A NOTE ON VISCOUS SPLITTING OF DEGENERATE CONVECTION-DIFFUSION EQUATIONS

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Abstract. We establish L^1 convergence of a viscous splitting method for nonlinear possibly strongly degenerate convection-diffusion problems. Since we allow the equations to be strongly degenerate, solutions can be discontinuous and they are not, in general, uniquely determined by their data. We thus consider entropy weak solutions realized by the vanishing viscosity method. This notion is broad enough to also include non-degenerate parabolic equations as well as hyperbolic conservation laws. It thus provides a suitable "L^1 type" framework for analyzing numerical schemes for convection-diffusion problems that are designed to handle *various* balances of convective and diffusive forces. We present a numerical example which shows that our splitting scheme has such "design".

0. Introduction

It is well known that accurate modeling of convective and diffusive processes is one of the most ubiquitous and challenging tasks in the numerical approximation of partial differential equations. This is partly because of the problems themselves, their widespread occurrence, as well as their close association with hyperbolic conservation laws. Nonlinear convection-diffusion equations arise in a variety of applications, ranging from models of turbulence [10], via traffic flow [48] and financial modeling [8], to two phase flow in porous media [11]. A convection-diffusion equation can also be viewed as a model problem for a system of convection-diffusion equations such as three phase flow in porous media [62] or the Navier-Stokes equations. Such equations also appear in, to mention a few, polymer chemistry [12], combustion modeling [51], modeling of semi-conductor devices [50], and in models for transport of solutes in ground water and surface water [6].

One objective of this paper is to establish convergence of a viscous splitting method for nonlinear possibly strongly degenerate convection-diffusion problems. First, convergence of the splitting will be established for the following initial value problem

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x \left[a(u)\partial_x u\right], \qquad u(x,0) = u_0(x), \qquad (x,t) \in Q_T, \tag{1}$$

where $u : Q_T \equiv \mathbb{R} \times \langle 0, T] \to \mathbb{R}$ denotes the unknown; $u_0 : \mathbb{R} \to [m, M]$, $f : [m, M] \to \mathbb{R}$, and $a : [m, M] \to [0, \infty\rangle$ are given smooth, bounded functions; and $\varepsilon > 0$ is a scaling parameter. Secondly, the initial-boundary value problem will be discussed and we show how to extend our method of proof to cover also this problem.

When (1) is non-degenerate, i.e., $a(u) > 0\ \forall u$, the mathematical theory is well known [54]. In particular, the initial value problem then admits a unique classical solution. This contrasts with the case where (1) is allowed to degenerate at certain points. The solution is then not necessarily smooth [1,2,3,53] and weak solutions must be sought. Almost all previous works on mathematical analysis focused mainly on the special case where the equation possesses only one or two point degeneracy and often only non-negative continuous solutions were considered [21,29,45,46]. The simplest examples are perhaps provided by the porous medium equation; $\partial_t u = \varepsilon \partial_x^2 (u^m)$, $m > 1$, and the convective porous medium equation; $\partial_t u + \partial_x (u^n) = \varepsilon \partial_x^2 (u^m)$, $n, m > 1$. To the best of our knowledge, the most general result on the existence and uniquness of weak solutions for (1) in the class of bounded and measurable functions is provided by Zhao [42]. The essential condition for uniqueness is that the function

$$A(u) = \int_0^u a(\xi)\, d\xi \tag{2}$$

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is strictly increasing. This condition is also sufficient for the existence of continuous solutions. Note that the restriction (2) does not rule out the possibility that $a(u)$ has an infinite number of zero points. For recent results on regularity properties of weak solutions we refer to Tassa [60] and the references cited therein. We refer to the survey texts [13,44] for a broad view of the theory of this type of degenerate parabolic equations.

The work of Volpert & Hudjaev [64] was the first devoted to degenerate equations that do not satisfy (2). Since the equations now are allowed to be strongly degenerate, the solutions can be discontinuous. We thus seek entropy weak solutions in the BV class that are realizable as the L^1 limit of smooth solutions u^μ of non-degenerate parabolic equations,

$$\partial_t u^\mu + \partial_x f(u^\mu) = \varepsilon \partial_x \left[a(u^\mu)\partial_x u^\mu\right] + \mu \partial_x^2 u^\mu,$$

as the regularization parameter $\mu > 0$ tends to zero. The existence of an entropy weak solution was demonstrated in [64], whereas the uniqueness of entropy weak solutions in the BV class was proved only recently by Wu & Yin [65]. Being a very general solution concept, the notion of an entropy weak solution includes all the previously mentioned parabolic equations as well as hyperbolic conservation laws; $\partial_t u + \partial_x f(u) = 0$ in the sense of Volpert [63], Kruzkov [47]. We refer to [43,61,66,67] for other works related to these mixed hyperbolic/parabolic equations. In particular, in [43] it is proved that if (x_0, t_0) is a point of approximate continuity of the entropy solution $u(x,t)$ such that $a(u(x_0,t_0)) > 0$, then $u(x,t)$ is a classical solution in a neighbourhood of (x_0,t_0). In addition, if the structure condition (2) holds, then the entropy solution is continuous.

From a computational point of view, when (1) is convection dominated, i.e., ε is small compared with other scales in (1), it is well known that Galerkin schemes exhibit non-physical oscillations, especially at the trailing end of moving shock fronts. In recent years there has been a significant activity on deriving splitting techniques [5,18,20,23,24,25,26,28,33,37,57] that can be used to combine numerical schemes for conservation laws with Galerkin schemes for parabolic equations with the purpose of eliminating such difficulties.

The most natural equation splitting can be written in the following abstract "product formula" form

$$u(x, n\Delta t) \approx \left[\mathcal{H}_{\Delta t} \circ \mathcal{S}^f_{\Delta t}\right]^n u_0, \qquad \Delta t > 0, \tag{3}$$

where $\mathcal{S}^f_t$ is the solution operator associated with the conservation law $\partial_t v + \partial_x f(v) = 0$ at time t and $\mathcal{H}_t$ is the operator associated with the nonlinear "heat" equation $\partial_t w = \varepsilon \partial_x \left[a(w)\partial_x w\right]$ at time t. A general convergence result for (3) was recently obtained by Karlsen & Risebro [37] in the case where $a(u) \equiv 1$. The present paper extends this result to any non-negative smooth $a(u)$. If the convective part of (1) is linear, one can effectively use time stepping along the characteristics to realize the operator $\mathcal{S}^f_t$, see [23,26,57]. A similar technique with a nonlinear convection part results in severe time step restrictions because the hyperbolic part of (1) may develop shocks. However, this problem has been solved by Espedal & Ewing [24] (see also e.g. [18,28,33]) who suggested to split the convective part of (1) into two parts: $f = \bar f + \tilde f$, where, in particular, $\bar f$ is linear in the shock region and thus makes time stepping along the characteristics feasible. This discussion motivates the following slightly more general splitting formula

$$u(x, n\Delta t) \approx \left[\mathcal{P}^{\tilde f}_{\Delta t} \circ \mathcal{S}^{\bar f}_{\Delta t}\right]^n u_0, \tag{4}$$

which is the product formula that will be analyzed later. Here $\mathcal{P}^{\tilde f}_t$ denotes the solution operator associated with the equation $\partial_t w + \partial_x \tilde f(w) = \varepsilon \partial_x \left[a(w)\partial_x w\right]$. We mention that the product formulas (3) and (4) are special cases of the corrected operator splitting approach introduced recently by Karlsen & Risebro [38] and Karlsen, Brusdal, Dahle, Evje & Lie [39] (see also [40,41] and §3).

The convergence analysis of numerical schemes has so far mainly been concerned with one or two point degenerated equations and often only the "convection free" case. We refer to [22,32,34,35,55,59] for analysis of some finite element and difference schemes applied within this context. It is well known that convergence of Galerkin schemes is difficult to establish when the equation is allowed to degenerate at certain points. The obvious reason is that the equation then fails to admit classical solutions. In order to get around the difficulties arising when trying to derive error estimates for Galerkin schemes for a degenerate equation, one consider instead an equation with a perturbed diffusion coefficient that is bounded away from zero. Consequently, the solutions are smooth and standard error analysis applies. The problem is then reduced to showing that the solution of the perturbed equation converges to the solution of the non-perturbed equation in a "L^2 type" topology consistent with that of finite element analysis.

Such convergence, along with corresponding error estimates for Galerkin schemes, has been demonstrated by Rose [56] for one-dimensional problems, but under restrictions that precludes, for example, realistic two phase flow applications where two-point degenerate diffusion function $a(u)$ and s-shaped flux function $f(u)$ are essential ingredients. These restrictions were recently removed by Koffi & Sharpely [27] who, in addition, extend the results to certain multi-dimensional equations. The theory developed in [27], which allow for both two-point degeneracy and s-shaped flux, has been applied by Chen, Espedal & Ewing [14] and Chen & Ewing [15] to establish convergence of Galerkin schemes for degenerate equations arising in the modelling of multiphase flow in groundwater hydrology. However, the theory in [27,56] is established under an assumption that implies diffusion dominated transport which thus contrasts with the typical multiphase flow application, in which one often encounters highly convection dominated situations. Furthermore, in the case of convection dominance the equations are "almost hyperbolic" and sometimes it seems reasonable to model such situations by hyperbolic conservation laws [7,9,31], i.e, by allowing the equations to be strongly degenerate.

Our intention is to demonstrate that the approximations generated by the product formula (4) converge to the entropy weak solution of the possibly strongly degenerate problem (1). As far as we have learned, this is the first time convergence is obtained for an approximation scheme for convection-diffusion problems of this generality. Numerical algorithms and the techniques used for their analysis tend to be very different in the two limiting cases of hyperbolic and parabolic equations (cf. e.g. [52]). The second and final objective of the present paper is to stress the following point of view: Since the solution concept found in [64] also include hyperbolic as well as non-degenerate parabolic equations, we claim that convergence analysis of numerical schemes designed for general convection-diffusion equations can advantageously be carried out within this "L^1 type" framework. The motivation for this approach comes from the fact that we then avoid following the standard route of analysing, often within fundamentally different topologies, the hyperbolic/convection dominated case and the parabolic case seperately. Furthermore, this approach is also consistent with the goal of designing numerical schemes that can handle accurately any combination of convective and diffusive forces.

The paper is organized as follows. For completeness we give a survey of the mathematical theory associated with (1) in §1. In §2 we establish convergence of the product formula (4) by making use of a standard compactness argument. In §3 we present and discuss numerical results of fully discrete versions of (3) and (4) applied to a convection dominated two phase flow problem. Finally, in §4 we briefly discuss theory associated with the initial-boundary value problem and then show how to extend our method of proof to cover this case.

