}{c}\frac{\partial p}{\partial t} + \frac{\partial p}{\partial n_f} = 0, \tag{1.13}$$

where $c = \alpha_f$ is defined in (1.9). Note that (1.13) is the standard first–order absorbing condition for the acoustic wave equation (1.9) (cf. [5]).

Similarly, for the case Γ_f is not a part of artificial boundary, the boundary condition (1.2) can be reduced to

$$\frac{\partial p}{\partial n_f} = 0, \qquad \text{on } \Gamma_f. \tag{1.14}$$

Finally, we modify the interface conditions in (1.6) and the initial conditions for $\mathbf{u}_f$. Note that equation (1.6.ii) can be written as

$$\sigma(\mathbf{u}_s)n_s = -\sigma(\mathbf{u}_f)n_f = pn_f, \quad \text{on } \Gamma_{sf}. \tag{1.15}$$

To simplify (1.6.i), differentiating it twice with respective to t and using (1.1.i) we have

$$\rho_f \frac{\partial^2 \mathbf{u}_s \cdot n_s}{\partial t^2} = -\rho_f \frac{\partial^2 \mathbf{u}_f \cdot n_f}{\partial t^2} = -\operatorname{div}(\sigma(\mathbf{u}_f)) \cdot n_f = \frac{\partial p}{\partial n_f}, \text{ on } \Gamma_{sf}, \tag{1.16}$$

where we have assumed that the source $\mathbf{g}_f$ vanishes on the interface Γ_{sf}.

Obviously, the initial conditions for $\mathbf{u}_f$ are replaced by

$$p(x,0) = p_0(x), \qquad \frac{\partial p(x,0)}{\partial t} = p_1(x), \quad \text{in } \Omega_f, \tag{1.17}$$

where

$$p_0(x) = -\lambda_f \operatorname{div} \mathbf{u}_{f,0}, \qquad p_1(x) = -\lambda_f \operatorname{div} \mathbf{u}_{f,1}. \tag{1.18}$$

Now, we collect (1.9)–(1.18) and summarize our mathematical model for elastic wave propagation in a fluid–solid composite medium as follows:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = g_f, \quad \text{in } \Omega_f, \tag{1.19.i}$$

$$\rho_s \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div}\ \sigma(\mathbf{u}) = \mathbf{g}_s, \quad \text{in } \Omega_s, \tag{1.19.ii}$$

$$\frac{\partial p}{\partial n_f} - \rho_f \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot n_s = 0, \quad \text{on } \Gamma, \tag{1.20.i}$$

$$\sigma(\mathbf{u})n_s - pn_f = 0, \quad \text{on } \Gamma, \tag{1.20.ii}$$

$$\frac{1}{c}\frac{\partial p}{\partial t} + \frac{\partial p}{\partial n_f} = 0, \quad \text{on } \Gamma_f, \tag{1.21.i}$$

$$\rho_s \mathcal{A}_s \frac{\partial \mathbf{u}}{\partial t} + \sigma(\mathbf{u})n_s = 0, \quad \text{on } \Gamma_s, \tag{1.21.ii}$$

$$p(x,0) = p_0(x), \quad \frac{\partial p}{\partial t}(x,0) = p_1(x), \quad \text{in } \Omega_f, \tag{1.22.i}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \frac{\partial \mathbf{u}}{\partial t}(x,0) = \mathbf{u}_1(x), \quad \text{in } \Omega_s. \tag{1.22.ii}$$

Where $\Gamma = \Gamma_{sf}$, $\Gamma_s = \partial\Omega_s \setminus \Gamma$ and $\Gamma_f = \partial\Omega_f \setminus \Gamma$. Note that the boundary conditions (1.21.i) and (1.21.ii) should be replaced by the (1.14) and (1.2) respectively if Γ_f and Γ_s are not parts of the artificial boundary. In the rest of this paper we will stick to boundary conditions (1.21.i) and (1.21.ii) for a convenient presentation.

§**2. Preliminaries.** The standard space notations are adopted in this paper (cf. [1]). For example, $H^k(D)$, $k \geq 0$ integer, denotes the standard Sobolev spaces over the domain D. When $k=0$, $H^0(D) = L^2(D)$. $\|\cdot\|_{H^k(D)}$ denotes the standard norms on $H^k(D)$. For a Banach space B, $L^r(0,T;B)$ stands for the space of L^r–integrable functions with range in B. $H^k([0,T];B)$ is the space of functions whose up to kth order derivatives with respect to t are in $L^2(0,T;B)$. $C^k([0,T];B)$ is the space of functions which are k–times continuously differentiable on $[0,T]$ with range in B and when $k=0$, we omit the superscript. The norm of $C^k([0,T];B)$ is defined in the obvious way. $\mathbf{B}$ denotes $(B)^N$, $N=2,3$. In addition, we also introduce the following special notations:

$$
\begin{aligned}
P_f &= \left\{ p;\ \frac{\partial^k p}{\partial t^k} \in L^\infty(0,T;H^{1-k}(\Omega_f)),\ k=0,1;\ \left.\frac{\partial p}{\partial t}\right|_{\Gamma_f} \in L^2(0,T;L^2(\Gamma_f)) \right\},\\
Q_f &= P_f \cap \left\{ q;\ \frac{\partial^k q}{\partial t^k} \in L^\infty(0,T;H^{2-k}(\Omega_f)),\ k=1,2;\ \left.\frac{\partial^2 q}{\partial t^2}\right|_{\Gamma_f} \in L^2(0,T;L^2(\Gamma_f)) \right\},\\
\mathbf{U}_s &= \left\{ \mathbf{u};\ \frac{\partial^k \mathbf{u}}{\partial t^k} \in L^\infty(0,T;\mathbf{H}^{1-k}(\Omega_s)),\ k=0,1;\ \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\Gamma_s} \in L^2(0,T;\mathbf{L}^2(\Gamma_s)) \right\},\\
\mathbf{V}_s &= \mathbf{U}_s \cap \left\{ \mathbf{v};\ \frac{\partial^k \mathbf{v}}{\partial t^k} \in L^\infty(0,T;\mathbf{H}^{2-k}(\Omega_s)),\ k=1,2;\ \left.\frac{\partial^2 \mathbf{v}}{\partial t^2}\right|_{\Gamma_s} \in L^2(0,T;\mathbf{L}^2(\Gamma_s)) \right\},\\
\widehat{Q}_f &= Q_f \cap L^\infty(0,T;H^2(\Omega_f)), \qquad \widehat{\mathbf{V}}_s = V_s \cap L^\infty(0,T;\mathbf{H}^2(\Omega_s)).
\end{aligned}
$$

We also make the following physical and mathematical assumptions throughout the paper unless it is stated otherwise.

(A1). $\rho_f = \text{constant} > 0$, $\quad \rho_s = \rho_s(x) \geq \overline{\rho}_s > 0$. λ_f, λ_s, μ_s are all positive constants.

(A2). $\Omega_f \subset \mathbb{R}^N$, $\Omega_s \subset \mathbb{R}^N$ for $N=2,3$ are bounded open sets with Lipschitz continuous boundary $\partial\Omega_f$ and $\partial\Omega_s$, respectively.

(A3). $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_s$, $\text{meas}(\Omega_f) \neq \emptyset$, $\text{meas}(\Omega_s) \neq \emptyset$. Assume that $\Gamma \neq \emptyset$. Note that it is acceptable if one of Γ_f and Γ_s is empty.

Next, we also introduce the following compactability conditions for the initial data.

(C1). $\mathbf{u}_0 \in \mathbf{H}^2(\Omega_s)$ and $\mathbf{u}_1 \in \mathbf{H}^1(\Omega_s)$ are said to be compatible on Γ_s if $\rho_s \mathcal{A}_s \mathbf{u}_1 + \sigma(\mathbf{u}_0) n_s = 0$, on Γ_s.

(C2). $p_0 \in H^1(\Omega_f)$ and $\mathbf{u}_0 \in \mathbf{H}^2(\Omega_s)$ are said to be compatible on Γ if $\sigma(\mathbf{u}_0) n_s - p_0 n_f = 0$, on Γ.

