On the Exactness of an S-Shaped Bifurcation Curve

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Abstract

For a class of two-point boundary value problems we prove exactness of S-shaped bifurcation curve. Our result applies to a problem from combustion theory, which involves nonlinearities like $e^{au/(u+a)}$ for $a > 0$.

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1 Introduction

We consider positive solutions of

$$u'' + \lambda f(u) = 0 \text{ on } (-1,1), \;\; u(-1) = u(1) = 0. \tag{1.1}$$

Here λ is a positive parameter, and we wish to describe all solutions of (1.1) for all values of λ. Our main example will be $f(u) = e^{\frac{au}{u+a}}$. This nonlinearity is connected with steady state of gas combustion according to the Arhenius law, see e.g. J. Bebernes and D. Eberly [2], and it has been studied before, see K.J. Brown, M.M.A. Ibrahim and R. Shivaji [3], R. Shivaji [9] and A. Castro and R. Shivaji [4]. For the above nonlinearity it was shown that for $a \leq 4$ there exists a unique positive solution for all λ, while for a large the solution diagram is roughly S-shaped, i.e. there is a range of λ for which there exist at least three solutions. Moreover, uniqueness of solution was proved for small and for large λ. In this paper we show that solution diagram consists of exactly one curve, which is exactly S-shaped, for a class of nonlinearities which includes the one above for $a > a_0$, where a_0 is defined below ($a_0 \simeq 4.35$). The bifurcation diagram is given in Pic. 1(a). A similar result was proved by S.-H. Wang [10] using the quadrature technique. In addition to obtaining an alternative proof, we do not require the boundness of $f(u)$, as was the case in [10]. This brings up a possibility of another type of S-shaped solution curves, as in Pic. 1(b). Moreover, when verifying the conditions of the theorem for $f(u) = e^{\frac{au}{u+a}}$, we introduce another technical improvement, which shortens the proof and produces a better critical constant.

We use tools from bifurcation theory, particularly the M.G. Crandall-P.H. Rabinowitz bifurcation theorem, which is recalled below, and the techniques from P. Korman, Y. Li and T. Ouyang [7].

We assume that $f(u) \in C^2[0, \bar{u}]$ for some $0 < \bar{u} \leq \infty$ and it satisfies

$$f(u) > 0 \text{ for all } 0 \leq u < \bar{u}, \tag{1.2}$$

and there is $\alpha \in (0, \bar{u})$ such that

$$f''(u) > 0 \text{ for } u \in (0, \alpha), \quad f''(u) < 0 \text{ for } u \in (\alpha, \bar{u}). \tag{1.3}$$

We also assume that $f(u)$ is "sublinear", i.e. either $\bar{u}$ is finite and $f(\bar{u}) = 0$, or else

$$\lim_{u \to \infty} \frac{f(u)}{u} = 0. \tag{1.4}$$

Remark. We denote $F(u) = \int_0^u f(t)\,dt$. If $u(x)$ is a positive solution of (1.1) then necessarily $F(u(0)) > 0$ and $f(u(0)) > 0$. Therefore, if condition $f(\bar{u}) = 0$ holds, then all possible $u(0)$ must lie in a single interval $(0, \bar{u})$.

For $f(u) = e^{\frac{au}{u+a}}$ one computes $\alpha = \frac{1}{2}a^2 - a$. In Theorem 2.1 below we make a further assumption on $f(u)$, which in particular implies that it is "sufficiently convex" in the beginning.

Next we recall the bifurcation theorem of M.G. Crandall and P.H. Rabinowitz [5].

Theorem 1.1 *[5] Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{x}) \in R \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space $N(F_x(\overline{\lambda}, \overline{x})) = span\,\{x_0\}$ be one-dimensional and codim $R(F_x(\overline{\lambda}, \overline{x})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$. If Z is a complement of span $\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ near $(\overline{\lambda}, \overline{x})$ form a curve $(\lambda(s), x(s)) = (\overline{\lambda} + \tau(s), \overline{x} + sx_0 + z(s))$, where $s \to (\tau(s), z(s)) \in R \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = 0$, $z'(0) = 0$.*

We shall write $u(x, \lambda)$ to denote solution of (1.1). Notice that taking the interval $(-1, 1)$ does not restrict the generality for the autonomous equation (1.1).