1. Preliminaries

In this section we recall the mathematical theory of degenerate parabolic equations developed in [64], which in turn depends heavily on the BV calculus developed in [63].

Let Ω be an open subset of $\mathbb{R}^d$ $(d > 1)$. The space $BV(\Omega)$ of functions of bounded variation consists of all integrable functions $u : \Omega \to \mathbb{R}$ whose first order partial derivatives $\frac{\partial u}{\partial y_i}$ $(i = 1, \dots, d)$ are represented by finite Borel measures. The total variation (semi-norm) $|u|_{BV(\Omega)}$ is by definition the sum of the total masses of these Borel measures. Moreover, $BV(\Omega)$ is a Banach space when equipped with the norm $||u||_{BV(\Omega)} = ||u||_{L^1(\Omega)} + |u|_{BV(\Omega)}$. Let us recall the regularity results concerning BV functions proved in [63]. For a BV function u, it occurs that with exception of a set of zero $(d-1)$ - dimensional Hausdorff measure, each point y of Ω is regular, i.e., either a point of approximate continuity (Lebesgue point) or a point of approximate jump (jump point). At a point of approximate jump y, there exists a unit normal $\nu(y) \in \mathbb{R}^d$, a left value $u_-(y)$ and a right value $u_+(y)$. Moreover, these jump points form an at most countable set of curves.

A useful concept proposed in [63] is the notion of averaged superposition.

Definition 1.1 [63]. *Let p be in $C^1(\mathbb{R})$ and u in $L^\infty(\Omega) \cap BV(\Omega)$. The averaged superposition of the function u by the function p is defined for H_{d-1} - almost every y in Ω by*

$$\hat{p}(u)(y) = \begin{cases} p(u(y)), & \text{if } y \text{ is an approximate continuity point of } u, \\ \int_0^1 p\left(\xi u_+(y) + (1-\xi) u_-(y)\right)\, d\xi, & \text{if } y \text{ is an approximate jump point of } u. \end{cases}$$

Here H_{d-1} denotes the $(d-1)$ - dimensional Hausdorff measure (see e.g. [68] for a definition of H_{d-1}).

Using the notion of averaged superposition, the following lemma can be proved.

Lemma 1.2 [63]. *Let u and v be in $L^\infty(\Omega) \cap BV(\Omega)$ and p in $C^1(\mathbb{R})$. Then the function $\hat{p}(u)$ is measurable and integrable with respect to the Borel measure $\frac{\partial v}{\partial y_i}$, so that the non-conservative product $\hat{p}(u)\frac{\partial v}{\partial y_i}$ makes sense as a finite Borel measure.*

This result is very useful since it leads to a notion of a weak solution to non-conservative partial differential equations such as (1), see Definition 1.5 below. The concept of averaged superposition (and generalizations of it) has also been applied as a tool in the study of systems of conservation laws in non-conservative form, see [19] and the references therein.

For later use, let us state the following formula for differentiation of an averaged superposition.

Lemma 1.3 [63]. *Let p be in $C^1(\mathbb{R})$ and u in $L^\infty(\Omega) \cap BV(\Omega)$. Then the differentiating rule*

$$\frac{\partial p(u(y))}{\partial y_i} = \frac{\partial \hat{p}(u)}{\partial u}\frac{\partial u}{\partial y_i} \tag{5}$$

holds. Here the equation (5) is to be understood in the sense of measures.

Furthermore, for applications to differential equations we need a formula for differentiating the product $\operatorname{sign}(u)v$. This case is considered in the next lemma.

Lemma 1.4 [63]. *Let u and $v = (v_1, \dots, v_d)$ be in $L^\infty(\Omega) \cap BV(\Omega)$ and let $\nabla \cdot v = \sum_{i=1}^d \frac{\partial v_i}{\partial y_i}$ be an absolutely continuous measure. Then $\operatorname{sign}(u)v$ is in $BV(\Omega)$ and, for any bounded Borel set E whose closure belongs to Ω, we have*

$$\int_E \nabla \cdot (\operatorname{sign}(u)v)\ dy = \int_E \operatorname{sign}(u)\,(\nabla \cdot v)\ dy + \int_{E\cap\Gamma_u} \left((\operatorname{sign}(u)v)_+ - (\operatorname{sign}(u)v)_-\right)\cdot \nu\, dH_{d-1}, \tag{6}$$

where Γ_u denotes the set of jump points of the function u, and ν denotes the unit normal to this set.

From now on we will consider the degenerate parabolic problem (1). Solutions to (1) can have discontinuities in the region of hyperbolicity $\{u : a(u) = 0\}$. We therefore seek solutions in the following sense.

Definition 1.5 [64]. *A function $u(x,t)$ in $L^\infty(Q_T) \cap BV(Q_T)$ is said to be an entropy weak solution of the initial value problem (1) provided the following two conditions are fulfilled:*

(C1). Let $r(u) = \sqrt{\varepsilon a(u)}$. Then $[\hat{r}(u)\partial_x u]$ should exist in the sense of distributions in $L^2_{loc}(Q_T)$. That is to say, there should exist a function g in $L^2_{loc}(Q_T)$ such that for all test functions $\phi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} [\hat{r}(u)\partial_x u]\,\phi(x,t)\,dt\,dx = \iint_{Q_T} g(x,t)\phi(x,t)\,dt\,dx.$$

(C2). For all non-negative functions $\phi \in C_0^\infty(Q_T)$ for which $\phi|_{t=T} = 0$ and all $k \in \mathbb{R}$, the following entropy inequality should hold:

$$\iint_{Q_T} \left(|u-k|\partial_t\phi + \operatorname{sign}(u-k)\,(f(u) - f(k) - \varepsilon\,[\hat{a}(u)\partial_x u])\,\partial_x\phi\right)\,dt\,dx + \int_{\mathbb{R}} |u_0 - k|\phi(x,0)\,dx \geq 0,$$

We remark that our definition differs slightly from the original one presented in [64]. The latter requires that the initial datum u_0 is assumed in the sense that

$$\lim_{t\to 0+} \int_K |u(x,t) - u_0(x)|\,dx = 0 \tag{7}$$

for every compactum $K \subset \mathbb{R}$. We have chosen to include this condition in the inequality (C2). By exploiting the arbitrariness of ϕ, it can easily be shown that the integral inequality (C2) implies (7).

Let us recall some basic consequences of (C1) and (C2), as pointed out in [64]. First, the condition (C1) implies that the measure $[\hat{a}(u)\partial_x u]$ is absolutely continuous, meaning that $[\hat{a}(u)\partial_x u]$ can be expressed by the integral of a function in L^1 (the Radon-Nikodym theorem). We shall always indentify $[\hat{a}(u)\partial_x]\, u$ with this L_1 function. Secondly, it is easy to see that the entropy condition (C2) implies that the equation

$$\partial_t u + \partial_x f(u) = \partial_x \left[\hat{a}(u)\partial_x u\right]$$

holds in the distributional sense. Consequently, the generalized derivative $\partial_x [\hat{a}(u)\partial_x u]$ is a locally finite measure, since $\partial_t u$ and $\partial_x f(u)$ are locally finite measures, and the differential equation (1) holds in the sense of equality of measures. Next, notice that for any $BV(Q_T)$ function u satisfying (C1) and (C2), one has

$$a(\xi) = 0, \qquad \forall \xi \in \mathrm{Int}(u_-, u_+), \tag{8}$$

whenever $u_- \neq u_+$. In other words, discontinuities are possible where there is degeneracy. From the same relation we see that a sufficient condition for there to be no discontinuities, except on a one-dimensional set of measure zero, is that $a(u) > 0$ on a dense set of values. On subsets $S \subset \Gamma_u$, the measure $\partial_x [\hat{a}(u)\partial_x u]$ is zero. Finally, the facts that $u(x,t) \in L^\infty(Q_T)$, $[\hat{r}(u)\partial_x u] \in L^2_{loc}(Q_T)$, and (8), imply that $[\hat{a}(u)\partial_x u] \in L^2_{loc}(Q_T)$ and hence $[\hat{a}(u)\partial_x u] \in L^1_{loc}(Q_T)$.

2. Viscous Splitting: The Initial Value Problem

We first construct the splitting approximations and subsequently establish the L^1 convergence of these approximations. Consistent with the discussion in the introduction and because of the increased generality, we introduce a smooth flux splitting of the form $f = \bar{f} + \tilde{f}$. A non-trivial realization of this flux splitting will be given in §3. Fix $T > 0$ and choose a time discretization $0 < t_1, \dots, t_N = T$ of $\langle 0, T]$. Define $\Delta t_n = t_{n+1} - t_n$ and $\Delta t = \max_n \Delta t_n$. Let now u^n denote the approximate solution to (1) at a fixed time $t = t_n$, $u^0 \equiv u_0$. Next, we explain how to construct u^{n+1} from u^n.

Let $v(x,t)$ denote the entropy weak solution to the hyperbolic conservation law

$$\partial_t v + \partial_x \bar{f}(v) = 0, \qquad v(x,0) = u^n(x), \qquad (x,t) \in \mathbb{R} \times \langle 0, \Delta t_n]. \tag{9}$$

Letting $\mathcal{S}^{\bar{f}}_t$ denote the solution operator associated with (9) at time t, we define $u^{n+1/2} = \mathcal{S}^{\bar{f}}_{\Delta t_n} u^n$.

From a numerical point of view, the possible roughness of the solution of the degenerate equation $\partial_t w + \partial_x \tilde{f}(w) = \varepsilon \partial_x [a(w)\partial_x w]$ requires an additional approximation or regularization accomplished by perturbing the diffusion coefficient $a(w)$ by μ. Consequently, we obtain a non-degenerate equation which possesses smooth solutions that are better suited to be approximated, by e.g. the Galerkin method, than the solutions of the non-perturbed (degenerate) equation.