Definition 2.1. Suppose that the initial datum functions and source functions satisfy $p_j \in H^{1-j}(\Omega_f)$, $\mathbf{u}_j \in \mathbf{H}^{1-j}(\Omega_s)$ $(j=0,1)$, $g_f \in L^2(0,T;L^2(\Omega_f))$, $\mathbf{g}_s \in L^2(0,T;\mathbf{L}^2(\Omega_s))$. Then a pair of functions $(p,\mathbf{u})$ is said to be a weak solution to problem (1.19)–(1.22) in $\overline{\Omega} \times [0,T]$ for given $T > 0$ if $(p,\mathbf{u}) \in P_f \times \mathbf{U}_s$ and satisfy (1.22) and the following two identities in

$\mathcal{D}'(0,T)$:

$$\text{(2.1.i)}\qquad \frac{d}{dt}\int_{\Omega_f}\frac{1}{c^2}\frac{\partial p}{\partial t}q\,dx+\int_{\Omega_f}\nabla p\cdot\nabla q\,dx+\int_{\Gamma_f}\frac{1}{c}\frac{\partial p}{\partial t}q\,d\tau-\frac{d^2}{dt^2}\int_{\Gamma}\rho_f\mathbf{u}\cdot n_s q\,d\tau$$
$$=\int_{\Omega_f}g_f q\,dx,\quad \forall q\in H^1(\Omega_f),$$

$$\text{(2.1.ii)}\quad \frac{d}{dt}\int_{\Omega_s}\rho_s\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{v}\,dx+\int_{\Omega_s}\sigma(\mathbf{u}):\varepsilon(\mathbf{v})\,dx+\int_{\Gamma_s}\rho_s\mathcal{A}_s\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{v}\,d\tau+\int_{\Gamma}p\mathbf{v}\cdot n_s\,d\tau$$
$$=\int_{\Omega_s}\mathbf{g}_s\cdot\mathbf{v}\,dx,\quad \forall\mathbf{v}\in\mathbf{H}^1(\Omega_s),$$

where $\mathcal{D}'(0,T)$ denotes the space of distributions over $(0,T)$.

Definition 2.2. A weak solution $(p,\mathbf{u})$ of problem (1.19)–(1.22) is said to be a quasi-strong solution if $(p,\mathbf{u})\in P_f\times\mathbf{V}_s$, and a strong solution if $(p,\mathbf{u})\in\widehat{Q}_f\times\widehat{\mathbf{V}}_s$.

Finally, we list some basic facts for later use.

Lemma 2.1. *Let B_0, B, and B_1 be reflective Banach spaces which satisfy $B_0\overset{c}{\hookrightarrow}B\hookrightarrow B_1$, where "$\hookrightarrow$" stands for embedding and "$\overset{c}{\hookrightarrow}$" means compact embedding. Then*

(i) $L^{r_0}(0,T;B_0)\cap\{\varphi;\ \varphi_t\in L^{r_1}(0,T;B_1)\}\overset{c}{\hookrightarrow}L^{r_0}(0,T;B)$,
(ii) $L^{r_0}(0,T;B)\cap\{\varphi;\ \varphi_t\in L^{r_0}(0,T;B)\}\overset{c}{\hookrightarrow}C([0,T];B)$,
(iii) $L^{\infty}(0,T;B_0)\cap\{\varphi;\ \varphi_t\in L^{r_2}(0,T;B_1)\}\overset{c}{\hookrightarrow}C([0,T];B)$,

for any $1\le r_0,\ r_1\le\infty$ and $1<r_2\le\infty$.

A proof of Lemma 2.1 can be found in [11] and [15].

Lemma 2.2. (Korn's inequality). *There exists a constant $C_0>0$, which depends only on D, such that*

$$\text{(2.2)}\qquad \|\varepsilon(\mathbf{v})\|^2_{L^2(D)}+\|\mathbf{v}\|^2_{L^2(D)}\ge C_0\|\mathbf{v}\|^2_{H^1(D)},\qquad \forall\mathbf{v}\in\mathbf{H}^1(D).$$

For a proof of Lemma 2.2, we refer to [4].

§**3. Existence of quasi–strong solutions.** In this section we will show existence of quasi–strong solutions for the initial boundary value problem (1.19)–(1.22). The method of compactness (cf. [11]) is used to prove the result, and standard Galerkin method is employed to construct approximate solutions. It is interesting to remark that from the physical model we can see that the differentiability of p (which is the divergence of $\mathbf{u}_f$) is one order lower than that of u and this observation is justified by our result.

Our main result of this section are given by the following theorem.

Theorem 3.1. *Suppose that the source and initial datum functions satisfy $g_f\in L^2(0,T;L^2(\Omega_f))$, $\mathbf{g}_s\in H^1(0,T;\mathbf{L}^2(\Omega_s))$, $p_j\in H^{1-j}(\Omega_f)$, $\mathbf{u}_j\in H^{2-j}(\Omega_s)$, for $j=0,1$, and p_0, $\mathbf{u}_0$, $\mathbf{u}_1$ satisfy* *the compatibility conditions (C1) and (C2). Then problem (1.19)–(1.22) possesses at least one quasi–strong solution* $(p,\mathbf{u})\in P_f\times \mathbf{V}_s$.

The proof of this theorem is divided into several steps. First, we choose sequences $\{g_f^m\}\in C^\infty([0,T];L^2(\Omega_f))$ and $\{\mathbf{g}_s^m\}\in C^\infty([0,T];\mathbf{L}^2(\Omega_s))$ such that

$$\text{(3.1.i)}\quad g_f^m \overset{m\to\infty}{\longrightarrow} g_f \text{ in } L^2(0,T;L^2(\Omega_f)),\qquad \|g_f^m\|_{L^2(0,T;L^2(\Omega_f))}\leq \|g_f\|_{L^2(0,T;L^2(\Omega_f))},$$

$$\text{(3.1.ii)}\quad \mathbf{g}_s^m \overset{m\to\infty}{\longrightarrow} \mathbf{g}_s \text{ in } H^1(0,T;\mathbf{L}^2(\Omega_s)),\qquad \|\mathbf{g}_s^m\|_{H^1(0,T;\mathbf{L}^2(\Omega_s))}\leq \|\mathbf{g}_s\|_{H^1(0,T;\mathbf{L}^2(\Omega_s))}.$$

Let $\{\varphi_j\}_{j=1}^\infty$ be a basis for $H^1(\Omega_f)$ and $\{\Psi_j\}_{j=1}^\infty$ a basis for $\mathbf{H}^2(\Omega_s)$. Set

$$P_f^m=\text{span}\{\varphi_1,\varphi_2,\dots,\varphi_m\},\qquad \mathbf{V}_s^m=\text{span}\{\Psi_1,\Psi_2,\dots,\Psi_m\},$$

for any positive integer m.

We then consider the following approximate problems (systems of second order ODEs): Find $T_m>0$, and $a_{mi}(t)\in C^2([0,T_m])$, $b_{mk}(t)\in C^2([0,T_m])$ for $i=1,2,\dots,m$; $j=1,2,\dots,m$ such that for

$$\text{(3.2.i)}\qquad p^m(x,t)=\sum_{i=1}^m a_{mi}(t)\varphi_i(x),\qquad \mathbf{u}^m(x,t)=\sum_{k=1}^m b_{mk}(t)\Psi_k(x)$$

there hold

$$\begin{aligned}\text{(3.2.ii)}\quad &\int_{\Omega_f}\frac{1}{c^2}\frac{\partial^2 p^m}{\partial t^2}\varphi_j\,dx+\int_{\Omega_f}\nabla p^m\cdot\nabla\varphi_j\,dx+\int_{\Gamma_f}\frac{1}{c}\frac{\partial p^m}{\partial t}\varphi_j\,d\tau\\ &\qquad -\int_\Gamma \rho_f\frac{\partial^2\mathbf{u}^m}{\partial t^2}\cdot n_s\varphi_j\,d\tau=\int_{\Omega_f} g_f^m\varphi_j\,dx,\quad j=1,2,\dots,m,\end{aligned}$$

$$\begin{aligned}\text{(3.2.iii)}\quad &\int_{\Omega_s}\rho_s\frac{\partial^2\mathbf{u}^m}{\partial t^2}\cdot\Psi_\ell\,dx+\int_{\Omega_s}\sigma(\mathbf{u}^m):\varepsilon(\Psi_\ell)\,dx+\int_{\Gamma_s}\rho_s\mathcal{A}_s\frac{\partial\mathbf{u}^m}{\partial t}\cdot\Psi_\ell\,d\tau\\ &\qquad +\int_\Gamma p^m\Psi_\ell\cdot n_s\,d\tau=\int_{\Omega_s}\mathbf{g}_s^m\cdot\Psi_\ell\,dx,\quad \ell=1,2,\dots,m,\end{aligned}$$