2 The Global Solution Curve

We consider positive solutions of

$$u'' + \lambda f(u) = 0 \text{ on } (-1, 1), \quad u(-1) = u(1) = 0. \tag{2.1}$$

Our main example will be $f(u) = e^{\frac{au}{u+a}}$. Here a is a fixed positive constant, and λ a positive parameter. We wish to describe all positive solutions of (2.1) as the parameter λ varies.

The linearized equation corresponding to (2.1) is

$$w'' + \lambda f'(u)w = 0 \text{ on } (-1, 1), \quad w(-1) = w(1) = 0. \tag{2.2}$$

If (2.2) has nontrivial solutions at some solution $u(x)$ of (2.1), we refer to (λ, u) as a critical point of (2.1).

Assume there is $\beta \geq \alpha$ such that

$$f^2(\beta) - 2F(\beta)f'(\beta) > 0. \tag{2.3}$$

The following lemma is an adaptation of Lemma 2.5 in [7].

Lemma 2.1 *Assume that $f(u)$ satisfies the condition (1.2), (1.3) and (1.4). Let (λ, u) be any critical point of (2.1), with $u(0) \geq \beta$, and $w(x)$ is the corresponding solution of (2.2). Then*

$$\int_0^1 f''(u)u_x w^2 dx > 0. \tag{2.4}$$

Proof. We shall derive a convenient expression for the integral in (2.4). Differentiating (2.2) gives

$$w_x'' + \lambda f'(u) w_x + \lambda f''(u) u_x w = 0. \tag{2.5}$$

From the equations (2.2) and (2.5)

$$(w w_x' - {w'}^2)' + \lambda f''(u) u_x w^2 = 0.$$

Integrating, we express

$$\begin{aligned} \lambda \int_0^1 f''(u) u_x w^2 dx &= -(w w'' - {w'}^2)|_0^1 \\ &= {w'}^2(1) - \lambda w^2(0) f'(u(0)). \end{aligned} \tag{2.6}$$

Differentiating (2.1) gives

$$u_x'' + \lambda f'(u) u_x = 0. \tag{2.7}$$

From (2.2) and (2.7)

$$(w u'' - u' w')' = 0.$$

This means that the quantity $wu'' - u'w'$ is constant over $[-1, 1]$. Evaluating it at $x = 0$, we conclude

$$w(x) u''(x) - u'(x) w'(x) = -\lambda w(0) f(u(0)) \text{ for all } x \in [-1, 1].$$

Evaluating this expression at $x = 1$ gives

$$w'(1) = \frac{\lambda w(0) f(u(0))}{u'(1)}. \tag{2.8}$$

Multiplying (2.1) by u', and integrating over $(0, x)$, we obtain

$${u'}^2(x) = 2\lambda [F(u(0)) - F(u(x))].$$

Setting here $x = 1$, and using the resulting formula in (2.8), we express

$${w'}^2(1) = \frac{\lambda w^2(0) f^2(u(0))}{2F(u(0))}.$$

Using this in (2.6) we finally obtain

$$\int_0^1 f''(u) u_x w^2 dx = \frac{w^2(0)}{2F(\rho)} I(\rho),$$

where we denote $\rho = u(0)$, and

$$I(\rho) = f^2(\rho) - 2F(\rho)f'(\rho).$$

To prove the lemma we need to show that $I(\rho) > 0$ for any $\rho \geq \beta$. Compute

$$I'(\rho) = -2F(\rho)f''(\rho) \geq 0 \ \text{ for } \ \rho \geq \beta,$$

and the lemma follows by the assumption (2.3).

In order to prove our main result we need to understand precisely how the solution curve changes its direction, which is determined by the function

$$h(u) = 2F(u) - uf(u). \tag{2.9}$$

We state our main result next.

Theorem 2.1 *Assume that $f(u)$ satisfies the conditions (1.2), (1.3) and (1.4). With $h(u) \equiv 2F(u) - uf(u)$ assume that*

$$h(\alpha) < 0. \tag{2.10}$$

Then the solution curve of (2.1) is exactly S-shaped, i.e. it starts at $\lambda = 0$, $u = 0$ it makes exactly two turns, and then continues for all $\lambda > 0$ without any more turns.