Let therefore $a_\mu(u) = a(u) + \mu$ for $\mu > 0$, and let $w(x,t)$ denote the solution of the parabolic problem

$$\partial_t w + \partial_x \tilde{f}(w) = \varepsilon \partial_x \left[a_\mu(w)\partial_x w\right], \qquad w(x,0) = u^{n+1/2}(x), \qquad (x,t) \in \mathbb{R} \times \langle 0, \Delta t_n]. \tag{10}$$

Let $\mathcal{P}^{\tilde{f},\mu}_t$ denote the solution operator associated with (10) at time t. With the current notation in hand, we can define the approximation u^{n+1} by the product formula

$$u^{n+1} = \left[\mathcal{P}^{\tilde{f},\mu}_{\Delta t_n} \circ \mathcal{S}^{\bar{f}}_{\Delta t_n}\right] u^n, \qquad (n = 0, ..., N-1). \tag{11}$$

It will be more convenient to work with functions defined in the whole upper plane, and not merely on the time strips $t = t_n$. Consider therefore the function $u_\eta(x,t)$, where η denotes $(\Delta t, \mu)$ and

$$u_\eta(x,t) = \left[\mathcal{P}^{\tilde{f},\mu}_{\Delta t_n} \circ \mathcal{S}^{\bar{f}}_{\Delta t_n}\right]^n u_0, \qquad (x,t) \in \mathbb{R} \times \langle t_n, t_{n+1}], \qquad n = 0, \dots, N. \tag{12}$$

Remark. *Let $\mathcal{H}^\mu_t$ denote the operator associated with the nonlinear "heat" equation $\partial_t w = \varepsilon \partial_x [a_\mu(w)\partial_x w]$. Choosing $\bar{f} = f$ (and thus $\tilde{f} = 0$), we recover the standard viscous splitting method* [37]

$$u^{n+1} = \left[\mathcal{H}^\mu_{\Delta t_n} \circ \mathcal{S}^{\bar{f}}_{\Delta t_n}\right] u^n. \tag{13}$$

We assume from now on that u_0 is in $W^{1,\infty}(\mathbb{R}) \cap BV(\mathbb{R}) \cap L^1(\mathbb{R})$. For the subsequent convergence analysis it will be necessary to know that the hyperbolic solution (9) does not develop singularities during the time interval $\langle 0, \Delta t_n]$. To this end, we recall that the blow-up time of (9) can be deduced from the formula

$$\left\| \partial_x u^{n+1/2} \right\|_{L^\infty(\mathbb{R})} \leq \frac{\|\partial_x u^n\|_{L^\infty(\mathbb{R})}}{1 - c_1 \|\partial_x u^n\|_{L^\infty(\mathbb{R})} \Delta t_n}, \qquad c_1 = \max_u |f''(u)|. \tag{14}$$

The solution of the non-degenerate parabolic equation (10) is always smooth and we shall assume that (10) possesses a smoothing property, say, of the form $\|\partial_x u^n\|_{L^\infty(\mathbb{R})} \leq c_2 / \left(\mu \sqrt{\varepsilon \Delta t}\right)$ for some finite constant c_2 independent of the time level n, and η (see e.g. [30]). Now, by choosing the time step Δt_n such that

$$\Delta t_n < \varepsilon \left(\frac{\mu}{c_1 c_2} \right)^2, \qquad (n = 1, \ldots, N-1), \tag{15}$$

we are *á priori* guaranteed that $\left\| \partial_x u^{n+1/2} \right\|_{L^\infty(\mathbb{R})}$ is bounded independently of n and η. Thus $u^{n+1/2}$ is in $W^{1,\infty}(\mathbb{R}) \cap BV(\mathbb{R})$, and not merely $BV(\mathbb{R})$. The assumption that the parabolic solution operator should "smooth out" the data at a certain rate is convenient but not necessary. Instead we could have included a separate smoothing step (using a standard mollifier) before we apply the hyperbolic solution operator to u^n. All the results found below would then have gone trough for $\mu > 0$ as well as in the case where we allow (10) to be degenerate, i.e., $\mu \equiv 0$. We now justify the term "approximate solution" by stating the following theorem.

Theorem 2.1. *Let $\{\eta = (\Delta t, \mu)\}$ be a sequence of discretization parameters tending to zero. Suppose that Δt and μ are chosen such that (15) holds. Then the sequence $\{u_\eta(x,t)\}$ of splitting solutions converges to the unique entropy weak solution $u(x,t)$ of the initial value problem (1). The convergence takes place in $L^1_{loc}(Q_T)$.*

Note, due to the method of proof, that in Theorem 2.1 the existence of a solution to (1) is a conclusion rather than an hypothesis. Theorem 2.1 will follow immediately from Lemmas 2.2, 2.3, and 2.4 below. Here we do not minimize the smoothness assumptions on f and a, but simply suppose that they are appropriately smooth.

Lemma 2.2. *There exists a subsequence $\{\eta_j\}$ such that the corresponding splitting sequence $\{u_{\eta_j}(x,t)\}$ converges to a function $u(x,t)$ in $L^\infty(Q_T) \cap BV(Q_T)$. The convergence takes place in $L^1_{loc}(Q_T)$.*

Proof. First, we claim that there exists a finite constant M, independent of the discretization parameters $\eta = (\Delta t, \mu)$ and time t, such that the following three estimates hold:

$$\|u_\eta(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq M, \qquad |u_\eta(\cdot,t)|_{BV(\mathbb{R})} \leq M, \qquad \|u_\eta(\cdot,t_2) - u_\eta(\cdot,t_1)\|_{L^1(\mathbb{R})} \leq M|t_2 - t_1|, \tag{16}$$

i.e., $u_\eta(x,t)$ is in $L^\infty(Q_T) \cap BV(Q_T)$ independently of η. The first claim follows from the well known fact that the solution operators $\mathcal{S}_t^{\bar{f}}$ and $\mathcal{P}_t^{\tilde{f},\mu}$ do not introduce new minima and maxima. The second claim follows from the fact that these solution operators are L^1 contractions,

$$\left\| \mathcal{S}_{\Delta t_n}^{\bar{f}} v_{0,1} - \mathcal{S}_{\Delta t_n}^{\bar{f}} v_{0,2} \right\|_{L^1(\mathbb{R})} \leq \|v_{0,1} - v_{0,2}\|_{L^1(\mathbb{R})}, \qquad \left\| \mathcal{P}_{\Delta t_n}^{\tilde{f},\mu} w_{0,1} - \mathcal{P}_{\Delta t_n}^{\tilde{f},\mu} w_{0,2} \right\|_{L^1(\mathbb{R})} \leq \|w_{0,1} - w_{0,2}\|_{L^1(\mathbb{R})},$$

and, thanks to translation invariance of (1), they do not increase the total variation of their data.

Let us finally demonstrate the validity of the last claim. Without loss of generality, assume that $t_1 = k\Delta t$ and $t_2 = l\Delta t$ for some integer k and l, where $k < l$. We will first establish weak Lipschitz continuity in time of the parabolic solution. To this end, integrating the parabolic differential equation (10) against a test function $\phi(x)$ over $\mathbb{R} \times \langle 0, \Delta t_n \rangle$ gives the result

$$\begin{aligned}
\left| \int_{\mathbb{R}} \left(\mathcal{P}_{\Delta t_n}^{\tilde{f},\mu} u^{n+1/2} - u^{n+1/2} \right) \phi(x)\, dx \right| &= \left| \int_0^{\Delta t_n} \int_{\mathbb{R}} \left(\varepsilon \partial_x \left[a_\mu(w) \partial_x w \right] - \partial_x \tilde{f}(w) \right) \phi(x)\, dx\, dt \right| \\
&= \left| \int_0^{\Delta t_n} \int_{\mathbb{R}} \left(\varepsilon \left[a_\mu(w) \partial_x w \right] \partial_x \phi(x) + \partial_x \tilde{f}(w) \phi(x) \right) dx\, dt \right| \\
&\leq \left(\|\varepsilon a_\mu(w) \partial_x w\|_{L^\infty(\mathbb{R} \times \langle 0, \Delta t_n \rangle)} |\phi|_{BV(\mathbb{R})} + \max_u \left| \tilde{f}' \right| \|\phi\|_\infty |u_0|_{BV(\mathbb{R})} \right) \Delta t,
\end{aligned}$$

where we have also taken into account the second part of (16). From [60] (see also [42,43]) we get that $\|\varepsilon a_\mu(w)\partial_x w\|_{L^\infty(\mathbb{R}\times\langle 0,\Delta t_n\rangle)} = \mathcal{O}(1)$ since $u^{n+1/2}$ is in $W^{1,\infty}(\mathbb{R})\cap BV(\mathbb{R})$ due to (15). We thus conclude that

$$\left(\|\varepsilon a_\mu(w)\partial_x w\|_{L^\infty(\mathbb{R}\times\langle 0,\Delta t_n\rangle)}|\phi|_{BV(\mathbb{R})} + \max_u\left|\tilde{f}'\right|\|\phi\|_\infty |u_0|_{BV(\mathbb{R})}\right)\Delta t = \mathcal{O}(1)\left(\|\phi\|_\infty + |\phi|_{BV(\mathbb{R})}\right)\Delta t.$$

Because of finite speed of propogation of the hyperbolic solution and and since the total variation of the splitting solution u_η is bounded, we have strong L^1 continuity in time; $\left\|\mathcal{S}^{\bar{f}}_{\Delta t_n}u^n - u^n\right\|_{L^1(\mathbb{R})} = \mathcal{O}(1)\Delta t$.

Using these continuity estimates, we obtain

$$\left|\int_{\mathbb{R}}\left(u^{n+1}-u^n\right)\phi(x)\,dx\right| = \mathcal{O}(1)\left(\|\phi\|_\infty + |\phi|_{BV(\mathbb{R})}\right)\Delta t.$$

Consequently, by summing over all n between k and l and using the triangle inequality repeatedly, we derive the weak continuity result

$$\left|\int_{\mathbb{R}}\left(u_\eta(x,t_2)-u_\eta(x,t_1)\right)\phi(x)\,dx\right| = \mathcal{O}(1)\left(\|\phi\|_\infty + |\phi|_{BV(\mathbb{R})}\right)(t_2-t_1). \tag{17}$$

This estimate can now be interpolated into the desired strong L^1 Lipshitz continuity in time by choosing $\phi(x) = \operatorname{sign}(u_\eta(x,t_2)-u_\eta(x,t_1))$ and observing that this ϕ has finite total variation.