$$\text{(3.3.i)}\qquad p^m(x,0)=p_0^m(x),\qquad \frac{\partial p^m}{\partial t}(x,0)=p_1^m(x),$$

$$\text{(3.3.ii)}\qquad \mathbf{u}^m(x,0)=\mathbf{u}_0^m(x)\qquad \frac{\partial\mathbf{u}^m}{\partial t}(x,0)=\mathbf{u}_1^m(x),$$

where p_0^m, p_1^m, u_0^m and u_1^m are any functions satisfying

$$\begin{aligned}&p_0^m\in P_f^m,\quad p_0^m\longrightarrow p_0\quad\text{in } H^1(\Omega_f),\qquad p_1^m\in P_f^m,\quad p_1^m\longrightarrow p_1\quad\text{in } L^2(\Omega_f),\\ &\mathbf{u}_0^m\in\mathbf{V}_s^m,\quad \mathbf{u}_0^m\longrightarrow\mathbf{u}_0\quad\text{in }\mathbf{H}^2(\Omega_s),\quad \mathbf{u}_1^m\in\mathbf{V}_s^m,\quad \mathbf{u}_1^m\longrightarrow\mathbf{u}_1\quad\text{in }\mathbf{H}^1(\Omega_s),\end{aligned}$$

where

$$\text{(3.3.iii)}\qquad \rho_s\mathcal{A}_s\mathbf{u}_1^m+\sigma(\mathbf{u}_0^m)n_s=0,\quad\text{on }\Gamma_s,$$

$$\text{(3.3.iv)}\qquad \sigma(\mathbf{u}_0^m)n_s-p_0^m n_f=0,\qquad\text{on }\Gamma.$$

Proposition 3.1. *For each fixed positive integer m, if $g_f^m \in C^r([0,T];L^2(\Omega_f))$, $\mathbf{g}_s^m \in C^{r+1}([0,T];L^2(\Omega_s))$ for any integer $r \geq 1$, then system (3.2.ii)–(3.3) has a solution pair $(\{a_{mi}(t)\}_{i=1}^m, \{b_{mk}(t)\}_{k=1}^m) \in [C^{r+2}([0,T])]^m \times [C^{r+3}([0,T])]^m$.*

Proof. It is easy to check that (3.2)–(3.3) can be rewritten as

$$\begin{bmatrix} E & H \\ 0 & A \end{bmatrix} \begin{bmatrix} \ddot{a}_m(t) \\ \ddot{b}_m(t) \end{bmatrix} + \begin{bmatrix} F & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \dot{a}_m(t) \\ \dot{b}_m(t) \end{bmatrix} + \begin{bmatrix} G & 0 \\ D & C \end{bmatrix} \begin{bmatrix} a_m(t) \\ b_m(t) \end{bmatrix} = \begin{bmatrix} g_f^m(t) \\ g_s^m(t) \end{bmatrix},$$

where the matrices A, B, C, D, E, F, G and H are defined by

$$\begin{aligned}
A_{k\ell} &= \int_{\Omega_s} \rho_s \Psi_k \cdot \Psi_\ell dx, & B_{k\ell} &= \int_{\Gamma_s} \rho_s \mathcal{A}_s \Psi_k \cdot \Psi_\ell d\tau, \\
C_{k\ell} &= \int_{\Omega_s} \sigma(\Psi_k) : \varepsilon(\Psi_\ell) dx, & D_{k\ell} &= \int_{\Gamma} \varphi_k \Psi \cdot n_s d\tau, \\
E_{k\ell} &= \int_{\Omega_f} \frac{1}{c^2} \varphi_k \varphi_\ell dx, & F_{k\ell} &= \int_{\Gamma_f} \frac{1}{c} \varphi_k \varphi_\ell d\tau, \\
G_{k\ell} &= \int_{\Omega_f} \nabla \varphi_k \cdot \nabla \varphi_\ell dx, & H_{k\ell} &= \int_{\Gamma} \rho_f \Psi_k \cdot n_s \varphi_\ell d\tau.
\end{aligned}$$

Notice that the matrices A and E are symmetric and positive definite. Then the proposition follows directly from applying the ordinary differential equation theory (cf. [10]) and a bootstrap argument.

Proposition 3.2. *Let $\{a_{mi}(t)\}_{i=1}^m \in [C^{r+2}([0,T])]^m$ and $\{b_{mk}(t)\}_{k=1}^m \in [C^{r+3}([0,T])]^m$ for some integer $r \geq 1$ be a solution of system (3.2.ii)–(3.3). Then the function pair $(p^m, \mathbf{u}^m)$ satisfies the following estimates:*

$$\|p^m\|_{L^\infty(0,T;H^1(\Omega_f))} \leq C, \tag{3.4.i}$$

$$\left\| \frac{\partial p^m}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega_f))} \leq C, \tag{3.4.ii}$$

$$\left\| \frac{\partial p^m}{\partial t} \right\|_{L^2(0,T;L^2(\Gamma_f))} \leq C, \tag{3.4.iii}$$

$$\|\mathbf{u}^m\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_s))} \leq C, \tag{3.5.i}$$

$$\left\| \frac{\partial \mathbf{u}^m}{\partial t} \right\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_s))} \leq C, \tag{3.5.ii}$$

$$\left\| \frac{\partial^2 \mathbf{u}^m}{\partial t^2} \right\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_s))} \leq C, \tag{3.5.iii}$$

$$\left\| \frac{\partial^2 \mathbf{u}^m}{\partial t^2} \right\|_{L^2(0,T;\mathbf{L}^2(\Gamma_s))} \leq C, \tag{3.5.iv}$$

where C is a positive constant which is independent of m.

Proof. First we multiply (3.2.ii) by $\frac{1}{\rho_f} a'_{mj}(t)$ and sum over $j = 1, 2, \dots, m$, we get

$$\tag{3.6} \frac{1}{2}\frac{d}{dt}\left(\left\|\frac{1}{c\sqrt{\rho_f}}\frac{\partial p^m}{\partial t}\right\|^2_{L^2(\Omega_f)}\right)+\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{1}{\sqrt{\rho_f}}\nabla p^m\right\|^2_{L^2(\Omega_f)}\right)+\left\|\frac{1}{\sqrt{c\rho_f}}\frac{\partial p^m}{\partial t}\right\|^2_{L^2(\Gamma_f)} - \int_\Gamma \frac{\partial^2 \mathbf{u}^m}{\partial t^2}\cdot n_s \frac{\partial p^m}{\partial t} d\tau = \int_{\Omega_f} \frac{1}{\rho_f} g_f^m \frac{\partial p^m}{\partial t} dx.$$

Next, differentiate both sides of equation (3.2.iii) with respect to t, multiply the new equation by $b''_{mk}(t)$ and sum over $k = 1, 2, \dots, n$, we get

$$\tag{3.7} \begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|\sqrt{\rho_s}\frac{\partial^2 \mathbf{u}^m}{\partial t^2}\right\|^2_{\mathbf{L}^2(\Omega_s)}\right)+\frac{d}{dt}\left(\left\|\sqrt{\mu_s}\varepsilon(\frac{\partial \mathbf{u}^m}{\partial t})\right\|^2_{\mathbf{L}^2(\Omega_s)}\right) \\ &+\frac{1}{2}\frac{d}{dt}\left(\left\|\sqrt{\lambda_s}\operatorname{div}\,\frac{\partial \mathbf{u}^m}{\partial t}\right\|^2_{L^2(\Omega_s)}\right)+\int_{\Gamma_s}\rho_s\mathcal{A}_s\frac{\partial^2 \mathbf{u}^m}{\partial t^2}\cdot\frac{\partial^2 \mathbf{u}^m}{\partial t^2}d\tau \\ &+\int_\Gamma \frac{\partial p^m}{\partial t}\frac{\partial^2 \mathbf{u}^m}{\partial t^2}\cdot n_s d\tau = \int_{\Omega_s}\frac{\partial \mathbf{g}_s^m}{\partial t}\cdot\frac{\partial^2 \mathbf{u}^m}{\partial t^2}dx. \end{aligned}$$