Proof. By the implicit function theorem there is a curve of positive solution of (2.1) starting at $\lambda = 0$, $u = 0$. This curve continues for increasing λ until a possible singular solution (λ_0, u) (i.e. (2.2) has nontrivial solutions), at which the Crandall-Rabinowitz Theorem 1.1 applies. A standard calculation shows that the function $\tau(s)$ defined in that theorem satisfies

$$\tau''(0) = -\lambda_0 \frac{\int_0^1 f''(u)w^3 dx}{\int_0^1 f(u)w dx}. \tag{2.11}$$

Indeed, differentiating the equation (2.1) twice in s, we have

$$u''_{ss} + \lambda f'(u)u_{ss} + 2\lambda' f'(u)u_s + \lambda f''(u){u_s}^2 + \lambda'' f(u) = 0.$$

Setting here $s = 0$, and using that $\tau'(0) = 0$ and $u_s|_{s=0} = w(x)$, we get

$$u''_{ss} + \lambda_0 f'(u)u_{ss} + \lambda_0 f''(u)w^2 + \tau''(0)f(u) = 0. \tag{2.12}$$

Multiplying (2.12) by w, and the equation (2.2) by u_{ss}, subtracting and integrating, we obtain (2.11). In P. Korman, Y. Li and T. Ouyang [7]. we showed that one may assume $w(x)$ to be positive on $(-1,1)$, and that the denominator in (2.11) is positive. It follows by (2.11) that when $u(0) < \alpha$ only turns to the left in (λ, u) "plane" are possible. Next we need a formula that gives the direction in which solution curve travels, derived in K.J. Brown, M.M.A. Ibrahim and R. Shivaji, [3]. With $\rho = u(0)$ and $h(u)$ as defined previously, they show that

$$\frac{d}{d\rho}\lambda(\rho)^{1/2} = \frac{1}{\sqrt{2}}\int_0^1 \frac{h(\rho)-h(\rho v)}{[F(\rho)-F(\rho v)]^{3/2}}dv. \tag{2.13}$$

We see that $\frac{d\lambda}{d\rho} < 0$ and the curve travels to the left, provided

$$h(\rho) < h(s) \text{ for all } s \in (0,\rho). \tag{2.14}$$

We now discuss the function $h(u)$. Since

$$h'(u) = f(u) - uf', \quad h''(u) = -uf'',$$

it follows that the function $h'(u)$ is decreasing on $(0,\alpha)$ and increasing on (α,∞). Since $h(0) = 0$, $h'(0) = f(0) > 0$, it follows that there exist u_1 and u_2, with $u_1 < \alpha < u_2$, such that $h'(u_1) = h'(u_2) = 0$ and

$$h' = f(u) - uf'(u) > 0 \text{ for } u \in (0,u_1) \cup (u_2,\infty), \tag{2.15}$$

$$h' = f(u) - uf'(u) < 0 \text{ for } u \in (u_1,u_2). \tag{2.16}$$

Indeed, existence of the first root $u_1 < \alpha$ follows immediately by (2.10). As for the second root u_2, if it did not exist, we would have

$$f(u) < uf'(u) \text{ for all } u > \alpha. \tag{2.17}$$

Integrating (2.17), we would have

$$f(u) > \frac{f(\alpha)}{\alpha}u \text{ for all } u > \alpha,$$

contradicting the assumption (1.4), in case $\bar{u} = \infty$. If $\bar{u} < \infty$, then we get a contradiction by plugging $\bar{u}$ into this inequality. Returning to the function $h(u)$, we notice that $h(0) = 0$ and $h(u)$ is concave on $(0,\alpha)$. By (2.15) $h(u)$ is decreasing on (α,u_2), at u_2 it takes its absolute minimum, and then increases for all $u > u_2$. The graph of $h(u)$ is given by Pic. 2. One sees that the condition (2.14) holds for all $u \in (\alpha, u_2)$ i.e. so long as $\alpha < u(0,\lambda) < u_2$ the solution curve travels to the left.

We now claim that for $\beta = u_2$ the condition (2.3) holds. Indeed, since $h(u_2) < 0$, we have

$$f(u_2)u_2 > 2F(u_2).$$

Hence

$$f^2(u_2) - 2F(u_2)f'(u_2) > f^2(u_2) - f(u_2)u_2 f'(u_2) = 0,$$

and the claim follows.