Now using (16) and the compactness of the embedding $L^1(O_T)\cap BV(O_T)\to L^1(O_T)$ (O_T open, bounded subset of Q_T) in a standard diagonal fashion, we achieve compactness of the sequence $\{u_\eta(x,t)\}$ in the space $L^1_{loc}(Q_T)$. Thus there exists a subsequence $\{\eta_j\}$ such that $\{u_{\eta_j}(x,t)\}$ converges in $L^1_{loc}(Q_T)$ to a function $u(x,t)$ in $L^\infty(Q_T)\cap BV(Q_T)$. There is no loss of generality in assuming that the entire sequence converges u. □

Remark. *Let us briefly comment on the time step restriction (15). When $u^{n+1/2}$ is (only) in $BV(\mathbb{R})$, the arguments leading to (17) can be slightly modified along the lines of* [37] *to yield a L^1 Hölder estimate in time with exponent $1/2$. The limit $u(\cdot,t)$ will therefore be in $BV(\mathbb{R})$ for each fixed t, but $u(x,t)$ is not necessarily in $BV(Q_T)$ and thus need not be a solution according to Definition 1.5. From a rigorous point of view, we hence claim that (15) is necessary in order to get convergence to the entropy weak solution. From a computational point of view, however, (15) is merely a technical assumption associated with the convergence analysis in the sense that (15) is not a stability condition, that is, convergence is obtained even without imposing this condition.*

Lemma 2.3. *Let $u(x,t)$ be the limit function from Lemma 2.2. Then $u(x,t)$ obeys (C1).*

Proof. Introduce, with a slight abuse of the notation, the two sequences $\{u_\eta(x,t)\}$ and $\{g_\eta(x,t)\}$,

$$u_\eta(x,t) = \left[\mathcal{P}^{\tilde{f},\mu}_{t-t_n}\circ\mathcal{S}^{\bar{f}}_{\Delta t_n}\right]u^n, \qquad g_\eta(x,t) = r(u_\eta)\,\partial_x u_\eta, \qquad (x,t)\in\mathbb{R}\times\langle t_n,t_{n+1}].$$

We note that also $u_\eta\to u$ in $L^1_{loc}(Q_T)$. Multiplying the differential equation for $u_\eta(x,t)$ on $\mathbb{R}\times\langle t_n,t_{n+1}\rangle$ by $u_\eta(x,t)$, integrating over Q_T, and then doing integration by parts in space, we get

$$\begin{aligned}
\|g_\eta\|^2_{L^2(Q_T)} &\le \iint_{Q_T}\varepsilon a_\mu(u_\eta)\left(\partial_x u_\eta\right)^2\,dt\,dx = -\iint_{Q_T}\varepsilon\partial_x\left[a_\mu(u_\eta)\,\partial_x u_\eta\right]u_\eta\,dt\,dx \\
&= -\sum_{n=0}^{N-1}\int_{\mathbb{R}}\int_{t_n}^{t_{n+1}}\left(\frac{1}{2}\partial_t\left(u_\eta\right)^2 + \partial_x\tilde{f}(u_\eta)u_\eta\right)dt\,dx \\
&= -\frac{1}{2}\sum_{n=0}^{N-1}\int_{\mathbb{R}}\left(\left(u_\eta\big|_{t=t_{n+1}}\right)^2 - \left(u_\eta\big|_{t=t_n}\right)^2\right)dx - \iint_{Q_T}\partial_x\tilde{f}(u_\eta)u_\eta\,dt\,dx \\
&= -\frac{1}{2}\sum_{n=0}^{N-1}\int_{\mathbb{R}}\left(\left[\left(u^{n+1}\right)^2-\left(u^n\right)^2\right] + \left[\left(u^n\right)^2 - \left(u^{n+1/2}\right)^2\right]\right)dx - \iint_{Q_T}\partial_x\tilde{f}(u_\eta)u_\eta\,dt\,dx \\
&= -\frac{1}{2}\int_{\mathbb{R}}\left[\left(u^N\right)^2-\left(u^0\right)^2\right]dx + \frac{1}{2}\sum_{n=0}^{N-1}\int_{\mathbb{R}}\left[\left(u^{n+1/2}\right)^2-\left(u^n\right)^2\right]dx - \iint_{Q_T}\partial_x\tilde{f}(u_\eta)u_\eta\,dt\,dx,
\end{aligned}$$

where we have, without loss of generality, assumed that $\left[a_{\mu}\left(u_{\eta}\right)\partial_{x}u_{\eta}\right]u_{\eta}=0$ as $|x|\to\infty$.

Since the solution operators $\mathcal{S}_t^{\bar{f}}$ and $\mathcal{P}_t^{\tilde{f},\mu}$ both are L^1 stable, we know that $\|u_\eta(\cdot,t)\|_{L^1(\mathbb{R})}\le\|u_\eta(\cdot,0)\|_{L^1(\mathbb{R})}$. Thus, in view of the facts that the splitting solution u_η is uniformly bounded and the data is in $L^1(\mathbb{R})$, the first term is clearly bounded independent of η,

$$\left|\frac{1}{2}\int_{\mathbb{R}}\left[\left(u^N\right)^2-\left(u^0\right)^2\right]\,dx\right|\le\left\|u^0\right\|_{L^\infty(\mathbb{R})}\left\|u^0\right\|_{L^1(\mathbb{R})}=\mathcal{O}(1). \tag{18}$$

Exploiting the L^1 Lipschitz continuity of the hyperbolic solution operator $\mathcal{S}_t^{\bar{f}}$ and that the splitting solution u_η is uniformly bounded, we obtain for the second term that

$$\left|\frac{1}{2}\sum_{n=0}^{N-1}\int_{\mathbb{R}}\left[\left(u^{n+1/2}\right)^2-\left(u^n\right)^2\right]\,dx\right|\le\|u_0\|_\infty\sum_{n=0}^{N-1}\left\|\mathcal{S}_{\Delta t_n}^{\bar{f}}u^n-u^n\right\|_{L^1(\mathbb{R})}=\mathcal{O}(1)T. \tag{19}$$

Since the variation of $u_\eta(\cdot,t)$ is uniformly bounded, we can also bound the third term independently of η,

$$\left|\iint_{Q_T}\partial_x\tilde{f}\left(u_\eta\right)u_\eta\,dt\,dx\right|=\mathcal{O}(1)\int_0^T\int_{\mathbb{R}}\left|\partial_x u_\eta\right|\,dx\,dt=\mathcal{O}(1)T. \tag{20}$$

From (18), (19), and (20) we can conclude that the following L^2 bound is valid: $\|g_\eta\|_{L^2(Q_T)}\le M(T)$, where $M(T)$ is a finite constant independent of η, but dependent on the data of the problem (1).

By virtue of the uniform L^2 estimate we conclude that $\{g_\eta\}$ is weakly compact in $L^2(Q_T)$, i.e., there exists a subsequence $\{g_{\eta_j}\}$ and a function $g\in L^2(Q_T)$ such that $g_{\eta_j}\rightharpoonup g$ weakly in $L^2(Q_T)$. Again, there is no loss of generality in assuming that the entire sequence $\{g_\eta\}$ converges weakly to g. Let G be defined such that $\partial G(u)/\partial u=\sqrt{\varepsilon a(u)}$ and let ϕ be a test function. We can then calculate as follows

$$\begin{aligned}\iint_{Q_T}g(x,t)\phi(x,t)\,dt\,dx&=\lim_{\eta\to0}\iint_{Q_T}\partial_x G(u_\eta)\phi\,dt\,dx=\lim_{\eta\to0}\iint_{Q_T}\left(-G(u_\eta)\partial_x\phi\right)\,dt\,dx=\iint_{Q_T}\left(-G(u)\partial_x\phi\right)\,dt\,dx\\&=\iint_{Q_T}\partial_x G(u)\phi\,dt\,dx=\iint_{Q_T}\frac{\partial\hat{G}(u)}{\partial u}\partial_x u\phi\,dt\,dx=\iint_{Q_T}\left[\hat{r}(u)\partial_x u\right]\phi\,dt\,dx.\end{aligned}$$

Here we have taken into account the differentiation rule of an averaged superposition (5). Consequently, we have shown that $[\hat{r}(u)\partial_x u]$ exists in the sense of distributions in $L^2_{loc}(Q_T)$. Thus (C1) holds. □

Lemma 2.4. *Let $u(x,t)$ be the limit function from Lemma 2.2. Then $u(x,t)$ obeys (C2).*

Proof. The proof will follow along the lines of [16,37]. Let us introduce, again with a slight abuse of the notation, the sequence $\{u_\eta(x,t)\}$,

$$u_\eta(x,t)=\begin{cases}\mathcal{S}_{2(t-t_n)}^{\bar{f}}u^n, & t\in\left\langle t_n,t_{n+\frac{1}{2}}\right],\\ \left[\mathcal{P}_{2(t-t_{n+\frac{1}{2}})}^{\tilde{f},\mu}\circ\mathcal{S}_{\Delta t_n}^{\bar{f}}\right]u^n, & t\in\left\langle t_{n+\frac{1}{2}},t_{n+1}\right].\end{cases} \tag{21}$$

Also now it should be clear that $u_\eta\to u$ in $L^1_{loc}(Q_T)$. Whenever notational convenient, we will adopt the notation $U_k(u)=|u-k|$, and $F_k(u;h)=\operatorname{sign}(u-k)\left(h(u)-h(k)\right)$ for any smooth function $h(u)$.

Let $w_n(t)=\mathcal{P}_t^{\tilde{f},\mu}u^{n+1/2}$ for $t\in\langle0,\Delta t_n]$. Recall that $w_n(t)$ is the smooth solution of the non-degenerate problem (10). Formally, integrating the obvious equation

$$\partial_t(w-k)+\partial_x\left(\tilde{f}(w)-\tilde{f}(k)\right)=\varepsilon\partial_x\left[a_\mu(w)\partial_x w\right]$$

against $\mathrm{sign}(w-k)\phi$ over $\mathbb{R}\times\langle 0,\Delta t_n\rangle$, doing integration by parts and exploiting the fact that $a(u)\geq 0$, we obtain that the following entropy inequality holds

$$\begin{aligned}\iint\limits_{\mathbb{R}\times\langle 0,\Delta t_n\rangle} & U_k(w_n)\,\partial_t\phi + F_k\left(w_n;\tilde{f}\right)\partial_x\phi + \varepsilon F_k(w_n;A_\mu)\partial_x^2\phi\,dt\,dx \\ &\geq \int_{\mathbb{R}} |w_n(\Delta t_n)-k|\,\phi(x,\Delta t_n)\,dx - \int_{\mathbb{R}}\left|u^{n+1/2}-k\right|\phi(x,0)\,dx, \qquad A_\mu(u)=\int_0^u a_\mu(\xi)\,d\xi,\end{aligned} \tag{22}$$

where $A_\mu(u)\to A(u)$ uniformly in u as $\mu\to 0$ (here $A(u)$ is given in (2)). We mention that the entropy inequality is rigorously derived in a more general context in the appendix. Let φ be test function defined by $\varphi(x,t)=\phi(x,\frac{t}{2})$. Having (22), we can now deduce the following "local in time" entropy inequality for $u_\eta(x,t)$