Adding (3.7) to (3.6) gives

$$\tag{3.8} \begin{aligned} &\frac{d}{dt}\left[\left\|\frac{1}{c\sqrt{\rho_f}}\frac{\partial p^m}{\partial t}\right\|^2_{L^2(\Omega_f)}+\left\|\frac{1}{\sqrt{\rho_f}}\nabla p^m\right\|^2_{\mathbf{L}^2(\Omega_f)}+\left\|\sqrt{\rho_s}\frac{\partial^2 \mathbf{u}^m}{\partial t^2}\right\|^2_{\mathbf{L}^2(\Omega_s)}\right. \\ &\left.+2\left\|\sqrt{\mu_s}\varepsilon\left(\frac{\partial \mathbf{u}^m}{\partial t}\right)\right\|^2_{\mathbf{L}^2(\Omega_s)}+\left\|\sqrt{\lambda_s}\operatorname{div}\,\left(\frac{\partial \mathbf{u}^m}{\partial t}\right)\right\|^2_{L^2(\Omega_s)}\right] \\ &+2\int_{\Gamma_s}\rho_s\mathcal{A}_s\frac{\partial^2 \mathbf{u}^m}{\partial t^2}\cdot\frac{\partial^2 \mathbf{u}^m}{\partial t^2}d\tau+2\left\|\frac{1}{\sqrt{c\rho_f}}\frac{\partial p^m}{\partial t}\right\|^2_{L^2(\Gamma_f)} \\ &=2\int_{\Omega_f} g_f^m\frac{\partial p^m}{\partial t}dx+2\int_{\Omega_s}\frac{\partial \mathbf{g}_s^m}{\partial t}\cdot\frac{\partial^2 \mathbf{u}^m}{\partial t^2}dx. \end{aligned}$$

Recalling that $\mathcal{A}_s$ is a positive definite $N \times N$ matrix, it follows from (3.1), (3.3), (3.8), Schwartz and Gronwall's inequalities that

$$\left\| \frac{1}{c\sqrt{\rho_f}} \frac{\partial p^m}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega_f))} \le C, \tag{3.9.i}$$

$$\left\| \frac{1}{\sqrt{\rho_f}} \nabla p^m \right\|_{L^\infty(0,T;L^2(\Omega_f))} \le C, \tag{3.9.ii}$$

$$\left\| \frac{1}{\sqrt{c\rho_f}} \frac{\partial p^m}{\partial t} \right\|_{L^2(0,T;L^2(\Gamma_f))} \le C, \tag{3.9.iii}$$

$$\left\| \sqrt{\rho_s} \frac{\partial^2 \mathbf{u}^m}{\partial t^2} \right\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_s))} \le C, \tag{3.10.i}$$

$$\left\| \sqrt{\mu_s} \varepsilon(\frac{\partial \mathbf{u}^m}{\partial t}) \right\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_s))} \le C, \tag{3.10.ii}$$

$$\left\| \sqrt{\lambda_s} \text{div } (\frac{\partial \mathbf{u}^m}{\partial t}) \right\|_{L^\infty(0,T;L^2(\Omega_s))} \le C, \tag{3.10.iii}$$

$$\left\| \sqrt{\rho_s} \frac{\partial^2 \mathbf{u}^m}{\partial t^2} \right\|_{L^2(0,T;\mathbf{L}^2(\Gamma_s))} \le C, \tag{3.10.iv}$$

where C is a positive constant which depends only on $\|p_0\|_{H^1(\Omega_f)}$, $\|p_1\|_{L^2(\Omega_f)}$, $\|\mathbf{u}_1\|_{\mathbf{H}^1(\Omega_s)}$, $\|g_f\|_{L^2(0,T:L^2(\Omega_f))}$, $\|\mathbf{g}_s\|_{H^1(0,T;\mathbf{L}^2(\Omega_s))}$ and $\|\frac{\partial^2 \mathbf{u}^m}{\partial t^2}(\cdot,0)\|_{\mathbf{L}^2(\Omega_s)}$.

To show C can be bounded from above by a constant independent of m, we need to bound $\|\frac{\partial^2 \mathbf{u}^m}{\partial t^2}(\cdot,0)\|_{\mathbf{L}^2(\Omega_s)}$ uniformly in m. To get such a bound, setting $t = 0$ in (3.2.iii) and using (3.3.iii) and (3.3.iv) we get

$$\begin{aligned} \int_{\Omega_s} \rho_s \frac{\partial^2 \mathbf{u}^m}{\partial t^2}(x,0) \cdot \Psi_\ell(x) dx &= \int_{\Omega_s} \mathbf{g}_s^m(x,0) \cdot \Psi_\ell(x) dx - \int_{\Omega_s} \sigma(\mathbf{u}_0^m) : \varepsilon(\Psi_\ell) dx \\ &\quad - \int_{\Gamma_s} \rho_s \mathcal{A}_s \mathbf{u}_1^m \cdot \Psi_\ell d\tau + \int_\Gamma p_0^m \Psi_\ell \cdot n_s d\tau, \\ &= \int_{\Omega_s} \mathbf{g}_s^m(x,0) \cdot \Psi_\ell(x) dx - \int_{\Omega_s} \text{div } \sigma(\mathbf{u}_0^m) \cdot \Psi_\ell dx. \end{aligned} \tag{3.11}$$

Lemma 2.1 (ii) and (3.1.ii) imply that

$$\begin{aligned} \left\| \frac{\partial^2 \mathbf{u}^m}{\partial t^2}(x,0) \right\|_{\mathbf{L}^2(\Omega_s)} &\le \|\mathbf{g}_s^m\|_{C([0,T],\mathbf{L}^2(\Omega_s))} + \|\text{div } \sigma(\mathbf{u}_0^m)\|_{\mathbf{L}^2(\Omega_s)} \\ &\le \|\mathbf{g}_s\|_{\mathbf{H}^1(0,T;\mathbf{L}^2(\Omega_s))} + \|\mathbf{u}_0\|_{\mathbf{H}^2(\Omega_s)}. \end{aligned}$$

So we have shown the constant C in (3.9)–(3.10) is independent of m. Now from

$$\frac{\partial \mathbf{u}^m}{\partial t}(x,t) = \int_0^t \frac{\partial^2 \mathbf{u}^m}{\partial t^2}(x,\tau) d\tau + \mathbf{u}_1^m(x),$$

we have

$$\left\| \frac{\partial \mathbf{u}^m}{\partial t} \right\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_s))} \le C. \tag{3.12}$$

It follows from (3.10.ii), (3.12) and Korn's inequality (2.2) that

$$\left\| \frac{\partial \mathbf{u}^m}{\partial t} \right\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_s))} \le C, \quad \text{which implies} \quad \|\mathbf{u}^m\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_s))} \le C. \tag{3.13}$$

By an argument similar to the one used to derive (3.12), using (3.9.i) we can show that

$$\|p^m\|_{L^\infty(0,T;L^2(\Omega_f))} \le C. \tag{3.14}$$

The proof is completed.

We are ready to show our main result of this section.

Proof of Theorem 3.1. From (3.4) and (ii) of Lemma 2.1 with $B_0 = H^1(\Omega_f)$, $B = L^2(\Omega_f)$, $B = L^2(\Omega_f)$ we get that $\{p^m\}_{m=1}^\infty$ is relatively compact in $C([0,T];L^2(\Omega_f))$; and from (3.5) and Lemma 2.1 with $B_0 = \mathbf{H}^1(\Omega_s)$, $B = \mathbf{L}^2(\Omega_s)$, $B_1 = \mathbf{L}^2(\Omega_s)$ and $B = \mathbf{H}^1(\Omega_s)$ we know that $\{\frac{\partial \mathbf{u}^m}{\partial t}\}_{m=1}^\infty$ is relatively compact in $C([0,T];\mathbf{L}^2(\Omega_s))$ and $\{\mathbf{u}^m\}_{m=1}^\infty$ is relatively compact in $C([0,T];\mathbf{H}^1(\Omega_s))$. Therefore there exist a subsequence of $\{p^m\}$ and a subsequence of $\{\mathbf{u}^m\}$ such that