We conclude that for $u(0) > u_2$ the Lemma 1 applies. We show next that this implies that only turns to the right in (λ, u) plane are possible. We proceed similarly to P. Korman, Y. Li and T. Ouyang [7]. Let $(\lambda_0, u_0(x))$ be a critical point of (2.1) with $u_0(0) > u_2$. The function $f''(u_0(x))$ changes sign exactly once on $(0,1)$, say at $x_0 > 0$. Clearly

$$(2.18)\quad f''(u_0(x)) < 0 \ \text{ for } \ x \in (0, x_0), \quad f''(u_0(x)) > 0 \ \text{ for } \ x \in (x_0, 1).$$

By stretching the function $w(x)$ we may assume that the functions $w(x)$ and $-u_0'(x)$ intersect at x_0. We claim that x_0 is the only point on $(0,1)$ where they intersect. Indeed, the functions $w(x)$ and $-u_0'$ are solutions of the same linear equation (2.2). If x_1 is another intersection point, adjacent to x_0, then we can find a constant $\mu \neq 1$ and a point $\overline{x} \in (x_0, x_1)$ so that $\mu w(\overline{x}) = -u_0(\overline{x})$ and $\mu w'(\overline{x}) = -u_0'(\overline{x})$, i.e. two distinct solutions with the same initial conditions, a contradiction. By Lemma 1 it follows that

$$(2.19)\qquad \int_0^1 f''(u_0(x))w^2 w\,dx < \int_0^1 f''(u_0(x))w^2(-u_x)\,dx < 0.$$

Indeed, $-u_x > w$ where $f'' > 0$, and $-u_x < w$ where $f'' < 0$. Hence the integrand on the right is pointwise greater than the one on the left. By (2.19) the numerator in (2.11) is negative, which means that only turns to the right are possible for $u > u_2$.

We now return to the curve of solutions, which started at $\lambda = 0$, $u = 0$. It is well-known that $u(0,\lambda)$ is increasing on this curve for all λ, see e.g. E.N. Dancer [6]. By the time $u(0,\lambda)$ reaches α, the curve already travels to the left. Since $f''(u) > 0$ for $u < \alpha$ it follows by (2.11) that only turns to the left are possible before $u(0,\lambda)$ reaches α, and hence exactly one such turn occurred. As $u(0,\lambda)$ keeps increasing between α and u_2, the function $h(u)$ keeps decreasing, and hence (2.14) continues to hold, and so the solution curve keeps travelling to the left until $u(0,\lambda)$ reaches u_2. When $u(0,\lambda) > u_2$

only turns to the right are possible, and indeed exactly one such turn will occur, for if the curve kept travelling to the left it would have nowhere to go (solutions of (2.1) are bounded for bounded λ). Hence solution curve is exactly S-shaped.

Finally, there is only one solution curve, since (in case $f(u) > 0$ for all u) on our curve of solutions the value $u(0)$ varies from 0 to ∞, while the value of $u(0)$ uniquely identifies the solution, see e.g., E.N. Dancer, [6]. (In case $f(u)$ vanishes at some $\bar{u}$ one argues similarly.)

Next we discuss a generalization of Theorem 2.1, obtained by replacing the sublinearity condition (1.4) by the following "weak sublinearity condition"

$$\left(\frac{f(u)}{u}\right)' \leq 0 \text{ for some } u = u_0 > \alpha. \tag{2.20}$$

Theorem 2.2 *Assume that $f(u)$ satisfies the conditions (1.2), (1.3) and (2.20). Then the solution curve of (2.1) is either exactly S-shaped as described in the previous theorem, or else after exactly two turns it tends to infinity at some finite $\bar{\lambda} > 0$.*

Proof. We begin by observing that our conditions allow $f(u)$ to be asymptotically linear as $u \to \infty$. Indeed, starting with any $f(u)$ satisfying (1.2) and (1.3) we may by adding a large constant obtain a function satisfying the condition (2.20) as well, at any $u_0 > \alpha$. This leaves $f(u)$ free to have any behavior at infinity, which is consistent with concavity, in particular it can be asymptotically linear. This implies that solution curve may go to infinity at some finite $\lambda = \bar{\lambda}$.

The proof proceeds similarly to the previous theorem. When it comes to the existence of u_2, the second root of $h(u)$, we observe that if it did not exist, we would have

$$f(u) < uf'(u) \text{ for all } u > \alpha,$$

contradicting the assumption (2.20). As before, the curve makes precisely one turn before $u(0)$ reaches α. We claim that the curve cannot go to infinity as it travels to the left. Indeed, setting $w(u) = \left(\frac{f(u)}{u}\right)'$, we observe that by (2.20) $w(u_0) \leq 0$. Since

$$w'(u) + \frac{2}{u}w(u) = \frac{f''(u)}{u} < 0,$$

we obtain by integrating the above equation

$$u^2 w(u) = u_0^2 w(u_0) + \int_{u_0}^{u} s f''(s)\, ds < 0 \ \text{ for all } u \geq u_0.$$

We conclude that $\frac{f(u)}{u}$ is decreasing for all $u > u_0$. Hence if $f(u)$ is asymptotical to $au + b$, with constant a and b, then $b > 0$. This implies that bifurcation from infinity is to the left, see e.g. [1]. We conclude that the solution curve cannot go to infinity on its way to the left.