$$\begin{aligned}&\int_{\mathbb{R}}\int_{t_{n+\frac12}}^{t_{n+1}}\left(\frac12 U_k(u_\eta)\,\partial_t\phi + F_k\left(u_\eta;\tilde{f}\right)\partial_x\phi + \varepsilon F_k(u_\eta;A_\mu)\,\partial_x^2\phi\right)dt\,dx \\ &\equiv \frac12\int_{\mathbb{R}}\int_0^{\Delta t_n}\left(U_k(w_n)\,\partial_\tau\varphi + F_k\left(w_n;\tilde{f}\right)\partial_x\varphi + \varepsilon F_k(w_n;A_\mu)\,\partial_x^2\varphi\right)d\tau\,dx \\ &\qquad \geq \frac12\int_{\mathbb{R}}\left|u^{n+1}-k\right|\phi(x,t_{n+1})\,dx - \frac12\int_{\mathbb{R}}\left|u^{n+1/2}-k\right|\phi\left(x,t_{n+\frac12}\right)dx,\end{aligned} \tag{23}$$

where we have used the substitution $\tau=2(t-t_{n+1/2})$. Adding this inequality and the similar inequality that is valid for $u_\eta(x,t)$ on $\mathbb{R}\times\left\langle t_n,t_{n+1/2}\right\rangle$, and summing over all $n=0,\dots,N-1$, yields the global inequality

$$\begin{aligned}\iint\limits_{Q_T}&\left(\frac12 U_k(u_\eta)\,\partial_t\phi + \left(S_N(t)F_k(u_\eta;\bar{f}) + T_N(t)F_k\left(u_\eta;\tilde{f}\right)\right)\partial_x\phi + \varepsilon T_N(t)F_k(u_\eta;A_\mu)\,\partial_x^2\phi\right)dt\,dx \\ &\quad + \frac12\int_{\mathbb{R}}|u_0-k|\,\phi(x,0)\,dx \geq 0,\end{aligned} \tag{24}$$

where we recall that $\phi|_{t=T}=0$. Here the characteristic functions $S_N(t)$ and $T_N(t)$ are defined as

$$S_N(t)=\sum_{n=0}^{N-1}\chi_{\left\langle t_n,t_{n+\frac12}\right]}(t), \qquad T_N(t)=\sum_{n=0}^{N-1}\chi_{\left\langle t_{n+\frac12},t_{n+1}\right]}(t). \tag{25}$$

Observe that the functions $S_N(t)$ and $T_N(t)$ both tend weakly in L^2 to $1/2$. In view of the dominated convergence theorem we can therefore pass to the limit as $\eta\to 0$ in (24), obtaining

$$\begin{aligned}\iint\limits_{Q_T}&\left(\frac12 U_k(u_\eta)\,\partial_t\phi + \left(S_N(t)F_k(u_\eta;\bar{f}) + T_N(t)F_k\left(u_\eta;\tilde{f}\right)\right)\partial_x\phi + \varepsilon T_N(t)F_k(u_\eta;A_\mu)\,\partial_x^2\phi\right)dt\,dx \\ &\xrightarrow{\eta\to0}\frac12\iint\limits_{Q_T}\left(|u-k|\partial_t\phi + \mathrm{sign}(u-k)(f(u)-f(k))\partial_x\phi + \varepsilon\,\mathrm{sign}(u-k)(A(u)-A(k))\partial_x^2\phi\right)dt\,dx, \\ &=\frac12\iint\limits_{Q_T}\left(|u-k|\partial_t\phi + \mathrm{sign}(u-k)\left(f(u)-f(k)-\varepsilon\partial_x\left[\mathrm{sign}(u-k)(A(u)-A(k))\right]\right)\partial_x\phi\right)dt\,dx.\end{aligned}$$

The limit $u(x,t)$ is in $BV(Q_T)$ and $[\hat{a}(u)\partial_x u]$ is an absolutely continuous measure. Thus, $\mathrm{sign}(u-k)(A(u)-A(k))$ is in $BV(Q_T)$ and we can apply Lemmas 1.3 and 1.4 to justify that the following equality

$$\partial_x\left[\mathrm{sign}(u-k)(A(u)-A(k))\right] = \mathrm{sign}(u-k)\partial_x\left[A(u)-A(k)\right] = \mathrm{sign}(u-k)\left[\hat{a}(u)\partial_x u\right]$$

holds in the sense of measures. This concludes the proof of the lemma and Theorem 2.1. $\square$

3. An Application to Two Phase Flow in Porous Media

In this section we consider an application of the theory developed in the previous section to a partial differential equation arising in the modelling of two phase flow in a porous medium. Consider a fluid in a one-dimensional homogenous porous medium consisting of two immiscible phases; a wetting phase, say, water and a non-wetting phase, say, oil. Let $u(x,t)$ denote the water saturation and thus $1-u$ the oil saturation. Then the equation modelling the immiscible displacement of oil by water takes the form (1); see [11] for a complete derivation of the mathematical model. In the present context, the function $f(u)$ is called the fractional flow function and it is determined by the relative permabilities and the viscosities of water and oil and by the gravitation. The fractional flow function has the characteristic features that it is non-convex and not necessarily monotone; see Figure 4.1 (left) for an example. The function $a(u)$ is called the capillary diffusion function. The capillary diffusion function is determined by the relative permabilities and the viscosities of water and oil, in addition to the capillary pressure function. The characteristic feature is the degenerate behaviour at $u=0$ and $u=1$; see Figure 4.1 (left) for an example.

For computational purposes, we recreate some of the features mentioned above by employing the following analytic expressions

$$f(u) = \frac{u^3}{u^3+(1-u)^3}\left[1-10(1-u)^3\right], \qquad a(u) = 4u(1-u), \tag{26}$$

consult Figure 4.1 (left). Furthermore, as initial saturation we use Riemann data; $u_0(x) = \chi_{[0.65,\infty)}(x)$.

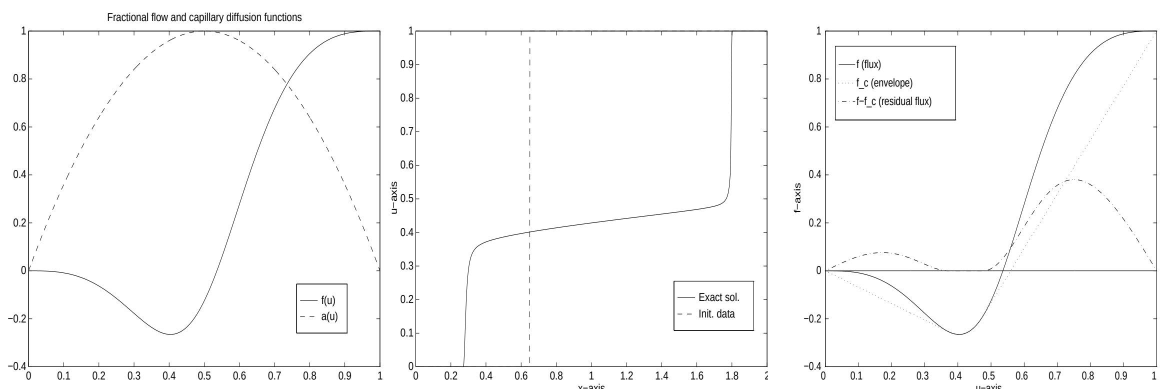

Figure 4.1. Left: The fractional flow function (including gravitation forces) and the capillary diffusion function. Middle: The initial function and the exact solution at time 0.5 with scaling parameter 0.01. Note the loss of regularity near the saturation values 0 and 1 (the solution is merely continuous). Right: The flux function (solid) and the flux splitting (dotted and dashdot) employed by F-OS. Observe that the residual flux only contains information about the structure of the fronts.

We will solve the problem (1) using the operator splitting formulas (13) and (11) with $\Delta t_n = \Delta t$ for all n, that is, we will use a fixed time step Δt. We abbreviate the scheme (13) by OS, and (11), which also take into account a flux splitting, by F-OS. In order to apply the splitting techniques we need a numerical method for the conservation law (9). We have used a front tracking method which is based on solving Riemann problems and "tracking" shock collisions. A description of this method is outside the scope of the present paper, we refer the reader instead to [17,36,49]. We also need to solve the parabolic equation (10). In the case where the convection term $\tilde{f}$ is zero (OS), we have used a standard Galerkin method. When $\tilde{f}$ is present (F-OS) we have employed a Petrov-Galerkin scheme. We refer to [39] for a description of these finite element schemes.

All solutions are computed up to time $T = 0.5$ and the scaling parameter ε is set to 0.01. In Figure 4.1 (middle) we have shown the exact solution. Notice the loss of regularity at $u=0$ and $u=1$. Let us first discuss the application of OS. In Figure 4.2 (left) we have shown a calculation on the interval $[0,2]$ using 200 mesh points, i.e., $\Delta x = 0.01$, and $\Delta t = 0.5$. In Figure 4.2 (right) we have used the same Δx, but the time step is reduced to $\Delta t = 0.025$. As is well known [37,38,39], we observe that OS produces too diffusive shock fronts when Δt is much larger than ε. On the other hand, OS clearly resolves the fronts properly when $\Delta t = \mathcal{O}(\varepsilon)$.

We now consider a particular realization of F-OS. To this end, we recall that the conservation law (9) with Riemann data $v(x,0) = \chi_{[0.65,\infty)}(x)$ admits a traveling wave solution $v(x,t) = v(x/t)$ of the form (see e.g. [58])

$$v\left(\frac{x}{t}\right) = \begin{cases} 0, & \text{for } \frac{x-0.65}{t} < f_c'(0), \\ (f_c')^{-1}\left(\frac{x}{t}\right), & \text{for } \frac{x-0.65}{t} \in [f_c'(0), f_c'(1)], \\ 1, & \text{for } \frac{x-0.65}{t} > f_c'(1), \end{cases} \tag{27}$$

where f_c denotes the lower convex envelope of f restricted to the interval $[0,1]$ and $(f_c')^{-1}$ the inverse of its derivative. Having the piecewise smooth solution (27) in mind, we define the flux splitting by letting $\bar{f} \equiv f_c$ and thus $\tilde{f} = f - f_c$; consult Figure 4.1 (right). In Figure 4.2 (right) we have shown a F-OS calculation using $\Delta x = 0.01$ and $\Delta t = 0.5$. Compared with Figure 4.2 (left), we see that F-OS resolves the shock fronts remarkably better than OS. It is fair to say that F-OS is, at least in the "visual" norm, free of temporal splitting error, which contrasts strongly with OS. In order to explain this behaviour we need to slightly change our point of view. Observe therefore that the two solutions of the conservation law (9) with the Riemann data $v(x,0) = \chi_{[0.65,\infty)}(x)$ and flux functions f and f_c, respectively, coincide. In other words, OS does not take into account the part of the flux function f that determines the self-sharpening nature of a front, i.e, the part that gives each shock front its characteristic structure. It is a matter of fact that OS uses the flux f_c instead of f and this hence explains why OS is too diffusive.