$$\begin{aligned}
&(3.15.\text{i}) && p^m \longrightarrow p && \text{in } C([0,T];L^2(\Omega_f)) \text{ strong and} \\
& && && \text{in } L^\infty(0,T;H^1(\Omega_f)) \text{ weak } *, \\
&(3.15.\text{ii}) && \frac{\partial p^m}{\partial t} \longrightarrow \frac{\partial p}{\partial t} && \text{in } L^\infty(0,T;L^2(\Omega_f)) \text{ weak } * \text{ and} \\
& && && \text{in } L^2(0,T;L^2(\Gamma_f)) \text{ weak.} \\
&(3.16.\text{i}) && \mathbf{u}^m \longrightarrow \mathbf{u} && \text{in } C([0,T];\mathbf{H}^1(\Omega_s)) \text{ strong,} \\
&(3.16.\text{ii}) && \frac{\partial \mathbf{u}^m}{\partial t} \longrightarrow \frac{\partial \mathbf{u}}{\partial t} && \text{in } C([0,T];\mathbf{L}^2(\Omega_s)) \text{ strong and} \\
& && && \text{in } L^\infty(0,T;\mathbf{H}^1(\Omega_s)) \text{ weak } *, \\
&(3.16.\text{iii}) && \frac{\partial^2 \mathbf{u}^m}{\partial t^2} \longrightarrow \frac{\partial^2 \mathbf{u}}{\partial t^2} && \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega_s)) \text{ weak } * \text{ and} \\
& && && \text{in } L^2(0,T;\mathbf{L}^2(\Gamma_s)) \text{ weak.}
\end{aligned}$$

Let j and ℓ be fixed in (3.2.ii) and (3.2.iii), and let $m > j$ and $m > \ell$. By (3.15) and (3.16) we get

$$\begin{aligned}
\int_{\Omega_f} \frac{1}{c^2} \frac{\partial^2 p^m}{\partial t^2} \varphi_j \, dx &= \frac{d}{dt} \int_{\Omega_f} \frac{1}{c^2} \frac{\partial p^m}{\partial t} \varphi_j \, dx \\
&\longrightarrow \frac{d}{dt} \int_{\Omega_f} \frac{1}{c^2} \frac{\partial p}{\partial t} \varphi_j \, dx \quad \text{in } \mathcal{D}'(0,T), \\
\int_{\Omega_f} \nabla p^m \cdot \nabla \varphi_j \, dx &\longrightarrow \int_{\Omega_f} \nabla p \cdot \nabla \varphi_j \, dx \qquad \text{in } L^\infty(0,T) \text{ weak}*,
\end{aligned}$$

$$\int_{\Gamma_f} \frac{1}{c}\frac{\partial p^m}{\partial t}\varphi_j\,d\tau \longrightarrow \int_{\Gamma_f} \frac{1}{c}\frac{\partial p}{\partial t}\varphi_j\,d\tau \qquad \text{in } L^2(0,T)\text{weak},$$

$$\begin{aligned}\int_\Gamma \rho_f \frac{\partial^2 \mathbf{u}^m}{\partial t^2}\cdot n_s\varphi_j\,d\tau &= \frac{d}{dt}\int_\Gamma \rho_f\frac{\partial^2 \mathbf{u}^m}{\partial t^2}\cdot n_s\varphi_j\,d\tau \\ &\longrightarrow \frac{d}{dt}\int_\Gamma \rho_f\frac{\partial \mathbf{u}}{\partial t}\cdot n_s\varphi_j\,d\tau \quad \text{in } \mathcal{D}'(0,T).\end{aligned}$$

Hence setting $m \longrightarrow \infty$ in (3.2.ii) we get

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_f} \frac{1}{c^2}\frac{\partial p}{\partial t}\varphi_j\,dx + \int_{\Omega_f} \nabla p\cdot\nabla\varphi_j\,dx + \int_{\Gamma_f} \frac{1}{c}\frac{\partial p}{\partial t}\varphi_j\,d\tau - \frac{d}{dt}\int_\Gamma \rho_f\frac{\partial \mathbf{u}}{\partial t}\cdot n_s\varphi_j\,d\tau \\ = \int_{\Omega_f} g_f\varphi_j\,dx, \quad \forall j\geq 1.\end{aligned}$$

Since $\{\varphi_j\}_{j=1}^\infty$ is a basis of $H^1(\Omega_f)$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_f} \frac{1}{c^2}\frac{\partial p}{\partial t}q\,dx + \int_{\Omega_f} \nabla p\cdot\nabla q\,dx + \int_{\Gamma_f} \frac{1}{c}\frac{\partial p}{\partial t}q\,d\tau - \frac{d}{dt}\int_\Gamma \rho_f\frac{\partial \mathbf{u}}{\partial t}\cdot n_s q\,d\tau \\ = \int_{\Omega_f} g_f q\,dx, \quad \forall q\in H^1(\Omega_f).\end{aligned} \tag{3.17}$$

Restricting $q \in H_0^1(\Omega_f)$ we conclude that $\frac{\partial^2 p}{\partial t^2} \in L^2(0,T;H^{-1}(\Omega_f))$, then (3.17) also implies that $\frac{\partial^2 \mathbf{u}}{\partial t^2}\cdot n_s \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma))$. Thus, we have shown that $p\in P_f$, $\mathbf{u}\in \mathbf{U}_s$ and $(p,\mathbf{u})$ satisfies (2.1.i). Similarly, by letting $m\longrightarrow\infty$ in (3.2.iii) we get that $(p,\mathbf{u})$ satisfies (2.1.ii).

Finally we show that $(p,\mathbf{u})$ also satisfies the initial conditions (1.22.i) and (1.22.ii). From (3.3.i), (3.3.ii), and (3.15.i), (3.16.i), (3.16.ii), we immediately get

$$\begin{aligned}&p(x,0) = p_0(x), \quad \text{in } \Omega_f,\\ &\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \frac{\partial \mathbf{u}}{\partial t}(x,0) = \mathbf{u}_1(x), \quad \text{in } \Omega_s.\end{aligned}$$

To show $\frac{\partial p}{\partial t}(x,0) = p_1(x)$, we notice that

$$\int_{\Omega_f} \frac{\partial p^m}{\partial t}\varphi_j\,dx \longrightarrow \int_{\Omega_f}\frac{\partial p}{\partial t}\varphi_j\;dx \quad \text{in } L^\infty(0,T) \text{ weak } *,$$

and since

$$\left|\frac{d}{dt}\int_{\Omega_f}\frac{\partial p^m}{\partial t}\varphi_j\,dx\right| = \left|\int_{\Omega_f}\frac{\partial^2 p^m}{\partial t^2}\varphi_j\;dx\right| \leq C\|\phi_j\|_{H^1(\Omega_f)}, \text{ for a.e. } t\in[0,T],$$

we have

$$\frac{d}{dt}\int_{\Omega_f}\frac{\partial p^m}{\partial t}\varphi_j\,dx \longrightarrow \frac{d}{dt}\int_{\Omega_f}\frac{\partial p}{\partial t}\varphi_j\;dx \quad \text{in } L^\infty(0,T) \text{ weak } *,$$

By Lemma 2.1 (ii) with $B=\mathbb{R}$

$$\int_{\Omega_f} \frac{\partial p^m}{\partial t}(x,0)\varphi_j(x)dx \longrightarrow \int_{\Omega_f} \frac{\partial p}{\partial t}(x,0)\varphi_j(x)dx.$$

Recalling we also have $\frac{\partial p^m}{\partial t}(x,0)=p_1^m(x) \longrightarrow p_1(x)$ in $L^2(\Omega_f)$, hence

$$\int_{\Omega_f} \frac{\partial p}{\partial t}(x,0)\varphi_j(x)dx = \int_{\Omega_f} p_1(x)\varphi_j(x)dx, \quad \forall j\geq 1.$$

Therefore

$$\frac{\partial p}{\partial t}(x,0)=p_1(x), \qquad \text{in } \Omega_f.$$

The proof is completed.

§**4. Uniqueness and regularity of quasi–strong solutions.** In this section, we first establish uniqueness of quasi–strong solutions which have been shown to exist in the last section. Then we study the regularity of the quasi–strong solution both in the time variable and the spatial variables. The regularity we get in the time variable is quite general, but no attempt is given to prove a very general spatial regularity result although we believe which can be done, instead, the spatial regularity is sought only in the case when the fluid domain Ω_f and the solid domain Ω_s are convex polygons, which is important for practical applications and numerical approximations.

First, we have the following uniqueness theorem. The usual test function trick for time–dependent problems plays a role in the proof.