Hence as before the curve will make exactly one more turn to the right, and then it may go to infinity at some finite $\bar{\lambda}$, see Pic. 1(b).

3 Applications

As our first application, we now consider $f(u) = e^{\frac{au}{u+a}}$. Then $f'(u) > 0$ for all $u > 0$, $f''(u) > 0$ for $0 < u < \frac{1}{2}a(a-2)$, $f''(u) < 0$ for $u > \frac{1}{2}a(a-2)$. If $0 < a \leq 4$, it was noticed previously, see e.g. R. Shivaji [9] that $f(u) > uf'(u)$ for all $u > 0$, and hence for any λ the problem (2.1) has a unique solution. (The previous writers were using the formula (2.13) to conclude uniqueness in this case. Alternatively, one could use Sturm comparison theorem to conclude that the linearized equation (2.2) can have no nontrivial solutions, and so the solution curve cannot turn).

Next we need the following lemma.

Lemma 3.1 *Let $a_0 > 4$ be solution of*

$$1 - \frac{2}{a} = 4 \int_2^a \frac{e^{-\tau+2}}{\tau^2} d\tau. \tag{3.1}$$

(Numerical evaluation shows that $a_0 \simeq 4.35$). Then in case $f(u) = e^{\frac{au}{u+a}}$ we have for all $a > a_0$

$$h(\alpha) < 0. \tag{3.2}$$

Proof. First of all notice that for all $a > a_0$

$$1 - \frac{2}{a} > 4 \int_2^a \frac{e^{-\tau+2}}{\tau^2} d\tau. \tag{3.3}$$

Indeed, denoting by $d(a)$ the difference between the left and right sides in (3.3), we have that $d(a_0) = 0$ and

$$d'(a) = \frac{2}{a^2} - 4\frac{e^{-a+2}}{a^2} > 0 \ \text{ for } \ a > 4.$$

This justifies (3.3) and also shows that the equation (3.1) has exactly one solution in $a > 4$ range. Compute

$$h(\alpha) = 2\int_0^{\frac{1}{2}a^2-a} e^{\frac{as}{s+a}}ds - \left(\frac{1}{2}a^2 - a\right)e^{a-2} \equiv -\frac{e^{a-2}a^2}{2}H,$$

where

$$\begin{aligned} H &= 1 - \frac{2}{a} - \frac{4}{a^2}\int_0^{\frac{1}{2}a^2-a} e^{\frac{as}{s+a}-a+2}ds \\ &= 1 - \frac{2}{a} - \frac{4}{a^2}\int_0^{\frac{1}{2}a^2-a} e^{-\frac{a^2}{a+s}+2}ds. \end{aligned}$$

We make a change of variables $\tau = \frac{a^2}{a+s}$, i.e. $s = \frac{a^2}{\tau} - a$, $ds = -\frac{a^2}{\tau^2}d\tau$. It follows that

$$H = 1 - \frac{2}{a} + 4\int_a^2 \frac{e^{-\tau+2}}{\tau^2} > 0 \tag{3.4}$$

in view of (3.3), and hence $h(\alpha) < 0$.

Remark. If the reader is uncomfortable with a (routine) use of computer to solve the equation (3.1), one can proceed as follows. In (3.4) set $t = \tau - 2$. Then

$$\begin{aligned} H &= 1 - \frac{2}{a} - 4\int_0^{a-2} \frac{e^{-t}}{(t+2)^2}dt \\ &= 1 - \frac{2}{a} - 4\int_0^{a-2} e^{-t}\left[\frac{1}{(t+2)^2} - \frac{1}{4}\right] - \int_0^{a-2} e^{-t}dt \\ &= -\frac{2}{a} + e^{-a+2} + \int_0^{a-2} e^{-t}\frac{4t+t^2}{(t+2)^2}dt, \end{aligned}$$

which is positive for large a.