Fortunately, the part that is thrown away can be identified as a residual flux term of the form $f_{\text{res}} = f - f_c$. Observe that f_{res} possesses a self-sharpening (anti-diffusive) whose purpose is to ensure the correct balance between convection and diffusion. This discussion motivates the following alternative formulation of (11) (and generalization of (13)), namely the corrected operator splitting formula

$$u^{n+1} = \left[\mathcal{P}_{\Delta t}^{\mu, f_{\text{res}}} \circ \mathcal{S}_{\Delta t}^{f}\right] u^n. \tag{28}$$

Obviously the corrected splitting formula (28) coincides with F-OS when $\Delta t = T$. Note that by taking this point of view, we have explained why F-OS is virtually without any splitting error. From the above discussion it should be clear that with a different initial saturation, F-OS may not produce accurate results. The reason is that in such case the flux splitting is no longer consistent with the solution, i.e., F-OS can no longer be interpreted as a splitting of the form (28). Let us conclude this section by mentioning, however, that the corrected splitting idea can be generalized so that (28) can handle any initial saturation (and flux function). As pointed out in [38], the resiudal flux term f_{res} can be constructed easily in a dynamical fashion if, for example, front tracking is used to realize the hyperbolic step (9). We refer to [38] for detailed description of this construction and [39,40,41] for applications of the corrected splitting approach.

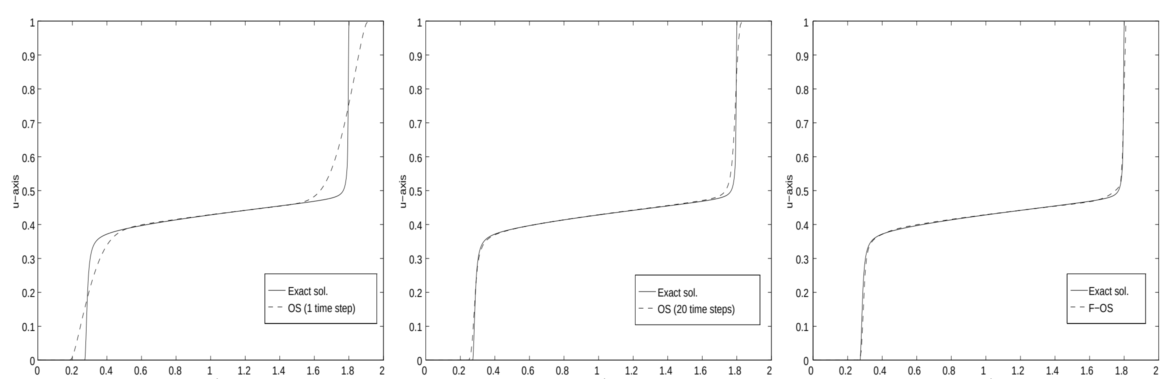

Figure 4.2. Left: Exact solution versus OS using 1 time step and 200 mesh points. Middle: Exact solution versus OS using 20 time steps and 200 mesh points (CPU time 1.43 sec). Right: The exact solution versus F-OS using 1 time step and 200 mesh points (CPU time 0.40 sec). We see that F-OS' temporal splitting error is negligible compared with OS.

4. Viscous Splitting: The Initial-Boundary Value Problem

The convection-diffusion problem (1) can also be restricted to some bounded interval I and then constrained by Dirichlet boundary conditions on ∂I. Let Q_T denote $I \times \langle 0, T]$, where we for simplicity let $I = \langle 0, 1\rangle$. From now we will consider the initial-boundary value problem

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x \left[a(u)\partial_x u\right], \qquad u(x,0) = u_0(x), \qquad u(0,t) = b_0, \qquad u(1,t) = b_1, \qquad (x,t) \in Q_T, \tag{29}$$

where b_0 and b_1 are finite constants. The notion of an entropy weak solution remains the same except that the boundary conditions have to be given a proper interpretation. As is well known in the context of conservation laws [4], the boundary conditions have to be formulated, in general, in a weak sense. It turns out that the vanishing viscosity method provides us with the correct (weak) formulation. For all non-negative functions $\phi \in C_0^\infty(Q_T)$ for which $\phi|_{t=T} = 0$, $\phi|_{x=i} \neq 0$ and all $k \in \mathbb{R}$, the following entropy inequality should now hold:

$$\begin{aligned} &\iint_{Q_T} \left(|u-k|\partial_t\phi + \operatorname{sign}(u-k)\left(f(u) - f(k) - [\hat{a}(u)\partial_x u]\right)\partial_x\phi\right) dt\,dx + \int_I |u_0 - k|\phi(x,0)\,dx \\ &\qquad \geq \sum_{i=0}^{1}\int_0^T \operatorname{sign}(b_i - k)\left(f(\gamma_i u) - f(k) - \varepsilon\gamma_i\left[\hat{a}(u)\partial_x u\right]\right)\phi\Big|_{x=i} n_i\,dt \\ &\qquad\quad + \sum_{i=0}^{1}\int_0^T \varepsilon\left(\operatorname{sign}(b_i - k) - \operatorname{sign}(\gamma_i u - k)\right)\left(A(\gamma_i u) - A(k)\right)\partial_x\phi\Big|_{x=i} n_i\,dt, \end{aligned} \tag{C2}$$

where $\gamma_i u = (\gamma u)(i,t)$ denotes the trace of $u(x,t)$ at $x = i$ and $n_i = (-1)^{i+1}$, $i = 0, 1$. Recall that the trace operator γ is well defined for any BV function. Furthermore, note that the existence of $\gamma_i\left[\hat{a}(u)\partial_x u\right]$ must be required in the definition. It suffices however to require that $\left[\hat{a}(u)\partial_x u\right](\cdot,t)$ is in $BV(\mathbb{R})$ for a.e. $t \in \langle 0, T]$.

Remark. *By exploiting the arbitrariness of ϕ, it is not difficult to deduce from the entropy inequality that $a(\xi) = 0$ $\forall\xi \in$ Int$(b_i, \gamma_i u)$, whenever $\gamma_i u \neq b_i$, and that the weak form of the boundary condition*

$$\left(\operatorname{sign}(b_i - k) - \operatorname{sign}(\gamma_i u - k)\right)\left(f(\gamma_i u) - f(k) - \varepsilon\gamma_i\left[\hat{a}(u)\partial_x u\right]\right)n_i \geq 0, \qquad i = 0, 1, \tag{30}$$

holds a.e. in $\langle 0, T]$. Letting $a \equiv 0$, we recover the correct formulation of the boundary condition for hyperbolic conservation laws [4]. Note that when f is linear and $a \equiv 0$, (30) implies that u is equal to the prescribed boundary data on the inflow boundary, whereas on the outflow boundary no boundary data are imposed.

The vanishing viscosity method was justified for the homogeneous boundary value problem by Wu & Wang [67]. The definition of an entropy weak solution used here, which takes into account possibly non-zero boundary data, represents a slight modification of the one introduced in [67]. We refer the reader to [66] for the treatment of multi-dimensional initial-boundary value problems.

As before we will investigate convergence of a sequence of splitting solutions produced by the formula

$$u_\eta(x,t) = \left[\mathcal{P}^{\tilde{f},\mu}_{\Delta t_n} \circ \mathcal{S}^{\bar{f}}_{\Delta t_n}\right]^n u_0, \qquad (x,t) \in I \times \langle t_n, t_{n+1}], \qquad n = 0, \ldots, N, \tag{31}$$

where $\mathcal{S}^{\bar{f}}_t$ and $\mathcal{P}^{\tilde{f},\mu}_t$ are the exact solution operators associated with (9) and (10), respectively, restricted to the interval I and with Dirichlet boundary data imposed on ∂I. We assume that $\Delta t = \max_n \Delta t_n$ and μ are chosen according to (15). Again we have the following three *á priori* bounds on the splitting solution (31)

$$\|u_\eta(\cdot,t)\|_{L^\infty(I)} \leq M, \qquad |u_\eta(\cdot,t)|_{BV(I)} \leq M, \qquad \|u_\eta(\cdot,t_2) - u_\eta(\cdot,t_1)\|_{L^1(I)} \leq M|t_2 - t_1|,$$

where M is some finite contant independent of the discretization parameters. Thus we obtain the strong compactness of $\{u_\eta\}$ in $L^1(Q_T)$. Assume that the entire sequence $\{u_\eta(x,t)\}$ converges to an element named $u(x,t)$ in $L^\infty(Q_T) \cap BV(Q_T)$. It remains to show that $u(x,t)$ is an entropy solution. Closely following the proof of Lemma 2.3, we can conclude that $[\hat{r}(u)\partial_x u]$ exists in the sense of distributions in $L^2(Q_T)$. Hence the limit $u(x,t)$ satisfies (C1). Furthermore, since the limit function $u(x,t)$ satisfies the differential equation in the sense of measures (see §2), we get that $[\hat{a}(u)\partial_x u](\cdot,t)$ is in $BV(\mathbb{R})$ for a.e. $t \in \langle 0, T]$.

Although the method of proof is essentially the same as for the initial value problem, the proof of (C2) in the present context is notably more technical because the crux is to represent the boundary terms as integrals over the whole domain Q_T, which in turn requires additional limiting arguments. To this end, we introduce two functions $\lambda_h^0(s)$ and $\lambda_h^1(s)$ representing smooth approximations to the characteristic functions $\chi_{\langle -\infty,0]}(s)$ and $\chi_{[1,\infty\rangle}(s)$, respectively. More concretely, these approximations are obtained as follows. First we define

$$\rho_h(s) = \int_{-\infty}^{s} \omega_h(\tau)\, d\tau, \qquad h > 0,$$

where $\omega_h(\tau)$ is a standard mollifier given by $\omega_h(\tau) = \frac{1}{h}\omega(\frac{\tau}{h})$, where $\omega(\tau) \in C_0^\infty(\mathbb{R}), \omega(\tau) \geq 0, \omega(\tau) = 0$ for $|\tau| \geq 1$ and $\int_{\mathbb{R}} \omega(\tau) d\tau = 1$. Then we define the two approximations by $\lambda_h^0(s) = 1 - \rho_h(s-2h)$ and $\lambda_h^1(s) = \rho_h(s-(1-2h))$. In particular we notice that the functions λ_h^i satisfy the following important properties:

$$\lambda_h^i(j) = \delta_{ij}, \qquad \partial_x \lambda_h^i(j) = 0, \qquad |\lambda_h^i| \leq 1, \qquad \lambda_h^i \to 0 \text{ a.e. in } I, \qquad i,j = 0,1, \tag{32}$$

where $\delta_{ij} = 1$ for $i = j$ and 0 for $i \neq j$. The following lemma will be exploited repeatedly.