Theorem 4.1. *Problem (1.19)–(1.22) has a unique quasi–strong solution under the same assumptions of Theorem 3.1.*

Proof. Existence has been proven in the last section. We only need to show uniqueness. Suppose there are two pairs of solutions $(p^{(1)},\mathbf{u}^{(1)})$ and $(p^{(2)},\mathbf{u}^{(2)})$ for problem (1.19)–(1.22). Let

$$p=p^{(2)}-p^{(1)}, \quad \mathbf{u}=\mathbf{u}^{(2)}-\mathbf{u}^{(1)}.$$

Thanks to the linearity of each equation in (1.19), we have that $(p,\mathbf{u})$ is a quasi–strong solution of problem (1.19)–(1.22) with zero force terms and initial data. For any fixed $\theta\in(0,T)$, integrating equations (2.1.i) and (2.1.ii) with respect to t over $(0,\theta)$ after setting

$$q(x,t)=-\int_t^\theta p(x,\eta)d\eta, \quad \mathbf{v}(x,t)=\frac{\partial \mathbf{u}}{\partial t}(x,t),$$

we get (noting that $q(x,\theta)=0$, $p(x,0)=0$, $\frac{\partial p}{\partial t}(x,0)=0$, $\mathbf{u}(x,0)=\frac{\partial \mathbf{u}}{\partial t}(x,0)=0$)

$$\text{(4.1.i)}\quad \int_0^\theta\int_{\Omega_f} -\frac{1}{c^2}\frac{\partial p}{\partial t}\frac{\partial q}{\partial t}dxdt+\int_0^\theta\int_{\Omega_f}\nabla p\cdot\nabla q dxdt-\int_0^\theta\int_{\Gamma_f}\frac{1}{c}p\frac{\partial q}{\partial t}d\tau dt$$
$$+\int_0^\theta\int_\Gamma \rho_f\frac{\partial \mathbf{u}}{\partial t}\cdot n_s\frac{\partial q}{\partial t}d\tau dt=0,$$

$$\text{(4.1.ii)}\quad \frac{1}{2}\left(\int_{\Omega_s}\rho_s\left|\frac{\partial \mathbf{u}}{\partial t}\right|^2dx\right)\Big|_{t=\theta}+\left(\int_{\Omega_s}\mu_s|\varepsilon(\mathbf{u})|^2dx\right)\Big|_{t=\theta}+\frac{1}{2}\left(\int_{\Omega_s}\lambda_s|\text{div }\mathbf{u}|^2dx\right)\Big|_{t=\theta}$$
$$+\int_0^\theta\int_{\Gamma_s}\rho_s\mathcal{A}_s\frac{\partial \mathbf{u}}{\partial t}\cdot\frac{\partial \mathbf{u}}{\partial t}d\tau dt+\int_0^\theta\int_\Gamma p\frac{\partial \mathbf{u}}{\partial t}\cdot n_s d\tau dt=0.$$

Since $\frac{\partial q}{\partial t}(x,t)=p(x,t)$, (4.1.i) can be rewritten as

$$\text{(4.2)}\quad -\frac{1}{2}\left(\int_{\Omega_f}\frac{1}{\rho_f c^2}p^2dx\right)\Big|_{t=\theta}-\frac{1}{2}\int_{\Omega_f}\frac{1}{\rho_f}\left|\nabla\int_0^\theta p(x,\eta)d\eta\right|^2dx$$
$$-\int_0^\theta\int_{\Gamma_f}\frac{1}{c\rho_f}p^2d\tau dt+\int_0^\theta\int_\Gamma\frac{\partial \mathbf{u}}{\partial t}\cdot n_s p d\tau dt=0.$$

Finally, subtracting (4.2) from (4.1.ii) gives us

$$\text{(4.3)}\qquad \|p\|^2_{L^2(\Omega_f)}+\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|^2_{\mathbf{L}^2(\Omega_s)}+\|\varepsilon(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega_s)}=0,\quad \forall\theta\in(0,T).$$

From (4.3) and Korn's inequality (2.2) we get

$$p(x,t)=0,\quad \mathbf{u}(x,t)=0$$

for almost every $(x,t)\in\Omega\times(0,T)$. This completes the proof.

Next, we establish some regularity results for the quasi–strong solution. The first result is about the regularity in time variable, which roughly says that the quasi–strong solution is smooth in time variable if the datum functions are smooth in time variable.

Theorem 4.2. *In addition to the assumptions of Theorem 3.1, suppose that there exist* $D_f\subset\subset\Omega_f$ *and* $D_s\subset\subset\Omega_s$ *with smooth boundaries* ∂D_f *and* ∂D_s *such that for* $k\geq 2$

(i) $p_j\in H^{k-j}(\Omega_f)$; $\text{supp}(p_j)\subset D_f$, $j=0,1$.

(ii) $\mathbf{u}_j\in\mathbf{H}^{k+1-j}(\Omega_s)$; $\text{supp}(\mathbf{u}_j)\subset D_s$, $j=0,1$.

(iii) $g_f\in H^{k-1}(0,T;L^2(\Omega_f))$; *and* $\text{supp}(\frac{\partial^\ell g_f}{\partial t^\ell}(x,0))\subset D_f$ *and* $\frac{\partial^{k-2-j}g_f}{\partial t^{k-2-j}}(x,0)\in H^j(\Omega_f)$ *for* $k\geq 3$, *where* $\ell=0,1,\cdots,k-3$, *and* $j=0,1$ *when* $k=3$ *and* $j=1,2$ *when* $k\geq 4$.

(iv) $\mathbf{g}_s\in H^k(0,T;\mathbf{L}^2(\Omega))$; *and* $\text{supp}(\frac{\partial^\ell \mathbf{g}_s}{\partial t^\ell}(x,0))\subset D_s$ *and* $\frac{\partial^{k-1-j}\mathbf{g}_s}{\partial t^{k-1-j}}(x,0)\in\mathbf{H}^j(\Omega_s)$ *for* $k\geq 2$, *where* $\ell=0,1,\cdots,k-2$, *and* $j=0,1$ *when* $k=2$ *and* $j=1,2$ *when* $k\geq 3$.

Then the quasi–strong solution $(p, \mathbf{u})$ *satisfies*

$$(4.4)\quad \frac{d^k p}{dt^k} \in L^\infty(0,T;L^2(\Omega_f)), \qquad \frac{d^j p}{dt^j} \in L^\infty(0,T;H^1(\Omega_f)), \quad j=1,2\cdots,k-1.$$

$$(4.5)\quad \frac{d^{k+1}\mathbf{u}}{dt^{k+1}} \in L^\infty(0,T;\mathbf{L}^2(\Omega_s)), \quad \frac{d^j \mathbf{u}}{dt^j} \in L^\infty(0,T;\mathbf{H}^1(\Omega_s)), \quad j=2,3,\cdots,k$$

Proof. We only give a sketch of the proof for the case $k=2$ since the proofs for $k>2$ can be written similarly by induction. When $k=2$, we have $g_f \in H^1(0,T;L^2(\Omega_f))$ and $\mathbf{g}_s \in H^2(0,T;L^2(\Omega_s))$. The idea of the proof is to show existence of a solution (p,u) which satisfies (4.4) and (4.5), then by Theorem 4.1 we know that the new solution coincides with the quasi–strong solution of (1.19)–(1.22). On the other hand, the proof of existence of a new solution is almost same as the proof of Theorem 3.1. The only difference is that here we need to derive better a priori estimates about the approximate solutions $\{(p^m,\mathbf{u}^m)\}_{m=1}^\infty$ which is possible because now we have better datum functions p_j, $\mathbf{u}_j$ $(j=1,2)$, g_f and $\mathbf{g}_s$.

Let $\{\varphi_j\}$ and $\{\Phi_j\}$ be the solutions of the following two problems, respectively,

$$\begin{aligned} -\Delta\varphi_j &= \gamma_j\varphi_j, && \text{in } \Omega_f,\\ \frac{\partial\varphi_j}{\partial n_f} + \varphi_j &= 0, && \text{on } \partial\Omega_f;\\ -\operatorname{div}(\sigma(\Psi_j)) &= \kappa_j\Psi_j, && \text{in } \Omega_s,\\ \sigma(\Psi_j)n_s + \Psi_j &= 0, && \text{on } \partial\Omega_s. \end{aligned}$$

Note that $\varphi_j \in H^1(\Omega_f)\cap H^2(D_f)$ and $\Psi_j \in \mathbf{H}^1(\Omega_s)\cap\mathbf{H}^3(D_s)$.