If $a > 4$ then $h'(u)$ has roots and in fact, $u_1, u_2 = \frac{1}{2}a^2 - a \pm \frac{a}{2}\sqrt{a^2-4a}$. If $a > a_0 \simeq 4.35$ then by Lemma 3.1 the condition (2.10) is satisfied, and hence the solution curve is exactly S-shaped. This is an improvement of the critical constant $a_0 \simeq 4.4967$ obtained by S.H. Wang [10]. We have thus proved the following theorem.

Theorem 3.1 *If $a > a_0$ then the solution curve for the problem*

$$u'' + \lambda e^{\frac{au}{u+a}} = 0 \quad \text{on } (-1,1),\ u(-1) = u(1) = 0$$

is exactly S-shaped.

Our second example involves $f(u) = 1 + u^2 - \epsilon u^p$, with constants $p > 2$, and $\epsilon > 0$. We compute $h(u) = u - \frac{1}{3}u^3 + \epsilon\frac{p-1}{p+1}u^{p+1}$, and $\alpha = \left[\frac{2}{\epsilon p(p-1)}\right]^{\frac{1}{p-2}}$. It follows

$$h(\alpha) = \left[\frac{2}{\epsilon p(p-1)}\right]^{\frac{1}{p-2}} - \frac{1}{3}\left[\frac{2}{\epsilon p(p-1)}\right]^{\frac{3}{p-2}} + \epsilon\frac{p-1}{p+1}\left[\frac{2}{\epsilon p(p-1)}\right]^{\frac{p+1}{p-2}}.$$

For ϵ small the leading terms in ϵ are second and third, and they have the same order in ϵ. We will have $h(\alpha) < 0$ for small ϵ, provided

$$\frac{1}{3}\left[\frac{2}{p(p-1)}\right]^{\frac{3}{p-2}} > \frac{p-1}{p+1}\left[\frac{2}{p(p-1)}\right]^{\frac{p+1}{p-2}}.$$

This is equivalent to

$$\frac{1}{3} > \frac{2}{p(p+1)},$$

which is true for $p > 2$. We conclude that for ϵ small, i.e. if

$$\epsilon^{\frac{2}{p-2}} < \frac{1}{3}\left[\frac{2}{p(p-1)}\right]^{\frac{2}{p-2}} - \frac{p-1}{p+1}\left[\frac{2}{p(p-1)}\right]^{\frac{p}{p-2}} \tag{3.5}$$

the solution curve is exactly S-shaped.

Theorem 3.2 *Assume that $p > 2$ and ϵ satisfies (3.5). Then the solution curve for*

$$u'' + \lambda(1 + u^2 - \epsilon u^p) = 0 \quad on\ (-1,1),\ u(-1) = u(1) = 0$$

is exactly S-shaped.

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References

[1] A. Ambrosetti and P. Hess, Positive solutions of asymptotically linear elliptic eigenvalue problems, *J. Math. Anal. Applic.* **73**, 411-422 (1980).

[2] J. Bebernes and D. Eberly, Mathematical Problems from Combustion Theory, *Springer-Verlag* (1989).

[3] K.J. Brown, M.M.A. Ibrahim and R. Shivaji, S-shaped bifurcation curves, *Nonlinear Anal. TMA* **5** (5), 475-486 (1981).

[4] A. Castro and R. Shivaji, Uniqueness of positive solutions for elliptic boundary value problems, *Proc. Royal Soc. Edinburgh* **98A**, 267-269 (1984).

[5] M.G. Crandall and P.H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Rational Mech. Anal.* **52**, 161-180 (1973).

[6] E.N. Dancer, On the structure of solutions of an equation in catalysis theory when a parameter is large, *J. Differential Equations* **37**, 404-437 (1980).

[7] P. Korman, Y. Li and T. Ouyang, Exact multiplicity results for boundary-value problems with nonlinearities generalizing cubic, *Proc. Royal Soc. Edinburgh* **126A**, 599-616 (1996).

[8] N. Mizoguchi and T. Suzuki, Equations of gas combustion: s-shaped bifurcation and mushrooms, *J. Differential Equations* **134**, 183-215 (1997).

[9] R. Shivaji, Remarks on an S-shaped bifurcation curve, *J. Math. Anal. Applic.* **111**, 374-387 (1985).

[10] S.-H. Wang, On S-shaped bifurcation curves, *Nonlinear Analysis TMA* **22**, 1475-1485 (1994).