Lemma 4.1 [67]. *Suppose that $u(x,t)$ is in $L^1(Q_T)$ and $u(\cdot,t)$ is in $BV(\mathbb{R})$ for a.e. $t \in \langle 0,T]$. Let $\Phi(t)$ be an element of $L^1(\langle 0,T])$. Then the following results hold:*

(a) *If $|u(x,t)| \leq \Phi(t)$ a.e. in Q_T and $\gamma_i v$ exists a.e. in $\langle 0,T]$, then for all suitable functions $\phi \in C_0^\infty(Q_T)$,*

$$\lim_{h \to 0} \iint\limits_{Q_T} \frac{\partial}{\partial x}\left(\phi \lambda_h^i\right) u\, dt\, dx = \int_0^T n_i \phi\big|_{x=i} \gamma_i u\, dt, \qquad i = 0,1.$$

(b) *If $\frac{\partial u}{\partial x}$ is an absolutely continous measure, then for all suitable functions $\phi \in C_0^\infty(Q_T)$,*

$$\iint\limits_{Q_T} \frac{\partial \phi}{\partial x} u\, dt\, dx = \sum_{i=0}^{1} \int_0^T n_i \phi\big|_{x=i} \gamma_i u\, dt - \iint\limits_{Q_T} \phi \frac{\partial u}{\partial x}\, dt\, dx, \qquad i = 0,1.$$

(c) *We have, for any smooth function $h(u)$, that $\gamma_i h(u) = h(\gamma_i u)$. Moreover, for almost all $k \in \mathbb{R}$, $\gamma_i \operatorname{sign}(u-k)\,(h(u) - h(k))$ exists a.e. in $\langle 0,T]$ and $\gamma_i \operatorname{sign}(u-k)\,(h(u)-h(k)) = \operatorname{sign}(\gamma_i u - k)\,(h(\gamma_i u) - h(k))$.*

Similarly to (23), we can formally deduce the following local entropy inequality (see the appendix for a rigorous treatment)

$$\begin{aligned}
&\int_I \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \left(\frac{1}{2} U_k(u_\eta)\, \partial_t \phi + F_k\left(u_\eta; \tilde{f}\right) \partial_x \phi + \varepsilon F_k(u_\eta; A_\mu)\, \partial_x^2 \phi \right) dt\, dx \\
&\quad \geq \frac{1}{2} \int_I \left|u^{n+1} - k\right| \phi(x, t_{n+1})\, dx - \frac{1}{2} \int_I \left|u^{n+1/2} - k\right| \phi\left(x, t_{n+\frac{1}{2}}\right) dx \\
&\qquad + \sum_{i=0}^{1} \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \operatorname{sign}(b_i - k) \left(\left(\tilde{f}(u_\eta) - \tilde{f}(k) - \varepsilon a_\mu(u_\eta) \partial_x u_\eta \right) \phi + \varepsilon \left(A_\mu(u_\eta) - A_\mu(k)\right) \partial_x \phi \right) \Big|_{x=i} n_i\, dt.
\end{aligned}$$

Adding this and the similar inequality valid for $u_\eta(x,t)$ on $I \times \langle t_n, t_{n+1/2}]$, and summing over all $n = 0, \ldots, N-1$, yields the global entropy inequality

$$\begin{aligned}
&\iint\limits_{Q_T} \left(\frac{1}{2} U_k(u_\eta)\, \partial_t \phi + \left(S_N(t) F_k(u_\eta; \bar{f}) + T_N(t) F_k\left(u_\eta; \tilde{f}\right) \right) \partial_x \phi + \varepsilon T_N(t) F_k(u_\eta; A_\mu)\, \partial_x^2 \phi \right) dt\, dx \\
&\quad + \frac{1}{2} \int_I |u_0 - k|\, \phi(x,0)\, dx \geq \sum_{i=0}^{1} \int_0^T \operatorname{sign}(b_i - k) S_N(t) \left(\bar{f}(\gamma_i u_\eta) - \bar{f}(k)\right) \phi\Big|_{x=i} n_i\, dt \\
&\quad + \sum_{i=0}^{1} \int_0^T \operatorname{sign}(b_i - k) T_N(t) \left(\left(\tilde{f}(u_\eta) - \tilde{f}(k) - \varepsilon a_\mu(u_\eta) \partial_x u_\eta \right) \phi + \varepsilon \left(A_\mu(u_\eta) - A_\mu(k)\right) \partial_x \phi \right) \Big|_{x=i} n_i\, dt \\
&\quad \equiv I_\eta^1 + I_\eta^2,
\end{aligned} \tag{33}$$

where the (characteristic) functions $S_N(t)$ and $T_N(t)$ are defined in (25). In view of the dominated convergence theorem, we can pass to the limit as follows

$$\iint\limits_{Q_T} \left(\frac{1}{2}U_k(u_\eta)\,\partial_t\phi + \left(S_N(t)F_k\left(u_\eta;\bar{f}\right) + T_N(t)F_k\left(u_\eta;\tilde{f}\right)\right)\partial_x\phi + \varepsilon T_N(t)F_k\left(u_\eta;A_\mu\right)\partial_x^2\phi\right)\,dt\,dx$$
$$\xrightarrow{\eta\to 0}\frac{1}{2}\iint\limits_{Q_T}\left(|u-k|\partial_t\phi + \operatorname{sign}(u-k)(f(u)-f(k))\partial_x\phi + \varepsilon\operatorname{sign}(u-k)(A(u)-A(k))\partial_x^2\phi\right)\,dt\,dx.$$

Since obviously $\operatorname{sign}(u-k)\,(A(u)-A(k))$ is in $BV(Q_T)$ and $[\hat{a}(u)\partial_x u]$ is absolutely continuous, we can use Lemmas 4.1 (b), (c), 1.4, and 1.3, to conclude that

$$\frac{1}{2}\iint\limits_{Q_T}\varepsilon\operatorname{sign}(u-k)\,(A(u)-A(k))\,\partial_x^2\phi\,dt\,dx$$
$$=\frac{1}{2}\sum_{i=0}^{1}\int_0^T\varepsilon\operatorname{sign}(\gamma_i u-k)\,(A(\gamma_i u)-A(k))\,\partial_x\phi\Big|_{x=i}n_i\,dt-\frac{1}{2}\iint\limits_{Q_T}\varepsilon\operatorname{sign}(u-k)\,[\hat{a}(u)\partial_x u]\,\partial_x\phi\,dt\,dx.$$

Let us now consider the boundary term I_η^1 associated with the hyperbolic conservation law. In view of Lemma 4.1 (a) and (c), we get

$$\begin{aligned}
I_\eta^1 &= \sum_{i=0}^{1}\int_0^T\operatorname{sign}(b_i-k)\gamma_i\left(S_N(t)\left(\bar{f}(u_\eta)-\bar{f}(k)\right)\right)\phi\Big|_{x=i}n_i\,dt\\
&= \lim_{h\to 0}\sum_{i=0}^{1}\iint\limits_{Q_T}\operatorname{sign}(b_i-k)S_N(t)\left(\bar{f}(u_\eta)-\bar{f}(k)\right)\partial_x\left(\phi\lambda_h^i\right)n_i\,dt\,dx\\
&\xrightarrow{\eta\to 0}\lim_{h\to 0}\frac{1}{2}\sum_{i=0}^{1}\iint\limits_{Q_T}\operatorname{sign}(b_i-k)\left(\bar{f}(u)-\bar{f}(k)\right)\partial_x\left(\phi\lambda_h^i\right)n_i\,dt\,dx\\
&= \frac{1}{2}\sum_{i=0}^{1}\int_0^T\operatorname{sign}(b_i-k)\left(\bar{f}(\gamma_i u)-\bar{f}(k)\right)\phi\Big|_{x=i}n_i\,dt.
\end{aligned}$$

Consider now the term I_η^2 associated with the parabolic equation. Using the two relations $\phi\lambda_h^i\big|_{x=j}=\delta_{i,j}\phi\big|_{x=j}$ and $\partial_x\left(\phi\lambda_h^i\right)\big|_{x=j}=\delta_{i,j}\,\partial_x\phi\big|_{x=j}$, we can rewrite I_η^2 as follows

$$\begin{aligned}
I_\eta^2 &= \sum_{i=0}^{1}\iint\limits_{Q_T}\operatorname{sign}(b_i-k)T_N(t)\partial_x\left(\left(\tilde{f}(u_\eta)-\tilde{f}(k)-\varepsilon a_\mu(u_\eta)\partial_x u_\eta\right)\phi\lambda_h^i+\varepsilon(A_\mu(u_\eta)-A_\mu(k))\partial_x\left(\phi\lambda_h^i\right)\right)n_i\,dt\,dx\\
&= \sum_{i=0}^{1}\iint\limits_{Q_T}\operatorname{sign}(b_i-k)T_N(t)\left(\partial_x\left(\tilde{f}(u_\eta)-\tilde{f}(k)-\varepsilon a_\mu(u_\eta)\partial_x u_\eta\right)\phi\lambda_h^i+\left(\tilde{f}(u_\eta)-\tilde{f}(k)\right)\partial_x(\phi\lambda_h^i)\right)n_i\,dt\,dx\\
&\qquad+\sum_{i=0}^{1}\iint\limits_{Q_T}\varepsilon\operatorname{sign}(b_i-k)T_N(t)\left(A_\mu(u_\eta)-A_\mu(k)\right)\partial_x^2\left(\phi\lambda_h^i\right)n_i\,dt\,dx\\
&= -\sum_{i=0}^{1}\iint\limits_{Q_T}\operatorname{sign}(b_i-k)T_N(t)\partial_t u_\eta(\phi\lambda_h^i)n_i\,dt\,dx+\sum_{i=0}\iint\limits_{Q_T}\operatorname{sign}(b_i-k)T_N(t)\left(\tilde{f}(u_\eta)-\tilde{f}(k)\right)\partial_x(\phi\lambda_h^i)n_i\,dt\,dx\\
&\qquad+\sum_{i=0}^{1}\iint\limits_{Q_T}\varepsilon\operatorname{sign}(b_i-k)T_N(t)\left(A_\mu(u_\eta)-A_\mu(k)\right)\partial_x^2(\phi\lambda_h^i)n_i\,dt\,dx\equiv -J_{\eta,h}^1+J_{\eta,h}^2+J_{\eta,h}^3,
\end{aligned}$$