Let P_f^m and $\mathbf{V}_s^m$ be same as in §3, that is,

$$P_f^m = \operatorname{span}\{\varphi_1,\varphi_2,\cdots,\varphi_m\}, \qquad \mathbf{V}_s^m = \operatorname{span}\{\Psi_1,\Psi_2,\cdots,\Psi_m\}.$$

And choose $\{g_f^m\} \in C^\infty([0,T];P_f^m)$ and $\{\mathbf{g}_s^m\} \in C^\infty([0,T];\mathbf{V}_s^m)$ such that

$$\begin{aligned} &(4.6.\text{i})\quad \operatorname{supp}(\mathbf{g}_s^m(x,0)) \subset D_s,\\ &(4.6.\text{ii})\quad g_f^m \longrightarrow g_f \text{ in } H^1(0,T;L^2(\Omega_f)), \quad \|g_f^m\|_{H^1(0,T;L^2(\Omega_f))} \le \|g_f\|_{H^1(0,T;L^2(\Omega_f))};\\ &(4.6.\text{iii})\quad \mathbf{g}_s^m \longrightarrow \mathbf{g}_s \text{ in } H^2(0,T;\mathbf{L}^2(\Omega_s)), \quad \|\mathbf{g}_s^m\|_{H^2(0,T;\mathbf{L}^2(\Omega))} \le \|\mathbf{g}_s\|_{H^2(0,T;\mathbf{L}^2(\Omega_s))},\\ &\qquad\qquad \|\mathbf{g}_s^m(\cdot,0)\|_{\mathbf{H}^1(\Omega_s)} \le \|\mathbf{g}_s(\cdot,0)\|_{\mathbf{H}^1(\Omega_s)}. \end{aligned}$$

Note that (4.6) replaces (3.1).

Let p_0^m, p_1^m, $\mathbf{u}_0^m$ and $\mathbf{u}_1^m$ be any functions satisfying

$$\begin{aligned} &(4.7.\text{i}) && p_0^m \in P_f^m, && \operatorname{supp}(p_0^m) \subset D_f, && p_0^m \longrightarrow p_0 && \text{in } H^2(\Omega_f),\\ &(4.7.\text{ii}) && p_1^m \in P_f^m, && \operatorname{supp}(p_1^m) \subset D_f, && p_1^m \longrightarrow p_1 && \text{in } H^1(\Omega_f),\\ &(4.7.\text{iii}) && \mathbf{u}_0^m \in \mathbf{V}_s^m, && \operatorname{supp}(\mathbf{u}_0^m) \subset D_s, && \mathbf{u}_0^m \longrightarrow \mathbf{u}_0 && \text{in } \mathbf{H}^3(\Omega_s),\\ &(4.7.\text{iv}) && \mathbf{u}_1^m \in \mathbf{V}_s^m, && \operatorname{supp}(\mathbf{u}_0^m) \subset D_s, && \mathbf{u}_1^m \longrightarrow \mathbf{u}_1 && \text{in } \mathbf{H}^2(\Omega_s). \end{aligned}$$

And let $\{(p^m, \mathbf{u}^m)\}_{m=1}^{\infty}$ still be defined by (3.2)–(3.3).

Clearly, the conclusions of Propositions 3.1 and 3.2 still hold. To get additional uniform in m estimates, differentiating both sides of (3.2.ii) with respect to t, multiplying the new equation by $\frac{1}{\rho_f} a''_{mj}(t)$, and sum over $k = 1, 2, \cdots, n$, we get

$$\begin{aligned}(4.8)\quad & \frac{1}{2}\frac{d}{dt}\left(\left\|\frac{1}{c\sqrt{\rho_f}}\frac{\partial^2 p^m}{\partial t^2}\right\|^2_{L^2(\Omega_f)}\right) + \frac{1}{2}\frac{d}{dt}\left(\left\|\frac{1}{\sqrt{\rho_f}}\nabla(\frac{\partial p^m}{\partial t})\right\|^2_{L^2(\Omega_f)}\right) \\ & + \left\|\frac{1}{\sqrt{c\rho_f}}\frac{\partial^2 p^m}{\partial t^2}\right\|^2_{L^2(\Gamma_f)} - \int_\Gamma \frac{\partial^3 \mathbf{u}^m}{\partial t^3}\cdot n_s \frac{\partial^2 p^m}{\partial t^2} d\tau = \int_{\Omega_f} \frac{1}{\rho_f}\frac{\partial g_f^m}{\partial t}\frac{\partial^2 p^m}{\partial t^2} dx.\end{aligned}$$

Then differentiating both sides of (3.2.iii) twice with respect to t, multiplying the resulted equation by $b'''_{mk}(t)$ and sum over $k = 1, 2, \ldots, n$, we get

$$\begin{aligned}(4.9)\quad & \frac{1}{2}\frac{d}{dt}\left(\left\|\sqrt{\rho_s}\frac{\partial^3 \mathbf{u}^m}{\partial t^3}\right\|^2_{\mathbf{L}^2(\Omega_s)}\right) + \frac{d}{dt}\left(\left\|\sqrt{\mu_s}\varepsilon(\frac{\partial^2 \mathbf{u}^m}{\partial t^2})\right\|^2_{\mathbf{L}^2(\Omega_s)}\right) \\ & + \frac{1}{2}\frac{d}{dt}\left(\left\|\sqrt{\lambda_s}\text{div }\frac{\partial^2 \mathbf{u}^m}{\partial t^2}\right\|^2_{L^2(\Omega_s)}\right) + \int_{\Gamma_s} \rho_s \mathcal{A}_s \frac{\partial^3 \mathbf{u}^m}{\partial t^3}\cdot\frac{\partial^3 \mathbf{u}^m}{\partial t^3} d\tau \\ & + \int_\Gamma \frac{\partial^2 p^m}{\partial t^2}\frac{\partial^3 \mathbf{u}^m}{\partial t^3}\cdot n_s d\tau = \int_{\Omega_s} \frac{\partial^2 \mathbf{g}_s^m}{\partial t^2}\cdot\frac{\partial^3 \mathbf{u}^m}{\partial t^3} dx.\end{aligned}$$

After adding (4.9) to (4.8) we obtain

$$\begin{aligned}(4.10)\quad & \frac{d}{dt}\Bigg[\left\|\frac{1}{c\sqrt{\rho_f}}\frac{\partial^2 p^m}{\partial t^2}\right\|^2_{L^2(\Omega_f)} + \left\|\frac{1}{\sqrt{\rho_f}}\nabla(\frac{\partial p^m}{\partial t})\right\|^2_{\mathbf{L}^2(\Omega_f)} + \left\|\sqrt{\rho_s}\frac{\partial^3 \mathbf{u}^m}{\partial t^3}\right\|^2_{\mathbf{L}^2(\Omega_s)} \\ & + 2\left\|\sqrt{\mu_s}\varepsilon\left(\frac{\partial^2 \mathbf{u}^m}{\partial t^2}\right)\right\|^2_{\mathbf{L}^2(\Omega_s)} + \left\|\sqrt{\lambda_s}\text{div }\left(\frac{\partial^2 \mathbf{u}^m}{\partial t^2}\right)\right\|^2_{L^2(\Omega_s)}\Bigg] \\ & + 2\int_{\Gamma_s} \rho_s \mathcal{A}_s \frac{\partial^3 \mathbf{u}^m}{\partial t^3}\cdot\frac{\partial^3 \mathbf{u}^m}{\partial t^3} d\tau + 2\left\|\frac{1}{\sqrt{c\rho_f}}\frac{\partial^2 p^m}{\partial t^2}\right\|^2_{L^2(\Gamma_f)} \\ & = 2\int_{\Omega_f} \frac{\partial g_f^m}{\partial t}\frac{\partial^2 p^m}{\partial t^2} dx + 2\int_{\Omega_s} \frac{\partial^2 \mathbf{g}_s^m}{\partial t^2}\cdot\frac{\partial^3 \mathbf{u}^m}{\partial t^3} dx.\end{aligned}$$