where we have also used the partial differential equation for $u_\eta(x,t)$ on $I \times \langle t_{n+1/2}, t_{n+1}]$, and moreover that $\partial_x\left(A_\mu(u_\eta) - A_\mu(k)\right) = a_\mu(u_\eta)\partial_x u_\eta$ which results in two terms canceling out. In the view of (32) and dominated convergence theorem, the limit $J^1_{\eta,h} \to 0$, as $h \to 0$ for each fixed $\eta > 0$, should be obvious. Furthermore, by letting $\eta \to 0$ and subsequently letting $h \to 0$, we obtain that

$$\begin{aligned} J^2_{\eta,h} &\xrightarrow{\eta\to 0} \frac{1}{2}\iint\limits_{Q_T} \operatorname{sign}(b_i - k)\left(\tilde f(u) - \tilde f(k)\right)\partial_x\left(\phi\lambda^i_h\right) n_i\, dt\, dx \\ &\xrightarrow{h\to 0} \frac{1}{2}\sum_{i=0}^{1}\int_0^T \operatorname{sign}(b_i - k)\left(\tilde f(\gamma_i u) - \tilde f(k)\right)\phi\Big|_{x=i} n_i\, dt, \end{aligned}$$

where have used Lemma 4.1 (a) and (c). Let us finally calculate the limit of $J^3_{\eta,h}$ as $\eta, h \to 0$ as follows

$$\begin{aligned} J^3_{\eta,h} &\xrightarrow{\eta\to 0} \frac{1}{2}\iint\limits_{Q_T} \varepsilon\operatorname{sign}(b_i - k)\left(A(u) - A(k)\right)\partial_x^2\left(\phi\lambda^i_h\right) n_i\, dt\, dx \\ &= \frac{1}{2}\sum_{i=0}^{1}\int_0^T \varepsilon\operatorname{sign}(b_i - k)\left(A(\gamma_i u) - A(k)\right)\partial_x\phi\Big|_{x=i} n_i\, dt - \frac{1}{2}\iint\limits_{Q_T} \varepsilon\operatorname{sign}(b_i - k)\left[\hat a(u)\partial_x u\right]\partial_x\left(\phi\lambda^i_h\right) n_i\, dt\, dx \\ &\xrightarrow{h\to 0} \frac{1}{2}\sum_{i=0}^{1}\int_0^T \varepsilon\operatorname{sign}(b_i - k)\left(A(\gamma_i u) - A(k)\right)\partial_x\phi\Big|_{x=i} n_i\, dt - \frac{1}{2}\int_0^T \varepsilon\operatorname{sign}(b_i - k)\gamma_i\left[\hat a(u)\partial_x u\right]\phi\Big|_{x=i} n_i\, dt, \end{aligned}$$

where we have once more used the dominated convergence theorem, Lemma 4.1 (b) and (c), the identity $\partial_x\left(\phi\lambda^i_h\right)\big|_{x=i} = \partial_x\phi\big|_{x=i}$, Lemma 1.3, and finally Lemma 4.1 (a) together with the fact that $[\hat a(u)\partial_x u]$ can be identified as an element in $L^1(Q_T)$ and, furthermore, is in $BV(\mathbb{R})$ (in space) for a.e. $t \in \langle 0, T]$. Summing up the recent calculations, we get that the limit $u(x,t)$ obeys (C2). Thus we have proven the following theorem.

Theorem 4.2. *Let $\{\eta = (\Delta t, \mu)\}$ be a sequence of discretization parameters tending to zero. Suppose that Δt and μ are chosen according to (15). Then the sequence $\{u_\eta(x,t)\}$ of splitting solutions converges to the unique entropy weak solution $u(x,t)$ of the initial-boundary value problem (29). The convergence takes place in $L^1(Q_T)$.*

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Appendix A.

Let $Q_T = I \times \langle 0,T]$, where $I = \langle 0,1\rangle$, and consider the non-degenerate initial-boundary value problem

$$\partial_t w + \partial_x g(w) = \varepsilon \partial_x \left[d(w)\partial_x w\right], \qquad w(x,0) = w_0(x), \qquad w(0,t) = b_0, \qquad w(1,t) = b_1. \qquad (x,t) \in Q_T, \tag{34}$$

where b_0 and b_1 are constants which are assumed to be consistent with $u_0(0)$ and $u_0(1)$, respectively. Under appropriate assumption on $g(w)$, $d(w)$, and $w_0(x)$, it is well known that (34) possesses a unique classical solution $w(x,t)$; see e.g. [54]. We wish to prove rigorously that $w(x,t)$ satisfies an (entropy) integral inequality.

Lemma A.1. *Let D denote the primitive of d; $D(w) = \int^w d(\xi)\, d\xi$. The smooth solution $w(x,t)$ of (34) satisfies the following integral inequality*

$$\begin{aligned}
&\iint_{Q_T} \left(|w-k|\partial_t\phi + \operatorname{sign}(w-k)(g(w)-g(k))\partial_x\phi + \operatorname{sign}(w-k)\left(D(w)-D(k)\right)\partial_x^2\phi\right)\, dt\, dx \\
&\qquad \geq \int_I |w(x,T)-k|\phi(x,T)\, dx - \int_I |w_0 - k|\phi(x,0)\, dx \\
&\qquad\quad + \sum_{i=0}^{1}\int_0^T \operatorname{sign}(b_i - k)\left(g(w)-g(k)-d(w)\partial_x w\right)\phi\Big|_{x=i} n_i\, dt \\
&\qquad\quad + \sum_{i=0}^{1}\int_0^T \operatorname{sign}(b_i-k)\left(D(w)-D(k)\right)\partial_x\phi\Big|_{x=i} n_i\, dt\, dx
\end{aligned} \tag{35}$$

for all non-negative functions $\phi \in C_0^\infty(Q_T)$ and all $k \in \mathbb{R}$. A similar integral inequality holds for the initial value problem.

Proof. To prove the integral inequality we introduce the smooth function sign_h, $\operatorname{sign}_h(\tau) = \tau/\sqrt{\tau^2+h}$, for $h > 0$. Note that $\operatorname{sign}_h(\tau) \to \operatorname{sign}(\tau)$ as $h \to 0$ for a.e. τ, and $\tau \operatorname{sign}_h'(\tau) \to 0$ as $h \to 0$ for a.e. τ and is, furthermore, uniformly bounded in h and τ. We multiply the obvious equation $\partial_t(w-k) + \partial_x(g(w)-g(k)) = \partial_x\left[d(w)\partial_x w\right]$

by $\mathrm{sign}_h(w-k)\phi$ and integrate over Q_T. Integration by parts then gives the following equality

$$
\begin{aligned}
&\int_I (w-k)\,\mathrm{sign}_h(w-k)\phi\Big|_{t=0}^{t=T}\,dx - \iint\limits_{Q_T} (w-k)\partial_t\left(\mathrm{sign}_h(w-k)\phi\right)\,dt\,dx\\
&+\int_0^T (g(w)-g(k))\,\mathrm{sign}_h(w-k)\phi\Big|_{x=0}^{x=1}\,dt - \iint\limits_{Q_T} (g(w)-g(k))\,\partial_x\left(\mathrm{sign}_h(w-k)\phi\right)\,dt\,dx\\
&\quad= \int_0^T d(w)\partial_x w\,\mathrm{sign}_h(w-k)\phi\Big|_{x=0}^{x=1}\,dt - \iint\limits_{Q_T} d(w)\partial_x w\partial_x\left(\mathrm{sign}_h(w-k)\phi\right)\,dt\,dx,
\end{aligned}
\tag{36}
$$

which we conveniently write as $I_h^1 - I_h^2 + I_h^3 - I_h^4 = I_h^5 - I_h^6$. The purpose now is to pass to the limit as $h\to 0$. Using the dominated convergence theorem and the properties of sign_h, we get for three of the terms that

$$
\begin{aligned}
&I_h^1 \xrightarrow{h\to 0} \int_I |w(x,T)-k|\phi(x,T)\,dx - \int_I |w_0-k|\phi(x,0)\,dx,\\
&I_h^5 - I_h^3 \xrightarrow{h\to 0} \sum_{i=0}^{1}\int_0^T \mathrm{sign}(b_i-k)\left(g(w)-g(k)-d(w)\partial_x w\right)\phi\Big|_{x=i} n_i\,dt.
\end{aligned}
$$

Exploiting the fact that $\tau\,\mathrm{sign}_h'(\tau)\to 0$ for a.e. τ, we get for the second term and (similarly) for the fourth term that

$$
\begin{aligned}
&I_h^2 = \iint\limits_{Q_T} (w-k)\,\mathrm{sign}_h'(w-k)\partial_t w\phi\,dt\,dx + \iint\limits_{Q_T} (w-k)\,\mathrm{sign}_h(w-k)\partial_t\phi\,dt\,dx \xrightarrow{h\to 0} \iint\limits_{Q_T} |w-k|\partial_t\phi\,dt\,dx,\\
&I_h^4 \xrightarrow{h\to 0} \iint\limits_{Q_T} \mathrm{sign}(w-k)\left(g(w)-g(k)\right)\partial_x\phi\,dt\,dx.
\end{aligned}
$$

Using the facts that ϕ, d, and sign_h' are non-negative, we get for the last term in (36) that

$$
\begin{aligned}
I_h^6 &= \iint\limits_{Q_T} d(w)\left(\partial_x w\right)^2\mathrm{sign}_h'(w-k)\phi\,dt\,dx + \iint\limits_{Q_T} d(w)\partial_x w\,\mathrm{sign}_h(w-k)\partial_x\phi\,dt\,dx\\
&\geq \iint\limits_{Q_T} d(w)\partial_x w\,\mathrm{sign}_h(w-k)\partial_x\phi\,dt\,dx = \iint\limits_{Q_T} \partial_x\left(D(w)-D(k)\right)\mathrm{sign}_h(w-k)\partial_x\phi\,dt\,dx\\
&= \sum_{i=0}^{1}\int_0^T \mathrm{sign}_h(b_i-k)\left(D(w)-D(k)\right)\partial_x\phi\Big|_{x=i} n_i\,dt - \iint\limits_{Q_T}\left(D(w)-D(k)\right)\partial_x\left(\mathrm{sign}_h(w-k)\partial_x\phi\right)\,dt\,dx\\
&\xrightarrow{h\to 0} \sum_{i=0}^{1}\int_0^T \mathrm{sign}(b_i-k)\left(D(w)-D(k)\right)\partial_x\phi\Big|_{x=i} n_i\,dt - \iint\limits_{Q_T} \mathrm{sign}(w-k)\left(D(w)-D(k)\right)\partial_x^2\phi\,dt\,dx,
\end{aligned}
$$

where the limit in the last line is similar to that of I_h^2 and I_h^4. This concludes the proof of (35) and thus the lemma. □

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