It follows from (3.2.ii) and (3.2.iii) that

$$\begin{aligned}\frac{\partial^2 p^m}{\partial t^2}(x, 0) &= \Delta p_0^m + g_f^m(x, 0), \\ \frac{\partial^2 \mathbf{u}^m}{\partial t^2}(x, 0) &= \text{div } \sigma(\mathbf{u}_0^m) + \mathbf{g}_s^m(x, 0), \\ \frac{\partial^3 \mathbf{u}^m}{\partial t^3}(x, 0) &= \text{div } \sigma(\mathbf{u}_1^m(x)) + \frac{\partial \mathbf{g}_s^m}{\partial t}(x, 0).\end{aligned}$$

Hence

$$\left\|\frac{\partial^2 p^m(x,0)}{\partial t^2}\right\|_{L^2(\Omega_f)} + \left\|\frac{\partial^3 \mathbf{u}^m(x,0)}{\partial t^3}\right\|_{\mathbf{L}^2(\Omega_s)} + \left\|\varepsilon\left(\frac{\partial^2 \mathbf{u}^m(x,0)}{\partial t^2}\right)\right\|_{\mathbf{L}^2(\Omega_s)} + \left\|\operatorname{div}\left(\frac{\partial^2 \mathbf{u}^m(x,0)}{\partial t^2}\right)\right\|_{L^2(\Omega_s)} \leq C. \tag{4.11}$$

By (4.10), (4.11) and Gronwall's inequality we get

$$\frac{\partial p^m}{\partial t} \text{ is bounded in } H^1(\Omega_f), \qquad \frac{\partial^2 p^m}{\partial t^2} \text{ is bounded in } L^2(\Omega_f),$$
$$\frac{\partial^2 \mathbf{u}^m}{\partial t^2} \text{ is bounded in } \mathbf{H}^1(\Omega_s), \qquad \frac{\partial^3 \mathbf{u}^m}{\partial t^3} \text{ is bounded in } \mathbf{L}^2(\Omega_s).$$

Finally, following the proof of Theorem 3.1 we can show that there is a subsequence of $\{(p^m, \mathbf{u}^m)\}$ with the limit $(p, \mathbf{u})$ which satisfies (1.22) and (2.1) and (4.4)–(4.5). The proof is completed after applying the uniqueness theorem, Theorem 4.1.

We end this section by showing some spatial regularities of the quasi–strong solution. In particular, for practical application and numerical approximation purposes we are interested in the case when Ω_f and Ω_s are convex polygonal domains.

Theorem 4.3. *In addition to the assumptions of Theorem 3.1, if Ω_s is either a smooth domain or a convex polygon, then the quasi–strong solution $(p, \mathbf{u}) \in P_f \times \widehat{\mathbf{V}}_s$.*

Proof. It is suffice to show that $\mathbf{u} \in L^\infty(0,T;\mathbf{H}^2(\Omega_s))$. Noting that Lemma 2.1 (ii) implies that $\mathbf{g}_s \in L^\infty(0,T;L^2(\Omega))$, and (2.1.ii) can be rewritten as

$$\int_{\Omega_s} \sigma(\mathbf{u}) : \varepsilon(\mathbf{v}) dx = \int_{\Omega_s} (\mathbf{g}_s - \rho_s \frac{\partial^2 \mathbf{u}}{\partial t^2}) \cdot \mathbf{v} dx - \int_{\Gamma_s} \rho_s \mathcal{A}_s \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} d\tau - \int_\Gamma p\mathbf{v} \cdot n_s d\tau, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega_s), \quad t \in (0,T). \tag{4.12}$$

This implies that for almost every $t \in (0,T)$, $\mathbf{u}(\cdot,t) \in \mathbf{H}^1(\Omega_s)$ is a weak solution of the following problem:

$$-\operatorname{div} \sigma(\mathbf{w}) = \mathbf{F}, \quad \text{in } \Omega_s, \tag{4.13.i}$$
$$\sigma(\mathbf{w}) n_s = \mathbf{h}_1, \quad \text{on } \Gamma_s, \tag{4.13.ii}$$
$$\sigma(\mathbf{w}) n_s = \mathbf{h}_2, \quad \text{on } \Gamma, \tag{4.13.iii}$$

where

$$\mathbf{F} = \mathbf{g}_s - \rho_s \frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad \mathbf{h}_1 = -\rho_s \mathcal{A}_s \frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{h}_s = p n_s. \tag{4.13.iv}$$

Since $\mathbf{F}(\cdot,t) \in \mathbf{L}^2(\Omega_s)$, $\mathbf{h}_1(\cdot,t) \in \mathbf{H}^{\frac{1}{2}}(\Gamma_s)$, $\mathbf{h}_2(\cdot,t) \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$, and Ω_s is assumed to be either smooth or a convex polygon, from [9] we know that $\mathbf{u}(\cdot,t) \in \mathbf{H}^2(\Omega_s)$ and for almost every $t \in (0,T)$ there holds the following estimate:

$$\|\mathbf{u}(\cdot,t)\|_{H^2(\Omega_s)} \le C\left[\|\mathbf{u}(\cdot,t)\|_{\mathbf{L}^2(\Omega_s)} + \|\mathbf{F}\|_{\mathbf{L}^2(\Omega_s)} + \left\|\rho_s \mathcal{A}_s \frac{\partial \mathbf{u}}{\partial t}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma_s)} + \|p n_s\|_{H^{\frac{1}{2}}(\Gamma)}\right]$$

Therefore

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}^2(\Omega_s)).$$

The proof is completed.

Similarly, we can show the following theorem.

Theorem 4.4. *In addition to the assumptions of Theorem 4.2, if both Ω_f and Ω_s are smooth domains or convex polygons, then besides (4.4) and (4.5), the quasi–strong solution also satisfies*

$$\frac{d^j p}{dt^j} \in L^\infty(0,T;H^2(\Omega_f)), \quad j = 0,1,2\cdots,k-2. \tag{4.14}$$

$$\frac{d^j \mathbf{u}}{dt^j} \in L^\infty(0,T;\mathbf{H}^2(\Omega_s)), \quad j = 0,1,2,\cdots,k-1 \tag{4.15}$$

References

1. R. A. Adams, *Sobolev Spaces*, Academic Press, New York, NY, 1975.
2. J. Boujot, *Mathematical formulation of fluid–structure interaction problems*, Math. Modeling and Numer. Anal. **21** (1987), 239–260.
3. L. Demkowicz, J. T. Oden, M. Ainsworth and P. Geng, *Solution of elastic scattering problems in linear acoustics using $h-p$ boundary element methods*, J. Comp. and Applied Math. **36** (1991), 29–63.
4. G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.
5. B. Engquist and A. Majda, *Absorbing boundary conditions for the numerical simulation of waves*, Math. Comp. **31** (1977), 629–651.
6. B. Engquist and A. Majda, *Radiation boundary conditions for acoustic and elastic wave calculations*, Comm. Pure Appl. Math. **32** (1979), 313–357.
7. X. Feng P. Lee and Y. Wei, *Finite element methods and domain decomposition algorithms for a fluid–solid interaction problem*, (preprint).
8. D. Givoli, *Numerical methods for problems in infinite domains*, Elsevie, Amsterdam/New York, 1992.
9. P. Grisvard, *Singularities in Boundary Value Problems*, Research Notes in Applied Mathematics, Vol. 22, MASSON and Springer–Verlag, 1992.
10. J. Hale, *Ordinary Differential Equations*, Wiley–Interscience, New York, 1969.
11. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non linéaires*, Dunod - Gauthier - Villars, Paris, 1969.
12. J. Lysmer and R. L. Kuhlmeyer, *Finite dynamic model for infinite media*, J. Engrg. Mech. Div., Proc. ASCE 95 **EM4** (1969), 859–877.
13. J.E.Santos, J.Douglas,Jr. and A.P.Calderón, *Finite element methods for a composite model in elastodynamics*, SIAM J. Numer. Anal. **25** (1988), 513–523.
14. J. E. Santos, J. Douglas, Jr., M. E. Morley, and O. M. Lovera, *Finite element methods for a model for full waveform acoustic logging*, IMA Journal of Numerical Analysis **8** (1988), 415–433.
15. J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. **146** (1987), 65–96.
16. D. Sheen, *A numerical method for approximating wave propagation in a porous medium saturated by a two-phase fluid*, Mat. Apl. Comput. **11** (1992), 209-225.